Homework 7

Michael Brodskiy
Professor: Q. Yan

April 9, 2023

1 Symmetric Quantum Well

1. (a) If $|\psi(x)|^2$ is symmetric (as specified), then $\psi(x)$ is symmetric about the origin as well. We know that $\psi(-x) = \psi(x)$ or $\psi(-x) = -\psi(x)$. If neither of these cases is true, then $|\psi(x)|^2$ can not be symmetric. For example, if $\psi(-x) \neq \psi(x)$, then:

$$|\psi(x)| \neq |\psi(-x)|$$

This means:

$$|\psi(x)|^2 \neq |\psi(-x)|^2$$

which contradicts the symmetry of $|\psi(x)|^2$

Additionally, if $\psi(-x) \neq -\psi(x)$, the process becomes similar, as $-\psi(x)$ becomes positive due to the absolute value sign.

Thus, it is necessary that $\psi(-x) = \psi(x)$ or $\psi(-x) = -\psi(x)$, so that $|\psi(x)|^2$ maintains its symmetry.

(b) Assuming the solution to the function is $A\cos(kx) + B\sin(kx)$, and knowing that $U(x)$ is zero:

$$\frac{\partial^2\psi}{\partial x^2} = -Ak^2\cos(kx) - Bk^x\sin(kx)$$

$$\frac{\partial^2\psi}{\partial x^2} = -k^2\psi(x)$$

Thus, we get:

$$\frac{\hbar^2 k^2}{2m}\psi(x) = E\psi(x)$$

$$k = \frac{\sqrt{2mE}}{\hbar}$$

Given the above functions, we know that k must be:

$$k = \frac{n\pi}{L}, \qquad n = 1, 2, 3, \cdots$$

Using boundary conditions at $\frac{L}{2}$:

$$\psi\left(\frac{L}{2}\right) = A\cos\left(\frac{n\pi}{2}\right) + B\sin\left(\frac{n\pi}{2}\right)$$

When n is odd, this means $A = 0$, as cos goes to zero, and, due to boundary conditions, $B = 0$ as well; when n is even, this means $B = 0$ as sin goes to zero, and, due to boundary conditions, $A = 0$ as well. Now applying normalization:

When n is odd:

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} |B \sin\left(\frac{2\pi x}{L}\right)|^2 \, dx = 1 \Rightarrow b = \sqrt{\frac{L}{2}}$$

When n is even:

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} |A \cos\left(\frac{\pi x}{L}\right)|^2 \, dx = 1 \Rightarrow A = \sqrt{\frac{L}{2}}$$

Thus, the equation becomes:

$$\begin{cases} \sqrt{\frac{2}{L}} \cos\left(\frac{n\pi x}{L}\right), & \text{for } n \text{ odd} \\ \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), & \text{for } n \text{ even} \end{cases}$$

(c) Given the wave functions from (b) and applying them to the wave function:

$$\begin{cases} \sqrt{\frac{2}{L}} \cos\left(\frac{n\pi x}{L}\right) \to -\frac{n\pi}{L}\sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \to -\frac{n^2\pi^2}{L^2}\sqrt{\frac{2}{L}} \cos\left(\frac{n\pi x}{L}\right) \\ \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \to \frac{n\pi}{L}\sqrt{\frac{2}{L}} \cos\left(\frac{n\pi x}{L}\right) \to -\frac{n^2\pi^2}{L^2}\sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \end{cases}$$

Putting this into the wave function, we get:

$$-\frac{\hbar^2}{2m}\left(-\frac{n^2\pi^2}{L^2}\sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)\right) = (E - U(x))\sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

Canceling out common terms, we get:

$$E = \frac{\hbar^2 n^2 \pi^2}{2mL^2} + U(x)$$

Because E is constant and independent of x, and $U(x)$ is zero within the well, the possible energies become:

$$E = \frac{\hbar^2 n^2 \pi^2}{2mL^2}$$

2 Speed of Electron in Bohr Model

2. (a) We know that $\alpha = \dfrac{e^2}{4\pi\varepsilon_o \hbar c}$

Due to the principles of angular momentum, we obtain:

$$mvr = n\hbar$$

The radius of the n-th state is:

$$r_n = \frac{4\pi\varepsilon_o\hbar^2 n^2}{me^2}$$

Using the two formulas and substituting, we get:

$$v = \frac{1}{n}\left(\frac{e^2}{4\pi\varepsilon_o\hbar}\right)$$

Multiplying by c, we get:

$$v = \frac{1}{n}\left(\frac{e^2}{4\pi\varepsilon_o\hbar}\right)\frac{c}{c}$$

$$v = \frac{c}{n}\left(\frac{e^2}{4\pi\varepsilon_o\hbar c}\right)$$

And thus, v becomes:

$$v = \frac{\alpha c}{n}$$

(b) Given a hydrogen-like atom with charge Ze, the coulomb force is Z^2 times its original value. The radius, however, is decreased by a factor of Z, meaning that the velocity becomes:

$$v_Z = \frac{Z\alpha c}{n}$$

3 Ionized Helium

3. Ionized helium energy levels would look as follows:

The major difference between hydrogen and helium is the result of the two protons in the nucleus, which modify the energy levels of each quanta. The energy of each n-th orbit is given by:

$$E_n = -13.6\frac{z^2}{n^2}$$

Thus, the energies for the different levels, given $z = 2$ instead of hydrogen's $z = 1$, we obtain:

$$E_{1,2,3,4} = -54.4, -13.6, -6.04, -3.4[\text{eV}]$$

Using the wavelength for emission spectra formula:

$$\frac{1}{\lambda} = Rz^2\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$$

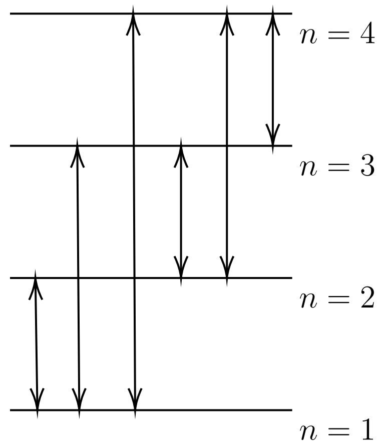

Figure 1: Helium Energy Transition Diagram

Using a calculator[1], the following wavelengths are obtained:

$$\lambda_{1\leftrightarrow 2} = 30.39[\text{nm}]$$
$$\lambda_{1\leftrightarrow 3} = 25.64[\text{nm}]$$
$$\lambda_{1\leftrightarrow 4} = 24.3[\text{nm}]$$
$$\lambda_{2\leftrightarrow 3} = 164.1[\text{nm}]$$
$$\lambda_{2\leftrightarrow 4} = 121.5[\text{nm}]$$
$$\lambda_{3\leftrightarrow 4} = 468.8[\text{nm}]$$

4 Optical Transition and Momentum Conservation

4. The momentum of a photon, given a difference in energy, is:

$$p = \frac{E_1 - E_2}{c}$$

Conservation of linear momentum means that the atom and photon will have equal but opposite momentums. Using the above momentum in the formula for energy, we get:

$$K_R = \frac{p^2}{2m} = \frac{(E_1 - E_2)^2}{2c^2 m}$$

Therefore, the provided formula is correct in describing the recoil energy:

[1]GNU Octave

$$K_R \cong \frac{(E_1 - E_2)^2}{2Mc^2}$$