Homework 3

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1. (a) We know that, for a spin-1 system, the spin can be:

$$\boxed{S_z = \hbar, 0, -\hbar}$$

Given the quantum state in the problem, we can find the probabilities of each as:

$$P_1 = \left(\frac{1}{\sqrt{30}}\right)^2$$

$$P_0 = \left(\frac{2}{\sqrt{30}}\right)^2$$

$$P_{-1} = \left(\frac{|5i|}{\sqrt{30}}\right)^2$$

This gives us:

$$\boxed{\begin{cases} P_1 & = 1/30 \\ P_0 & = 2/15 \\ P_{-1} & = 5/6 \end{cases}}$$

We can calculate the expectation value of S_z by using:

$$\langle \psi | S_z | \psi \rangle$$

In matrix notation, this gives us:

$$< S_z > = \left[\frac{\hbar}{30}\right] \begin{pmatrix} 1 & 2 & -5i \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 5i \end{pmatrix}$$

We proceed to evaluate:

1

$$< S_z >= \left[\frac{\hbar}{30}\right] (1 \quad 0 \quad 5i) \begin{pmatrix} 1 \\ 2 \\ 5i \end{pmatrix}$$

$$< S_z >= \left[\frac{\hbar}{30}\right] (1 - (25))$$

$$\boxed{< S_z >= -\frac{24\hbar}{30} = -\frac{12\hbar}{15}}$$

(b) Similar to the expectation value of S_z in part (a), we apply the matrix notation of S_x to write:

$$< S_x >= \left[\frac{\hbar}{30\sqrt{2}}\right] (1 \quad 2 \quad -5i) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 5i \end{pmatrix}$$

$$< S_x >= \left[\frac{\hbar}{30\sqrt{2}}\right] (2 \quad 1-5i \quad 2) \begin{pmatrix} 1 \\ 2 \\ 5i \end{pmatrix}$$

$$< S_x >= \left[\frac{\hbar}{30\sqrt{2}}\right] (4 + 10i - 10i)$$

$$\boxed{< S_x >= \frac{4\hbar}{30\sqrt{2}} = \frac{2\hbar}{15\sqrt{2}}}$$

2. We may begin by converting $|\psi_f\rangle$ into the S_z basis to simply probability calculations. We know:

$$|1\rangle = \frac{1}{2}\left(|1\rangle + i\sqrt{2}|0\rangle - |-1\rangle\right)$$

$$|0\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |-1\rangle\right)$$

$$|-1\rangle = \frac{1}{2}\left(|1\rangle - i\sqrt{2}|0\rangle - |-1\rangle\right)$$

Applying this to the given state function, we may find:

$$|\psi_f\rangle = \frac{1+i}{2\sqrt{7}}\left[|1\rangle + i\sqrt{2}|0\rangle - |-1\rangle\right] + \frac{2}{\sqrt{14}}\left[|1\rangle + |-1\rangle\right] - \frac{i}{2\sqrt{7}}\left[|1\rangle - i\sqrt{2}|0\rangle - |-1\rangle\right]$$

We distribute and simplify:

$$|\psi_f\rangle = \left[\frac{1+i}{2\sqrt{7}} + \frac{2}{\sqrt{14}} - \frac{i}{2\sqrt{7}}\right]|1\rangle + \left[\frac{i-1}{\sqrt{14}} - \frac{1}{\sqrt{14}}\right]|0\rangle + \left[-\frac{1+i}{2\sqrt{7}} + \frac{2}{\sqrt{14}} + \frac{i}{2\sqrt{7}}\right]|-1\rangle$$

$$|\psi_f\rangle = \left[\frac{1}{2\sqrt{7}} + \frac{\sqrt{2}}{\sqrt{7}}\right]|1\rangle + \left[\frac{i-2}{\sqrt{14}}\right]|0\rangle + \left[-\frac{1}{2\sqrt{7}} + \frac{\sqrt{2}}{\sqrt{7}}\right]|-1\rangle$$

$$|\psi_f\rangle = \left[\frac{1+2\sqrt{2}}{2\sqrt{7}}\right]|1\rangle + \left[\frac{i-2}{\sqrt{14}}\right]|0\rangle + \left[\frac{2\sqrt{2}-1}{2\sqrt{7}}\right]|-1\rangle$$

We know that the probability of a normalized state (like ψ_i) can be expressed as:

$$P = |\langle\psi_f|\psi_i\rangle|^2$$

We take the complex conjugate to get:

$$\langle\psi_f| = \left[\frac{1+2\sqrt{2}}{2\sqrt{7}}\right]|1\rangle - \left[\frac{i+2}{\sqrt{14}}\right]|0\rangle + \left[\frac{2\sqrt{2}-1}{2\sqrt{7}}\right]|-1\rangle$$

We proceed to take the inner product of the two:

$$\langle\psi_f|\psi_i\rangle = \left[\frac{1+2\sqrt{2}}{2\sqrt{7}}\right]\left[\sqrt{\frac{1}{6}}\right] + \left[\frac{i+2}{\sqrt{14}}\right]\left[\sqrt{\frac{2}{6}}\right] + \left[\frac{2\sqrt{2}-1}{2\sqrt{7}}\right]\left[i\sqrt{\frac{3}{6}}\right]$$

