Lecture 1

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- Linear Circuits
  - Circuits which may contain resistors, capacitors, inductors, independent voltage sources, and/or independent current sources
  - Voltage and current equations take the form of $v(t) = V\sin(\omega t)$ and $I(t) = I\cos(\omega t + \phi)$, respectively
  - $\omega = 2\pi f$, where f is 60[Hz] in the United States
  - All voltages and currents across such components must be sinusoids as well
    * ω stays the same, only the amplitudes (V or I) and phase shifts (ϕ) differ from component to component
    * To make a more compact representation of these values, we define a phasor, represented as:

$$V_{avg}(t) = V_{peak}\cos(\omega t + \phi) \to \hat{V} = V_{peak}\angle\phi$$

    * A phase shift with time difference τ has $\phi = \omega\tau$
      · Note, shifted to the left would mean $\phi > 0$, while shifted to the right would mean $\phi < 0$
    * Since operations with phasors often involve integration, we define a root-mean square (RMS) value of a phasor, which gives us:

$$\hat{V}_1 = \frac{V}{\sqrt{2}}\angle\phi$$

and

$$\hat{V}_2 = \frac{V}{\sqrt{2}}\angle - \phi$$

    * With $\hat{V}_1$ being a left-shifted phasor and $\hat{V}_2$ being a right-shifted phasor

- The RMS is defined with:

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T (V\sin(\omega t))^2\,dt} = \frac{1}{\sqrt{2}}V$$

- We can say $\hat{V}_1$ is leading $\hat{V}_2$ because it reaches its peak earlier (alternatively, it may be said that $\hat{V}_2$ is lagging $\hat{V}_1$)

- Phasor Diagram:

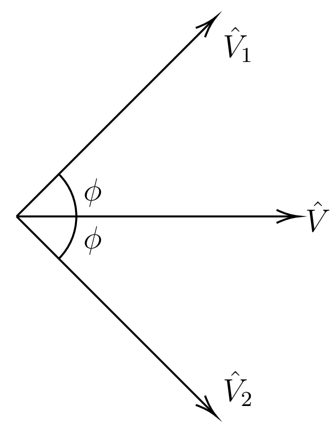

Figure 1: Sample Phasor Diagram

- Power
  - $p(t) = v(t)i(t)$ represents the instantaneous power dissipated (or consumed) by a component, where $v(t)$ is the voltage drop across a component, and $i(t)$ is the current flow through a component
  - Power follows the passive sign convention, which means that it is defined by the orientation of voltage drop and current flow
    - Current must enter the box at the positive voltage terminal to represent consumed power

- In a given circuit, let $v(t) = V_{max}\cos(\omega t + \theta_v)$ and $i(t) = I_{max}\cos(\omega t + \theta_I)$, the instantaneous power can be represented as:

$$p(t) = V_{max}I_{max}\cos(\omega t + \theta_v)\cos(\omega t + \theta_I)$$
$$p(t) = \frac{V_{max}I_{max}}{2}\left[\cos(\theta_v - \theta_I) + \cos(2\omega t + \theta_v + \theta_I)\right]$$

  - It can be observed that the average power may be expressed as:

$$p_{avg}(t) = \frac{1}{2}V_{max}I_{max}\cos(\theta_v - \theta_I)$$

- From here, we can notice that:

$$\hat{V} = \frac{V_{max}}{\sqrt{2}}\angle\phi_v \text{ and } \hat{I} = \frac{I_{max}}{\sqrt{2}}\angle\phi_I$$

- Thus, if we multiply the magnitudes of the two by cos of the angle difference, we get:

$$|\hat{V}||\hat{I}|\underbrace{\cos(\theta_v - \theta_I)}_{\text{power factor}} = p_{avg}$$

- This can be written as:

$$p_{avg} = \text{Re}\left\{\hat{V}_{rms}\hat{I}^*_{rms}\right\} = \frac{1}{2}V_{max}I_{max}\angle\theta_v - \theta_I$$

- The reactive power, Q, is defined as the imaginary part:

$$Q = \text{Im}\left\{\hat{V}_{rms}\hat{I}^*_{rms}\right\} = |\hat{V}_{rms}||\hat{I}_{rms}|\sin(\phi)$$

- The apparent (essentially overall) power, S may be written as:

$$\hat{S} = \hat{V}_{rms}\hat{I}_{rms}$$
$$|\hat{S}| = |\hat{V}_{rms}||\hat{I}_{rms}|$$

- The relationship between the three powers may be expressed as shown below:

