Lecture 1

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- Key Features of Quantum Mechanics

  1. Probabilistic outcome of measurements
     - Compute probabilities exactly, and that is the most complete information possible
  2. Dual wave-particle nature of mature
     - Which one we observe depends on the experiment performed
  3. Conjugate variables (from classical mechanics) develop "uncertainty" relations
     - Wave theory relation:

$$\Delta x \Delta p \geq \hbar$$
$$\Delta E \Delta t \geq \hbar$$

     - Classical mechanics is "contained" in Quantum mechanics, which includes classical electricity and magnetism
  4. Every particle and object built from particles, including light, falls into one of two classes:
     - Fermions (spin is any odd multiple of $\hbar/2$)
       * Electrons are an example
     - Bosons (spin is an integer multiple of $\hbar$, including 0)
       * Photons are an example
  5. The properties of Quantum mechanics not falling into features 1-4 are largely familiar from classical physics

- Similar but Different

  - Every electron has exactly the same spin, charge, and mass
  - Every photon has exactly the same spin, charge, and mass

- This indicates no inequality among particles of the same type

- Time-Evolution in Quantum Mechanics
  - Time-evolution is given by the Hamiltonian, $\mathcal{H}$
  - Hamilton's Equations give us:

$$\frac{d\vec{q}}{dt} = \frac{\partial \mathcal{H}}{\partial \vec{p}}$$

  - We may write the Hamiltonian as:

$$\mathcal{H} = \frac{\vec{p}^2}{2m} \longrightarrow \frac{\partial \mathcal{H}}{\partial \vec{p}} = \frac{\vec{p}}{m} = \vec{v}$$

  - Alternatively, we may write the force as:

$$\vec{F} = \frac{d\vec{p}}{dt} = -\frac{\partial \mathcal{H}}{\partial \vec{q}}$$

- Stern-Gerlach Experiments
  - Took place in 1922 with Otto Stern and Walther Gerlach
  - The act of observing a quantum particle affects its measurable properties in a way foreign to our experience
  - The experiment looks like this:

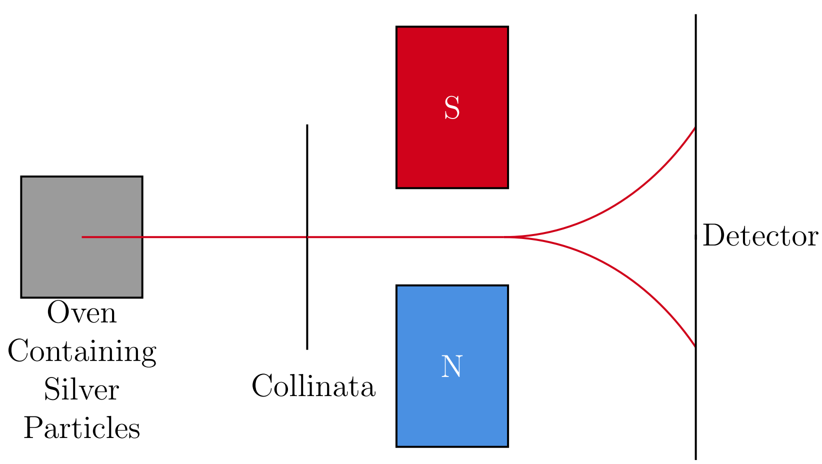

Figure 1: Set Up of Stern-Gerlach Experiment

- Assuming an atom has a monopole moment, $\vec{\mu}$, the potential energy of the interaction with a magnetic field $\vec{B}$ is $E = \vec{\mu}\vec{B}$
- Consider a classical description of the atom's moment:

$$\mu = IA$$

  * Where I is the electrical current and A is the area of the loop
- A particle of charge q traveling at speed v in a circle of radius r gives us:

$$\mu = \frac{qvr}{2} = \frac{qL}{2m}$$

  * Where $L = mvr$ is the orbitable angular momentum
- Particles carry an intrinsic angular momentum, $\vec{S}$, called spin

