Homework 8

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1. We may write the correlation coefficient as:

$$\rho_{XY} = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

Given that $Y = X + 2Z$, we obtain:

$$\text{Cov}(X, Y) = \text{Cov}(X, X + 2Z)$$

We break this apart to get:

$$\text{Cov}(X, X + 2Z) = \text{Cov}(X, X) + 2\text{Cov}(X, Z)$$
$$\text{Cov}(X, X + 2Z) = \text{Var}(X) + 2\text{Cov}(X, Z)$$

Additionally, we get:

$$\text{Var}(Y) = \text{Var}(X + 2Z)$$
$$\text{Var}(X + 2Z) = \text{Var}(X) + 4\text{Var}(Z) + 4\text{Cov}(X, Z)$$

Thus, the expression becomes:

$$\rho_{XY} = \frac{\text{Var}(X) + 2\text{Cov}(X, Z)}{\sqrt{\text{Var}(X)[\text{Var}(X) + 4\text{Var}(Z) + 4\text{Cov}(X, Z)]}}$$

The final step is to calculate the covariance between X and Z. Since it is stated that the two are independent, we arrive at:

$$\text{Cov}(X, Z) = 0$$

And thus:

$$\rho_{XY} = \frac{\text{Var}(X)}{\sqrt{\text{Var}(X)[\text{Var}(X) + 4\text{Var}(Z)]}}$$

We plug in our known values to get:

$$\rho_{XY} = \frac{16}{\sqrt{16[(16) + 4(4)]}}$$

$$\boxed{\rho_{XY} = \frac{\sqrt{2}}{2}}$$

Since the correlation coefficient is not zero, $\underline{X \text{ and } Y \text{ are not independent}}$.

Now, given $W = 2X - Z$, we want to find:

$$E[W] = E[2X - Z]$$

$$\text{Var}(W) = \text{Var}(2X - Z)$$

$$\text{Cov}(W, Y) = \text{Cov}(2X - Z, X + 2Z)$$

For the expectation value, we simply decompose to get:

$$E[W] = 2E[X] - E[Z]$$

$$E[W] = 2(2) - 1$$

$$\boxed{E[W] = 3}$$

The variance can be expanded to get:

$$\text{Var}(W) = 4\text{Var}(X) + \text{Var}(Z) - \cancel{4\text{Cov}(X, Z)}$$

$$\text{Var}(W) = 4(16) + 4$$

$$\boxed{\text{Var}(W) = 68}$$

Finally, we find the covariance as:

$$\text{Cov}(W, Y) = \text{Cov}(2X, X + 2Z) - \text{Cov}(Z, X + 2Z)$$

$$\text{Cov}(W, Y) = 2\text{Var}(X) + 3\text{Cov}(X, Z) - 2\text{Var}(Z)$$

Plugging in values, we get:

$$\text{Cov}(W, Y) = 2(16) + \cancel{3\text{Cov}(X, Z)} - 2(4)$$

$$\boxed{\text{Cov}(W, Y) = 24}$$

2. (a) We may begin by expressing $E[Y]$ as:

$$E[Y] = 2E[X_1] - E[X_2^2]$$

This gives us:

$$E[Y] = 2(0) - \int_{-1.5}^{1.5} \frac{x^2}{3}\, dx$$

$$E[Y] = -\frac{x^3}{9}\Big|_{-1.5}^{1.5}$$

$$\boxed{E[Y] = -\frac{3}{4}}$$

(b) Similarly, we separate the variance to get:

$$\mathrm{Var}(Y) = 4\mathrm{Var}(X_1) + \mathrm{Var}(X_2^2)$$

Using our standard uniform formula, we find:

$$\mathrm{Var}(X_1) = \frac{(3)^2}{12}$$

$$\mathrm{Var}(X_1) = \frac{3}{4}$$

We can then express $\mathrm{Var}(X_2^2)$ as:

$$\mathrm{Var}(X_2^2) = E[X_2^4] - E[X_2^2]^2$$

We compute this and enter already calculated values to get:

$$\mathrm{Var}(X_2^2) = \int_{-1.5}^{1.5} \frac{x^4}{3}\, dx - (.75)^2$$

$$\mathrm{Var}(X_2^2) = \frac{x^5}{15}\Big|_{-1.5}^{1.5} - (.75)^2$$

$$\mathrm{Var}(X_2^2) = 1.0125 - .5625$$

$$\mathrm{Var}(X_2^2) = .45$$

We sum to get:

$$\mathrm{Var}(Y) = 4(.75) + .45$$

$$\boxed{\mathrm{Var}(Y) = 3.45}$$

(c) Finally, we can calculate the covariance as:

$$\text{Cov}(Y, X_1) = \text{Cov}(2X_1 - X_2^2, X_1)$$

We expand to get:

$$\text{Cov}(2X_1 - X_2^2, X_1) = 2\text{Var}(X_1) - \text{Cov}(X_2^2, X_1)$$

Given that X_2 and X_1 are independent, the covariance term becomes zero and we get:

$$\text{Cov}(Y, X_1) = 2\text{Var}(X_1)$$
$$\text{Cov}(Y, X_1) = 2(.75)$$
$$\boxed{\text{Cov}(Y, X_1) = 1.5}$$

