Homework 9

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1. We begin by writing the ground state of the oscillator as:

$$\phi_0 * (x) = \sqrt[4]{\left(\frac{m\omega}{\pi\hbar}\right)} e^{-m\omega x^2/(2\hbar)}$$

We take $\beta^2 = m\omega/\hbar$ to get:

$$\phi_0 * (x) = \sqrt[4]{\left(\frac{\beta^2}{\pi}\right)} e^{-\beta^2 x^2/2}$$

(a) We may begin by calculating the expectation value of position as:

$$\langle x \rangle = \int_{-\infty}^{\infty} \phi_0^*(x) \cdot x \cdot \phi_0(x)\, dx$$

We may observe that this gives us:

$$\langle x \rangle = \int_{-\infty}^{\infty} x|\phi_0 * (x)|^2\, dx$$

Which ultimately means, due to even symmetry, that:

$$\boxed{\langle x \rangle = 0}$$

Similarly, we compute the momentum:

$$\langle p \rangle = -i\hbar \int_{-\infty}^{\infty} \phi_0^*(x) \frac{d}{dx} \phi_0(x)\, dx$$

We enter the function to get:

$$\langle p \rangle = -i\sqrt{\frac{m\omega\hbar}{\pi}} \int_{-\infty}^{\infty} \left(e^{-\beta^2 x^2/2}\right)\left(-\beta^2 x e^{-\beta^2 x^2/2}\right)\, dx$$

$$\langle p \rangle = -i\sqrt{\frac{m\omega\hbar}{\pi}} \int_{-\infty}^{\infty} -\beta^2 x e^{-\beta^2 x^2}\, dx$$

And, once again, we get:

$$\boxed{\langle p \rangle = 0}$$

We then find the expectations of the squares:

$$\langle x^2 \rangle = \sqrt{\frac{\beta^2}{\pi}} \int_{-\infty}^{\infty} x^2 e^{-\beta^2 x^2}\, dx$$

$$\langle x^2 \rangle = \frac{1}{2\beta^2}$$

$$\boxed{\langle x^2 \rangle = \frac{\hbar}{2m\omega}}$$

And then for the momentum:

$$\langle p^2 \rangle = (-i\hbar)^2 \int_{-\infty}^{\infty} \phi_0^*(x) \frac{d^2}{dx^2} \left[\phi_0(x)\right] dx$$

$$\langle p^2 \rangle = -\hbar^2 \sqrt{\frac{\beta^2}{\pi}} \int_{-\infty}^{\infty} e^{-\beta^2 x^2/2} \frac{d^2}{dx^2} \left[e^{-\beta^2 x^2/2}\right] dx$$

$$\langle p^2 \rangle = -\hbar^2 \sqrt{\frac{\beta^2}{\pi}} \int_{-\infty}^{\infty} e^{-\beta^2 x^2/2} \frac{d}{dx} \left[-\beta^2 x e^{-\beta^2 x^2/2}\right] dx$$

$$\langle p^2 \rangle = -\hbar^2 \sqrt{\frac{\beta^2}{\pi}} \int_{-\infty}^{\infty} e^{-\beta^2 x^2/2} \left[\beta^4 x e^{-\beta^2 x^2/2} - \beta^2 e^{-\beta^2 x^2/2}\right] dx$$

$$\langle p^2 \rangle = -\hbar^2 \sqrt{\frac{\beta^2}{\pi}} \int_{-\infty}^{\infty} \beta^4 x e^{-\beta^2 x^2} - \beta^2 e^{-\beta^2 x^2}\, dx$$

Finally, evaluating gives us:

$$\langle p^2 \rangle = -\hbar^2 \sqrt{\frac{\beta^2}{\pi}} \left[\sqrt{\frac{\beta^2 \pi}{4}} - \sqrt{\beta^2 \pi}\right]$$

$$\langle p^2 \rangle = -\hbar^2 \left[\frac{\beta^2}{2} - \beta^2\right]$$

$$\langle p^2 \rangle = \frac{\beta^2 \hbar^2}{2}$$

$$\boxed{\langle p^2 \rangle = \frac{m\omega\hbar}{2}}$$

(b) Using the ladder operators, we can write out:

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)$$
$$a^\dagger = \frac{1}{\sqrt{2m\omega\hbar}}(ip + m\omega x)$$
$$a = \frac{1}{\sqrt{2m\omega\hbar}}(-ip + m\omega x)$$