$$\vec{\mu} = \frac{gq\vec{S}}{2m}$$

  * Where g is the gyroscopic ratio
- Noting that silver atoms were used is important, as different atoms give different results. Considering the shell filling of silver, we know that it extends to a singular atom in the $5s$ shell.
  * Since the mass of the nucleus $\geq 2000m_e$, we find a ratio of magnetic moments as:

$$\frac{\vec{\mu}_{nuc}}{\vec{\mu}_{e^-}} << 1$$

  * Hence, we have $\vec{\mu}_{\text{Ag}} = -g\frac{e}{2m_e}\vec{S}$, where e is the magnitude of an electron's charge
  * This produces a force of $F_z = -g\frac{e}{2m_e}S_z \cdot \frac{2B_z}{2z}$
  * Thus, the deflection of the beam in the Stern-Gerlach experiment is a measure of the projection of the intrinsic spin onto the z-axis
  * Note, the heat of the oven randomizes the direction of $\vec{\mu}$, and, classicly, we have $S_z = |\vec{S}|\cos(\theta)$ and should be continuous over an S_z range given by: $-|\vec{S}| \leq S_z \leq |\vec{S}|$
    · But, only two beams are observed!
    · Only two S_z components are possible, since $S_z = \pm\hbar/2$
- We know:

$$\hbar \approx= 1.0546 \cdot 10^{-34}[\text{J s}] = 6.5821 \cdot 10^{-16}[\text{eVs}]$$

$$\hbar = \frac{h}{2\pi} \text{ (Planck's Constant)}$$

* This indicates a quantization of an electron's intrinsic angular momentum along the z axis
* In the case that $S_z = +\hbar/2$, we call this spin "up"
* In the case that $S_z = -\hbar/2$, we call this spin "down"
* The quantity S_z itself is called an "observable"
* The device depicted within the Stern-Gerlach experiment is called an "analyzer", since it sorts the input into two possible outputs
* New notation: $S_z = +\hbar/2$ can be written in Dirac notation as:

$$\left|+\frac{\hbar}{2}\right\rangle \Longrightarrow |+\rangle$$

* The negative (down) spin becomes:

$$\left|-\frac{\hbar}{2}\right\rangle \Longrightarrow |-\rangle$$

* A "ket" can be depicted as follows:

$$|\cdot\rangle$$

  · Or, in its general form:

$$|\psi\rangle$$

* Note that for up and down spin, occasionally one may encounter:

$$|+\rangle = |\uparrow\rangle$$
$$|-\rangle = |\downarrow\rangle$$

– Experiment 1:

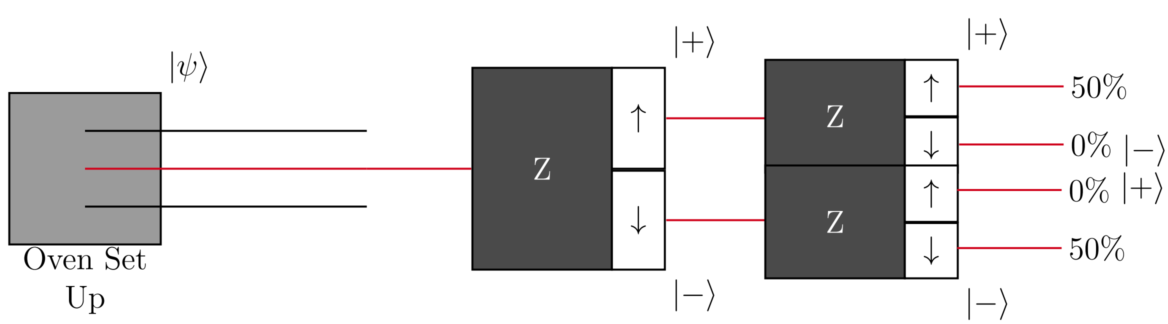

Figure 2: Experimental Setup

    - Although both Stern-Gerlach analyzers in the experiment are the same, they play different roles
        - The first analyzer prepares the beam and the second analyzer measures the beam
        - The first analyzer is often referred to as a state preparation device

- Experiment 2:

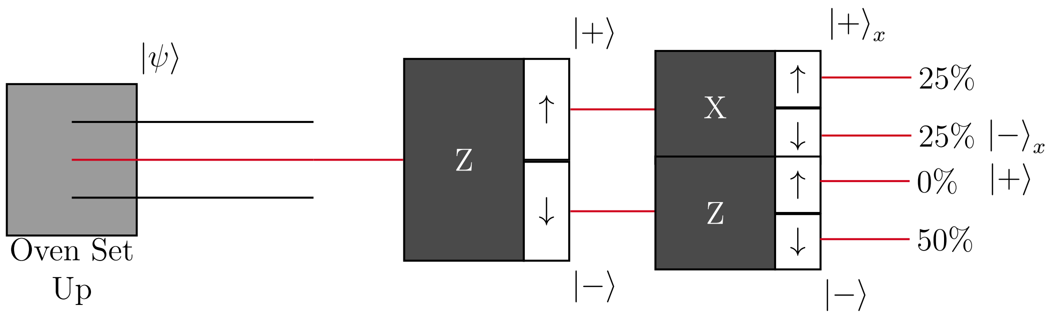

Figure 3: Experimental Setup

    - The second analyzer is rotated by 90° with respect to the first and aligned with the x-axis
        - Only two possible outputs from second analyzer
        - Results would be unchanged if we used the lower part of the first analyzer
        - One can not predict which of the second analyzer parts any particular atom will emerge from
        - These results highlight the probablistic nature of quantum mechanics; quantum mechanics is a complete description of reality (as far as we know)

- Experiment 3:

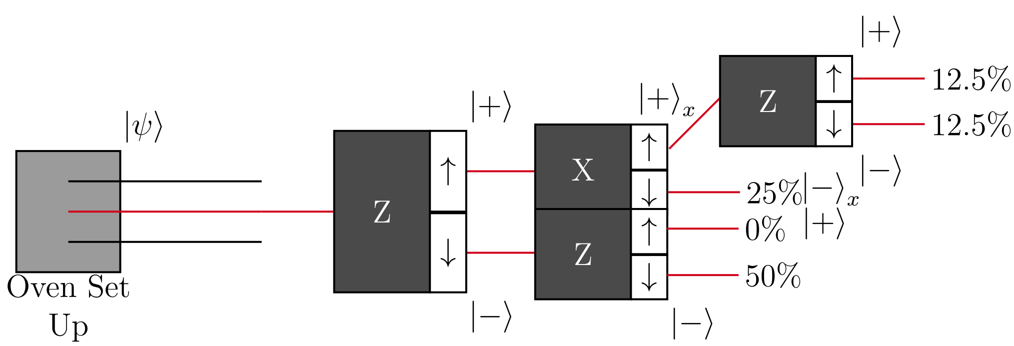

Figure 4: Experimental Setup

* This tells us that the S_x and S_z are incompatible observables, which means we can not know the values of both simultaneously

- Experiment 4:

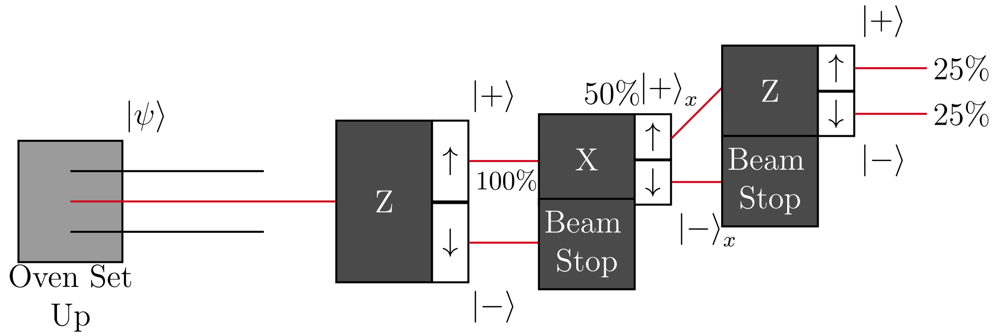

Figure 5: Experimental Setup Part One

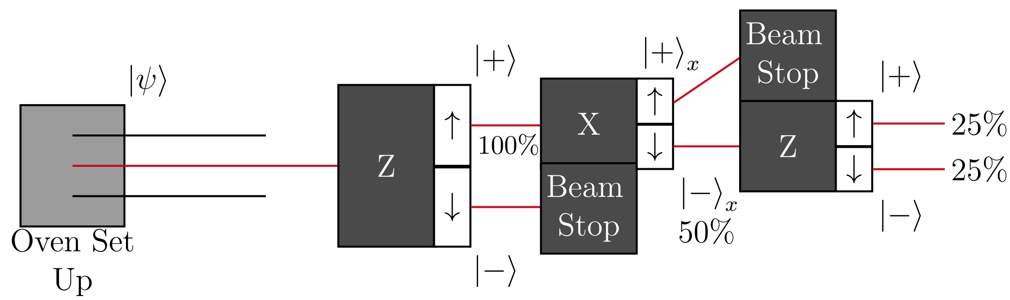

Figure 6: Experimental Setup Part Two

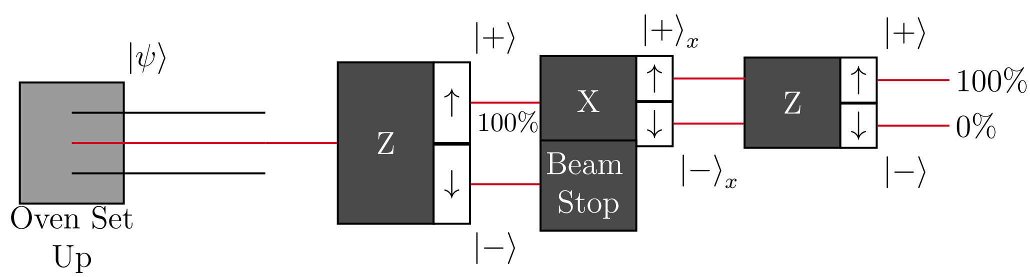

Figure 7: Experimental Setup Part Three

* Results are akin to the double slit experiment and destructive interference

- Quantum State Vectors
  - The kets ($|\psi\rangle$) obey many rules of ordinary spacial vectors
  - $|\psi\rangle$ is a quantum state vector and is part of a vector space called a Hilbert Space
  - The dimensionality of the Hilbert Space is determined by the physics at hand
  - In the Stern-Gerlach experiment the Hilbert Space has just two states ($|+\rangle$ and $|-\rangle$)
    * These states form a basis like unit vectors $\hat{i}, \hat{j}, \hat{k}$ for 3-D space. These basis vectors are normalized, orthogonal, and complete
      · Normailization: $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$
      · Orthogonality: $\hat{i} \cdot \hat{j} = \hat{i} \cdot \hat{k} = \hat{j} \cdot \hat{k} = 0$
      · Completeness: $\vec{A} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$
      · Note: The dot product (or scalar product) displayed in the normalization and orthogonality conditions is central to these properties
    * For the S_z measurement, the basis states are: $|+\rangle$ and $|-\rangle$ and are referred to as the "S_z basis"
      · Since this is a complete basis, a general state is (where a and b are complex numbers):

$$|\psi\rangle = a\,|+\rangle + b\,|-\rangle$$

    * To discuss orthogonality and normalization (orthonormality) we need to understand how scalar products apply to kets
    * The complex-conjugated quantum state vector is called a 'bra' in the Dirac notation of quantum mechanics (note the asterisk indicates the complex conjugate):

$$\langle\psi| = a^*\,\langle+| + b^*\,\langle-|$$

  - The scalar product in quantum mechanics is defined as:

$$(\langle\psi|)(|\psi\rangle) = \langle\psi|\psi\rangle$$

    * "bra-ket" or bracket
    * For example:

$$(\langle+|)(|-\rangle) = \langle+|-\rangle$$

    * This defines an inner product between two state vectors
    * In analogy to 3-D spatial vectors, this is also called a projection

