

Homework 5

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1. (a) To be a valid CDF, we know that the terms continuously build until they sum to 1. In this case, all of the terms become 1 at $v = 10$. Thus, we can differentiate to find the PDF:

$$f_V(v) = \frac{d}{dv}[F_V(v)]$$
$$f_V(v) = 2c(v - 2), 2 \leq v < 10$$

We then know:

$$\int_2^{10} 2c(v - 2) dv = 1$$

We can solve to get:

$$2c \left[\frac{v^2}{2} - 2v \right]_2^{10} = 1$$
$$2c [(50 - 2) - (20 - 4)] = 1$$
$$64c = 1$$

Which finally gets us:

$$\boxed{c = \frac{1}{64}}$$

- (b) We can then find the probability that the response time is greater than 5[ms] as:

$$P(v > 5) = 1 - F_V(v)$$
$$P(v > 5) = 1 - \frac{1}{64}(5 - 2)^2$$

$$\boxed{P(v > 5) = \frac{55}{64}}$$

(c) We can then find the response time probability for between 5 and 8 milliseconds:

$$P(5 \leq v < 8) = F_V(8) - F_V(5)$$
$$P(5 \leq v < 8) = \frac{1}{64}[(8-2)^2 - (5-2)^2]$$

$$\boxed{P(5 \leq v < 8) = \frac{27}{64}}$$

(d) We can find this to be:

$$P(v > 7 | 5 \leq v \leq 8) = \frac{P(7 < v \leq 8)}{P(5 \leq v \leq 8)}$$

We find the probability of the numerator:

$$P(7 < v \leq 8) = F_V(8) - F_V(7)$$
$$P(7 < v \leq 8) = \frac{1}{64}[(8-2)^2 - (7-2)^2]$$
$$P(7 < v \leq 8) = \frac{11}{64}$$

This gives us:

$$P(v > 7 | 5 \leq v \leq 8) = \frac{11/64}{27/64}$$

$$\boxed{P(v > 7 | 5 \leq v \leq 8) = \frac{11}{27}}$$

(e) To find the applicable value, we may write:

$$1 - F_V(a) = .36$$

We expand this to write:

$$1 - \frac{1}{64}(a-2)^2 = .36$$

We then solve:

$$a = \sqrt{64(.64)} + 2$$
$$a = \pm 6.4 + 2$$

Since the time has to be positive, we find:

$$\boxed{a = 8.4[\text{ms}]}$$

2. (a) To be a valid PDF, we know:

$$\int_{-\infty}^{\infty} ae^{-.2|x|} dx = 1$$

We expand:

$$\int_{-\infty}^0 ae^{.2x} dx + \int_0^{\infty} ae^{-.2x} dx = 1$$

This gives us:

$$\frac{ae^{.2x}}{.2} \Big|_{-\infty}^0 - \frac{ae^{-.2x}}{.2} \Big|_0^{\infty} = 1$$

We continue to solve:

$$(5a - 0) - (0 - 5a) = 1$$

$$10a = 1$$

$$\boxed{a = .1}$$

- (b) We know that the expectation value can be expressed as:

$$E[x] = \int_{-\infty}^{\infty} xf_X(x) dx$$

This gives us:

$$E[x] = \frac{1}{10} \left[\int_{-\infty}^0 xe^{.2x} dx + \int_0^{\infty} xe^{-.2x} dx \right]$$

$$\boxed{E[x] = 0[\mu V]}$$

We may observe that, due to symmetry about the x axis, the expected value is zero.

- (c) Once again, we break up the function to find the CDF in two regions. We begin with the first region:

$$F_X(x < 0) = \int_{-\infty}^x .1e^{.2x} dx$$

This gives us:

$$F_X(x < 0) = .5e^{.2x} \Big|_{-\infty}^x$$

$$F_X(x < 0) = .5e^{.2x}$$

We then find the second region:

$$F_X(x \geq 0) = \int_{-\infty}^0 .1e^{.2x} dx + \int_0^x .1e^{-.2x} dx$$

This gives us:

$$F_X(x \geq 0) = .5e^{.2x} + [-.5e^{-.2x}] \Big|_0^x$$

$$F_X(x \geq 0) = 1 - .5e^{-.2x}$$

Finally, we may express this as:

$$F_X(x) = \begin{cases} .5e^{.2x}, & x < 0 \\ 1 - .5e^{-.2x}, & x \geq 0 \end{cases}$$