This lets us write:

$$\langle n|x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}} \left\langle n|a^\dagger + a\middle|n\right\rangle$$

We continue to expand:

$$\langle n|x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}} \left[\left\langle n|a^\dagger\middle|n\right\rangle + \langle n|a|n\rangle\right]$$

Since we know:

$$a^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle$$
$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle$$

We get:

$$\langle n|x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}} \left[\left\langle n|\sqrt{n+1}\middle|n+1\right\rangle + \left\langle n|\sqrt{n}\middle|n-1\right\rangle\right]$$

Since $\langle m|n\rangle = \delta_{mn} = 0$, we conclude:

$$\boxed{\langle x\rangle = 0}$$

We proceed to evaluate the momentum, since we know:

$$p = i\sqrt{\frac{m\omega\hbar}{2}}(a^\dagger - a)$$

This gives us:

$$\langle n|p|n\rangle = i\sqrt{\frac{m\omega\hbar}{2}} \left\langle n|a^\dagger - a\middle|n\right\rangle$$

Again, we apply the ladder operator-n relationship to get:

$$\langle n|p|n\rangle = i\sqrt{\frac{m\omega\hbar}{2}} \left[\left\langle n|\sqrt{n+1}\middle|n+1\right\rangle - \left\langle n|\sqrt{n}\middle|n-1\right\rangle\right]$$

Once again, we find:

$$\boxed{\langle p \rangle = 0}$$

We continue to find the squares as:

$$\langle x^2 \rangle = \frac{\hbar}{2m\omega} \left\langle n \middle| a^{\dagger 2} + a^\dagger a + a a^\dagger + a^2 \middle| n \right\rangle$$

Since we know that $\left\langle n \middle| a^{\dagger 2} \middle| n \right\rangle = \langle n | a^2 | n \rangle = 0$, we get:

$$\langle x^2 \rangle = \frac{\hbar}{2m\omega} \left\langle n \middle| a a^\dagger + a^\dagger a \middle| n \right\rangle$$

$$\langle x^2 \rangle = \frac{\hbar}{2m\omega} \langle n | n + (n+1) | n \rangle$$

$$\langle x^2 \rangle = \frac{\hbar}{2m\omega} (2n+1)$$

$$\boxed{\langle x^2 \rangle = \frac{\hbar}{m\omega} (n+1/2)}$$

Given that the process is the same, just with a different coefficient, we may write:

$$\boxed{\langle p^2 \rangle = m\omega\hbar(n+1/2)}$$

We may observe that the obtained results are in accordance with the values calculated in (a)

(c) First and foremost, we know that:

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

We first find the position as:

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$\Delta x = \sqrt{\langle x^2 \rangle}$$

$$\Delta x = \sqrt{\frac{\hbar}{m\omega}(n+1/2)}$$

Similarly, we find the momentum as:

$$\Delta p = \sqrt{m\omega\hbar(n+1/2)}$$

Multiplying the two together, we get:

$$\Delta x \Delta p = \left(\sqrt{\frac{\hbar}{m\omega}(n+1/2)} \right) \left(\sqrt{m\omega\hbar(n+1/2)} \right)$$

$$\boxed{\Delta x \Delta p = \hbar(n + 1/2)}$$

Since we know the minimum value of n is 0, we conclude that $\boxed{\Delta x \Delta p \geq \hbar/2}$

2. (a) We can normalize by writing:

$$\langle \psi | \psi \rangle = 1$$
$$A^2 \left[\langle 0| + 2e^{-i\pi/2} \langle 1| \right] \left[|0\rangle + 2e^{i\pi/2} |1\rangle \right] = 1$$
$$A^2 [1 + 4] = 1$$
$$\boxed{A = \sqrt{\frac{1}{5}}}$$

(b) Inserting the normalization constant calculated in (a), in tandem with the time evolution formula, we get:

$$|\psi(t)\rangle = \frac{1}{\sqrt{5}} \left[e^{-iE_o t/\hbar} |0\rangle + 2e^{(i\pi/2) - iE_1 t/\hbar} |1\rangle \right]$$

We can simplify this to:

$$\boxed{|\psi(t)\rangle = \frac{e^{-i\omega t/2}}{\sqrt{5}} \left[|0\rangle + 2e^{(i\pi/2) - i\omega t} |1\rangle \right]}$$