* Given the analogs, we may write that normality is defined as:

$$\langle +|+\rangle = \langle -|-\rangle = 1$$

* Furthermore, orthogonality may be defined as:

$$\langle +|-\rangle = \langle -|+\rangle = 0$$

* Completeness is then defined as:

$$|\psi\rangle = a\,|+\rangle + b\,|-\rangle$$

* With these components, we can then write:

$$\langle +|\psi\rangle = \langle +|\,(a\,|+\rangle + b\,|-\rangle) = \langle +|a|+\rangle + \langle +|b|-\rangle = a\,\langle +|+\rangle + b\,\langle +|-\rangle$$

  · This can then be simplified with our properties to get:

$$\langle +|\psi\rangle = a$$

  · Likewise, we may write:

$$\langle -|\psi\rangle = b$$

* Using these results, we may rewrite the wave function as:

$$|\psi\rangle = \langle +|\psi\rangle\,|+\rangle + \langle -|\psi\rangle\,|-\rangle$$
$$|\psi\rangle = |+\rangle\,\langle +|\psi\rangle + |-\rangle\,\langle -|\psi\rangle$$
$$|\psi\rangle = \underbrace{(|+\rangle\,\langle +| + |-\rangle\,\langle -|)}_{1}\,|\psi\rangle$$

  · We can note that the identity is equivalent to one

* We can also reverse the projection to write:

$$\langle \psi|+\rangle = \langle +|a^*|+\rangle + \langle -|b^*|+\rangle = a^*\,\langle +|+\rangle + b^*\,\langle -|+\rangle = a^*$$

  · This gives us the result that (note this holds for any state):

$$\langle \psi|+\rangle = \langle +|\psi\rangle^* \Rightarrow \langle \phi|\psi\rangle = \langle \psi|\phi\rangle^*$$

  · In quantum mechanics, state vectors must be normalized since they describe a situation where the quantum state has probability 1

$$\langle \psi|\psi\rangle = (a^*\,\langle +| + b^*\,\langle -|)(a\,|+\rangle + b\,|-\rangle)$$
$$= a^*a\,\langle +|+\rangle + a^*b\,\langle +|-\rangle + ab^*\,\langle -|+\rangle + b^*b\,\langle -|-\rangle$$
$$\langle \psi|\psi\rangle = a^*a + b^*b = |a|^2 + |b|^2 = 1$$

- Equivalently, we may write:

$$|\langle +|\psi\rangle|^2 + |\langle -|\psi\rangle|^2 = 1$$

- In quantum mechanics, the probability that the state $|\psi\rangle$ is measured as spin up is $P_{S_z=+\frac{\hbar}{2}} = |\langle +|\psi\rangle|^2$
- Equivalently, the probability that the state $|\psi\rangle$ is measures as spin down is $P_{S_z=-\frac{\hbar}{2}} = |\langle -|\psi\rangle|^2$
  - We can use a shorthand notation of P_+ for $S_z = +\frac{\hbar}{2}$ and P_- for $S_z = -\frac{\hbar}{2}$
  - We may observe that the coefficients of the quantum state function contain the information regarding the probability
  - This leads to Postulate 4 of Quantum Mechanics (for a 1/2-spin system)
    - The probability of obtaining the value $\pm\hbar/2$ in a measurement of the observable, S_z, on a system in the state $|\psi\rangle$ is $P_\pm = |\langle\pm|\psi\rangle|^2$, where $|\pm\rangle$ is the basis ket of S_z corresponding to the result $\pm\hbar/2$
    - $\langle -|\psi\rangle$ is referred to as the probability amplitude (or just amplitude)
- For the Stern-Gerlach experiments in which a different axis was tested after the first analyzer, we may write:

$$|_x\langle +|+\rangle|^2 = |_x\langle -|+\rangle|^2 = |_x\langle +|-\rangle|^2 = |_x\langle -|-\rangle|^2 = \frac{1}{2}$$

- Since $|+\rangle$ and $|-\rangle$ form a complete basis, we can write:

$$|+\rangle_x = a\,|+\rangle + b\,|-\rangle$$
$$|-\rangle_x = c\,|+\rangle + d\,|-\rangle$$