(d) We can calculate the former as:

$$P[X > 0] = 1 - P[X \leq 0]$$

$$P[X > 0] = 1 - \int_{-\infty}^0 f_X(x) dx$$

This gives us:

$$P[X > 0] = 1 - [.5e^{.2x}] \Big|_{-\infty}^0$$

$$P[X > 0] = .5$$

Similarly, we may find:

$$P[X > 2] = 1 - P[X \leq 2]$$

This gives us:

$$P[X > 2] = 1 - P[X > 0] - \int_0^2 .1e^{-.2x} dx$$

We continue to evaluate:

$$P[X > 2] = 1 - .5 - [-.5e^{-.2x}]_0^2$$

$$P[X > 2] = 1 - .5 - [-.5e^{-.4} - (-.5)]$$

$$P[X > 2] = .5e^{-.4}$$

$$P[X > 2] = .3352$$

(e) Given that the PDF is symmetrical, we may simply write:

$$P[|X| > 2] = 2P[X > 2]$$

$$P[|X| > 2] = 2(.3352)$$

$$\boxed{P[|X| > 2] = .6704}$$

3. (a) Given that the PDF can be found as the differential of the CDF, we get:

$$f_X(x) = \frac{d}{dx} [F_X(x)]$$

$$f_X(x) = \frac{1}{16} \frac{d}{dx} [(x-2)^2], \quad 2 \leq x < 6$$

This gives us:

$$\boxed{f_X(x) = \frac{x-2}{8}, \quad 2 \leq x < 6}$$

(b) We can find the expected value using the formula:

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

This gives us:

$$E[X] = \int_{-\infty}^{\infty} x \left(\frac{x-2}{8} \right) dx$$

$$E[X] = \frac{1}{8} \int_2^6 (x^2 - 2x) dx$$

We evaluate to find:

$$E[X] = \frac{1}{8} \left[\frac{x^3}{3} - x^2 \right]_2^6$$

$$E[X] = \frac{1}{8} \left[(72 - 36) - \left(\frac{8}{3} - 4 \right) \right]$$

$$\boxed{E[X] = \frac{112}{24} = 4.6\bar{6}[\text{ms}]}$$

Using this mean value, we can find the variance as:

$$\text{Var}[X] = \int_2^6 (x - 4.6\bar{6})^2 f_X(x) dx$$

We evaluate to get:

$$\boxed{\text{Var}[X] = .88\bar{8}[\text{ms}^2]}$$

(c) Using the PDF, we may find $P[X > 4] = 1 - P[X \leq 4]$. This gives us:

$$P[X > 4] = 1 - \int_2^4 f_X(x) dx$$

$$P[X > 4] = 1 - \int_2^4 \frac{x-2}{8} dx$$

We evaluate to get:

$$P[X > 4] = 1 - \frac{1}{16}(x^2 - 2x) \Big|_2^4$$

$$P[X > 4] = 1 - \frac{1}{16}[(16 - 8) - (4 - 4)]$$

$$\boxed{P[X > 4] = \frac{1}{2} = .5}$$

From here, we may determine the conditional probability of:

$$P[X > 5 | X > 4] = \frac{P[X > 5]}{P[X > 4]}$$

From here, we find:

$$P[X > 5] = 1 - \frac{1}{16}(x^2 - 2x) \Big|_2^5$$

$$P[X > 5] = 1 - \frac{1}{16}[(25 - 10) - (4 - 4)]$$

$$\boxed{P[X > 5] = \frac{1}{16} = .0625}$$

We then find:

$$P[X > 5 | X > 4] = \frac{.0625}{.5}$$

$$\boxed{P[X > 5 | X > 4] = .125}$$

(d) We know that the PDF will scale such that:

$$f_{X|A}(x) = \frac{f_X(x)}{P[A]}$$

Thus, we can find:

$$P[A] = 1 - \frac{1}{16}(x^2 - 2x) \Big|_2^3$$

$$P[A] = 1 - \frac{1}{16}[(9 - 6) - (4 - 4)] \Big|_2^3$$

$$P[A] = \frac{13}{16} = .8125$$

Thus, we scale the initial pdf to get:

$$f_{X|A}(x) = \frac{16}{13} \left[\frac{x-2}{8} \right]$$

$$\boxed{f_{X|A}(x) = \frac{2x-4}{13}, \quad 3 \leq x < 6}$$

4. (a) We know that the PDF is continuous and equivalent over the whole interval, and, thus, we may write:

$$f_X(x) = \frac{1}{a - (-a)}, \quad -a \leq x \leq a$$

$$\boxed{f_X(x) = \frac{1}{2a}, \quad -a \leq x \leq a}$$

- (b) Given that we know the mean will be zero, we may write:

$$\text{Var}[X] = \int_{-a}^a (x-0)^2 \frac{1}{2a} dx$$

$$\text{Var}[X] = \int_{-a}^a \frac{x^2}{2a} dx$$

We then evaluate:

$$\text{Var}[X] = \frac{x^3}{6a} - a^a$$

$$\text{Var}[X] = \left[\frac{a^3}{6a} - \left(-\frac{a^3}{6a} \right) \right]$$

$$\text{Var}[X] = \left[\frac{a^2}{6} + \frac{a^2}{6} \right]$$

$$\boxed{\text{Var}[X] = \frac{a^2}{3}}$$

- (c) We can write:

$$P \left[X > \frac{a}{2} | X > 0 \right] = \frac{P \left[X > \frac{a}{2} \right]}{P[X > 0]}$$