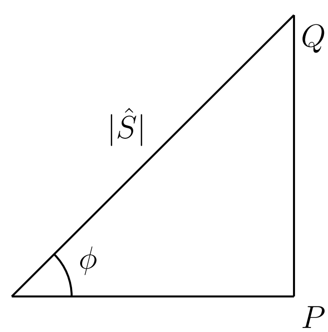

Figure 2: Power Relationship

- This lets us derive several equations[1]:

$$|S|^2 = P^2 + Q^2$$
$$|\hat{S}| = |\hat{V}||\hat{I}|$$
$$\hat{S} = |\hat{V}||\hat{I}|\cos(\phi) + j|\hat{V}||\hat{I}|\sin(\phi)$$

[1]Note: from this point, the RMS subscript is dropped, as it is implied that power relationships always deal with RMS

* Although all of these are power values, each generally has its own unit (a bit illogically):

P	Watts (W)
Q	Vars
S	Volt-Ampères (VA)

- Given a network with $\hat{V}$ and $\hat{I}$, then:
  - If $\hat{V}$ leads $\hat{I}$, then the power factor is lagging (current is behind the voltage), which results in:
    * $\phi > 0$
    * $P > 0$
    * $Q > 0$
    * Thus, real and reactive power are consumed
  - If $\hat{V}$ lags $\hat{I}$, then the power factor is leading (current is ahead of voltage), which results in:
    * $\phi < 0$
    * $P > 0$
    * $Q < 0$
    * Thus, real power is consumed, while reactive power is generated (or drawn)
- Given a box shown below, we can make some important derivations:

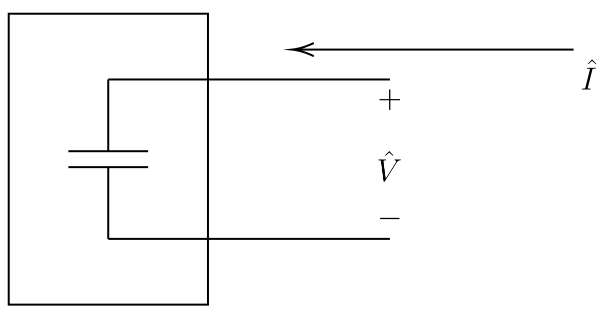

Figure 3: Capacitor Box

  - We know:

$$i(t) = C\frac{dV_c}{dt}$$

  - Given:

$$v_c(t) = V\cos(\omega t) \Rightarrow \hat{V}_c = V\angle 0°$$
$$i(t) = CV\omega(-\sin(\omega t) = -\omega CV\cos(\omega t - \pi/2) \Rightarrow \hat{I} = \omega CV\angle\pi/2 = j\omega cV$$

- The complex power can be defined as:

$$S = V\angle 0^\circ(-j\omega CV) = V^2\omega C(-j)$$

- As we know, capacitors do not consume real power, and, thus, there is only a reactive component. Thus, we say:

$$Q = -\omega CV^2$$

- We may conclude that capacitors are sources (suppliers) of reactive power (inductors would have the opposite reaction — they would only *consume* reactive power)

• Summary

  - In time domain:

$$v(t) = R \cdot i(t)$$
$$v_L(t) = L\frac{di}{dt}$$
$$i_C(t) = C\frac{dV_c}{dt}$$

  - In phasor domain:

$$\hat{V} = R \cdot \hat{I}$$
$$\hat{V}_L = j\omega L\hat{I}, \text{ current lags voltage}$$
$$\hat{I}_C = j\omega L\hat{V}, \text{ current leads voltage}$$

  - These may be used in combination
    * For example, given a resistor and inductor in series, we may write:

$$v(t) = i(t) \cdot R + L\frac{di}{dt}$$
$$\hat{V} = \hat{I}(R + j\omega L)$$

  - Instantaneous power is:

$$p(t) = v(t)i(t) = \frac{1}{2}V_{max}I_{max}\left[\cos(\theta_v - \theta_I) + \cos(2\omega t + \theta_v + \theta_I)\right]$$

    * Note the frequency of instantaneous power is twice that of voltage or current

$$p_{avg} = \frac{1}{2}V_{max}I_{max}\underbrace{\cos(\overbrace{\theta_v - \theta_I}^{\phi})}_{\text{power factor}}$$

– The units of all powers are:

$$P \to \text{Watts (W)}$$
$$Q \to \text{Vars}$$
$$S \to \text{Volt-Ampères (VA)}$$