Homework 6

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1. First and foremost, we may begin by writing the function in terms of exponentials:

$$A\sin\left(\frac{p_o x}{\hbar}\right) = \frac{A}{2i}\left[e^{ip_o x/\hbar} - e^{-ip_o x/\hbar}\right]$$

From here, we may observe that this is a superposition of momentum eigenstates:

$$\frac{A}{2i}\left[e^{ip_o x/\hbar} - e^{-ip_o x/\hbar}\right] = \frac{1}{\sqrt{2}}\left[|p_o\rangle - |{-p_o}\rangle\right]$$

As such, we see that the wave function <u>is not an eigenstate of momentum</u>. Furthermore, we may observe that measuring momentum gives us either p_o or $-p_o$. We can then calculate the expectation value as:

$$\langle p\rangle = \langle\psi(x)|p|\psi(x)\rangle$$

We may expand to write:

$$\langle p\rangle = \frac{1}{2}\left[(\langle p_o| - \langle -p_o|)p(|p_o\rangle - |{-p_o}\rangle)\right]$$

We continue to evaluate to get:

$$\langle p\rangle = \frac{1}{2}\left[\langle p_o|p|p_o\rangle + \langle -p_o|p|{-p_o}\rangle\right]$$

$$\langle p\rangle = \frac{1}{2}\left[p_o + (-p_o)\right]$$

$$\boxed{\langle p\rangle = 0}$$

Now, we proceed to find the momentum probability density function, which we know may be expressed as:

$$\phi(p) = \langle p|\psi(x)\rangle$$

$$\phi(p) = \frac{1}{2\pi\hbar\sqrt{2}} \int_{-\infty}^{\infty} \left[e^{i(p_o-p)x/\hbar} - e^{-i(p_o+p)x/\hbar}\right] dx$$

Per our Fourier transform rules, we may see that this becomes:

$$\boxed{\phi(p) = \frac{1}{\sqrt{2}} \left[\delta(p-p_o) - \delta(p+p_o)\right]}$$

From here, we may find the probability density as:

$$P(p) = |\phi(p)|^2$$

$$\boxed{P(p) = \frac{1}{2} \left[\delta(p-p_o) + \delta(p+p_o)\right]}$$

We may compute the uncertainty by finding $\langle p^2\rangle$, which simply yields a similar result as the expectation value of p:

$$\langle p^2\rangle = \frac{1}{2}\left(p_o^2 + (-p_o)^2\right)$$

$$\boxed{\langle p^2\rangle = p_o^2}$$

From here, we find:

$$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2}$$

$$\Delta p = \sqrt{p_o^2}$$

$$\boxed{\Delta p = p_o}$$

2. We begin by normalizing the wave function:

$$\int_{-\infty}^{\infty} \psi^*(x)\psi(x)\, dx = 1$$

This gives us:

$$A^2 \int_{-\infty}^{\infty} e^{-x^2/2\alpha^2}\, dx = 1$$

Integrating gives us:

$$\int_{-\infty}^{\infty} e^{-x^2/2\alpha^2}\, dx = \alpha\sqrt{2\pi}$$

This the gives us:

$$A^2 = \frac{1}{\alpha\sqrt{2\pi}}$$

$$\boxed{A = \frac{1}{\sqrt{\alpha\sqrt{2\pi}}}}$$

The wave function then becomes:

$$\psi(x) = \frac{1}{\sqrt{\alpha\sqrt{2\pi}}} e^{(ip_o x/\hbar)-(x^2/4\alpha^2)}$$

(a) Now with a normalized function, we use the position representation to write:

$$\langle p \rangle = \int_{-\infty}^{\infty} \psi^*(x) \left(-i\hbar\frac{d}{dx}\right) \psi(x)\, dx$$

We expand this to get:

$$\langle p \rangle = \frac{1}{\alpha\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(ip_o x/\hbar)-(x^2/4\alpha^2)} \left(-i\hbar\frac{d}{dx}\right) e^{(ip_o x/\hbar)-(x^2/4\alpha^2)}\, dx$$

We continue to solve:

$$\langle p \rangle = \frac{-i\hbar}{\alpha\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{ip_o}{\hbar} - \frac{x}{2\alpha^2}\right] e^{-x^2/2\alpha^2}\, dx$$

$$\langle p \rangle = \frac{-i\hbar}{\alpha\sqrt{2\pi}} \left[\frac{ip_o\alpha}{\hbar}\sqrt{2\pi}\right]$$

$$\boxed{\langle p \rangle = p_o}$$

(b) We then use momentum representation by finding the momentum wave function:

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x) e^{-ipx/\hbar}\, dx$$

$$\phi(p) = \frac{1}{\sqrt{\sqrt{2\pi}\alpha}\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \left[e^{(ip_o x/\hbar)-(x^2/4\alpha^2)}\right] e^{-ipx/\hbar}\, dx$$

$$\phi(p) = \frac{1}{\sqrt{\sqrt{2\pi}\alpha}\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{(i(p_o-p)x/\hbar)-(x^2/4\alpha^2)}\, dx$$

This gives us:

$$\phi(p) = \frac{1}{\sqrt{\sqrt{2\pi}\alpha}\sqrt{2\pi\hbar}} 2\alpha\sqrt{\pi} e^{-\alpha^2(p-p_o)^2/\hbar^2}$$

$$\phi(p) = \left(\frac{2\alpha^2}{\pi\hbar^2}\right)^{1/2} e^{-\alpha^2(p-p_o)^2/\hbar^2}$$

We can then calculate the momentum expectation value as:

$$\langle p \rangle = \int_{-\infty}^{\infty} \phi^*(p) p \phi(p)\, dp$$

$$\langle p \rangle = \left(\frac{2\alpha^2}{\pi\hbar^2}\right)^{1/2} \int_{-\infty}^{\infty} p e^{-2\alpha^2(p-p_o)^2/\hbar^2}\, dp$$

Using u-substitution, we get:

$$\langle p \rangle = \left(\frac{2\alpha^2}{\pi\hbar^2}\right)^{1/2} \left[\frac{\hbar}{\alpha\sqrt{2}}(p_o\sqrt{\pi})\right]$$

$$\boxed{\langle p \rangle = p_o}$$

3. (a) When the energy of the particles is less than the height of the potential energy step, we have:

$$\phi_E(x) = \begin{cases} e^{ikx} + Be^{-ikx}, \\ x < 0 \\ Ce^{-qx}, \\ x > 0 \end{cases}$$

The boundary conditions give us:

$$\phi(0^-) = \phi(0^+) \Rightarrow A + B = C$$

$$\left.\frac{d\phi(x)}{dx}\right|_{x=0^-} = \left.\frac{d\phi(x)}{dx}\right|_{x=0^+} \Rightarrow ikA - ikB = -qC$$

We may solve for the ratio of A to B to find the reflection. We may do this by plugging the first condition equation into the second to get:

$$ikA - ikB = -q(A + B)$$

We proceed to solve for B/A to get:

$$ikA + qA = ikB - qB$$

$$\frac{B}{A} = \frac{ik + q}{ik - q}$$

The absolute squares of these gives us the reflection coefficient as:

$$R = \frac{|B|^2}{|A|^2}$$

$$R = \frac{k^2 + q^2}{k^2 + q^2}$$

$$\boxed{R = 1}$$

Thus, we see that the entire beam is reflected, and that no particles are transmitted to the detector.

(b) Similarly to part (a), we may find that:

$$\phi_E(x) = \begin{cases} e^{ik_1 x} + Be^{-ik_1 x}, \\ x < 0 \\ Ce^{ik_2 x}, \\ x > 0 \end{cases}$$

Imposing the boundary conditions gives us:

$$A + B = C$$
$$ik_1 A - ik_1 B = ik_2 C$$

Once again, we solve to get:

$$ik_1 A - ik_1 B = ik_2(A + B)$$
$$ik_1 A - ik_2 A = ik_1 B + ik_2 B$$
$$\frac{B}{A} = \frac{k_1 - k_2}{k_1 + k_2}$$

Once again, we square this to get:

$$R = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}$$
$$R = \frac{k_1^2 - 2k_1 k_2 + k_2^2}{k_1^2 + 2k_1 k_2 + k_2^2}$$

Since we know:

$$k_1 = \sqrt{\frac{2mE}{\hbar^2}} \quad \text{and} \quad k_2 = \sqrt{\frac{2m(E - V_o)}{\hbar^2}}$$

This gives us:

$$R = \frac{\sqrt{E}^2 - 2\sqrt{E}\sqrt{E - V_o} + \sqrt{E - V_o}^2}{\sqrt{E}^2 + 2\sqrt{E}\sqrt{E - V_o} + \sqrt{E - V_o}^2}$$

$$\boxed{R = \frac{2E - 2\sqrt{E^2 - EV_o} - V_o}{2E + 2\sqrt{E^2 - EV_o} - V_o}}$$

We may observe that the above coefficient is less than 100%, since the numerator has more being subtracted from $2E$ than the denominator.