We begin by finding:

$$P[X > 0] = 1 - \frac{0 - (-a)}{2a}$$

$$P[X > 0] = \frac{1}{2}$$

We then find:

$$P\left[X > \frac{a}{2}\right] = 1 - \frac{\frac{a}{2} - (-a)}{a - (-a)}$$

$$P\left[X > \frac{a}{2}\right] = \frac{1}{4}$$

We then divide to get:

$$P\left[X > \frac{a}{2} | X > 0\right] = \frac{.25}{.5}$$

$$\boxed{P\left[X > \frac{a}{2} | X > 0\right] = \frac{1}{2}}$$

- (d) Given the uniform probability, we may simply double our value of $P\left[X > \frac{a}{2}\right]$ to find:

$$P\left[|X| > \frac{a}{2}\right] = 2P\left[X > \frac{a}{2}\right]$$

This then gives us the probability of a fault trigger as:

$$\boxed{P\left[|X| > \frac{a}{2}\right] = \frac{1}{2}}$$

- (e) We begin by setting the equation appropriately to get:

$$2X^2 > \frac{a^2}{3}$$

Solving this gives us:

$$|X| > \frac{a}{\sqrt{6}}$$

Thus, we want to find:

$$P\left[|X| > \frac{a}{\sqrt{6}}\right] = 2P\left[X > \frac{a}{\sqrt{6}}\right]$$

This gives us:

$$P\left[|X| > \frac{a}{\sqrt{6}}\right] = 2\left[1 - \frac{\frac{a}{\sqrt{6}} - (-a)}{2a}\right]$$

$$\boxed{P\left[|X| > \frac{a}{\sqrt{6}}\right] = .5918}$$

6. (a) For an exponential random variable, we know:

$$\lambda = \frac{1}{\mu}$$

Which gives us:

$$\lambda = \frac{1}{2}$$

Thus, we may write the PDF as:

$$f_X(x) = .5e^{-.5x}, \quad x > 0$$

- (b) First, we want to find the probability of the request taking less than two seconds:

$$P[X < 2] = \int_0^2 .5e^{-.5x} dx$$

$$P[X < 2] = -e^{-.5x} \Big|_0^2$$

$$P[X < 2] = [-e^{-1} - (-1)]$$

$$P[X < 2] = .6321$$

From here, since each request is independent, we simply find:

$$P[X_3 < 2] = (P[X < 2])^3$$

$$P[X_3 < 2] = (.6321)^3$$

$$P[X_3 < 2] = .2526$$

- (c) i. Although the λ value stays the same as the above, we define n as the quantity of requests, or 3, such that:

$$Y = \text{Erlang}(3, .5)$$

- ii. Using the Erlang distribution, we write:

$$F_Y(y) = .0625y^2e^{-.5y}$$

- iii. We can find the mean value as:

$$E[Y] = \int_0^\infty .0625y^2e^{-.5y} dy$$

We solve to get:

$$E[Y] = 6$$

Then, we find the variance:

$$\text{Var}[Y] = \int_0^{\infty} .0625(y-6)^2 y^2 e^{-.5y} dy$$

Solving gives us:

$$\boxed{\text{Var}[Y] = 12}$$

iv. We can find the probability as:

$$P[Y < 5] = \int_0^5 .0625 y^2 e^{-.5y} dy$$

Solving gives us:

$$\boxed{P[Y < 5] = .45618}$$

8. (a) Given that we want 95% of all readings to be within 1° of μ , we can use the inverse normal function to find a z-score of:

$$\frac{x - \mu}{\sigma} = \pm 1.96$$

Furthermore, we know that:

$$x - \mu = \pm 1$$

We combine the two to get:

$$\pm 1.96\sigma = \pm 1$$

$$\boxed{\sigma = .5102}$$

- (b) Using our z-score formula, we may write:

$$z = \frac{25 - \mu}{1} = 25 - \mu$$

We know that:

$$P[Y \leq 25^\circ] = P[Z \leq 25 - \mu]$$

Given the probability for $Y \leq 25^\circ$, we can use the inverse normal function to write:

$$z = 1.4985$$

This gives us:

$$25 - \mu = 1.4985$$

$$\boxed{\mu = 23.501^\circ}$$

(c) Similarly, we may find:

$$z = -1.9954$$

From this, we write:

$$-\mu = -1.9954$$

$$\boxed{\mu = 1.9954^\circ}$$

(d) If the probability above or below a certain point is equal to $1/2$, then we are at the mean. Therefore, we may write:

$$\boxed{\mu = 25^\circ}$$

9. (a)

(b)

(c)

(d)

10. (a)

(b)

(c)

11.