Homework 4

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1. (a) Given the measurement, we know that:

$$|\psi(0)\rangle = |+\rangle_x$$

Or, alternatively:

$$\boxed{|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)}$$

(b) First and foremost, we may write the Hamiltonian as:

$$\vec{H} = \mu\vec{B}$$

Furthermore, we know that a quantum state will evolve with time according to:

$$|\psi(t)\rangle = |\psi(0)\rangle\, e^{-\frac{iEt}{\hbar}}$$

With E representing the energy of the particle. Given that the particle is in a $\hat{\mathbf{z}}$ only orientation, we may write the Hamiltonian as:

$$\vec{H} = \mu B_o S_z$$

Using the Larmor precession frequency, ω_o, we find the eigenvalues of this state to be:

$$E_\pm = \lambda_\pm = \pm\frac{\omega_o\hbar}{2}$$

Thusly, combining the time-evolving state formula along with the above energy eigenvalues allows us to obtain the time-evolving state at $t = T$ as:

$$\boxed{|\psi(T)\rangle = \frac{1}{\sqrt{2}}\left[e^{-\frac{i\omega_o T}{2}}\,|+\rangle + e^{\frac{i\omega_o T}{2}}\,|-\rangle\right]}$$

(c) First, we can take the $|\psi(T)\rangle$ as the new $|\psi(0)\rangle$, such that:

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left[e^{-\frac{i\omega_o T}{2}}|+\rangle + e^{\frac{i\omega_o T}{2}}|-\rangle\right]$$

Given that we want this aligned with the y axis due to the new basis, we may rewrite the above as:

$$|\psi(0)\rangle = \frac{1}{2}\left[\left(e^{-\frac{i\omega_o T}{2}} - ie^{\frac{i\omega_o T}{2}}\right)|+\rangle_y + \left(e^{-\frac{i\omega_o T}{2}} + ie^{\frac{i\omega_o T}{2}}\right)|-\rangle_y\right]$$

We can then substitute into our formula for time evolution to get:

$$|\psi(t)\rangle = \frac{1}{2}\left[\left(e^{-\frac{i\omega_o T}{2}} - ie^{\frac{i\omega_o T}{2}}\right)e^{-\frac{iE_+ t}{\hbar}}|+\rangle_y + \left(e^{-\frac{i\omega_o T}{2}} + ie^{\frac{i\omega_o T}{2}}\right)e^{-\frac{iE_- t}{\hbar}}|-\rangle_y\right]$$

Note that, because of the similar form of the magnetic field, the eigenvalues take the same scalar values, just different orientations, for the Hamiltonian. Thus, we get:

$$|\psi(t)\rangle = \frac{1}{2}\left[\left(e^{-\frac{i\omega_o T}{2}} - ie^{\frac{i\omega_o T}{2}}\right)e^{-\frac{i\omega_o t}{2}}|+\rangle_y + \left(e^{-\frac{i\omega_o T}{2}} + ie^{\frac{i\omega_o T}{2}}\right)e^{\frac{i\omega_o t}{2}}|-\rangle_y\right]$$

We then take $t \to T$ to get the state at time T:

$$|\psi(T)\rangle = \frac{1}{2}\left[\left(e^{-i\omega_o T} - i\right)|+\rangle_y + \left(1 + ie^{i\omega_o T}\right)|-\rangle_y\right]$$

We then use our probability formula:

$$P_a = |\langle a|\psi(t)\rangle|^2$$

We can take $a \to |+\rangle_x$ and $t \to T$ to write:

$$P_{+x} = |\ {}_x\langle +|\psi(T)\rangle|^2$$

We expand and use matrix notation to get:

$$P_{+x} = \left|\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\frac{1}{2}\left[\left(e^{-i\omega_o T} - i\right)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} + \left(1 + ie^{i\omega_o T}\right)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}\right]\right|^2$$

