Lecture 6

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- Harmonic Oscillator
  - Classical

$$F = -kx$$
$$V(x) = \frac{1}{2}kx^2$$
$$F = -\frac{dV}{dx}$$

  - Quantum

$$E = \frac{p^2}{2m} + \frac{1}{2}kx^2$$
$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$
$$\omega = \sqrt{\frac{k}{m}}$$

    * Energy eigenvalues can be found as:

$$E_n = \hbar\omega(n + 1/2)$$

    * We may find:

$$H = \hbar\omega(aa^\dagger + 1/2)$$

    * And from here, we find:

$$[a, a^\dagger] = 1$$

      · This indicates that the operators a and $a^\dagger$ raise and lower the energy eigenstates

- We can write this as:

$$a\,|E\rangle \propto |E - \hbar\omega\rangle$$
$$a^\dagger\,|E\rangle \propto |E + \hbar\omega\rangle$$

- These are called "ladder operators"
- Note that there is an asymmetry in the ladder, since $aa^\dagger \neq a^\dagger a$
- Since there is a lowest energy state in the harmonic oscillator well, states can not be lowered in energy indefinitely, such that:

$$a\,|E_{lowest}\rangle = 0$$

- This is called the ladder termination condition

$$H\,|E_{lowest}\rangle = \hbar\omega(aa^\dagger + 1/2)\,|E_{lowest}\rangle = \frac{\hbar\omega}{2}\,|E_{lowest}\rangle$$

- Thus, we may conclude that the lowest energy is $\hbar\omega/2$
- Since this is finite, we say the quantum mechanical ground state has a zero-point energy of $\hbar\omega/2$