# Lecture 5

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- Associated Legendre Functions
  - For $l = 0, 1, 2, 3 \cdots$
$$\left[(1-z^2)\frac{d^2}{dz^2} - 2z\frac{d}{dz} + l(l+1) - \frac{m^2}{1-z^2}\right] P(z)0$$
  - With:
$$P_l^m(z) = P_l^{-m}(z) = (1-z^2)^{m/2}\frac{d^m}{dz^m}P_l(z)$$
  - Since $P_l(z)$ is an $l$-th order polynomial, then $P_l^m(z)$ vanishes if $m > l$
  - Properties of $P_l^m(z)$:
    * $P_l^m(z) = 0$ if $|m| > l$
    * $P_l^m(z) = P_l^{-m}(z)$
    * $P_l^m(\pm 1) = 0$ for $m \neq 0$
    * $P_l^m(-z) = (-1)^{l-m}P_l^m(z)$
    * $\displaystyle\int_{-1}^{1} P_l^m(z)P_q^m(z)\,dz = \frac{2}{2l+2}\cdot\frac{(l+m)!}{(l-m)!}\delta_{lq}$
  - We can obtain our $\Theta(\theta)$ function as:
$$\Theta_l^m(\theta) = (-1)^m\frac{(2l+1)}{2}\frac{(l-m)!}{(l+m)!}P_l^k(\cos(\theta)), \quad m \geq 0$$

    And:
$$\Theta_l^{-m}(\theta) = (-1)^m\Theta_l^m(\theta)$$

– Here, we arrive at a point where the associated Legendre Functions are needed:

$$P_o^o = 1$$
$$P_1^o = \cos(\theta)$$
$$P_1^1 = \sin(\theta)$$
$$P_2^o = \frac{1}{2}(3\cos^3(\theta) - 1)$$
$$P_3^o = \frac{1}{2}(5\cos^5(\theta) - 3\cos(\theta))$$

– Spherical Harmonics

$$Y_l^m(\theta, \phi) = (-1)^{(m+|m|)/2}\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}P_l^m(\cos(\theta))e^{im\phi}$$
$$Y_l^m(\theta, \phi) = (-1)(Y_l^m(\theta, \phi))^*$$

Our first few values may be written as:

$$Y_o^o(\theta, \phi) = \frac{1}{\sqrt{4\pi}}$$
$$Y_1^o(\theta, \phi) = \sqrt{\frac{3}{4\pi}}\cos(\theta)$$
$$Y_1^{\pm 1}(\theta, \phi) = \mp\sqrt{\frac{3}{8\pi}}\sin(\theta)e^{\pm i\phi}$$

– Important Properties:

* Orthonormality:

$$\langle l_1 m_1 | l_2 m_2 \rangle = \delta_{l_1 l_2}\delta_{m_1 m_2}$$

* Completeness:

$$\psi(\theta, \phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} C_{lm} Y_l^m(\theta, \phi)$$
$$C_{lm} = \langle lm | \psi \rangle = \int_0^{2\pi}\int_0^{\pi}(Y_l^m(\theta, \phi))^*\psi(\theta, \phi)\, d\Omega$$

* Parity:

$$Y_l^m(\pi - \theta, \phi + \pi) = (-1)^l Y_l^m(\theta, \phi)$$