Homework 8

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1. First and foremost, we have the wave function as:

$$\psi_{321}(r,\theta,\phi)=-\frac{\sqrt{3}}{27\sqrt{\pi}}\sqrt[3]{\frac{Z}{3a_o}}\left(\frac{Zr}{a_o}\right)^2 e^{-Zr/3a_o}\sin(\theta)\cos(\theta)e^{i\phi}$$

We can apply the differential form L_z to get:

$$L_z\psi_{321}(r,\theta,\phi)=-i\hbar\frac{\partial}{\partial\phi}\left[\frac{\sqrt{3}}{27\sqrt{\pi}}\sqrt[3]{\frac{Z}{3a_o}}\left(\frac{Zr}{a_o}\right)^2 e^{-Zr/3a_o}\sin(\theta)\cos(\theta)e^{i\phi}\right]$$

$$L_z\psi_{321}(r,\theta,\phi)=-i\hbar\frac{\sqrt{3}}{27\sqrt{\pi}}\sqrt[3]{\frac{Z}{3a_o}}\left(\frac{Zr}{a_o}\right)^2 e^{-Zr/3a_o}\sin(\theta)\cos(\theta)\frac{\partial}{\partial\phi}\left[e^{i\phi}\right]$$

$$L_z\psi_{321}(r,\theta,\phi)=\hbar\frac{\sqrt{3}}{27\sqrt{\pi}}\sqrt[3]{\frac{Z}{3a_o}}\left(\frac{Zr}{a_o}\right)^2 e^{-Zr/3a_o}\sin(\theta)\cos(\theta)e^{i\phi}$$

We may observe that this gives us $L_z\psi_{321}=\hbar\psi_{321}$, and, therefore, ψ_{321} is an eigenstate of L_z with eigenvalue $\hbar$ (as expected with $m=1$). We now proceed to check $\vec{L}^2$:

$$\vec{L}^2\psi_{321}(r,\theta,\phi)=-\hbar^2\left[\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2}\right]\psi_{321}(r,\theta,\phi)$$

We pull out r-dependent terms as constants (let us express these as $\mathbf{R}$), which gives us:

$$\vec{L}^2\psi_{321}(r,\theta,\phi)=-\hbar^2 R\left[\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left[\sin(\theta)\frac{\partial}{\partial\theta}\sin(\theta)\cos(\theta)\right]e^{i\phi}+\cot(\theta)\frac{\partial^2}{\partial\phi^2}e^{i\phi}\right]$$

$$\vec{L}^2\psi_{321}(r,\theta,\phi)=-\hbar^2 Re^{i\phi}\left[\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left[\sin(\theta)\cos^2(\theta)-\sin^3(\theta)\right]-\cot(\theta)\right]$$

$$\vec{L}^2\psi_{321}(r,\theta,\phi) = -\hbar^2 Re^{i\phi}\left[\frac{1}{\sin(\theta)}\left[\cos^3(\theta) - 5\cos(\theta)\sin^2(\theta)\right] - \cot(\theta)\right]$$

$$\vec{L}^2\psi_{321}(r,\theta,\phi) = -\hbar^2 Re^{i\phi}\left[\cos^2(\theta)\cot(\theta) - 5\cos(\theta)\sin(\theta) - \cot(\theta)\right]$$

Using trigonometric identities, we may simplify to:

$$\vec{L}^2\psi_{321}(r,\theta,\phi) = -\hbar^2 Re^{i\phi}\left[-6\cos(\theta)\sin(\theta)\right]$$

And finally:

$$\vec{L}^2\psi_{321}(r,\theta,\phi) = 6\hbar^2 Re^{i\phi}\left[\cos(\theta)\sin(\theta)\right]$$

We may see that this indicates that ψ_{321} is, indeed, an eigenfunction of $\vec{L}^2$, with eigenvalue $6\hbar^2$ (which would be expected for $l = 2$, since $[2(2+1)]$ gives us 6). Finally, we check the Hamiltonian, which can be expressed as:

$$H = -\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin(\theta)}\frac{\partial^2}{\partial\phi^2}\right)\right] + V(r)$$

Before we evaluate, we can simplify this to:

$$H = -\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\vec{L}^2}{r^2\hbar^2}\right] - \frac{Ze^2}{4\pi\varepsilon_o r}$$

We take $\mu \to m_e$ to give us:

2. (a)
   (b)
   (c)
   (d)