  - Per the results of the Stern-Gerlach experiments, we may write:

$$|a|^2 = |b|^2 = |c|^2 = |d|^2 = \frac{1}{2}$$

  - This means:

$$|+\rangle_x = \frac{1}{\sqrt{2}}\left(|+\rangle + e^{i\alpha}\,|-\rangle\right)$$
$$|-\rangle_x = \frac{1}{\sqrt{2}}\left(|+\rangle + e^{-i\beta}\,|-\rangle\right)$$

  - Therefore, only the relative phase of the weights are important

    * If experiment 1 is repeated, but with both analyzers aligned in the x direction, then one may find:

$$P_{+x} = |_x \langle + | + \rangle_x|^2 = 1$$
$$P_{-x} = |_x \langle - | + \rangle_x|^2 = 0$$

    * Equivalently, we may write:

$$\frac{1}{2}\left(1 + e^{i(\alpha-\beta)}\right) = 0$$
$$e^{i(\alpha-\beta)} = 1$$

    * And finally, we get our condition:

$$e^{i\alpha} = e^{i\beta}$$

    * We are free to choose $\alpha = \beta = 0$
    * Thus, we can express x-axis ket alignment as:

$$|\pm\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle \pm |-\rangle)$$

- Superposition of States

$$|\psi\rangle = a\,|+\rangle + b\,|-\rangle$$

  - The above is a superposition state (or a coherent state) because of the importance of the relative phase of the two terms
  - A contrasting state with different possible outcomes but no definite phase relation is called a mixed state
  - A mixed state is not described by a quantum state $|\psi\rangle$, but instead a density matrix
  - In matrix form, we can write:

$$|\pm\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$$

    * Note the top refers to the coefficient of the 'up' z state and the bottom refers to the coefficient of the 'down' z state. Given this, the basis states are:

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- * A general state may be written as:

$$|\psi\rangle = \begin{pmatrix} \langle +|\psi\rangle \\ \langle -|\psi\rangle \end{pmatrix}$$

  * For Hilbert Spaces whose dimensions are greater than 2, the matrix notation is generally more convenient

- Repeating the earlier analysis but for the y state, we get:

$$|\pm\rangle_y = \frac{1}{\sqrt{2}}(|+\rangle \pm i\,|-\rangle)$$

- In matrix notation this is equivalent to:

$$|\pm\rangle_y = \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$$

- General Quantum States
  - Suppose we have an N-dimensional Hilbert Space with possible outcomes $|a_1\rangle$, $|a_2\rangle$,..., $|a_N\rangle$
  - We can then form the relation:

$$\langle a_i|a_j\rangle = \delta_{ij} \quad \text{orthonormality}$$

    * With the Kroenecker Delta function being:

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

  - We can determine completeness using:

$$\begin{aligned} |\psi\rangle &= \sum_{i=1}^{N} (\langle a_i|\psi\rangle)\,|a_i\rangle \\ &= \sum_{i=1}^{N} |\psi\rangle\,(\langle a_i|a_i\rangle) \\ &= |\psi\rangle \end{aligned}$$

    * Here we see the identity matrix

- Postulates of Quantum Mechanics

- These can not be proven, but have been successfully tested in many experiments

1. The state of a quantum system, including all of the information you can know about it, is represented by the $|\psi\rangle$
2. A physical observable is represented mathematically by an operator A that acts on kets
3. The only possible result of a measurement of an observable is one of the eigenvalues of the corresponding operator A
4. The probability of obtaining the eigenvalue a_n in the measurement of the observable, A, on the system in the state $|\psi\rangle$ is given by:

$$P_{a_n} = |\langle a_n|\psi\rangle|^2$$

   - Where $i\,|a_n\rangle$ is the normalized eigenvector of A corresponding to the eigenvalue a_n

5. After a measurement of A that yields the result a_n, the quantum system is in a new state that is the normalized projection of the original ket onto onto the ket (or kets) corresponding to the result of the measurement

$$|\psi\prime\rangle = \frac{P_n\,|\psi\rangle}{\sqrt{\langle\psi|P_n|\psi\rangle}}$$

6. The time-evolution of a quantum system is determined by the Hamiltonian or total energy equation $H(t)$ through the Schrödinger equation:

$$i\hbar\frac{d}{dt}\,|\psi(t)\rangle = H(t)\,|\psi(t)\rangle$$