Lecture 1

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- Key Features of Quantum Mechanics

  1. Probabilistic outcome of measurements
     - Compute probabilities exactly, and that is the most complete information possible
  2. Dual wave-particle nature of mature
     - Which one we observe depends on the experiment performed
  3. Conjugate variables (from classical mechanics) develop "uncertainty" relations
     - Wave theory relation:

$$\Delta x \Delta p \geq \hbar$$
$$\Delta E \Delta t \geq \hbar$$

     - Classical mechanics is "contained" in Quantum mechanics, which includes classical electricity and magnetism
  4. Every particle and object built from particles, including light, falls into one of two classes:
     - Fermions (spin is any odd multiple of $\hbar/2$)
       * Electrons are an example
     - Bosons (spin is an integer multiple of $\hbar$, including 0)
       * Photons are an example
  5. The properties of Quantum mechanics not falling into features 1-4 are largely familiar from classical physics

- Similar but Different

  - Every electron has exactly the same spin, charge, and mass
  - Every photon has exactly the same spin, charge, and mass

- This indicates no inequality among particles of the same type

- Time-Evolution in Quantum Mechanics
  - Time-evolution is given by the Hamiltonian, $\mathcal{H}$
  - Hamilton's Equations give us:

$$\frac{d\vec{q}}{dt} = \frac{\partial\mathcal{H}}{\partial\vec{p}}$$

  - We may write the Hamiltonian as:

$$\mathcal{H} = \frac{\vec{p}^2}{2m} \longrightarrow \frac{\partial\mathcal{H}}{\partial\vec{p}} = \frac{\vec{p}}{m} = \vec{v}$$

  - Alternatively, we may write the force as:

$$\vec{F} = \frac{d\vec{p}}{dt} = -\frac{\partial\mathcal{H}}{\partial\vec{q}}$$

- Stern-Gerlach Experiments
  - Took place in 1922 with Otto Stern and Walther Gerlach
  - The act of observing a quantum particle affects its measurable properties in a way foreign to our experience
  - The experiment looks like this:

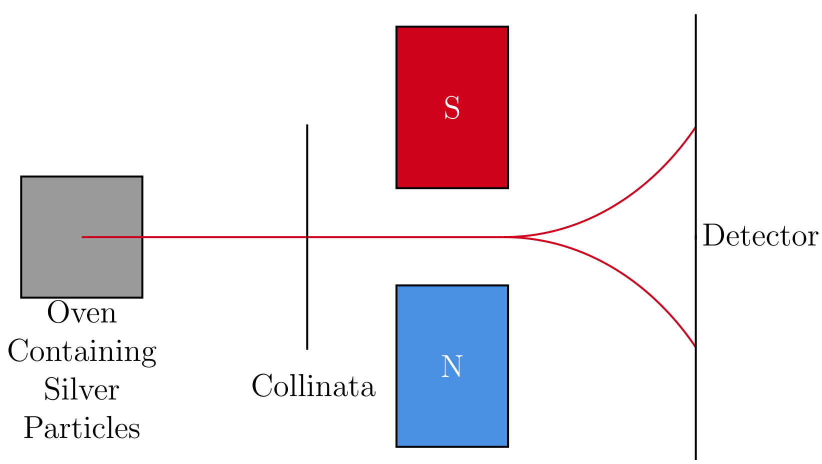

Figure 1: Set Up of Stern-Gerlach Experiment

– Assuming an atom has a monopole moment, $\vec{\mu}$, the potential energy of the interaction with a magnetic field $\vec{B}$ is $E = \vec{\mu}\vec{B}$
– Consider a classical description of the atom's moment:

$$\mu = IA$$

  * Where I is the electrical current and A is the area of the loop
– A particle of charge q traveling at speed v in a circle of radius r gives us:

$$\mu = \frac{qvr}{2} = \frac{qL}{2m}$$

  * Where $L = mvr$ is the orbitable angular momentum
– Particles carry an intrinsic angular momentum, $\vec{S}$, called spin

$$\vec{\mu} = \frac{gq\vec{S}}{2m}$$

  * Where g is the gyroscopic ratio
– Noting that silver atoms were used is important, as different atoms give different results. Considering the shell filling of silver, we know that it extends to a singular atom in the $5s$ shell.
  * Since the mass of the nucleus $\geq 2000m_e$, we find a ratio of magnetic moments as:

$$\frac{\vec{\mu}_{nuc}}{\vec{\mu}_{e^-}} << 1$$

  * Hence, we have $\vec{\mu}_{\text{Ag}} = -g\frac{e}{2m_e}\vec{S}$, where e is the magnitude of an electron's charge
  * This produces a force of $F_z = -g\frac{e}{2m_e}S_z \cdot \frac{2B_z}{2z}$
  * Thus, the deflection of the beam in the Stern-Gerlach experiment is a measure of the projection of the intrinsic spin onto the z-axis
  * Note, the heat of the oven randomizes the direction of $\vec{\mu}$, and, classicly, we have $S_z = |\vec{S}|\cos(\theta)$ and should be continuous over an S_z range given by: $-|\vec{S}| \leq S_z \leq |\vec{S}|$
    · But, only two beams are observed!
    · Only two S_z components are possible, since $S_z = \pm\hbar/2$
– We know:

$$\hbar \approx= 1.0546 \cdot 10^{-34}[\text{J s}] = 6.5821 \cdot 10^{-16}[\text{eVs}]$$

$$\hbar = \frac{h}{2\pi} \text{ (Planck's Constant)}$$

* This indicates a quantization of an electron's intrinsic angular momentum along the z axis
* In the case that $S_z = +\hbar/2$, we call this spin "up"
* In the case that $S_z = -\hbar/2$, we call this spin "down"
* The quantity S_z itself is called an "observable"
* The device depicted within the Stern-Gerlach experiment is called an "analyzer", since it sorts the input into two possible outputs
* New notation: $S_z = +\hbar/2$ can be written in Dirac notation as:

$$\left|+\frac{\hbar}{2}\right\rangle \Longrightarrow |+\rangle$$

* The negative (down) spin becomes:

$$\left|-\frac{\hbar}{2}\right\rangle \Longrightarrow |-\rangle$$

* A "ket" can be depicted as follows:

$$|\cdot\rangle$$

  · Or, in its general form:

$$|\psi\rangle$$

* Note that for up and down spin, occasionally one may encounter:

$$|+\rangle = |\uparrow\rangle$$
$$|-\rangle = |\downarrow\rangle$$

– Experiment 1:

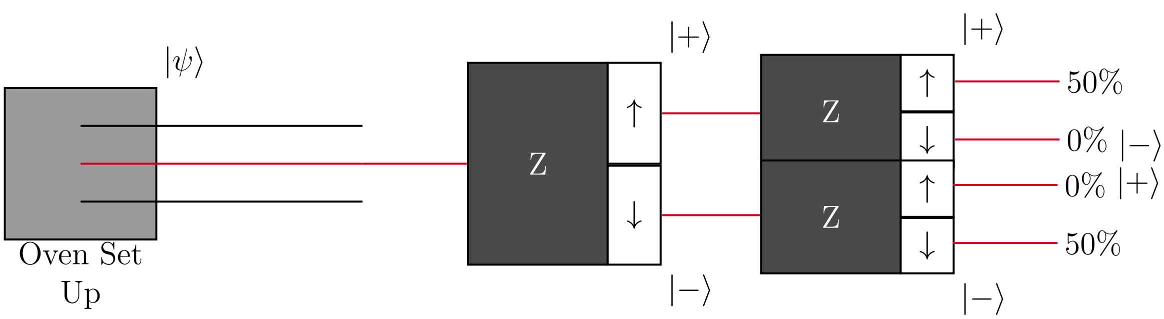

Figure 2: Experimental Setup

* Although both Stern-Gerlach analyzers in the experiment are the same, they play different roles
    * The first analyzer prepares the beam and the second analyzer measures the beam
    * The first analyzer is often referred to as a state preparation device

– Experiment 2:

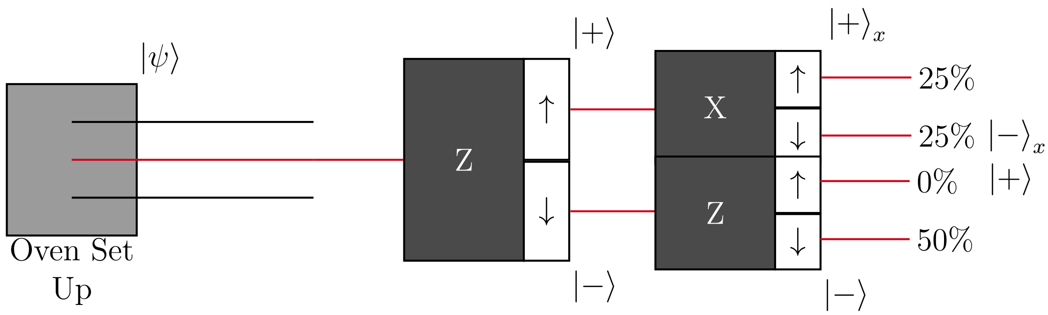

Figure 3: Experimental Setup

* The second analyzer is rotated by 90° with respect to the first and aligned with the x-axis
    * Only two possible outputs from second analyzer
    * Results would be unchanged if we used the lower part of the first analyzer
    * One can not predict which of the second analyzer parts any particular atom will emerge from
    * These results highlight the probablistic nature of quantum mechanics; quantum mechanics is a complete description of reality (as far as we know)

– Experiment 3:

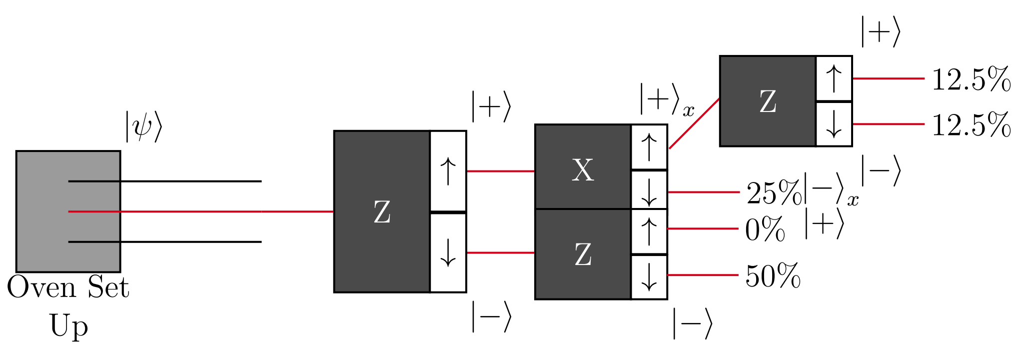

Figure 4: Experimental Setup

* This tells us that the S_x and S_z are incompatible observables, which means we can not know the values of both simultaneously

- Experiment 4:

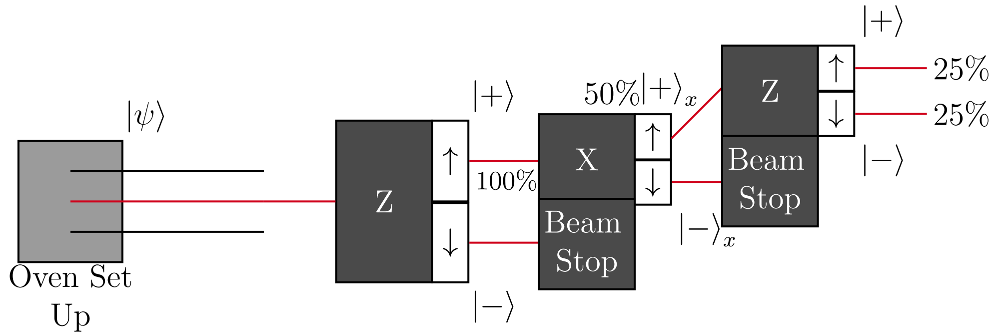

Figure 5: Experimental Setup Part One

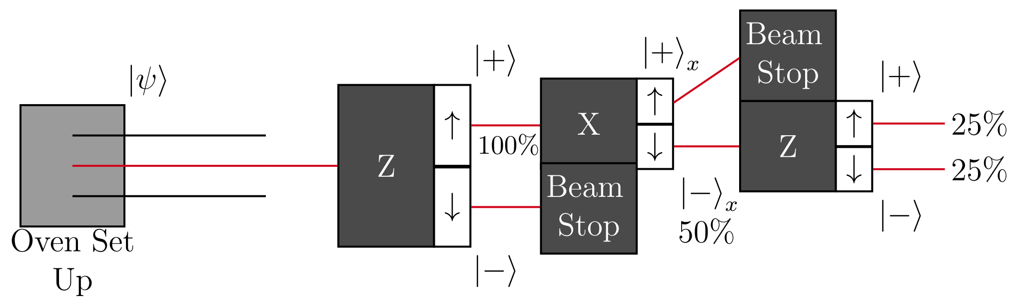

Figure 6: Experimental Setup Part Two

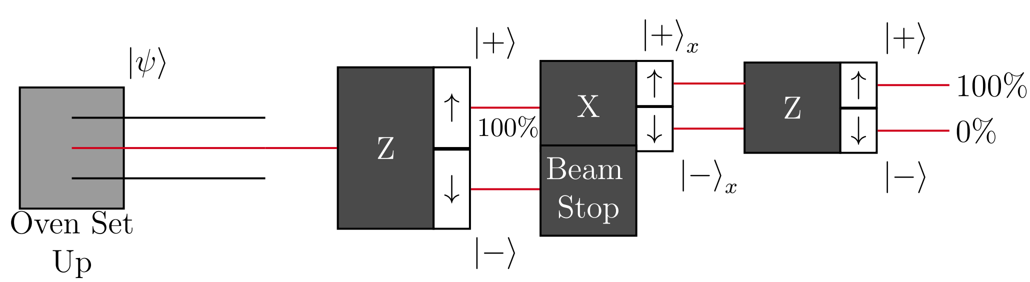

Figure 7: Experimental Setup Part Three

* Results are akin to the double slit experiment and destructive interference

- Quantum State Vectors
  - The kets ($|\psi\rangle$) obey many rules of ordinary spacial vectors
  - $|\psi\rangle$ is a quantum state vector and is part of a vector space called a Hilbert Space
  - The dimensionality of the Hilbert Space is determined by the physics at hand
  - In the Stern-Gerlach experiment the Hilbert Space has just two states ($|+\rangle$ and $|-\rangle$)
    * These states form a basis like unit vectors $\hat{i}, \hat{j}, \hat{k}$ for 3-D space. These basis vectors are normalized, orthogonal, and complete
      · Normailization: $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$
      · Orthogonality: $\hat{i} \cdot \hat{j} = \hat{i} \cdot \hat{k} = \hat{j} \cdot \hat{k} = 0$
      · Completeness: $\vec{A} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$
      · Note: The dot product (or scalar product) displayed in the normalization and orthogonality conditions is central to these properties
    * For the S_z measurement, the basis states are: $|+\rangle$ and $|-\rangle$ and are referred to as the "S_z basis"
      · Since this is a complete basis, a general state is (where a and b are complex numbers):

$$|\psi\rangle = a\,|+\rangle + b\,|-\rangle$$

    * To discuss orthogonality and normalization (orthonormality) we need to understand how scalar products apply to kets
    * The complex-conjugated quantum state vector is called a 'bra' in the Dirac notation of quantum mechanics (note the asterisk indicates the complex conjugate):

$$\langle\psi| = a^*\,\langle+| + b^*\,\langle-|$$

  - The scalar product in quantum mechanics is defined as:

$$(\langle\psi|)(|\psi\rangle) = \langle\psi|\psi\rangle$$

    * "bra-ket" or bracket
    * For example:

$$(\langle+|)(|-\rangle) = \langle+|-\rangle$$

    * This defines an inner product between two state vectors
    * In analogy to 3-D spatial vectors, this is also called a projection

* Given the analogs, we may write that normality is defined as:

$$\langle +|+\rangle = \langle -|-\rangle = 1$$

* Furthermore, orthogonality may be defined as:

$$\langle +|-\rangle = \langle -|+\rangle = 0$$

* Completeness is then defined as:

$$|\psi\rangle = a\,|+\rangle + b\,|-\rangle$$

* With these components, we can then write:

$$\langle +|\psi\rangle = \langle +|\,(a\,|+\rangle + b\,|-\rangle) = \langle +|a|+\rangle + \langle +|b|-\rangle = a\,\langle +|+\rangle + b\,\langle +|-\rangle$$

