The Wave-Like Properties of Particles

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Contents

1 De Broglie's Hypothesis

- After Einstein's theory, it was determined that light has dual particle-wave nature
- In 1924, Louis de Broglie proposes a hypothesis:
  - Any object moving with a momentum p is associated with a wave of wavelength λ, where:

$$\boxed{\lambda = \frac{h}{p}}$$

  - λ refers to the "De Broglie" wavelength, h is the Planck constant, and p is the momentum
  - For experimental measurement of the wave–like behavior of particles, the double and single-slit experiments were performed

2 Experimental Evidence for De Broglie Waves

- Particle Diffraction Experiment
  - For light of wavelength λ incident on a slit of width a, the diffraction pattern has a minimum at angles:

$$\boxed{a\sin(\theta) = n\lambda, \qquad n = 1, 2, 3, \cdots}$$

  - Each of the atoms acts as a scatter
  - The scattered electron waves interfere
  - The crystal serves as a diffraction grating
  - The maxima occurs at angle:

$$\boxed{d\sin(\phi) = n\lambda}$$

  - Where λ is the de Broglie wavelength

2.1 Double-Slit Experiment

- Question: Through which slit does the particle pass?
- Result: No diffraction pattern on the screen
- if we check which slit the particle passes through:
  - Particle behavior is measured

- We can not observe its wave nature simultaneously! (Principle of complementarity)

- Conclusion:
  - The electron will behave as a wave or a particle

3 Heisenberg Uncertainty Relationships

- Applying the uncertainty relationship to de Broglie waves:

$$p = \frac{h}{\lambda} \Rightarrow dp = -\frac{h}{\lambda^2}\, d\lambda \Rightarrow \Delta p = \frac{h}{\lambda^2}\Delta\lambda$$

- Finally, this yields:

$$\Delta x \Delta p \approx \varepsilon h$$

- From quantum mechanics:

$$\Delta x \Delta p \geq \frac{h}{4\pi}$$

$$\varepsilon = \frac{1}{4\pi}$$

$$\Delta x \Delta p \geq \frac{1}{2}\hbar$$

- Where $\hbar = \dfrac{h}{2\pi}$
- When a coin is flipped, or a dice is rolled:
  - No way to predict a single flip/roll
  - But, we can predict the distribution of the results from a large # of flips or rolls
  - Quantum Theory allows for the same behavior

- Wave Function
  - What is the amplitude of the de Broglie wave?
  - Checking classical waves:
    * Waves in the ocean: Height of water level
    * Sound wave: Volume density of molecules

    * Light waves: $\overrightarrow{E}, \overrightarrow{B}$ field
    * de Broglie waves: The probability of finding a particle at a given (x, t)
        · This is known as ψ, the wave function
        · In n-dimensional space, it becomes $\psi(x_1, x_2, \cdots, x_n, t)$
        · In classical physics, the intensity (I) of any wave is proportional to $|A|$
        · For quantum mechanics, we have the probability of final particle $P \propto |\psi|^2$
    * The requirement for wave function ψ is that $|\psi|^2 \geq 0$
    * Any physical measurement is related to $P \propto |\psi|^2$
    * ψ are generally complex #'s
    * Properties of Complex Numbers
        · $\psi = Re(\psi) + iIm(\psi)$
        · The complex conjugate is: $\psi^* = Re(\psi) - iIm(\psi)$

- In the complex plane:
  - The phase factor is $\boxed{z = |z|e^{i\theta}}$
  - The wave function of a free particle is $Ae^{i(kx-\omega t)}$ or $A[\cos(kx - \omega t)+i]sin(kx-\omega t)]$

- Behavior of a wave function:
  - Reflection and transmission at a boundary
  - Penetration of the reflected wave
  - Continuity at the boundary

- The mathematical solution for wave functions
  - The wave function itself must be continuous
  - The slope of the wave function must be continuous ("boundary condition")
  - By confing a particle by 2 boundaries, we've learned:
    * It may be anywhere in space
    * A definite/continuous valued λ, p, E

- In the left/right region:

$$U = qV = (-e)(-V_o) = eV_o$$

- A wave confined in a well with infinite-height barriers is known as a standing wave

- The two nodes $(A = 0)$ are at two boundaries

- With this standing wave:

$$\lambda_n = \frac{2L}{n-1}, \qquad n = 1, 2, 3 \cdots$$

- Where n is the number of nodes, and L is the width of the well
- From de Broglie theory, the energy becomes:

$$E_n = n^2 \left(\frac{h^2}{8mL^2} \right)$$

- Where the expression in the parenthesis is E_o
- The n term makes it so that energy is quantized
- This is the general nature of quantum particles
- A particle confined in space $\rightarrow$ energy is quantized
- Schrödinger's Equation:
  - The behavior of the wave function is controlled by a differential eqution called "Schrödinger's equation"
  - The role is similar to Newton's 2^{nd} law
  - It is a second order differential equation
  - Using a free particle, with A as the amplitude, to derive the equation (at a given time $t = t_o$), $\psi(x)$ is:

$$\boxed{\psi(x) = A\sin(kx), \qquad k = \frac{2\pi}{\lambda}}$$

$$\boxed{\frac{d\psi(x)}{dx} = Ak\cos(kx)}$$

$$\boxed{\frac{d^2\psi(x)}{dx} = -Ak^2\sin(kx) = -k^2\psi(x)}$$

  - Using the kinetic energy, $K = \dfrac{p^2}{2m} = \left(\dfrac{\hbar}{\lambda}\right)^2 \dfrac{1}{2m}$

$$K = \frac{\hbar^2 k^2}{2m}$$

  - Thus, we obtain:

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = K\psi(x)}$$

- Using total energy:

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x)}$$

- This is the unidimensional, static[1] Schrödinger equation

- Probability density and normalization
  - We defined $P(x)\,dx = |\psi(x)|^2\,dx$
  - $P(x)\,dx$ represents the probability density

$$\int_{x_1}^{x_2} P(x)\,dx \Rightarrow P(x_1 : x_2)$$

  - Is the probability of finding the particle in range x_1 to x_2

$$\int_{-\infty}^{\infty} P(x)\,dx = 1 \;\forall \text{ particles}$$

  - This means:

$$\boxed{\int_{-\infty}^{\infty} |\psi(x)|^2\,dx = 1}$$

  - The average location of the particle is given by:

$$\boxed{\frac{\sum n_1x_1 + n_2x_2 + \cdots + n_ix_i}{\sum n_1 + n_2 + \cdots + n_i}}$$

  - On a much smaller interval, we can use:

$$x_{avg} = \frac{\displaystyle\int_{-\infty}^{\infty} xP(x)\,dx}{\displaystyle\int_{-\infty}^{\infty} P(x)\,dx} = \int_{-\infty}^{\infty} x|\psi(x)|^2\,dx$$

  - For any function of x, the average is:

$$[f(x)]_{avg} = \int_{-\infty}^{\infty} f(x)|\psi(x)|^2\,dx$$

- Given a case of constant potential energy:

$$\psi(x) = A\sin(kx) + B\cos(kx)$$

[1]time independent

- Given a case where $E < U_o$[2]

$$\psi(x) = Ae^{-kx}$$

- Light emission or absorption with quantum wave systems
  - By emitting energy in photons, the particle can move from a higher state to a lower state
  - By absorbing energy from photons, the particle can move from a lower to a higher state

[2] e^{kx} has to be removed because it diverges