3. (a) To find the joint PDF, we may use the following formula:

$$f_{XY}(x, y) = \frac{1}{\sqrt{2\pi(1-\rho^2)\sigma_x\sigma_y}} e^{-\frac{1}{2}\left[\frac{(x-\mu_x)^2}{\sigma_x^2} - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \frac{(y-\mu_y)^2}{\sigma_y^2}\right]}$$

We enter our given values to get:

$$f_{XY}(x, y) = \frac{1}{\sqrt{2\pi(1-.25)(4)(2)}} e^{-\frac{1}{2}\left[\frac{(x-1)^2}{16} - \left(\frac{x-1}{4}\right)\left(\frac{y-2}{2}\right) + \frac{(y-2)^2}{4}\right]}$$

$$\boxed{f_{XY}(x, y) = \frac{1}{\sqrt{12\pi}} e^{-\frac{1}{2}\left[\frac{(x-1)^2}{16} - \left(\frac{x-1}{4}\right)\left(\frac{y-2}{2}\right) + \frac{(y-2)^2}{4}\right]}}$$

(b) Given $V = 2X - 3Y$, we may write:

$$E[V] = E[2X - 3Y]$$
$$\text{Var}(V) = \text{Var}(2X - 3Y)$$

We can obtain the former by writing:

$$E[V] = 2E[X] - 3E[Y]$$
$$E[V] = 2(1) - 3(2)$$
$$\boxed{E[V] = -4}$$

Similarly, we expand the variance to write:

$$\text{Var}(V) = 4\text{Var}(X) + 9\text{Var}(Y) - 12\text{Cov}(X, Y)$$
$$\text{Var}(V) = 4\text{Var}(X) + 9\text{Var}(Y) - 12\rho_{XY}\sigma_x\sigma_y$$

$$\text{Var}(V) = 4(4)^2 + 9(2)^2 - 12\left(\frac{1}{2}\right)(4)(2)$$
$$\text{Var}(V) = 64 + 36 - 48$$
$$\boxed{\text{Var}(V) = 52}$$

(c) Since we know that a linear combination of normal variables also yields a normal variable, we can write:

$$V = \text{Norm}(-4, 52)$$

Accordingly, we find:

$$P[V > 4] = 1 - P\left[\frac{V - \mu_v}{\sigma_v} \leq \frac{4 - \mu_v}{\sigma_v}\right]$$

This gives us:

$$P[V > 4] = 1 - P\left[Z \leq 1.1094\right]$$
$$\boxed{P[V > 4] = .1336}$$

4. We may begin by finding the marginal PDF of X:

$$f_X(x) = \int_0^{2x} 1\,dy$$
$$f_X(x) = 2x, \quad 0 \leq x \leq 1$$

We can then find the expected value as:

$$E[X] = \int_0^1 2x^2\,dx$$
$$E[X] = \frac{2}{3}x^3\Big|_0^1$$
$$E[X] = \frac{2}{3}$$

From here, we find the variance as:

$$\text{Var}(X) = \int_0^1 \left(x - \frac{2}{3}\right)^2 (2x)\,dx$$
$$\boxed{\text{Var}(X) = \frac{1}{18}}$$

Similarly, we may find the marginal PDF of y as:

$$f_Y(y) = \int_{y/2}^{1} 1\, dx$$

$$f_Y(y) = 1 - \frac{y}{2}, \quad 0 \leq y \leq 2$$

We then find the expectation value as:

$$E[Y] = \int_0^2 y\left(1 - \frac{y}{2}\right)\, dy$$

$$E[Y] = \int_0^2 y - \frac{y^2}{2}\, dy$$

$$E[Y] = \frac{y^2}{2} - \frac{y^3}{6}\Big|_0^2$$

$$E[Y] = \frac{2}{3}$$

We then find the variance as:

$$\text{Var}(Y) = \int_0^2 \left(y - \frac{2}{3}\right)^2 \left(1 - \frac{y}{2}\right)\, dy$$

$$\boxed{\text{Var}(Y) = \frac{2}{9}}$$

Then, to compute the covariance, we obtain:

$$E[XY] = \int_0^1 \int_0^{2x} xy\, dy\, dx$$

This gives us:

$$E[XY] = \left(\frac{x^4}{2}\right)\Big|_0^1$$

$$E[XY] = \frac{1}{2}$$

This leads to the covariance as:

$$\text{Cov}(X,Y) = \frac{1}{2} - \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)$$

$$\boxed{\text{Cov}(X,Y) = \frac{1}{18}}$$

We then use the above calculated values to find the total packet variance, or:

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X,Y)$$

Entering our known values, we get:

$$\text{Var}(X+Y) = \frac{1}{18} + \frac{2}{9} + 2\left(\frac{1}{18}\right)$$

$$\boxed{\text{Var}(X+Y) = \frac{7}{18}}$$

5. (a) We may observe that, based on the given information, the Null Hypothesis may be written as:

$$\boxed{H_o : p = .2 \quad \text{(failure rate remains .2)}}$$

Furthermore, the alternative hypothesis becomes:

$\boxed{H_1 : p < .2 \quad \text{(failure rate is below .2)}}$

(b) Given a fixed number of trials, we may observe that the number of failures must follow a binomial distribution. We may write this as:

$$X = \text{Binomial}(20, .2)$$

We may expand this for H_o to write:

$$\boxed{P[X = k] = \binom{20}{k} .2^k .8^{20-k}}$$

For H_1, the distribution shifts since the probability would be below $p = .2$, meaning that p decreases and $1 - p$ increases.

(c) We know that our expectation value may be written as:

$$E[X|H_o] = np$$

$$E[X|H_o] = (20)(.2)$$

$$\boxed{E[X|H_o] = 4 \text{ failures}}$$

(d) There are two types of errors we may observe:

- Type I Error, labeled α — This occurs when the null hypothesis is rejected despite being true. Given this problem, we would find that the probability of failure is <u>not</u> $p = .2$, despite the error rate actually being $p = .2$
- Type II Error, labeled β — This occurs when the null hypothesis is not rejected, despite being false. In this case, we would find that the error rate probability is $p = .2$, even though the actual failure rate is less.

(e) We are given a threshold to reject H_o if $X \leq 1$, and accept H_o if $X > 1$. Accordingly, a Type I Error (α) will occur with probability:

$$\alpha = P[X = 1] + P[X = 0]$$
$$\alpha = .057646 + .011529$$
$$\boxed{\alpha = .069175}$$

(f) Making a Type II Error would mean that the actual probability is $p = .1$, but we find the $p = .2$. This means that the probability of making a Type II Error is the probability of the null hypothesis being accepted with $p = .2$. We can write a z-score as:

$$z = \frac{.2 - .1}{4(.8)/\sqrt{20}}$$
$$z = .25$$

Then taking an inverse CDF function, we find:

$$\beta = \text{invCDF}(.25)$$
$$\boxed{\beta = .4013}$$

The power is thus:

$$\text{Power} = 1 - \beta$$
$$\boxed{\text{Power} = .5987}$$

(g) Since we are required to keep the sample size and difference between probabilities constant, our only option to increase the power is to increase the significance value, α. We can check our α value from (e) to get:

$$Z = \text{invCDF}\left(1 - \frac{\alpha}{2}\right)$$
$$Z = \text{invCDF}(.9654)$$
$$Z = 1.8171$$

We then use this z-score to obtain:

$$\bar{x} = .2 - 1.8171\left(\frac{1.7889}{\sqrt{20}}\right)$$
$$\bar{x} = -.5269$$

We then find the β value by using:

$$P\left[Z \geq \frac{-.5269 - .1}{1.7889\sqrt{20}}\right]$$

$$P\left[Z \geq -.078355\right] = \beta$$

$$\boxed{\beta = .4688}$$

Thus, we may see that we are not using an adequate α value for enough power.

