Homework 4

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1. (a) To normalize the state, we can take the magnitudes of each component to get:

$$\sqrt{A^2 + (-A)^2 + (A)^2} = 1$$

This gives us:

$$\sqrt{3A^2} = 1$$
$$A\sqrt{3} = 1$$
$$\boxed{A = \frac{1}{\sqrt{3}}}$$

Thus, we may write the initial state as:

$$\boxed{|\psi(t=0)\rangle = \frac{1}{\sqrt{3}}(|\phi_1\rangle - |\phi_2\rangle + i\,|\phi_3\rangle)}$$

(b) We know that the possible measurements are the eigenvalues of the corresponding eigenstates. Let us refer to them as the respective energies of each state, E_n. Therefore, we can measure any of the following observables:

$$|\phi_1\rangle \to E_1, |\phi_2\rangle \to E_2, |\phi_3\rangle \to E_3$$

Note that, for an infinite square well, the possible energy potentials for a Spin-1 system are:

$$\boxed{E_1 = \frac{\pi^2\hbar^2}{2mL^2},\ E_2 = \frac{2\pi^2\hbar^2}{mL^2},\ E_3 = \frac{9\pi^2\hbar^2}{2mL^2},}$$

We can find the probabilities of each as:

$$P_n = |\langle\phi_n|\psi(t)\rangle|^2$$

We may observe that, because the magnitude of each coefficient is same, they all occur with the same probability, or:

$$\boxed{P_{E_n} = 1/3}$$

(c) We can find the average energy by using the probability as weights:

$$< E >= P_{E_1}E_1 + P_{E_2}E_2 + P_{E_3}E_3$$

This gives us:

$$< E >= \frac{1}{3}[E_1 + E_2 + E_3]$$

Using the energy values from above, we write:

$$< E >= \frac{1}{3}\left[\frac{\pi^2\hbar^2}{2mL^2} + \frac{2\pi^2\hbar^2}{mL^2} + \frac{9\pi^2\hbar^2}{2mL^2}\right]$$

$$< E >= \frac{1}{3}\left[\frac{7\pi^2\hbar^2}{mL^2}\right]$$

$$\boxed{< E >= \frac{7\pi^2\hbar^2}{3mL^2}}$$

(d) Using our time evolution formula in tandem with our ϕ_n eigenstates and corresponding E_n eigenvalues, we may write the time evolution as:

$$\boxed{|\psi(t)\rangle = \frac{1}{\sqrt{3}}\left[e^{-\frac{i\pi^2\hbar t}{2mL^2}}|\phi_1\rangle - e^{-\frac{2i\pi^2\hbar t}{mL^2}}|\phi_2\rangle + ie^{-\frac{9i\pi^2\hbar t}{2mL^2}}|\phi_3\rangle\right]}$$

(e) Plugging this in to the above equation, we find:

$$\left|\psi\left(\frac{\hbar}{E_1}\right)\right\rangle = \frac{1}{\sqrt{3}}\left[e^{-i}|\phi_1\rangle - e^{-\frac{iE_2}{E_1}}|\phi_2\rangle + ie^{-\frac{iE_3}{E_1}}|\phi_3\rangle\right]$$

$$\left|\psi\left(\frac{\hbar}{E_1}\right)\right\rangle = \frac{1}{\sqrt{3}}\left[e^{-i}|\phi_1\rangle - e^{-4i}|\phi_2\rangle + ie^{-9i}|\phi_3\rangle\right]$$

We may observe, however, that energy states are stationary, and, therefore, the probabilities remain the same.

2. Given that we can infer the well is of length L based on the formula, we may normalize by writing:

$$\int_0^L |\psi|^2\,dx = 1$$

This gives us:

$$\int_0^L (AxL - Ax^2)^2 \, dx = 1$$

We continue to solve:

$$\int_0^L A^2x^4 - 2A^2x^3L + A^2x^2L^2 \, dx = 1$$

$$\left[\frac{A^2x^5}{5} - \frac{A^2x^4L}{2} + \frac{A^2x^3L^2}{3}\right]\Big|_0^L = 1$$

$$\left[\frac{A^2L^5}{5} - \frac{A^2L^5}{2} + \frac{A^2L^5}{3}\right] = 1$$

$$\frac{A^2L^5}{30} = 1$$

$$\boxed{A = \pm\sqrt{\frac{30}{L^5}}}$$

Thus, we write the equation of state as:

$$\boxed{|\psi(x, t = 0)\rangle = x\sqrt{\frac{30}{L^5}}(L - x)}$$

We know that, for a particle of mass m in an infinite square well of length L, we have:

$$\psi(x, 0) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

We may solve for the constant as:

$$c_n = \langle E_n| \, |\psi(0)\rangle|$$

$$c_n = \int_0^L \phi^*(x)\psi(x, 0) \, dx$$

$$c_n = \int_0^L \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) Ax(L - x) \, dx$$

Finally, solving this gets us:

$$c_n = \frac{4\sqrt{15}}{n^3\pi^3}[1 - (-1)^n] \to \frac{8\sqrt{15}}{n^3\pi^3}, \quad n = \text{odd}$$

We move on to find the time-dependent form as:

$$\psi(x,t) = \sum_n c_n e^{-iE_n t/\hbar}\phi_n(x)$$

This gives us a final form of:

$$\psi(x,t) = \frac{8\sqrt{30}}{\pi^3\sqrt{L}} \sum_{n=\text{odd}} \frac{1}{n^3} e^{-iE_n t/\hbar} \sin\left(\frac{n\pi x}{L}\right)$$

Expanding the energy gives us:

$$\boxed{\psi(x,t) = \frac{8}{\pi^3}\sqrt{\frac{30}{L}} \sum_{n=1,3,5,\ldots} \left(\frac{1}{n^3}\right)\left(\sin\left(\frac{n\pi x}{L}\right)\right) e^{-\frac{in^2\pi^2\hbar t}{2mL^2}}}$$

Finally, we may calculate the expectation value as:

$$< x >= \int_0^L \psi^*(x,t) \cdot x \cdot \psi(x,t)\, dx$$

We may rewrite this as:

$$< x > (t) = \langle \psi(x,t)|x|\psi(x,t)\rangle = \sum_{m,n} c_m^* c_n e^{i(E_m - E_n)t/\hbar} \langle E_m|x|E_n\rangle$$

Given that only diagonal elements survive, we take $m = n$ to get:

$$< x > (t) = \sum_{n=1}^{\infty} c_n^* c_n \langle E_n|x|E_n\rangle$$

$$< x > (t) = \frac{L}{2}\sum_{n=1}^{\infty} |c_n|^2$$

Finally, this simplifies to:

$$\boxed{< x > (t) = \frac{L}{2}}$$

This result is expected, as, a particle in a box oscillating with a wave function would, on average, be in the center since there is no bias/preference for one side.

3. We know that the wave function may be written as:

$$\phi(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

Now with $L \to 3L$, we find:

$$\phi(x) = \sqrt{\frac{2}{3L}} \sin\left(\frac{n\pi x}{3L}\right)$$

This means that, for the ground and first excited states, we have:

$$\phi_{n=1}(x) = \sqrt{\frac{2}{3L}} \sin\left(\frac{\pi x}{3L}\right)$$
$$\phi_{n=2}(x) = \sqrt{\frac{2}{3L}} \sin\left(\frac{2\pi x}{3L}\right)$$

From here, we may calculate the probabilities of each state as:

$$P_{n=1} = |\langle\phi_1(x)|\,|\psi(x)\rangle|^2$$

Given that the wave function is real, we know $\phi^*_{n=1}(x) = \phi_{n=1}(x)$. This gives us:

$$P_{n=1} = \left(\int_0^L \phi_{n=1}(x)\psi(x)\,dx + \int_L^{3L} \phi_{n=1}(x)(0)\,dx\right)^2$$
$$P_{n=1} = \left(\frac{2}{\sqrt{3}L}\int_0^L \sin\left(\frac{\pi x}{3L}\right)\sin\left(\frac{\pi x}{L}\right)\,dx\right)^2$$

Entering this into a solver, we get:

$$P_{n=1} = \left(\frac{9}{8\pi}\right)^2$$
$$\boxed{P_{n=1} = \frac{81}{64\pi^2}}$$

Similarly, we may find:

$$P_{n=2} = |\langle\phi_2(x)|\,|\psi(x)\rangle|^2$$

We proceed in a similar manner to the ground state, to write:

$$P_{n=2} = \left(\int_0^L \phi_{n=2}(x)\psi(x)\,dx\right)^2$$
$$P_{n=2} = \left(\frac{2}{\sqrt{3}L}\int_0^L \sin\left(\frac{2\pi x}{3L}\right)\sin\left(\frac{\pi x}{L}\right)\,dx\right)^2$$

Once again, we use a solver to get:

$$P_{n=2} = \left(\frac{9}{5\pi}\right)^2$$

$$\boxed{P_{n=2} = \frac{81}{25\pi^2}}$$