Homework 7

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1. We begin by writing out the eigenstates as:

$$|n\rangle = \frac{1}{\sqrt{2\pi}}e^{in\phi}$$

We may take the inner product of two <u>different</u> states ($n \neq m$) to get:

$$\langle m|n\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{i(n-m)\phi}\,d\phi$$

We evaluate to get:

$$\langle m|n\rangle = \frac{1}{2\pi i(n-m)}\left[e^{i(n-m)\phi}\right]\Big|_0^{2\pi}$$
$$\langle m|n\rangle = \frac{1}{2\pi i(n-m)}\left[e^{2\pi i(n-m)}-1\right]$$

Since the exponent is an integer multiple of 2π, we get:

$$\langle m|n\rangle = \frac{1}{2\pi i(n-m)}[1-1]$$
$$\boxed{\langle m|n\rangle = 0}$$

Furthermore, the inner product of the same eigenstate gives us:

$$\langle n|n\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\phi$$
$$\langle n|n\rangle = \frac{2\pi}{2\pi}$$
$$\boxed{\langle n|n\rangle = 1}$$

Accordingly, we observe that <u>the states are orthonormal</u>:

$$\boxed{\langle m|n\rangle = \delta_{mn}}$$

2. (a) We begin by normalizing:

$$\langle\psi|\psi\rangle = \int_0^{2\pi} \left|\frac{N}{2+\cos(3\phi)}\right|^2 d\phi$$

We expand to get:

$$\langle\psi|\psi\rangle = \int_0^{2\pi} \frac{|N|^2}{4+4\cos(3\phi)+\cos^2(3\phi)}\, d\phi$$

Using a solver, we obtain:

$$\langle\psi|\psi\rangle = |N^2|\left(\frac{4\pi}{3\sqrt{3}}\right)$$

As such, we get:

$$\boxed{N = \sqrt{\left(\frac{4\pi}{3\sqrt{3}}\right)^{-1}}}$$

The wave function becomes:

$$\boxed{\psi(\phi) = \left(\frac{4\pi}{3\sqrt{3}}\right)^{-1/2} \frac{1}{2+\cos(3\phi)}}$$

(b) We plot the function to get:

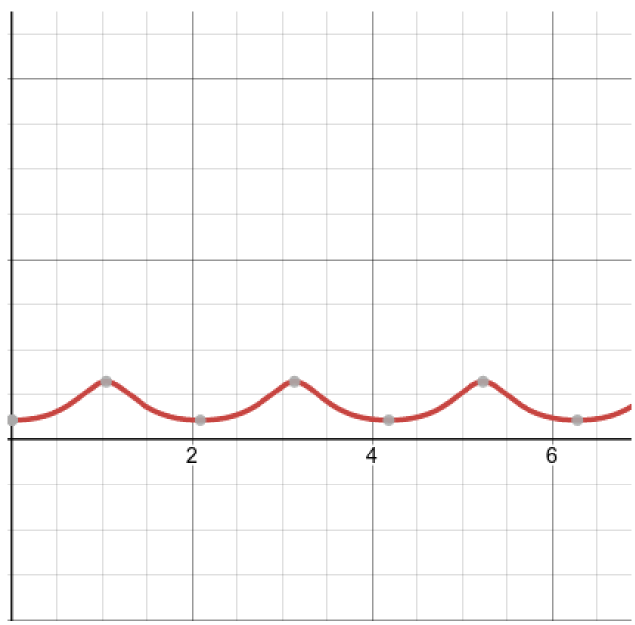

Figure 1: Plot of $\psi(\phi)$

(c) We may write the expectation value as:

$$\langle L_z \rangle = \langle \psi | L_z | \psi \rangle$$

This gives us:

$$\langle \psi | L_z | \psi \rangle = \int_0^{2\pi} \psi^*(\phi) L_z \psi(\phi)\, d\phi$$

We expand to get:

$$\langle \psi | L_z | \psi \rangle = \left[\frac{4\pi}{3\sqrt{3}}\right]^{-1} \int_0^{2\pi} \frac{1}{2+\cos(3\phi)} \left(-i\hbar \frac{d}{d\phi}\right) \frac{1}{2+\cos(3\phi)}\, d\phi$$

Now, we solve:

$$\langle \psi | L_z | \psi \rangle = i\hbar \frac{3\sqrt{3}}{4\pi} \int_0^{2\pi} \frac{1}{2+\cos(3\phi)} \left[\frac{3\sin(3\phi)}{(2+\cos(3\phi))^2}\right] d\phi$$

$$\langle \psi | L_z | \psi \rangle = i\hbar \frac{3\sqrt{3}}{4\pi} \int_0^{2\pi} \frac{3\sin(3\phi)}{(2+\cos(3\phi))^3}\, d\phi$$

Entering this into a solver, we obtain:

$$\boxed{\langle L_z \rangle = 0}$$

Evidently, we can see that the integrand must evaluate to zero, since, otherwise, the expectation value would be imaginary.

3. For $l = 1$, the spherical harmonics are:

$$Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}} \cos(\theta)$$

$$Y_1^{-1}(\theta,\phi) = \sqrt{\frac{3}{8\pi}} \sin(\theta) e^{-i\phi}$$

$$Y_1^{1}(\theta,\phi) = -\sqrt{\frac{3}{8\pi}} \sin(\theta) e^{i\phi}$$

We expand to get:

$$Y_1^{-1}(\theta,\phi) = \sqrt{\frac{3}{8\pi}} \sin(\theta)[\cos(\phi) - i\sin(\phi)]$$

$$Y_1^{1}(\theta,\phi) = -\sqrt{\frac{3}{8\pi}} \sin(\theta)[\cos(\phi) + i\sin(\phi)]$$

Finally, we transform from spherical to rectangular coordinates to get:

$$\boxed{Y_1^0(x,y,z) = \sqrt{\frac{3}{4\pi}}\frac{z}{\sqrt{x^2+y^2+z^2}}}$$

$$\boxed{Y_1^{-1}(x,y,z) = \sqrt{\frac{3}{8\pi}}\frac{x-iy}{\sqrt{x^2+y^2+z^2}}}$$

$$\boxed{Y_1^{1}(x,y,z) = -\sqrt{\frac{3}{8\pi}}\frac{x+iy}{\sqrt{x^2+y^2+z^2}}}$$

Combining the $m = \pm 1$ functions, gives us x or y in the numerator:

$$\boxed{\frac{1}{\sqrt{2}}\left[Y_1^{-1} - Y_1^{1}\right] = \sqrt{\frac{3}{4\pi}}\frac{x}{\sqrt{x^2+y^2+z^2}}}$$

$$\boxed{\frac{1}{\sqrt{2}}\left[Y_1^{-1} + Y_1^{1}\right] = \sqrt{\frac{3}{4\pi}}\frac{y}{\sqrt{x^2+y^2+z^2}}}$$

We see that these form the real spherical harmonics, or the p_x, p_y, and p_z orbitals.