(c) We may write:

$$\langle x \rangle = \langle \psi | x | \psi \rangle$$
$$\langle x \rangle = \sqrt{\frac{\hbar}{2m\omega}} \langle \psi | a^\dagger + a | \psi \rangle$$
$$\langle x \rangle = \frac{1}{5} \sqrt{\frac{\hbar}{2m\omega}} \left[(\langle 0| + 2e^{(-i\pi/2) + i\omega t} \langle 1|)(a^\dagger + a)(|0\rangle + 2e^{(i\pi/2) - i\omega t} |1\rangle \right]$$
$$\langle x \rangle = \frac{1}{5} \sqrt{\frac{\hbar}{2m\omega}} \left[2e^{(i\pi/2) - i\omega t} \langle 0|a|1\rangle + 2e^{(-i\pi/2) + i\omega t} \left\langle 1|a^\dagger \, |0\rangle \right| \right]$$
$$\langle x \rangle = \frac{1}{5} \sqrt{\frac{\hbar}{2m\omega}} \left[2e^{(i\pi/2) - i\omega t} \sqrt{1} + 2e^{(-i\pi/2) + i\omega t} \sqrt{1} \right]$$

We may see that this can be simplified to:

$$\langle x \rangle = \frac{2}{5} \sqrt{\frac{\hbar}{2m\omega}} \left[ie^{-i\omega t} + ie^{i\omega t} \right]$$
$$\boxed{\langle x \rangle = \sqrt{\frac{8\hbar}{25m\omega}} \sin(\omega t)}$$

Similarly, we take momentum as:

$$\langle p \rangle = \langle \psi | p | \psi \rangle$$

$$\langle p \rangle = i\sqrt{\frac{m\omega\hbar}{2}} \langle \psi | a^\dagger - a | \psi \rangle$$

$$\langle p \rangle = \frac{i}{5}\sqrt{\frac{m\omega\hbar}{2}} \left[(\langle 0| + 2e^{(-i\pi/2)+i\omega t} \langle 1|)(a^\dagger - a)(|0\rangle + 2e^{(i\pi/2)-i\omega t} |1\rangle) \right]$$

We can skip to the same step as the position, since the process is the same, except with a different sign:

$$\langle p \rangle = \frac{2i}{5}\sqrt{\frac{m\omega\hbar}{2}} \left[-ie^{-i\omega t} - ie^{i\omega t} \right]$$

$$\boxed{\langle p \rangle = \sqrt{\frac{8m\omega\hbar}{25}} \cos(\omega t)}$$

By Ehrenfest's theorem, we know that classical laws must still be obeyed by quantum particles, such that:

$$\langle p \rangle = m\frac{d}{dt}\langle x \rangle$$

Inserting the expectation value of x calculated above, we find:

$$\langle p \rangle = m\sqrt{\frac{8\hbar}{25m\omega}} \frac{d}{dt}[\sin(\omega t)]$$

Differentiating gives us:

$$\langle p \rangle = m\sqrt{\frac{8\hbar}{25m\omega}} \omega \cos(\omega t)$$

We simplify to get:

$$\langle p \rangle = \sqrt{\frac{8m\omega\hbar}{25}} \cos(\omega t)$$

Thus, <u>we confirmed Ehrenfest's theorem</u>

3. By the measured, we observe that the particle is in a superposition state consisting of $n = 0, 1$. Since each occurs with equal probability, the normalization constant must be $1/\sqrt{2}$. Applying our time-evolution formula, we may write:

$$|\psi(t)\rangle = \frac{e^{-i\omega t/2}}{\sqrt{2}} \left[e^{i\theta_0} |0\rangle + e^{i\theta_1 - i\omega t} |1\rangle \right]$$

Using the same process as (2), we may skip the steps to write:

$$\langle x \rangle = \sqrt{\frac{\hbar}{2m\omega}} \cos(\omega t + \Delta\theta_{01})$$

Equating this to the position measurement, we find:

$$-\sin(\omega t) = \cos(\omega t + \Delta\theta_{01})$$

Thus, we see that $\Delta\theta_{01} = \pi/2$. Accordingly, we may return to our momentum formula from (2) to get:

$$\langle p \rangle = \frac{i}{2}\sqrt{\frac{m\omega\hbar}{2}} \left[e^{i\omega t + i\pi/2} - e^{i\omega t - i\pi/2}\right]$$

$$\langle p \rangle = -\sqrt{\frac{m\omega\hbar}{2}} \sin(\omega t + \pi/2)$$

$$\boxed{\langle p \rangle = -\sqrt{\frac{m\omega\hbar}{2}} \cos(\omega t)}$$