  · This can then be simplified with our properties to get:

$$\langle +|\psi\rangle = a$$

  · Likewise, we may write:

$$\langle -|\psi\rangle = b$$

* Using these results, we may rewrite the wave function as:

$$\begin{aligned}|\psi\rangle &= \langle +|\psi\rangle\,|+\rangle + \langle -|\psi\rangle\,|-\rangle\\ |\psi\rangle &= |+\rangle\,\langle +|\psi\rangle + |-\rangle\,\langle -|\psi\rangle\\ |\psi\rangle &= \underbrace{(|+\rangle\,\langle +| + |-\rangle\,\langle -|)}_{1}\,|\psi\rangle\end{aligned}$$

  · We can note that the identity is equivalent to one

* We can also reverse the projection to write:

$$\langle \psi|+\rangle = \langle +|a^*|+\rangle + \langle -|b^*|+\rangle = a^*\,\langle +|+\rangle + b^*\,\langle -|+\rangle = a^*$$

  · This gives us the result that (note this holds for any state):

$$\langle \psi|+\rangle = \langle +|\psi\rangle^* \Rightarrow \langle \phi|\psi\rangle = \langle \psi|\phi\rangle^*$$

  · In quantum mechanics, state vectors must be normalized since they describe a situation where the quantum state has probability 1

$$\begin{aligned}\langle \psi|\psi\rangle &= (a^*\,\langle +| + b^*\,\langle -|)(a\,|+\rangle + b\,|-\rangle)\\ &= a^*a\,\langle +|+\rangle + a^*b\,\langle +|-\rangle + ab^*\,\langle -|+\rangle + b^*b\,\langle -|-\rangle\\ \langle \psi|\psi\rangle &= a^*a + b^*b = |a|^2 + |b|^2 = 1\end{aligned}$$

· Equivalently, we may write:

$$|\langle +|\psi\rangle|^2 + |\langle -|\psi\rangle|^2 = 1$$

– In quantum mechanics, the probability that the state $|\psi\rangle$ is measured as spin up is $P_{S_z=+\frac{\hbar}{2}} = |\langle +|\psi\rangle|^2$

– Equivalently, the probability that the state $|\psi\rangle$ is measures as spin down is $P_{S_z=-\frac{\hbar}{2}} = |\langle -|\psi\rangle|^2$

* We can use a shorthand notation of P_+ for $S_z = +\dfrac{\hbar}{2}$ and P_- for $S_z = -\dfrac{\hbar}{2}$
* We may observe that the coefficients of the quantum state function contain the information regarding the probability
* This leads to Postulate 4 of Quantum Mechanics (for a 1/2-spin system)
    · The probability of obtaining the value $\pm\hbar/2$ in a measurement of the observable, S_z, on a system in the state $|\psi\rangle$ is $P_\pm = |\langle \pm|\psi\rangle|^2$, where $|\pm\rangle$ is the basis ket of S_z corresponding to the result $\pm\hbar/2$
    · $\langle -|\psi\rangle$ is referred to as the probability amplitude (or just amplitude)

– For the Stern-Gerlach experiments in which a different axis was tested after the first analyzer, we may write:

$$|_x\langle +|+\rangle|^2 = |_x\langle -|+\rangle|^2 = |_x\langle +|-\rangle|^2 = |_x\langle -|-\rangle|^2 = \frac{1}{2}$$

– Since $|+\rangle$ and $|-\rangle$ form a complete basis, we can write:

$$|+\rangle_x = a\,|+\rangle + b\,|-\rangle$$
$$|-\rangle_x = c\,|+\rangle + d\,|-\rangle$$

* Per the results of the Stern-Gerlach experiments, we may write:

$$|a|^2 = |b|^2 = |c|^2 = |d|^2 = \frac{1}{2}$$

* This means:

$$|+\rangle_x = \frac{e^{i\alpha}}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)$$
$$|-\rangle_x = \frac{1}{\sqrt{2}}\left(|+\rangle + e^{i\beta}\,|-\rangle\right)$$