Homework 2

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1. We may begin diagonalizing the matrix by writing:

$$|S_x - \lambda \mathbb{1}| = 0$$

We expand this to:

$$\left| \begin{pmatrix} -\lambda & \hbar/2 \\ \hbar/2 & -\lambda \end{pmatrix} \right| = 0$$

We may write the determinant of the matrix as:

$$\lambda^2 - \frac{\hbar^2}{4} = 0$$

This gives us the eigenvalues as:

$$\boxed{\lambda = \pm\frac{\hbar}{2}}$$

We can then find the eigenvectors as:

$$\begin{pmatrix} -\lambda & \hbar/2 \\ \hbar/2 & -\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

We matrix multiply to get:

$$\begin{pmatrix} -\lambda x_1 + (\hbar x_2)/2 \\ (\hbar x_1)/2 - \lambda x_2 \end{pmatrix} = 0$$

We can solve this system of equations to get:

$$x_1 = \frac{\hbar x_2}{2\lambda} \quad \text{and} \quad x_2 = \frac{\hbar x_1}{2\lambda}$$

Plugging in our eigenvalues from earlier, we find solution points to be:

$$x_1 = \pm x_2$$

Thus, since we want normalized eigenvectors, we may write this as:

$$\boxed{x_\pm = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}}$$

2. Similar to problem (1), we may begin by writing:

$$|A - \lambda \mathbb{1}| = 0$$

This gives us:

$$\left| \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & -(\lambda + .25) & .5 \\ 0 & .5 & -(\lambda + .25) \end{pmatrix} \right| = 0$$

We take the determinant to find:

$$(1-\lambda)\left([\lambda + .25]^2 - .25\right) = 0$$
$$(1-\lambda)\left(\lambda^2 + .5\lambda - .375\right) = 0$$

We see that one solution is $\lambda = 1$. We continue to solve for the other two by using the quadratic formula:

$$\lambda = \frac{-.5 \pm \sqrt{.25 - 4(1)(-.375)}}{2}$$
$$\lambda = \frac{-1 \pm 2\sqrt{1}}{4}$$
$$\lambda = 1/4, -3/4$$

Thus, we see the eigenvalues are:

$$\boxed{\lambda = 1, 1/4, -3/4}$$

We then continue to find the eigenvectors:

$$\begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & -(\lambda + .25) & .5 \\ 0 & .5 & -(\lambda + .25) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

Carrying out the matrix multiplication gives us:

$$\begin{pmatrix} (1-\lambda)x_1 \\ -x_2(\lambda+.25)+.5x_3 \\ .5x_2 - x_3(\lambda+.25) \end{pmatrix} = 0$$

Plugging in our solutions for λ, we see that the possible combinations of solutions become:

$$x_1 = 1 \quad \text{and} \quad x_2, x_3 = 0 \quad \text{OR} \quad x_1 = 0 \quad \text{and} \quad x_2 = \pm x_3$$

Thus, we can get normalized eigenvector solutions as:

$$\boxed{x_A = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}}$$

3. We know the matrix representations of each orientation to be:

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

We can thus write the given expression as:

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Let us remove the multiple of $\hbar^2/4$ for now to simplify calculations. This gives us:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Carrying out matrix multiplication, we get:

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$\begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}$$

We return the $\hbar$ factor to get:

$$S_xS_y - S_yS_x = \frac{i\hbar^2}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We know that S_z can be represented as:

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We multiply by $i\hbar$ to get:

$$i\hbar S_z = \frac{i\hbar^2}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We can then conclude:

$$\boxed{\therefore S_x S_y - S_y S_x = i\hbar S_z}$$