Homework 1

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1. (a) We can normalize each state vector by multiplying the coefficients of each state vector by some constant c. This gives us:

   i.
   $$|a|^2 + |b|^2 = 1 \Longrightarrow (3c)^2 + (4c)^2 = 1$$

   This gives us:

   $$\begin{aligned} 25c^2 &= 1 \\ c^2 &= \frac{1}{25} \\ c &= \pm\frac{1}{5} \end{aligned}$$

   Thus, we may normalize the first quantum state vector by writing:

   $$\boxed{|\psi_1\rangle = \frac{3}{5}|+\rangle + \frac{4}{5}|-\rangle}$$

   ii. We continue with a similar approach to get:

   $$\begin{aligned} (c)^2 + (2c)^2 &= 1 \\ c^2 &= \frac{1}{5} \\ c &= \pm\frac{1}{\sqrt{5}} \end{aligned}$$

   This gives us a normalized state vector:

   $$\boxed{|\psi_2\rangle = \frac{1}{\sqrt{5}}|+\rangle + \frac{2i}{\sqrt{5}}|-\rangle}$$

iii. We continue with the same procedure. Note that the magnitude of the exponential is 1, such that:

$$(3c)^2 + (c)^2 = 1$$
$$c^2 = 10$$
$$c = \pm\sqrt{10}$$

This gives us:

$$\boxed{|\psi_3\rangle = \frac{3}{\sqrt{10}}|+\rangle + \frac{e^{\frac{\pi i}{3}}}{\sqrt{10}}|-\rangle}$$

(b) We may write the probability expression as: $P_{\pm} = |\langle\pm|\,|\psi\rangle\,|^2$

i. We use a from the normalized state vector. Since we know that $\langle+|\psi\rangle = a$ and $\langle-|\psi\rangle = b$, we may get:

$$P_+ = a^2$$
$$\boxed{P_+ = \frac{9}{25}}$$
$$P_- = b^2$$
$$\boxed{P_- = \frac{16}{25}}$$

We then need to find the probabilities in the x and y axis orientations. For the S_x orientation, we know:

$$S_x \Rightarrow \frac{1}{\sqrt{2}}(|+\rangle \pm |-\rangle)$$

This gives us:

$$P_{+x} = \left|\frac{1}{\sqrt{2}}(\langle+| + \langle-|)\frac{1}{5}(3\,|+\rangle + 4\,|-\rangle)\right|^2$$
$$P_{+x} = \frac{1}{50}\left|3\,\langle+|+\rangle + 4\,\langle-|-\rangle)\right|^2$$
$$\boxed{P_{+x} = \frac{49}{50}}$$

Consequently, we write:

$$\boxed{P_{-x} = 1 - \frac{49}{50} = \frac{1}{50}}$$

Finally, we know that the S_y orientation may be written as:

$$S_y = \frac{1}{\sqrt{2}}(\left|+\right\rangle \pm i\left|-\right\rangle)$$

This gives us:

$$P_{+y} = \Big|\frac{1}{\sqrt{2}}(\left\langle+\right| + i\left\langle-\right|)\frac{1}{5}(3\left|+\right\rangle + 4\left|-\right\rangle)\Big|^2$$

$$P_{+y} = \frac{1}{50}\Big|3\left\langle+|+\right\rangle + 4i\left\langle-|-\right\rangle\Big|^2$$

$$\boxed{P_{+y} = \frac{25}{50} = \frac{1}{2}}$$

And, consequently, we get:

$$\boxed{P_{-y} = 1 - \frac{25}{50} = \frac{1}{2}}$$

ii. Similarly, we take the coefficients to write:

$$P_{+} = a^2$$

$$\boxed{P_{+} = \frac{1}{5}}$$

$$P_{-} = b^2$$

$$\boxed{P_{-} = \frac{4}{5}}$$

We then check the S_x orientation to write:

$$P_{+x} = \Big|\frac{1}{\sqrt{2}}(\left\langle+\right| + \left\langle-\right|)\frac{1}{\sqrt{5}}(\left|+\right\rangle + 2i\left|-\right\rangle)\Big|^2$$

$$P_{+x} = \frac{1}{10}\Big|\left\langle+|+\right\rangle + 2i\left\langle-|-\right\rangle\Big|^2$$

$$\boxed{P_{+x} = \frac{5}{10} = \frac{1}{2}}$$

And, consequently:

$$\boxed{P_{-x} = 1 - \frac{5}{10} = \frac{1}{2}}$$

We then check the S_y orientation to get:

$$P_{+y} = \Big|\frac{1}{\sqrt{2}}(\left\langle+\right| + i\left\langle-\right|)\frac{1}{\sqrt{5}}(\left|+\right\rangle + 2i\left|-\right\rangle)\Big|^2$$

$$P_{+y} = \frac{1}{10}\Big|\left\langle+|+\right\rangle - 2\left\langle-|-\right\rangle\Big|^2$$

$$\boxed{P_{+y} = \frac{1}{10}}$$

And, consequently:

$$\boxed{P_{-y} = 1 - \frac{1}{10} = \frac{9}{10}}$$

iii. Finally, we find the last quantum state vector probabilities as:

$$P_{+} = a^2$$

$$\boxed{P_{+} = \frac{9}{10}}$$

$$P_{-} = b^2$$

$$\boxed{P_{-} = \frac{1}{10}}$$

We then check the S_x orientation:

$$P_{+x} = \left|\frac{1}{\sqrt{2}}(\langle +| + \langle -|)\frac{1}{\sqrt{10}}(3\,|+\rangle + e^{\frac{\pi i}{3}}\,|-\rangle)\right|^2$$

$$P_{+x} = \frac{1}{20}\left|3\,\langle +|+\rangle + e^{\frac{\pi i}{3}}\,\langle -|-\rangle\,\right|^2$$

We may convert to rectangular to get:

$$P_{+x} = \frac{1}{20}\left|3 + \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\,\right|^2$$

$$P_{+x} = \frac{1}{20}\left|\frac{7}{2} + \frac{\sqrt{3}}{2}i\right|^2$$

$$\boxed{P_{+x} = \frac{52}{80} = \frac{13}{20}}$$

Consequently, we write:

$$\boxed{P_{-x} = 1 - \frac{13}{20} = \frac{7}{20}}$$

Finally, we check the S_y orientation to get:

$$P_{+y} = \left|\frac{1}{\sqrt{2}}(\langle +| + i\,\langle -|)\frac{1}{\sqrt{10}}(3\,|+\rangle + e^{\frac{\pi i}{3}}\,|-\rangle)\right|^2$$

$$P_{+y} = \frac{1}{20}\left|(\langle +| + i\,\langle -|)\left(3\,|+\rangle + \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)|-\rangle\right)\,\right|^2$$

$$\boxed{P_{+y} = \frac{10 - 3\sqrt{3}}{20}}$$

Consequently, we get:

$$\boxed{P_{-y} = 1 - \frac{10 - 3\sqrt{3}}{20}} = \frac{10 + 3\sqrt{3}}{20}$$

(c) We can write the matrix forms of the up and down vectors as:

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad \text{and} \quad |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

i. Using the above, we may multiply to write:

$$|\psi_1\rangle = \frac{3}{5\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} + \frac{4}{5\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\boxed{|\psi_1\rangle = \begin{pmatrix}\dfrac{7}{5\sqrt{2}}\\[2ex] -\dfrac{1}{5\sqrt{2}}\end{pmatrix}}$$

ii. We use the same strategy to write:

$$|\psi_2\rangle = \frac{1}{\sqrt{5}\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} + \frac{2i}{\sqrt{5}\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\boxed{|\psi_2\rangle = \frac{1}{\sqrt{10}}\begin{pmatrix}1+2i\\1-2i\end{pmatrix}}$$

iii. For the final quantum state vector, let us rewrite in rectangular for easier understanding:

$$|\psi_3\rangle = \frac{3}{\sqrt{10}}|+\rangle + \left(\frac{1}{2\sqrt{10}} + \frac{\sqrt{3}i}{2\sqrt{10}}\right)|-\rangle$$

$$|\psi_3\rangle = \frac{3}{\sqrt{10}\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} + \left(\frac{1}{2\sqrt{10}} + \frac{\sqrt{3}i}{2\sqrt{10}}\right)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$|\psi_3\rangle = \frac{3}{\sqrt{20}}\begin{pmatrix}1\\1\end{pmatrix} + \left(\frac{1}{2\sqrt{20}} + \frac{\sqrt{3}i}{2\sqrt{20}}\right)\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\boxed{|\psi_3\rangle = \frac{1}{2\sqrt{20}}\begin{pmatrix}7+\sqrt{3}i\\5-\sqrt{3}i\end{pmatrix}}$$

(d) To find the probabilities in matrix form, we may multiply the matrix form of the quantum state vector with the respective 'up' or 'down' matrix form, and square the result

i. Using this, we may write:

$$P_+ = \left[\frac{1}{\sqrt{2}}\begin{pmatrix}\frac{7}{5\sqrt{2}} & -\frac{1}{5\sqrt{2}}\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\right]^2$$

$$P_+ = \left[\frac{7}{10} - \frac{1}{10}\right]^2$$

$$\boxed{P_+ = \frac{36}{100} = \frac{9}{25}}$$

$$P_- = \left[\frac{1}{\sqrt{2}}\begin{pmatrix}\frac{7}{5\sqrt{2}} & -\frac{1}{5\sqrt{2}}\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}\right]^2$$

$$P_- = \left[\frac{7}{10} + \frac{1}{10}\right]^2$$

$$\boxed{P_- = \frac{64}{100} = \frac{16}{25}}$$

We then use the S_x orientation to write:

$$P_{+x} = \left[\begin{pmatrix}\frac{7}{5\sqrt{2}} & -\frac{1}{5\sqrt{2}}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right]^2$$

$$\boxed{P_{+x} = \frac{49}{50}}$$

And, consequently, we get:

$$\boxed{P_{-x} = \frac{1}{50}}$$

Then we check the S_y orientation:

$$P_{+y} = \left[\frac{1}{\sqrt{2}}\begin{pmatrix}\frac{7}{5\sqrt{2}} & -\frac{1}{5\sqrt{2}}\end{pmatrix}\begin{pmatrix}1\\i\end{pmatrix}\right]^2$$

$$P_{+y} = \frac{1}{100}\left[\begin{pmatrix}7 & -1\end{pmatrix}\begin{pmatrix}1\\i\end{pmatrix}\right]^2$$

$$\boxed{P_{+y} = \frac{50}{100} = \frac{1}{2}}$$

Consequently, we get:

$$\boxed{P_{-y} = 1 - \frac{50}{100} = \frac{1}{2}}$$

ii. We repeat the same for the next quantum state function:

$$P_+ = \left[\frac{1}{\sqrt{20}}\begin{pmatrix}1+2i & 1-2i\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\right]^2$$

$$P_+ = \left[\frac{1}{\sqrt{20}} + \frac{1}{\sqrt{20}}\right]^2$$

$$\boxed{P_+ = \frac{4}{20} = \frac{1}{5}}$$

$$P_- = \left[\frac{1}{\sqrt{20}}\begin{pmatrix}1+2i & 1-2i\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}\right]^2$$

$$P_- = \left[\frac{|4i|}{\sqrt{20}}\right]^2$$

$$\boxed{P_- = \frac{16}{20} = \frac{4}{5}}$$

We then check the S_x direction:

$$P_{+x} = \frac{1}{10}\left[\begin{pmatrix}1+2i & 1-2i\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right]^2$$

$$\boxed{P_{+x} = \frac{5}{10} = \frac{1}{2}}$$

Consequently, we may find:

$$\boxed{P_{-x} = 1 - \frac{5}{10} = \frac{1}{2}}$$

Finally, we find the S_y orientation probability:

$$P_{+y} = \frac{1}{20}\left[\begin{pmatrix}1+2i & 1-2i\end{pmatrix}\begin{pmatrix}1\\i\end{pmatrix}\right]^2$$

$$P_{+y} = \frac{1}{20}\left[3+3i\right]^2$$

$$\boxed{P_{+y} = \frac{18}{20} = \frac{9}{10}}$$

Consequently, we get:

$$\boxed{P_{-y} = 1 - \frac{18}{20} = \frac{1}{10}}$$

iii.

$$P_+ = \left[\frac{1}{2\sqrt{40}}\begin{pmatrix}7+\sqrt{3}i & 5-\sqrt{3}i\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\right]^2$$

$$P_+ = \left[\frac{7}{2\sqrt{40}} + \frac{5}{2\sqrt{40}}\right]^2$$

$$\boxed{P_+ = \frac{36}{40} = \frac{9}{10}}$$

$$P_+ = \left[\frac{1}{2\sqrt{40}}\begin{pmatrix}7+\sqrt{3}i & 5-\sqrt{3}i\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}\right]^2$$

$$P_- = \left[\frac{|2+2\sqrt{3}i|}{\sqrt{40}}\right]^2$$

$$\boxed{P_- = \frac{4}{40} = \frac{1}{10}}$$

We then check the S_x orientation probability:

$$P_{+x} = \frac{1}{80}\left[\begin{pmatrix}7+\sqrt{3}i & 5-\sqrt{3}i\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right]^2$$

$$\boxed{P_{+x} = \frac{52}{80} = \frac{13}{20}}$$

Consequently, we find:

$$\boxed{P_{-x} = 1 - \frac{52}{80} = \frac{7}{20}}$$

We then find the S_y orientation:

$$P_{+y} = \left[\frac{1}{2\sqrt{40}}\begin{pmatrix}7+\sqrt{3}i & 5-\sqrt{3}i\end{pmatrix}\begin{pmatrix}1\\i\end{pmatrix}\right]^2$$

$$P_{+y} = \frac{1}{160}\left[\begin{pmatrix}7+\sqrt{3}i & 5-\sqrt{3}i\end{pmatrix}\begin{pmatrix}1\\i\end{pmatrix}\right]^2$$

$$P_{+y} = \frac{1}{160}\left[\left(7+\sqrt{3}i+5i+\sqrt{3}\right)\right]^2$$

$$P_{+y} = \frac{1}{160}\left[(7+\sqrt{3})+(5+\sqrt{3})i\right]^2$$

$$\boxed{P_{+y} = \frac{80+24\sqrt{3}}{160} = \frac{10+3\sqrt{3}}{20}}$$

Consequently, we find:

$$\boxed{P_{+y} = 1 - \frac{80 + 24\sqrt{3}}{160} = \frac{10 - 3\sqrt{3}}{20}}$$

We may observe that whether bra-ket notation or matrix form is used, the probability remains the same.

2. (a) We know that the probability in the S_z orientation for such a quantum state function is simply the squares of the coefficients. This gives us:

$$P_+ = a^2$$

$$\boxed{P_+ = \left(\frac{2}{\sqrt{13}}\right)^2 = \frac{4}{13}}$$

Consequently, we may find:

$$P_- = b^2$$

$$\boxed{P_- = \left(\frac{3}{\sqrt{13}}\right)^2 = \frac{9}{13}}$$

(b) We may find the S_x orientation by using matrix form. This will give us:

$$|\psi\rangle = \frac{1}{\sqrt{13}}\begin{pmatrix} 2 \\ 3i \end{pmatrix}$$

We then multiply by the S_x matrix:

$$|\pm\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$$

This gives us:

$$P_{+x} = \frac{1}{26}\left|\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3i \end{pmatrix}\right|^2$$

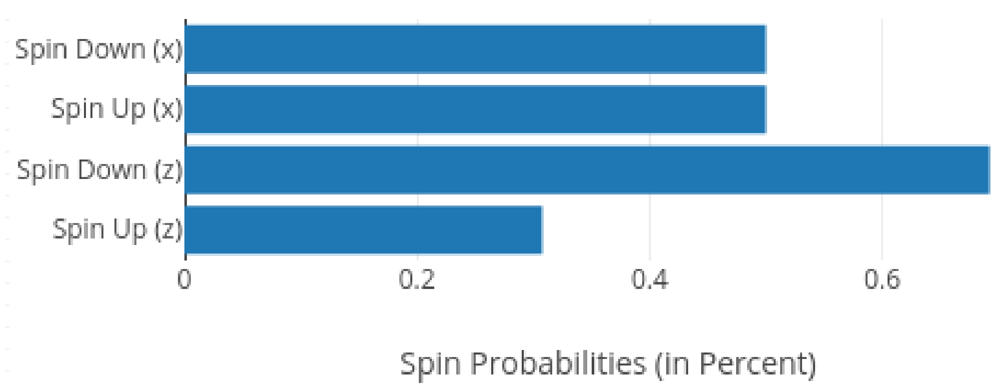
$$P_{+x} = \frac{1}{26}\left|2 + 3i\right|^2$$

$$\boxed{P_{+x} = \frac{13}{26} = \frac{1}{2}}$$

Consequently, we find:

$$\boxed{P_{-x} = 1 - \frac{13}{26} = \frac{1}{2}}$$

(c) The probabilities may be plotted as follows:



3. We may observe that the results from (a) and (b) in part 2 are swapped in part (3)

(a) Given that the orientation is towards the x axis, we may write the state function in matrix as:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[\frac{2}{\sqrt{13}}\begin{pmatrix}1\\1\end{pmatrix} + \frac{3i}{\sqrt{13}}\begin{pmatrix}1\\-1\end{pmatrix}\right]$$

$$|\psi\rangle = \frac{1}{\sqrt{26}}\begin{pmatrix}2+3i\\2-3i\end{pmatrix}$$

We then multiply by the S_z orientation to get:

$$P_+ = \frac{1}{26}\left|\begin{pmatrix}2+3i\\2-3i\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix}\right|^2$$

$$P_+ = \frac{1}{26}\left|2+3i\right|^2$$

$$\boxed{P_+ = \frac{13}{26} = \frac{1}{2}}$$

Consequently, we may find:

$$\boxed{P_- = 1 - \frac{13}{26} = \frac{1}{2}}$$

(b) We then use a similar process to find:

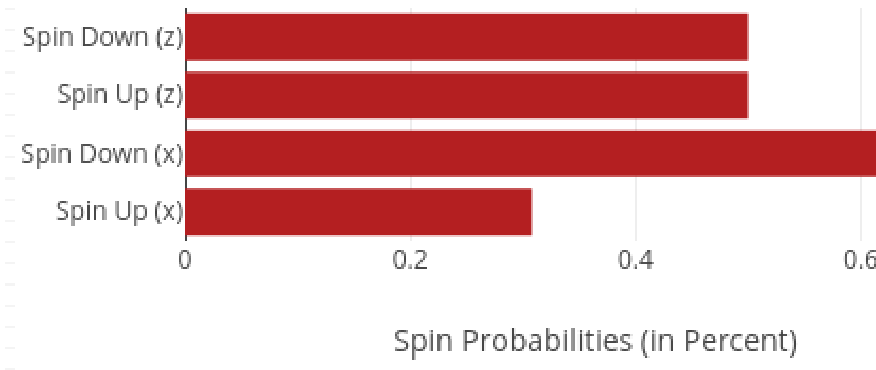
$$P_{+x} = \frac{1}{52}\left|\begin{pmatrix}2+3i\\2-3i\end{pmatrix}\begin{pmatrix}1 & 1\end{pmatrix}\right|^2$$

$$\boxed{P_{+x} = \frac{16}{52} = \frac{4}{13}}$$

And, consequently:

$$\boxed{P_{-x} = 1 - \frac{16}{52} = \frac{9}{13}}$$

(c) We may then re-plot to get:



4. (a) First and foremost, we may observe that the S_z probabilities are identical for each of the given quantum states. These probabilities are:

$$\boxed{P_+ = \left(\frac{4}{5}\right)^2 = \frac{16}{25}}$$

$$\boxed{P_- = \left(\frac{3}{5}\right)^2 = \frac{9}{25}}$$

We may then rewrite each state in terms of matrix form to get:

$$|\psi_1\rangle \frac{1}{5}\begin{pmatrix} 4 \\ 3i \end{pmatrix}$$

$$|\psi_2\rangle \frac{1}{5}\begin{pmatrix} 4 \\ -3i \end{pmatrix}$$

$$|\psi_1\rangle \frac{1}{5}\begin{pmatrix} -4 \\ 3i \end{pmatrix}$$

We can then calculate probabilities for the S_x and S_y orientations. Continuing, we check each wave function:

i. Function 1:
We begin with S_x:

$$P_{+x} = \frac{1}{50}\left|\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 3i \end{pmatrix}\right|^2$$

$$P_{+x} = \frac{1}{50}\Big|(4+3i)\Big|^2$$

$$\boxed{P_{+x} = \frac{1}{2}}$$

Consequently, we write:

$$\boxed{P_{-x} = \frac{1}{2}}$$

We then check the y orientation, which gives us:

$$P_{+y} = \frac{1}{50}\Big|\begin{pmatrix}1 & i\end{pmatrix}\begin{pmatrix}4 \\ 3i\end{pmatrix}\Big|^2$$

$$\boxed{P_{+y} = \frac{1}{50}}$$

Consequently, we may get:

$$\boxed{P_{-y} = 1 - \frac{1}{50} = \frac{49}{50}}$$

ii. Function 2:
We begin with S_x:

$$P_{+x} = \frac{1}{50}\Big|\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}4 \\ -3i\end{pmatrix}\Big|^2$$

$$P_{+x} = \frac{1}{50}\Big|(4-3i)\Big|^2$$

$$\boxed{P_{+x} = \frac{1}{2}}$$

Consequently, we write:

$$\boxed{P_{-x} = \frac{1}{2}}$$

We then check the y orientation, which gives us:

$$P_{+y} = \frac{1}{50}\Big|\begin{pmatrix}1 & i\end{pmatrix}\begin{pmatrix}4 \\ -3i\end{pmatrix}\Big|^2$$

$$\boxed{P_{+y} = \frac{49}{50}}$$

Consequently, we may get:

$$\boxed{P_{-y} = 1 - \frac{49}{50} = \frac{1}{50}}$$

iii. Function 3:

$$P_{+x} = \frac{1}{50}\left|\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}-4 \\ 3i\end{pmatrix}\right|^2$$

$$P_{+x} = \frac{1}{50}\left|(-4+3i)\right|^2$$

$$\boxed{P_{+x} = \frac{1}{2}}$$

Consequently, we write:

$$\boxed{P_{-x} = \frac{1}{2}}$$

We then check the y orientation, which gives us:

$$P_{+y} = \frac{1}{50}\left|\begin{pmatrix}1 & i\end{pmatrix}\begin{pmatrix}-43i\end{pmatrix}\right|^2$$

$$\boxed{P_{+y} = \frac{49}{50}}$$

Consequently, we may get:

$$\boxed{P_{-y} = 1 - \frac{49}{50} = \frac{1}{50}}$$

(b) We may observe that the S_z and S_x probabilities remained invariant. On the other hand, the S_y probabilities were dependent on the phase; that is, for a $\pi/2$ change in phase, the probabilties switch (the up probability becomes the down probability and vice versa). We may observe that the probabilities are equivalent for $|\psi_2\rangle$ and $|\psi_3\rangle$.