

# Homework 3

Michael Brodskiy

Professor: G. Fiete

February 4, 2025

1. (a) We know that, for a spin-1 system, the spin can be:

$$\boxed{S_z = \hbar, 0, -\hbar}$$

Given the quantum state in the problem, we can find the probabilities of each as:

$$P_1 = \left( \frac{1}{\sqrt{30}} \right)^2$$

$$P_0 = \left( \frac{2}{\sqrt{30}} \right)^2$$

$$P_{-1} = \left( \frac{|5i|}{\sqrt{30}} \right)^2$$

This gives us:

$$\boxed{\begin{cases} P_1 &= 1/30 \\ P_0 &= 2/15 \\ P_{-1} &= 5/6 \end{cases}}$$

We can calculate the expectation value of  $S_z$  by using:

$$\langle \psi | S_z | \psi \rangle$$

In matrix notation, this gives us:

$$\langle S_z \rangle = \left[ \frac{\hbar}{30} \right] \begin{pmatrix} 1 & 2 & -5i \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 5i \end{pmatrix}$$

We proceed to evaluate:

$$\langle S_z \rangle = \left[ \frac{\hbar}{30} \right] \begin{pmatrix} 1 & 0 & 5i \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 5i \end{pmatrix}$$

$$\langle S_z \rangle = \left[ \frac{\hbar}{30} \right] (1 - (25))$$

$$\boxed{\langle S_z \rangle = -\frac{24\hbar}{30} = -\frac{12\hbar}{15}}$$

(b) Similar to the expectation value of  $S_z$  in part (a), we apply the matrix notation of  $S_x$  to write:

$$\langle S_x \rangle = \left[ \frac{\hbar}{30\sqrt{2}} \right] \begin{pmatrix} 1 & 2 & -5i \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 5i \end{pmatrix}$$

$$\langle S_x \rangle = \left[ \frac{\hbar}{30\sqrt{2}} \right] \begin{pmatrix} 2 & 1 - 5i & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 5i \end{pmatrix}$$

$$\langle S_x \rangle = \left[ \frac{\hbar}{30\sqrt{2}} \right] (4 + 10i - 10i)$$

$$\boxed{\langle S_x \rangle = \frac{4\hbar}{30\sqrt{2}} = \frac{2\hbar}{15\sqrt{2}}}$$

2. We may begin by converting  $|\psi_f\rangle$  into the  $S_z$  basis to simplify probability calculations. We know:

$$|1\rangle = \frac{1}{2} (|1\rangle + i\sqrt{2}|0\rangle - |-1\rangle)$$

$$|0\rangle = \frac{1}{\sqrt{2}} (|1\rangle + |-1\rangle)$$

$$|-1\rangle = \frac{1}{2} (|1\rangle - i\sqrt{2}|0\rangle - |-1\rangle)$$

Applying this to the given state function, we may find:

$$|\psi_f\rangle = \frac{1+i}{2\sqrt{7}} [|1\rangle + i\sqrt{2}|0\rangle - |-1\rangle] + \frac{2}{\sqrt{14}} [|1\rangle + |-1\rangle] - \frac{i}{2\sqrt{7}} [|1\rangle - i\sqrt{2}|0\rangle - |-1\rangle]$$

We distribute and simplify:

$$|\psi_f\rangle = \left[ \frac{1+i}{2\sqrt{7}} + \frac{2}{\sqrt{14}} - \frac{i}{2\sqrt{7}} \right] |1\rangle + \left[ \frac{i-1}{\sqrt{14}} - \frac{1}{\sqrt{14}} \right] |0\rangle + \left[ -\frac{1+i}{2\sqrt{7}} + \frac{2}{\sqrt{14}} + \frac{i}{2\sqrt{7}} \right] |-1\rangle$$

$$|\psi_f\rangle = \left[ \frac{1}{2\sqrt{7}} + \frac{\sqrt{2}}{\sqrt{7}} \right] |1\rangle + \left[ \frac{i-2}{\sqrt{14}} \right] |0\rangle + \left[ -\frac{1}{2\sqrt{7}} + \frac{\sqrt{2}}{\sqrt{7}} \right] |-1\rangle$$

$$|\psi_f\rangle = \left[ \frac{1+2\sqrt{2}}{2\sqrt{7}} \right] |1\rangle + \left[ \frac{i-2}{\sqrt{14}} \right] |0\rangle + \left[ \frac{2\sqrt{2}-1}{2\sqrt{7}} \right] |-1\rangle$$

We know that the probability of a normalized state (like  $\psi_i$ ) can be expressed as:

$$P = |\langle \psi_f | \psi_i \rangle|^2$$

We take the complex conjugate to get:

$$\langle \psi_f | = \left[ \frac{1+2\sqrt{2}}{2\sqrt{7}} \right] \langle 1| - \left[ \frac{i+2}{\sqrt{14}} \right] \langle 0| + \left[ \frac{2\sqrt{2}-1}{2\sqrt{7}} \right] \langle -1|$$