5. **Extra Credit**

(a) The tree diagram can be drawn as:

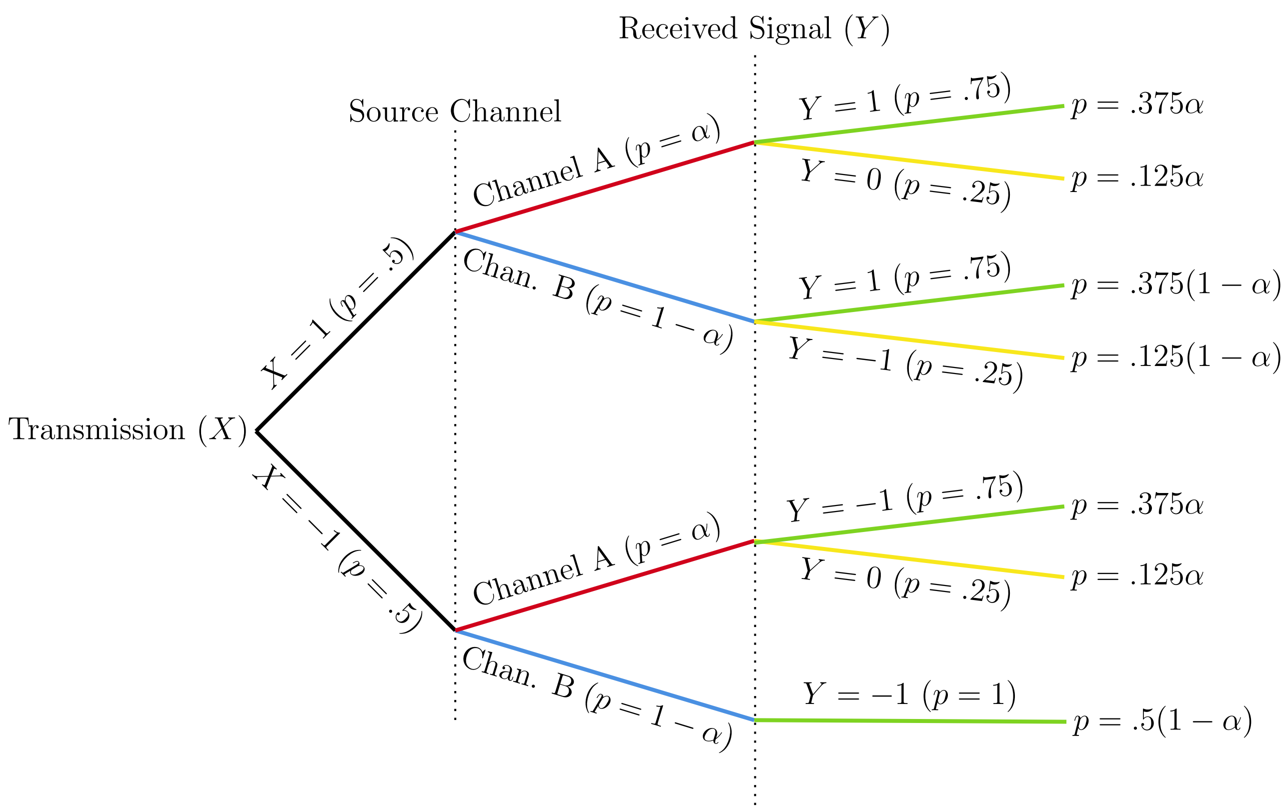

Figure 1: Tree Diagram for Transmission (X) and Receipt (Y)

(b) Based on the tree diagram, we may conclude that $\boxed{Y = \{-1, 0, 1\}}$ with corresponding probabilities $\boxed{p = \{.625 - .25\alpha, .25\alpha, .375\}}$

(c) We want to find $P[A|Y = -1]$, which gives us:

$$P[A|Y = -1] = \frac{P[Y = -1|A]P[A]}{P[Y = -1]}$$

By the tree diagram, we may find:

$$P[Y = -1|A] = .375\alpha$$

$$P[Y=-1] = .625 - .25\alpha$$

Furthermore, we are given:

$$P[A] = \alpha$$

Thus, we obtain:

$$\boxed{P[A|Y=-1] = \frac{.375\alpha^2}{.625 - .25\alpha}}$$

(d) Similarly, we write:

$$P[B|Y=-1] = 1 - \frac{P[Y=-1|B]P[B]}{P[Y=-1]}$$

This gives us:

$$P[Y=-1|B] = .625(1-\alpha)$$
$$P[B] = (1-\alpha)$$

As such, we get:

$$\boxed{P[B|Y=-1] = \frac{.625(1-\alpha)^2}{.625 - .25\alpha}}$$

(e) Given that Channel A is used when:

$$P[A|Y=-1] > P[B|Y=-1]$$

We can enter our expressions to obtain:

$$\frac{.375\alpha^2}{.625 - .25\alpha} > \frac{.625(1-\alpha)^2}{.625 - .25\alpha}$$

We may continue to simplify the expression:

$$.375\alpha^2 > .625(1-\alpha)^2$$
$$.375\alpha^2 > .625(\alpha^2 - 2\alpha + 1)$$
$$.25\alpha^2 - 2\alpha + 1 < 0$$
$$\alpha^2 - 8\alpha + 4 < 0$$
$$\alpha < 4 \pm 2\sqrt{3}$$

