Lecture 6

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- Harmonic Oscillator
  - Classical

$$F = -kx$$
$$V(x) = \frac{1}{2}kx^2$$
$$F = -\frac{dV}{dx}$$

  - Quantum

$$E = \frac{p^2}{2m} + \frac{1}{2}kx^2$$
$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$
$$\omega = \sqrt{\frac{k}{m}}$$

    * Energy eigenvalues can be found as:

$$E_n = \hbar\omega(n + 1/2)$$

    * We may find:

$$H = \hbar\omega(aa^\dagger + 1/2)$$

    * And from here, we find:

$$[a, a^\dagger] = 1$$

      · This indicates that the operators a and $a^\dagger$ raise and lower the energy eigenstates

· We can write this as:

$$a|E\rangle \propto |E - \hbar\omega\rangle$$
$$a^\dagger |E\rangle \propto |E + \hbar\omega\rangle$$

· These are called "ladder operators"
· Note that there is an asymmetry in the ladder, since $aa^\dagger \neq a^\dagger a$
· Since there is a lowest energy state in the harmonic oscillator well, states can not be lowered in energy indefinitely, such that:

$$a|E_{lowest}\rangle = 0$$

· This is called the ladder termination condition

$$H|E_{lowest}\rangle = \hbar\omega(aa^\dagger + 1/2)|E_{lowest}\rangle = \frac{\hbar\omega}{2}|E_{lowest}\rangle$$

· Thus, we may conclude that the lowest energy is $\hbar\omega/2$
· Since this is finite, we say the quantum mechanical ground state has a zero-point energy of $\hbar\omega/2$

- Excited States
  - We may obtain:

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$$
$$\phi_n(x) = \frac{1}{\sqrt{n!}}\left[\sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{\hbar}{m\omega}\frac{d}{dx}\right)^n\right]\phi_0(x)$$

- Dirac Notation

$$|n\rangle = |\phi_n\rangle = |E_n\rangle = |(n+1/2)\hbar\omega\rangle$$

  - With $\phi_n(x) = \langle x|n\rangle$
  - Since $\langle n|n\rangle = 1$, and $\int_{-\infty}^{\infty} |x\rangle\langle x|\, dx = \mathbb{1}$

$$1 = \langle n|\int_{-\infty}^{\infty}|x\rangle\langle x|\, dx\,|n\rangle = \int_{-\infty}^{\infty}\phi_n^*(x)_n(x)\,dx$$

  - By orthonormality, we have:

$$\delta_{mn} = \langle m|n\rangle = \int_{-\infty}^{\infty}\phi_m^*(x)\phi_n(x)\,dx$$

- Since the Hermitian operator states are eigenstates of the Hamiltonian, they form a complete set of states, such that:

$$\sum_{n=0}^{\infty} |n\rangle \langle n| = \mathbb{1}$$

- A general state $|\psi\rangle$ can be written as:

$$|\psi\rangle = \sum_{n=0}^{\infty} (\underbrace{\langle n|\psi\rangle}_{c_n}) |n\rangle$$

- We know:

$$c_n = \int_{-\infty}^{\infty} \phi_n^*(x)\psi(x)\, dx$$

- The probability of the state $|\psi\rangle$ having energy E_n is:

$$P_{E_n} = |\langle n|\psi\rangle|^2 = |c_n|^2$$