And now, we simply evaluate:

$$P_{+x} = \frac{1}{16}\left|\begin{pmatrix}1 & 1\end{pmatrix}\left[\left(e^{-i\omega_o T} - i\right)\begin{pmatrix}1\\ i\end{pmatrix} + \left(1 + ie^{i\omega_o T}\right)\begin{pmatrix}1\\ -i\end{pmatrix}\right]\right|^2$$

$$P_{+x} = \frac{1}{16}\left|\left(e^{-i\omega_o T} - i\right)\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1\\ i\end{pmatrix} + \left(1 + ie^{i\omega_o T}\right)\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1\\ -i\end{pmatrix}\right|^2$$

$$P_{+x} = \frac{1}{16}\left|\left(e^{-i\omega_o T} - i\right)(1+i) + \left(1 + ie^{i\omega_o T}\right)(1-i)\right|^2$$

$$P_{+x} = \frac{1}{16}\Big| \left(e^{-i\omega_o T} - i + ie^{-i\omega_o T} + 1\right) + \left(1 + ie^{i\omega_o T} - i + e^{i\omega_o T}\right) \Big|^2$$

Per our trigonometric identities, this gives us:

$$P_{+x} = \frac{1}{4}\Big| i\cos(\omega_o T) + \cos(\omega_o T) - i + 1 \Big|^2$$

We then take the magnitude and square to get:

$$P_{+x} = \frac{1}{4}(1 + \cos^2(\omega_o T))$$

Using our double-angle formula, we get:

$$\boxed{P_{+x} = \frac{3}{4} + \frac{\cos(2\omega_o T)}{4}}$$

2. With the given Hamiltonian, we may observe:

$$\lambda = E_1, E_2$$

And the corresponding eigenstates are:

$$\lambda = E_1 \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \lambda = E_2 \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Similarly, we may see that the eigenvalues for $\hat{A}$ are:

$$\lambda = \pm a$$

And the corresponding eigenstates are:

$$\lambda = a \rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \lambda = -a \rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Thus, we may conclude:

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

In terms of the Hamiltonian's eigenstates, we may observe:

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

We know that the time-evolution of the state may be written as:

$$|\psi(t)\rangle = |\psi(0)\rangle\, e^{-\frac{iEt}{\hbar}}$$

Thus, we get:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-iE_2 t/\hbar} \\ e^{-iE_1 t/\hbar} \end{bmatrix}$$

We want to find the expectation value of $\hat{A}$, so we write:

$$< \hat{A} >= \left\langle \psi(t) | \hat{A} \middle| \psi(t) \right\rangle$$

Writing this in matrix form, we get:

$$< \hat{A} >= \frac{1}{2} \begin{bmatrix} e^{iE_2 t/\hbar} & e^{iE_1 t/\hbar} \end{bmatrix} \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} \begin{bmatrix} e^{-iE_2 t/\hbar} \\ e^{-iE_1 t/\hbar} \end{bmatrix}$$

Now, we evaluate:

$$< \hat{A} >= \frac{1}{2} \begin{bmatrix} ae^{iE_1 t/\hbar} & ae^{iE_2 t/\hbar} \end{bmatrix} \begin{bmatrix} e^{-iE_2 t/\hbar} \\ e^{-iE_1 t/\hbar} \end{bmatrix}$$

$$< \hat{A} >= \frac{1}{2} \left[ae^{i(E_1 - E_2)t/\hbar} + ae^{i(E_2 - E_1)t/\hbar} \right]$$

We use our trigonometric identities to write:

$$< \hat{A} >= a \cos\left(\frac{E_2 - E_1}{\hbar} t \right)$$

As such, we may see:

$$\boxed{\omega = \frac{E_2 - E_1}{\hbar} \Rightarrow f = \frac{E_2 - E_1}{h}}$$

3. (a) We determine this by finding the eigenvalues and corresponding eigenstates of the Hamiltonian. This gives us:

$$\left| \begin{bmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{bmatrix} \right| = 0$$

We solve:

$$(2 - \lambda)^2 - 1 = 0$$

$$4 - 4\lambda + \lambda^2 - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 3)(\lambda - 1) = 0$$

Thus, we get (note we reintroduce the factor of E_o):

$$\boxed{\lambda = 1E_o, 3E_o}$$

As such, we see that we can measure either 1 or 3 for the energy. The corresponding eigenstates become:

$$\boxed{\lambda = 1 \rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and } \lambda = 3 \rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

