

The Wave-Like Properties of Particles

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1 De Broglie's Hypothesis

- After Einstein's theory, it was determined that light has dual particle-wave nature
- In 1924, Louis de Broglie proposes a hypothesis:
 - Any object moving with a momentum p is associated with a wave of wavelength λ , where:

$$\lambda = \frac{h}{p}$$

- λ refers to the “De Broglie” wavelength, h is the Planck constant, and p is the momentum
- For experimental measurement of the wave-like behavior of particles, the double and single-slit experiments were performed

2 Experimental Evidence for De Broglie Waves

- Particle Diffraction Experiment
 - For light of wavelength λ incident on a slit of width a , the diffraction pattern has a minimum at angles:

$$a \sin(\theta) = n\lambda, \quad n = 1, 2, 3, \dots$$

- Each of the atoms acts as a scatter
- The scattered electron waves interfere
- The crystal serves as a diffraction grating
- The maxima occurs at angle:

$$d \sin(\phi) = n\lambda$$

- Where λ is the de Broglie wavelength

2.1 Double-Slit Experiment

- Question: Through which slit does the particle pass?
- Result: No diffraction pattern on the screen
- if we check which slit the particle passes through:
 - Particle behavior is measured

- We can not observe its wave nature simultaneously! (Principle of complementarity)
- Conclusion:
 - The electron will behave as a wave or a particle

3 Heisenberg Uncertainty Relationships

- Applying the uncertainty relationship to de Broglie waves:

$$p = \frac{h}{\lambda} \Rightarrow dp = -\frac{h}{\lambda^2} d\lambda \Rightarrow \Delta p = \frac{h}{\lambda^2} \Delta \lambda$$

- Finally, this yields:

$$\Delta x \Delta p \approx \varepsilon h$$

- From quantum mechanics:

$$\Delta x \Delta p \geq \frac{h}{4\pi}$$

$$\varepsilon = \frac{1}{4\pi}$$

$$\Delta x \Delta p \geq \frac{1}{2} \hbar$$

- Where $\hbar = \frac{h}{2\pi}$
- When a coin is flipped, or a dice is rolled:
 - No way to predict a single flip/roll
 - But, we can predict the distribution of the results from a large # of flips or rolls
 - Quantum Theory allows for the same behavior
- Wave Function
 - What is the amplitude of the de Broglie wave?
 - Checking classical waves:
 - * Waves in the ocean: Height of water level
 - * Sound wave: Volume density of molecules

- * Light waves: \vec{E}, \vec{B} field
- * de Broglie waves: The probability of finding a particle at a given (x, t)
 - This is known as ψ , the wave function
 - In n -dimensional space, it becomes $\psi(x_1, x_2, \dots, x_n, t)$
 - In classical physics, the intensity (I) of any wave is proportional to $|A|$
 - For quantum mechanics, we have the probability of final particle $P \propto |\psi|^2$
- * The requirement for wave function ψ is that $|\psi|^2 \geq 0$
- * Any physical measurement is related to $P \propto |\psi|^2$
- * ψ are generally complex #'s
- * Properties of Complex Numbers
 - $\psi = Re(\psi) + iIm(\psi)$
 - The complex conjugate is: $\psi^* = Re(\psi) - iIm(\psi)$
- In the complex plane:
 - The phase factor is $\boxed{z = |z|e^{i\theta}}$
 - The wave function of a free particle is $Ae^{i(kx - \omega t)}$ or $A[\cos(kx - \omega t) + i\sin(kx - \omega t)]$
- Behavior of a wave function:
 - Reflection and transmission at a boundary
 - Penetration of the reflected wave
 - Continuity at the boundary
- The mathematical solution for wave functions
 - The wave function itself must be continuous
 - The slope of the wave function must be continuous (“boundary condition”)
 - By confining a particle by 2 boundaries, we’ve learned:
 - * It may be anywhere in space
 - * A definite/continuous valued λ, p, E
- In the left/right region:

$$U = qV = (-e)(-V_o) = eV_o$$

- A wave confined in a well with infinite-height barriers is known as a standing wave
- The two nodes ($A = 0$) are at two boundaries

- With this standing wave:

$$\lambda_n = \frac{2L}{n}, \quad n = 1, 2, 3 \dots$$

- Where n is the number of nodes, and L is the width of the well
- From de Broglie theory, the energy becomes:

$$E_n = n^2 \left(\frac{h^2}{8mL^2} \right)$$

- Where the expression in the parenthesis is E_o
- The n term makes it so that energy is quantized
- This is the general nature of quantum particles
- A particle confined in space \rightarrow energy is quantized
- Schrödinger's Equation:
 - The behavior of the wave function is controlled by a differential equation called "Schrödinger's equation"
 - The role is similar to Newton's 2nd law
 - It is a second order differential equation
 - Using a free particle, with A as the amplitude, to derive the equation (at a given time $t = t_o$), $\psi(x)$ is:

$$\psi(x) = A \sin(kx), \quad k = \frac{2\pi}{\lambda}$$

$$\frac{d\psi(x)}{dx} = Ak \cos(kx)$$

$$\frac{d^2\psi(x)}{dx^2} = -Ak^2 \sin(kx) = -k^2\psi(x)$$

- Using the kinetic energy, $K = \frac{p^2}{2m} = \left(\frac{\hbar}{\lambda} \right)^2 \frac{1}{2m}$

$$K = \frac{\hbar^2 k^2}{2m}$$

- Thus, we obtain:

$$-\frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} = K\psi(x)$$

- Using total energy:

$$\boxed{-\frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x)}$$

- This is the unidimensional, static¹ Schrödinger equation

- Probability density and normalization

- We defined $P(x) dx = |\psi(x)|^2 dx$
- $P(x) dx$ represents the probability density

$$\int_{x_1}^{x_2} P(x) dx \Rightarrow P(x_1 : x_2)$$

- Is the probability of finding the particle in range x_1 to x_2

$$\int_{-\infty}^{\infty} P(x) dx = 1 \quad \forall \text{ particles}$$

- This means:

$$\boxed{\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1}$$

- The average location of the particle is given by:

$$\boxed{\frac{\sum n_1 x_1 + n_2 x_2 + \dots + n_i x_i}{\sum n_1 + n_2 + \dots + n_i}}$$

- On a much smaller interval, we can use:

$$x_{avg} = \frac{\int_{-\infty}^{\infty} x P(x) dx}{\int_{-\infty}^{\infty} P(x) dx} = \int_{-\infty}^{\infty} x |\psi(x)|^2 dx$$

- For any function of x , the average is:

$$[f(x)]_{avg} = \int_{-\infty}^{\infty} f(x) |\psi(x)|^2 dx$$

- Given a case of constant potential energy:

$$\psi(x) = A \sin(kx) + B \cos(kx)$$

¹time independent

- Given a case where $E < U_o$ ²

$$\psi(x) = Ae^{-kx}$$

² e^{kx} has to be removed because it diverges