We then simplify:

$$\langle\psi_f|\psi_i\rangle = \frac{1+2\sqrt{2}}{2\sqrt{42}} + \frac{i+2}{\sqrt{42}} + \left[\frac{1}{\sqrt{7}} - \frac{1}{2\sqrt{14}}\right]i$$

For ease of calculation, we may convert to decimal form:

$$\langle\psi_f|\psi_i\rangle = .604 + .3986i$$

We can then find the probability as:

$$|\langle\psi_f|\psi_i\rangle|^2 = .604^2 + .3986^2$$

$$\boxed{P = .5237}$$

3. (a) We know that $\hat{A}$ and $\hat{B}$ commute if $\hat{A}\hat{B} = \hat{B}\hat{A}$. Thus, we begin by calculating $\hat{A}\hat{B}$:

$$\hat{A}\hat{B} = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \begin{pmatrix} b_1 & 0 & 0 \\ 0 & 0 & b_2 \\ 0 & b_2 & 0 \end{pmatrix}$$

This gives us:

$$\hat{A}\hat{B} = \begin{pmatrix} a_1 b_1 & 0 & 0 \\ 0 & 0 & a_2 b_2 \\ 0 & a_3 b_2 & 0 \end{pmatrix}$$

3

Now, we calculate $\hat{B}\hat{A}$:

$$\hat{B}\hat{A} = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & 0 & b_2 \\ 0 & b_2 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$$

This gives us:

$$\hat{B}\hat{A} = \begin{pmatrix} b_1 a_1 & 0 & 0 \\ 0 & 0 & b_2 a_3 \\ 0 & b_2 a_2 & 0 \end{pmatrix}$$

We may see that $\hat{A}\hat{B} \neq \hat{B}\hat{A}$ and, therefore, the operators <u>do not commute</u>

(b) We begin working with $\hat{A}$:

$$|\hat{A} - \lambda \mathbb{1}| = 0$$

This gives us:

$$(a_1 - \lambda)(a_2 - \lambda)(a_3 - \lambda) = 0$$

Thus, we see that, since this is a diagonal matrix, we get $\boxed{\lambda = a_1, a_2, \text{ and } a_3}$ as the eigenvalues. Thus, we may observe that the normalized eigenvectors are:

$$\boxed{\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

We then proceed to do the same with $\hat{B}$:

$$\left| \begin{bmatrix} b_1 - \lambda & 0 & 0 \\ 0 & -\lambda & b_2 \\ 0 & b_2 & -\lambda \end{bmatrix} \right| = 0$$

This gives us:

$$(b_1 - \lambda)(\lambda^2 - b_2^2) = 0$$

We can thus see that the eigenvalues are:

$$\boxed{\lambda = b_1, \pm b_2}$$

From here, we can observe that the normalized eigenvectors become:

$$\boxed{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}}$$

(c) Let us assume that the basis eigenvectors are:

$$|1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad |2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, |3\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We calculate the measure of $\hat{B}$ by writing:

$$\hat{B}\,|2\rangle = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & 0 & b_2 \\ 0 & b_2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\hat{B}\,|2\rangle = \begin{pmatrix} 0 \\ 0 \\ b_2 \end{pmatrix}$$

Rewriting this in terms of the basis vectors of $\hat{B}$, we may find:

$$\hat{B}\,|2\rangle = \frac{b_2}{2} \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right)$$

$$\boxed{\hat{B}\,|2\rangle = \frac{b_2}{\sqrt{2}} \left(|B_2\rangle + |B_3\rangle \right)}$$

Since we may see that the two occur with equal probabilities, we may say that, for the operator $\hat{B}$, the probabilities are $P_{B_2} = P_{B_3} = .5$, while $P_{B_1} = 0$ (where B_1 corresponds to b_1, B_2 to b_2 and B_3 to $-b_2$ eigenvalue solutions)

On the other hand, we may find the probabilities of subsequently measuring a (after $|2\rangle$) to be:

$$P_{\pm b_2, a_1} = |\langle a_1 | B_2 \rangle|^2 \rightarrow |\langle 1| \frac{1}{\sqrt{2}} (|2\rangle \pm |3\rangle)|^2$$

$$P_{\pm b_2, a_2} = |\langle a_2 | B_2 \rangle|^2 \rightarrow |\langle 2| \frac{1}{\sqrt{2}} (|2\rangle \pm |3\rangle)|^2$$

$$P_{\pm b_2, a_3} = |\langle a_3 | B_2 \rangle|^2 \rightarrow |\langle 3| \frac{1}{\sqrt{2}} (|2\rangle \pm |3\rangle)|^2$$

Since we know that the inner product of the same basis vectors produces one and two different basis vectors produces zero due to orthonormality, we find:

$$\boxed{P_{\pm b_2, a_1} = 0}$$

$$P_{\pm b_2, a_2} = \left| \frac{\pm 1}{\sqrt{2}} \right|^2$$

$$P_{\pm b_2, a_3} = \left| \frac{\pm 1}{\sqrt{2}} \right|^2$$

And finally:

$$P_{\pm b_2, a_2} = \frac{1}{2}$$

$$P_{\pm b_2, a_3} = \frac{1}{2}$$

We may note that the sign on the $|3\rangle$ term is irrelevant, as we only care about the magnitude.

(d) Note that the result from part (a) indicates that we can not know the measurements of A and B at the same time with certainty. This is confirmed by the results from part (c), as we find that measuring $\hat{B}\,|2\rangle$ before $\hat{A}$ has no effect on the result of $\hat{A}$