We proceed to take the inner product of the two:

$$\langle \psi_f | \psi_i \rangle = \left[ \frac{1+2\sqrt{2}}{2\sqrt{7}} \right] \left[ \sqrt{\frac{1}{6}} \right] + \left[ \frac{i+2}{\sqrt{14}} \right] \left[ \sqrt{\frac{2}{6}} \right] + \left[ \frac{2\sqrt{2}-1}{2\sqrt{7}} \right] \left[ i\sqrt{\frac{3}{6}} \right]$$

We then simplify:

$$\langle \psi_f | \psi_i \rangle = \frac{1+2\sqrt{2}}{2\sqrt{42}} + \frac{i+2}{\sqrt{42}} + \left[ \frac{1}{\sqrt{7}} - \frac{1}{2\sqrt{14}} \right] i$$

For ease of calculation, we may convert to decimal form:

$$\langle \psi_f | \psi_i \rangle = .604 + .3986i$$

We can then find the probability as:

$$|\langle \psi_f | \psi_i \rangle|^2 = .604^2 + .3986^2$$

$$\boxed{P = .5237}$$

3. (a) We know that  $\hat{A}$  and  $\hat{B}$  commute if  $\hat{A}\hat{B} = \hat{B}\hat{A}$ . Thus, we begin by calculating  $\hat{A}\hat{B}$ :

$$\hat{A}\hat{B} = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \begin{pmatrix} b_1 & 0 & 0 \\ 0 & 0 & b_2 \\ 0 & b_2 & 0 \end{pmatrix}$$

This gives us:

$$\hat{A}\hat{B} = \begin{pmatrix} a_1b_1 & 0 & 0 \\ 0 & 0 & a_2b_2 \\ 0 & a_3b_2 & 0 \end{pmatrix}$$

Now, we calculate  $\hat{B}\hat{A}$ :

$$\hat{B}\hat{A} = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & 0 & b_2 \\ 0 & b_2 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$$

This gives us:

$$\hat{B}\hat{A} = \begin{pmatrix} b_1 a_1 & 0 & 0 \\ 0 & 0 & b_2 a_3 \\ 0 & b_2 a_2 & 0 \end{pmatrix}$$

We may see that  $\hat{A}\hat{B} \neq \hat{B}\hat{A}$  and, therefore, the operators do not commute

(b) We begin working with  $\hat{A}$ :

$$|\hat{A} - \lambda \mathbb{1}| = 0$$

This gives us:

$$(a_1 - \lambda)(a_2 - \lambda)(a_3 - \lambda) = 0$$

Thus, we see that, since this is a diagonal matrix, we get  $\lambda = a_1, a_2, \text{ and } a_3$  as the eigenvalues. Thus, we may observe that the normalized eigenvectors are:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

We then proceed to do the same with  $\hat{B}$ :

$$\left| \begin{bmatrix} b_1 - \lambda & 0 & 0 \\ 0 & -\lambda & b_2 \\ 0 & b_2 & -\lambda \end{bmatrix} \right| = 0$$

This gives us:

$$(b_1 - \lambda)(\lambda^2 - b_2^2) = 0$$

We can thus see that the eigenvalues are:

$$\lambda = b_1, \pm b_2$$

From here, we can observe that the normalized eigenvectors become:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

(c) Let us assume that the basis eigenvectors are:

$$|1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad |2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad |3\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We calculate the measure of  $\hat{B}$  by writing:

$$\begin{aligned} \hat{B} |2\rangle &= \begin{pmatrix} b_1 & 0 & 0 \\ 0 & 0 & b_2 \\ 0 & b_2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ \hat{B} |2\rangle &= \begin{pmatrix} 0 \\ 0 \\ b_2 \end{pmatrix} \end{aligned}$$

Rewriting this in terms of the basis vectors of  $\hat{B}$ , we may find:

$$\begin{aligned} \hat{B} |2\rangle &= \frac{b_2}{2} \left( \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right) \\ \hat{B} |2\rangle &= \frac{b_2}{\sqrt{2}} (|B_2\rangle + |B_3\rangle) \end{aligned}$$

Since we may see that the two occur with equal probabilities, we may say that, for the operator  $\hat{B}$ , the probabilities are  $P_{B_2} = P_{B_3} = .5$

On the other hand, we may find  $a$  to be:

$$\begin{aligned} \hat{A} \begin{pmatrix} 0 \\ 0 \\ b_2 \end{pmatrix} &= \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ b_2 \end{pmatrix} \\ \hat{A} \begin{pmatrix} 0 \\ 0 \\ b_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ b_2 a_3 \end{pmatrix} \end{aligned}$$

We may see that this is equivalent to an integer multiple of the eigenvector  $|A_3\rangle$  of operator  $\hat{A}$ . Thus, we may find that:

$$\hat{A}\hat{B} |2\rangle = b_2 a_3 |A_3\rangle$$

And, therefore, the measurement produces  $a_3$  with a probability of 1

- (d) Note that the result from part (a) indicates that we can not know the measurements of  $A$  and  $B$  at the same time with certainty. This is confirmed by the results from part (c), as we find that measuring  $\hat{B} |2\rangle$  before  $\hat{A}$  has no effect on the result of  $\hat{A}$