(c) We may plot the reflection coefficient (which is unity until the boundary) as:

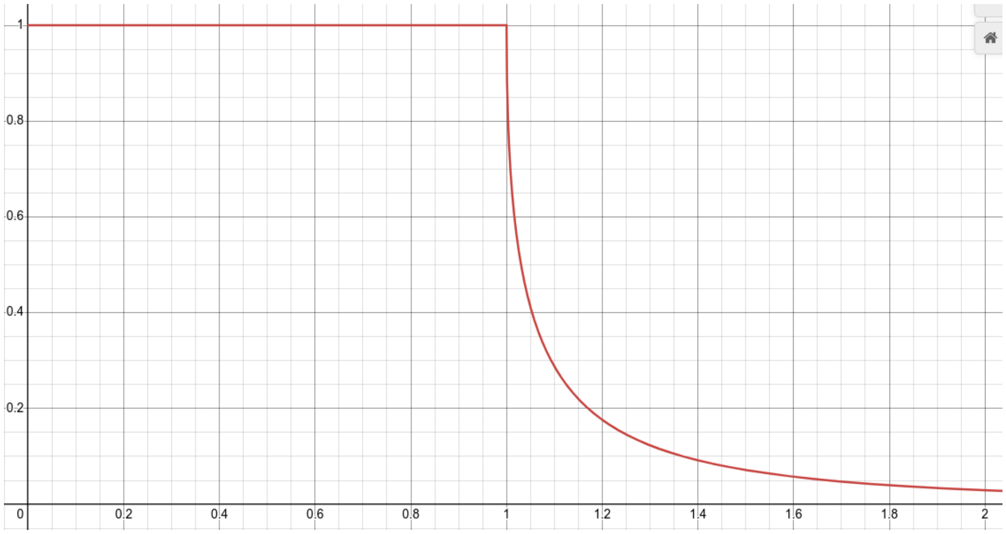

Figure 1: Plot of Reflection Coefficient as a Function of Incident Energy

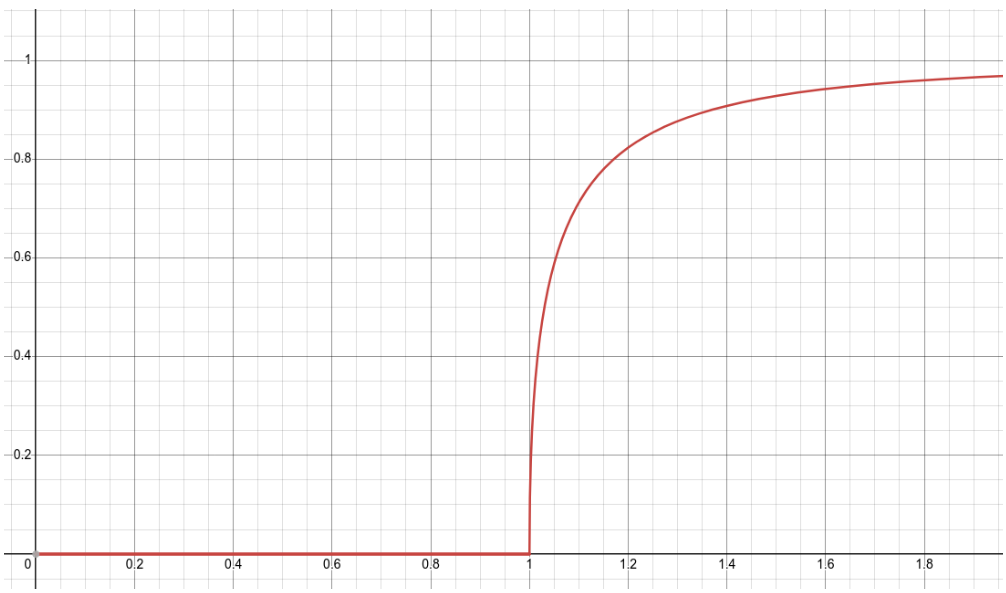

Figure 2: Plot of Transmission Coefficient as a Function of Incident Energy

We may see that the reflection is unity and transmission is zero until the boundary is crossed. Then, the reflection decreases in a manner inversely proportional to the incident energy.