Lecture 2

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January 27, 2025

- Predictions for experiment: what are the possible results of the measurement of spin projected onto an arbitrary direction, $\hat{n}$, and what are the predicted probabilities?
- An operator is a mathematical object that acts on a ket and transforms it into a new ket:

$$A\,|\psi\rangle = |\phi\rangle$$

- A ket that is not changed by the operator except to be multiplied by a constant is an eigenvector (below, a is an eigenvalue):

$$A\,|\psi\rangle = a\,|\psi\rangle$$

- A physical observable is represented mathematically by an operator, A, that acts on kets
- The only possible result of a measurement of an observable is one of the eigenvalues, a_n, of the corresponding operator, A. Thus, the equatiion below is an eigenvalue equation:

$$A\,|\psi\rangle = a\,|\psi\rangle$$

  - We have observed such an equation already:

$$S_z\,|+\rangle = \frac{\hbar}{2}\,|+\rangle$$
$$S_z\,|-\rangle = \frac{-\hbar}{2}\,|-\rangle$$

  - Since we had the matrix representations of the up and down states, there must also be one for the operator S_z:

$$|+\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$S_z = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

- From this, we get:

$$S_z \left|+\right\rangle \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$S_z \left|-\right\rangle \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix} = \frac{-\hbar}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- We can thus conclude:

$$S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- We can observe that an operator is always diagonal in its own basis; eigenvectors are unit vectors in their own basis

* Matrix Representation of Operators

  - A general operator may be expressed as:

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots \\ A_{21} & A_{22} & A_{23} & \cdots \\ A_{31} & A_{32} & A_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

  - This refers to:

$$A_{ij} = \langle i|A|j\rangle$$

  - The action on a general ket is:

$$A\left|\psi\right\rangle = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots \\ A_{21} & A_{22} & A_{23} & \cdots \\ A_{31} & A_{32} & A_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} A_{11}c_1 + A_{12}c_2 + A_{13}c_3 + \cdots \\ A_{21}c_1 + A_{22}c_2 + A_{23}c_3 + \cdots \\ A_{31}c_1 + A_{32}c_2 + A_{33}c_3 + \cdots \\ \vdots \end{pmatrix}$$

  - If we can write the new ket, we get:

$$\left|\phi\right\rangle = A\left|\psi\right\rangle = \sum_i b_i \left|i\right\rangle = \sum_i \sum_j A_{ij} c_j$$

* Diagonalization of Operators

  - Generally, one knows the matrix representation of an operator but wishes to know the possible results of a measurement

- The eigenvalue equation:

$$A |a_n\rangle = a_n |a_n\rangle$$

- which in the two-dimensional Hilbert space is:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = a_n \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

- where c_1, c_2 are unknown coefficients of $|a_n\rangle$
- We can get the following equations:

$$(A_{11} - a_n)c_1 + A_{12}c_2 = 0$$
$$A_{21}c_1 + (A_{22} - a_n)c_2 = 0$$

  * This has solutions for unknowns c_1 and c_2 only when the determinant of the coefficients vanishes:

$$\begin{vmatrix} A_{11} - a_n & A_{12} \\ A_{21} & A_{22} - a_n \end{vmatrix} = 0$$

  * This can be written in terms of the identity matrix as:

$$\det(A - \lambda \mathbb{1}) = 0$$

    · Where:

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- Matrix Representation Summary:
  - We can write our observables as:

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

  - And we can write our kets as:

$$|\pm\rangle_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$$

$$|\pm\rangle_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$$

- Spin Component in a General Direction
  - In Cartesian, the unit vector $\hat{n}$ is:

$$\hat{n} = \sin(\theta)\cos(\phi)\hat{\imath} + \sin(\theta)\sin(\phi)\hat{\jmath} + \cos(\theta)\hat{k}$$

  - The spin components along this direction are found by projection of spin onto this vector:

$$\vec{S} \cdot \hat{n} = S_x \sin(\theta)\cos(\phi) + S_y \sin(\theta)\sin(\phi) + S_z \cos(\theta)$$

    * This is equivalent to:

$$\frac{\hbar}{2}\begin{pmatrix} \cos(\theta) & \sin(\theta)e^{-i\phi} \\ \sin(\theta)e^{i\phi} & -\cos(\theta) \end{pmatrix}$$

  - Following the diagonalization procedure, the eigenvalues are $\pm\hbar/2$ with eigenvalues:

$$|+\rangle_n = \cos\left(\frac{\theta}{2}\right)|+\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|-\rangle$$
$$|-\rangle_n = \sin\left(\frac{\theta}{2}\right)|+\rangle - \cos\left(\frac{\theta}{2}\right)e^{i\phi}|-\rangle$$

  - This can represent any possible ket in a spin-1/2 system if one allows for all possible angles:

$$0 \leq \theta < \pi \quad \text{and} \quad 0 \leq \phi < 2\pi$$

  - Using the above, we can find the probability of a cascade of analyzers with the $\hat{n}$ direction and then x direction gives us:

$$P_{+x} = |\,{}_x\langle +|+\rangle_n|^2 = \frac{1}{2}[1 + \sin(\theta)\cos(\phi)]$$
$$P_{-x} = |\,{}_x\langle -|+\rangle_n|^2 = \frac{1}{2}[1 - \sin(\theta)\cos(\phi)]$$