Homework 3

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1. (a) We know that, for a spin-1 system, the spin can be:

$$\boxed{S_z = \hbar, 0, -\hbar}$$

Given the quantum state in the problem, we can find the probabilities of each as:

$$P_1 = \left(\frac{1}{\sqrt{30}}\right)^2$$
$$P_0 = \left(\frac{2}{\sqrt{30}}\right)^2$$
$$P_{-1} = \left(\frac{|5i|}{\sqrt{30}}\right)^2$$

This gives us:

$$\boxed{\begin{cases} P_1 & = 1/30 \\ P_0 & = 2/15 \\ P_{-1} & = 5/6 \end{cases}}$$

We can calculate the expectation value of S_z by using:

$$\langle\psi|S_z|\psi\rangle$$

In matrix notation, this gives us:

$$< S_z >= \left[\frac{\hbar}{30}\right] \begin{pmatrix} 1 & 2 & -5i \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 5i \end{pmatrix}$$

We proceed to evaluate:

$$< S_z >= \left[\frac{\hbar}{30}\right] \begin{pmatrix} 1 & 0 & 5i \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 5i \end{pmatrix}$$

$$< S_z >= \left[\frac{\hbar}{30}\right] (1 - (25))$$

$$\boxed{< S_z >= -\frac{24\hbar}{30} = -\frac{12\hbar}{15}}$$

(b) Similar to the expectation value of S_z in part (a), we apply the matrix notation of S_x to write:

$$< S_x >= \left[\frac{\hbar}{30\sqrt{2}}\right] \begin{pmatrix} 1 & 2 & -5i \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 5i \end{pmatrix}$$

$$< S_x >= \left[\frac{\hbar}{30\sqrt{2}}\right] \begin{pmatrix} 2 & 1-5i & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 5i \end{pmatrix}$$

$$< S_x >= \left[\frac{\hbar}{30\sqrt{2}}\right] (4 + 10i - 10i)$$

$$\boxed{< S_x >= \frac{4\hbar}{30\sqrt{2}} = \frac{2\hbar}{15\sqrt{2}}}$$

2.

3. (a) We know that $\hat{A}$ and $\hat{B}$ commute if $\hat{A}\hat{B} = \hat{B}\hat{A}$. Thus, we begin by calculating $\hat{A}\hat{B}$:

$$\hat{A}\hat{B} = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \begin{pmatrix} b_1 & 0 & 0 \\ 0 & 0 & b_2 \\ 0 & b_2 & 0 \end{pmatrix}$$

This gives us:

$$\hat{A}\hat{B} = \begin{pmatrix} a_1 b_1 & 0 & 0 \\ 0 & 0 & a_2 b_2 \\ 0 & a_3 b_2 & 0 \end{pmatrix}$$

Now, we calculate $\hat{B}\hat{A}$:

$$\hat{B}\hat{A} = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & 0 & b_2 \\ 0 & b_2 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$$

This gives us:

$$\hat{B}\hat{A} = \begin{pmatrix} b_1 a_1 & 0 & 0 \\ 0 & 0 & b_2 a_3 \\ 0 & b_2 a_2 & 0 \end{pmatrix}$$

We may see that $\hat{A}\hat{B} \neq \hat{B}\hat{A}$ and, therefore, the operators <u>do not commute</u>

(b) We begin working with $\hat{A}$:

$$|\hat{A} - \lambda \mathbb{1}| = 0$$

This gives us:

$$(a_1 - \lambda)(a_2 - \lambda)(a_3 - \lambda) = 0$$

Thus, we see that, since this is a diagonal matrix, we get $\boxed{\lambda = a_1, a_2, \text{ and } a_3}$ as the eigenvalues. Thus, we may observe that the normalized eigenvectors are:

$$\boxed{\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

We then proceed to do the same with $\hat{B}$:

$$\left| \begin{bmatrix} b_1 - \lambda & 0 & 0 \\ 0 & -\lambda & b_2 \\ 0 & b_2 & -\lambda \end{bmatrix} \right| = 0$$

This gives us:

$$(b_1 - \lambda)(\lambda^2 - b_2^2) = 0$$

We can thus see that the eigenvalues are:

$$\boxed{\lambda = b_1, \pm b_2}$$

From here, we can observe that the normalized eigenvectors become:

$$\boxed{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}}$$

(c) Let us assume that the basis eigenvectors are:

$$|1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad |2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, |3\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We calculate the measure of $\hat{B}$ by writing:

$$\hat{B}\,|2\rangle = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & 0 & b_2 \\ 0 & b_2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\hat{B}\,|2\rangle = \begin{pmatrix} 0 \\ 0 \\ b_2 \end{pmatrix}$$

Rewriting this in terms of the basis vectors of $\hat{B}$, we may find:

$$\hat{B}\,|2\rangle = \frac{b_2}{2}\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right)$$

$$\boxed{\hat{B}\,|2\rangle = \frac{b_2}{\sqrt{2}}\left(|B_2\rangle + |B_3\rangle\right)}$$

Since we may see that the two occur with equal probabilities, we may say that, for the operator $\hat{B}$, the probabilities are $P_{B_2} = P_{B_3} = .5$

On the other hand, we may find a to be:

$$\hat{A}\begin{pmatrix} 0 \\ 0 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ b_2 \end{pmatrix}$$

$$\hat{A}\begin{pmatrix} 0 \\ 0 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ b_2 a_3 \end{pmatrix}$$

We may see that this is equivalent to an integer multiple of the eigenvector $|A_3\rangle$ of operator $\hat{A}$. Thus, we may find that:

$$\hat{A}\hat{B}\,|2\rangle = b_2 a_3\,|A_3\rangle$$

And, therefore, <u>the measurement produces a_3 with a probability of 1</u>

(d) Note that the result from part (a) indicates that we can not know the measurements of A and B at the same time with certainty. This is confirmed by the results from part (c), as we find that measuring $\hat{B}\,|2\rangle$ before $\hat{A}$ has no effect on the result of $\hat{A}$