Lecture 6

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- Harmonic Oscillator
  - Classical

$$F = -kx$$
$$V(x) = \frac{1}{2}kx^2$$
$$F = -\frac{dV}{dx}$$

  - Quantum

$$E = \frac{p^2}{2m} + \frac{1}{2}kx^2$$
$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$
$$\omega = \sqrt{\frac{k}{m}}$$

    * Energy eigenvalues can be found as:

$$E_n = \hbar\omega(n + 1/2)$$

    * We may find:

$$H = \hbar\omega(aa^\dagger + 1/2)$$

    * And from here, we find:

$$[a, a^\dagger] = 1$$

      · This indicates that the operators a and $a^\dagger$ raise and lower the energy eigenstates

    - We can write this as:

$$a\,|E\rangle \propto |E - \hbar\omega\rangle$$
$$a^\dagger\,|E\rangle \propto |E + \hbar\omega\rangle$$

    - These are called "ladder operators"
    - Note that there is an asymmetry in the ladder, since $aa^\dagger \neq a^\dagger a$
    - Since there is a lowest energy state in the harmonic oscillator well, states can not be lowered in energy indefinitely, such that:

$$a\,|E_{lowest}\rangle = 0$$

    - This is called the ladder termination condition

$$H\,|E_{lowest}\rangle = \hbar\omega(aa^\dagger + 1/2)\,|E_{lowest}\rangle = \frac{\hbar\omega}{2}\,|E_{lowest}\rangle$$

    - Thus, we may conclude that the lowest energy is $\hbar\omega/2$
    - Since this is finite, we say the quantum mechanical ground state has a zero-point energy of $\hbar\omega/2$

- Excited States
  - We may obtain:

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}\,|0\rangle$$
$$\phi_n(x) = \frac{1}{\sqrt{n!}}\left[\sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{\hbar}{m\omega}\frac{d}{dx}\right)^n\right]\phi_0(x)$$

- Dirac Notation

$$|n\rangle = |\phi_n\rangle = |E_n\rangle = |(n + 1/2)\hbar\omega\rangle$$

  - With $\phi_n(x) = \langle x|n\rangle$
  - Since $\langle n|n\rangle = 1$, and $\int_{-\infty}^{\infty} |x\rangle\langle x|\;dx = \mathbb{1}$

$$1 = \langle n|\int_{-\infty}^{\infty} |x\rangle\langle x|\;dx\,|n\rangle = \int_{-\infty}^{\infty} \phi_n^*(x)_n(x)\,dx$$

  - By orthonormality, we have:

$$\delta_{mn} = \langle m|n\rangle = \int_{-\infty}^{\infty} \phi_m^*(x)\phi_n(x)\,dx$$

- Since the Hermitian operator states are eigenstates of the Hamiltonian, they form a complete set of states, such that:

$$\sum_{n=0}^{\infty} |n\rangle \langle n| = \mathbb{1}$$

- A general state $|\psi\rangle$ can be written as:

$$|\psi\rangle = \sum_{n=0}^{\infty} \underbrace{(\langle n|\psi\rangle)}_{c_n} |n\rangle$$

- We know:

$$c_n = \int_{-\infty}^{\infty} \phi_n^*(x)\psi(x)\,dx$$

- The probability of the state $|\psi\rangle$ having energy E_n is:

$$P_{E_n} = |\langle n|\psi\rangle|^2 = |c_n|^2$$

- Quantum Hermitian Operator
  - We have define the following values:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$
$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right)$$
$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - i\frac{\hat{p}}{m\omega}\right)$$
$$H = \hbar\omega\,(n + 1/2)$$

  - We know that, for bosons:

$$[a, a^\dagger] = 1$$

  - Matrix Representation
    * First, we must choose a basis. The Hamiltonian is diagonal in its own basis:

$$H = \begin{pmatrix} \hbar\omega/2 & 0 & \cdots & 0 \\ 0 & 3\hbar\omega/2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \hbar\omega(n+1/2) \end{pmatrix}$$

* Since $c_n = \langle n|\psi\rangle$, we have:

$$|\psi\rangle = \begin{pmatrix} c_o \\ c_1 \\ \vdots \\ c_n \end{pmatrix}$$

* We can now find a matrix representation of the ladder operators:

$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle$$
$$a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

* We find:

$$a = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{2} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \sqrt{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{n} \end{pmatrix}$$

$$a^\dagger = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{2} & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \sqrt{n} \end{pmatrix}$$

* Furthermore, by definition of the Hermitian matrices, we find:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & \sqrt{2} & 0 & \cdots & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \cdots & 0 \\ 0 & 0 & \sqrt{3} & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \sqrt{n} \\ 0 & 0 & 0 & 0 & \sqrt{n} & 0 \end{pmatrix}$$

$$\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}} \begin{pmatrix} 0 & -1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & -\sqrt{2} & 0 & \cdots & 0 \\ 0 & \sqrt{2} & 0 & -\sqrt{3} & \cdots & 0 \\ 0 & 0 & \sqrt{3} & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & -\sqrt{n} \\ 0 & 0 & 0 & 0 & \sqrt{n} & 0 \end{pmatrix}$$

∗ This can be written formulaically as:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a)$$

$$\hat{p} = \sqrt{\frac{\hbar m\omega}{2}}(a^\dagger - a)$$