Since the probability must be in the range 0 to 1, we obtain:

$$\alpha < .5359$$

As such, we may observe that channel A is used (given $Y = -1$) when:

$$\boxed{0 < \alpha < .5359}$$

(f) Writing similar expressions for each of the probabilities, we get:

$$P[A|Y = 1] = \alpha^2$$
$$P[B|Y = 1] = (1 - \alpha)^2$$

And then:

$$P[A|Y = 0] = \alpha$$
$$P[B|Y = 0] = 0$$

As such, we find:

$$\alpha^2 > (1 - \alpha)^2$$
$$\alpha > 1 - \alpha$$
$$\alpha_{Y=1} > .5$$

Thus, we see that Channel A is used for $Y = 1$ when $\alpha = .6$. Furthermore, Channel A is always used when $Y = 0$.

(g) Using the MAP decision rule, we may observe that channel A is used in every case, since $\alpha > 0, .5, .5359$

6. (a) Using the law of total probability, we may write:

$$P[A] = P[A|B]P[B] + P[A|B']P[B']$$

Thus, we may obtain:

$$P[K = m] = P[K = m|H_1]P[H_1] + P[K = m|H_o]P[H_o]$$

Our values are given as:

$$P[K = m|H_1] = (1 - v_1)v_1^m$$
$$P[K = m|H_o] = (1 - v_o)v_o^m$$
$$P[H_1] = p$$
$$P[H_o] = 1 - p$$

As such, our expression becomes:

$$\boxed{P[K = m] = p(1 - v_1)v_1^m + (1 - p)(1 - v_o)v_o^m}$$

(b) To find this value, we may apply Bayes Rule:

$$P[A|B] = P[B|A] \cdot \frac{P[A]}{P[B]}$$

Using our known values, we may write:

$$P[H_n|K = m] = P[K = m|H_n] \cdot \frac{P[H_n]}{P[K = m]}$$

We may thus obtain:

$$\boxed{P[H_1|K = m] = \frac{p(1 - v_1)v_1^m}{p(1 - v_1)v_1^m + (1 - p)(1 - v_o)v_o^m}}$$

And similarly:

$$\boxed{P[H_o|K = m] = \frac{(1 - p)(1 - v_o)v_o^m}{p(1 - v_1)v_1^m + (1 - p)(1 - v_o)v_o^m}}$$

(c) Using the MAP decision rule, we obtain:

$$P[H_1|K = m] \geq P[H_o|K = m]$$

This gives us:

$$p(1 - v_1)v_1^m \geq (1 - p)(1 - v_o)v_o^m$$

We rearrange to get:

$$\left(\frac{v_1}{v_o}\right)^m \geq \frac{(1 - p)(1 - v_o)}{p(1 - v_1)}$$

$$m \ln\left(\frac{v_1}{v_o}\right) \geq \ln\left(\frac{(1 - p)(1 - v_o)}{p(1 - v_1)}\right)$$

Finally, we find:

$$m \geq \frac{\ln\left(\frac{(1-p)(1-v_o)}{p(1-v_1)}\right)}{\ln\left(\frac{v_1}{v_o}\right)}$$

Thus, we see that the threshold value is:

$$\boxed{n_o = \frac{\ln\left(\frac{(1-p)(1-v_o)}{p(1-v_1)}\right)}{\ln\left(\frac{v_1}{v_o}\right)}}$$

Where $m \geq n_o$ yields H_1 and $m < n_o$ yields H_o

(d) We use our results from part (c) and the given values to get:

$$n_o = \frac{\ln\left(\frac{(.5)(.9)}{.5(.1)}\right)}{\ln\left(\frac{.9}{.1}\right)}$$

$$\boxed{n_o = 1}$$

We may write the probability of error as:

$$P[\text{error}] = .5(.1)(.9)^1 + .5(.9)(.1)^1$$

$$\boxed{P[\text{error}] = .09}$$

7. (a) First and foremost, with the given information, we know that $X = A$ with probability $P[S_1] = p$, and, conversely, $X = 0$ with probability $P[S_o] = 1 - p$. We know the received signal is received according to:

$$Y = X + Z$$

Given that Z is a normal Gaussian distribution with $Z = \text{Norm}(0, \sigma^2)$, we know that, when X reads a high temperature, we get:

$$Y = A + Z$$
$$Y = \text{Norm}(A, \sigma^2)$$

Accordingly, we may write:

$$\boxed{f_{Y|S_1}(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-A)^2}{2\sigma^2}}}$$

Similarly, when the reading is low, