Lecture 3

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- Hermitian Operators
  - So far, we have only considered operators acting on kets:

$$|\phi\rangle = A\,|\psi\rangle$$

  - If the operator acts on a bra it must act to the left:

$$\langle\epsilon| = \langle\psi|\,A$$

  - However, the bra, $\langle\epsilon|$, is <u>not</u> the bra that corresponds to the ket, $|\phi\rangle = A\,|\psi\rangle$
  - The bra $\langle\phi|$ is found by defining a new operator A^+ that obeys:

$$\langle\phi| = \langle\psi|\,A^+$$

    * A^+ is called the Hermitian adjoint of A. Consider the inner product:

$$\langle\phi|\beta\rangle = \langle\beta|\phi\rangle^*$$
$$\left\langle\psi|A^+|\beta\right\rangle = (\langle\beta|A|\psi\rangle)^*$$

    * This relates the matrix elements of A and A^+
    * Therefore, A^+ is found by transposing and complex conjugating the matrix representing A
  - An operator, A, is Hermitian if it is equal to its Hermitian adjoint, A^+
  - If an operator is Hermitian, then its bra, $\langle\psi|\,A$ is equal to the bra $\langle\phi|$ that corresponds to the ket $|\phi\rangle = A\,|\psi\rangle$
    * In quantum mechanics, all operators that correspond to physical observables are Hermitian
  - Hermitian matrices have real eigenvalues, which ensures results of measurements are always real-values

  - The eigenvectors of Hermitian matrices comprise a complete set of basis states, which ensures the eigenvectors of any observable are a valid basis

- Projection Operators
  - Recall for a spin-1/2 system we had the identity relation:

$$|+\rangle\langle+| + |-\rangle\langle-| = \mathbb{1}$$

  - We can express this in matrix notation as:

$$\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

  - This gives us the 2x2 identity matrix
  - The individual operators, $|+\rangle\langle+|$ and $|-\rangle\langle-|$, are called projection operators:

$$P_+ = |+\rangle\langle+| = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$$

$$P_- = |-\rangle\langle-| = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$$

  - Thus, for a general state, we may write $P_+ + P_- = \mathbb{1}$
  - From here, we may write:

$$P_+|\psi\rangle = |+\rangle\langle+|\psi\rangle = (\langle+|\psi\rangle)|+\rangle$$
$$P_-|\psi\rangle = |-\rangle\langle-|\psi\rangle = (\langle-|\psi\rangle)|-\rangle$$

  - The effect of the projection operator on a given state is to produce a new, normalized state

$$|\psi'\rangle = P_+|\psi\rangle$$

  - The projection postulate thus becomes:

$$|\psi'\rangle = \frac{P_+|\psi\rangle}{\sqrt{\langle\psi|P_+|\psi\rangle}} = |+\rangle$$

  - This indicates a “collapse” of the quantum state vector

- Measurement
  - In quantum mechanics, one must perform multiple identical measurements on identically prepared systems to infer the probabilities of outcomes

- For example, if one performs N measurements of the projections of $|\psi\rangle$ and obtains $+\hbar/2$ N_+ times, then:

$$\lim_{N\to\infty} \frac{N_+}{N} = |\, \langle + | \psi \rangle \,|^2$$

- It is useful to characterize statistical data sets by their mean and standard deviation

$$< S_z >= \frac{\hbar}{2} P_+ + \left(-\frac{\hbar}{2} \right) P_- = \langle \psi | S_z | \psi \rangle$$