We apply our probability formula to get:

$$P_1 = |\left\langle 1 \middle| \hat{A} \right\rangle|^2$$

Writing the eigenstate for $|1\rangle$ in terms of the S_z basis gives us:

$$|1\rangle = \frac{1}{\sqrt{2}}[|a_1\rangle - |a_2\rangle]$$

Similarly, we get:

$$|3\rangle = \frac{1}{\sqrt{2}}[|a_1\rangle + |a_2\rangle]$$

Using the two states above, we may write:

$$|a_1\rangle = \frac{1}{\sqrt{2}}[|1\rangle + |3\rangle]$$

$$|a_2\rangle = \frac{1}{\sqrt{2}}[|1\rangle - |3\rangle]$$

We normalize the initial state by finding $C = 5$ to get:

$$|\psi(0)\rangle = \frac{3}{5}|a_1\rangle + \frac{4}{5}|a_2\rangle$$

We then write the probability in matrix form to get:

$$P_1 = \frac{1}{2}\left|[|a_1\rangle - |a_2\rangle]\left[\frac{3}{5}|a_1\rangle + \frac{4}{5}|a_2\rangle\right]\right|^2$$

We solve to get:

$$P_1 = \frac{1}{2}\left| -\frac{1}{5}\right|^2$$

$$\boxed{P_1 = \frac{1}{50}}$$

Then we find:

$$P_3 = \frac{1}{2}\Big|[|a_1\rangle + |a_2\rangle]\left[\frac{3}{5}|a_1\rangle + \frac{4}{5}|a_2\rangle\right]\Big|^2$$

$$P_3 = \frac{1}{2}\Big|\frac{7}{5}\Big|^2$$

$$\boxed{P_3 = \frac{49}{50}}$$

(b) Next, we calculate the expectation value:

$$< \hat{A} >= \left\langle \psi(t) \middle| \hat{A} \middle| \psi(t) \right\rangle$$

Writing $\psi(0)$ in terms of the Hamiltonian eigenstates gives us:

$$|\psi(0)\rangle = -\frac{1}{5\sqrt{2}}|1\rangle + \frac{7}{5\sqrt{2}}|3\rangle$$

Applying this to the time-evolution formula, we get:

$$|\psi(t)\rangle = -\frac{1}{5\sqrt{2}}|1\rangle e^{-iE_ot/\hbar} + \frac{7}{5\sqrt{2}}|3\rangle e^{-3iE_ot/\hbar}$$

We then rewrite in terms of a_1 and a_2 eigenstates to get:

$$|\psi(t)\rangle = -\frac{1}{10}\left[e^{-iE_ot/\hbar} - 2e^{-3iE_ot/\hbar}\right]|a_1\rangle + \frac{1}{10}\left[e^{-iE_ot/\hbar} + 2e^{-3iE_ot/\hbar}\right]|a_2\rangle$$

By the definitions of operators, we know:

$$\hat{A}|a_i\rangle = a_i|a_i\rangle$$

So we obtain:

$$\begin{aligned} < \hat{A} >= a_1 &\left[-\frac{1}{10}e^{ie_ot/\hbar} + \frac{7}{10}e^{3ie_ot/\hbar}\right]\left[-\frac{1}{10}e^{-ie_ot/\hbar} + \frac{7}{10}e^{-3ie_ot/\hbar}\right] + \\ a_2 &\left[\frac{1}{10}e^{ie_ot/\hbar} + \frac{7}{10}e^{3ie_ot/\hbar}\right]\left[\frac{1}{10}e^{-ie_ot/\hbar} + \frac{7}{10}e^{-3ie_ot/\hbar}\right] \end{aligned}$$

From here, we continue to simplify:

$$< \hat{A} >= a_1\left[\frac{1}{2} - \frac{7}{100}e^{-2iE_ot/\hbar} - \frac{7}{100}e^{2iE_ot/\hbar}\right] + a_2\left[\frac{1}{2} + \frac{7}{100}e^{-2iE_ot/\hbar} + \frac{7}{100}e^{2iE_ot/\hbar}\right]$$

$$< \hat{A} >= \frac{(a_1 + a_2)}{2} - \frac{7a_1}{50}\cos\left(\frac{2E_ot}{\hbar}\right) + \frac{7a_2}{50}\cos\left(\frac{2E_ot}{\hbar}\right)$$

As such, we can finally obtain:

$$\boxed{< \hat{A} >= \frac{(a_1 + a_2)}{2} + \frac{7(a_2 - a_1)}{50}\cos\left(\frac{2E_ot}{\hbar}\right)}$$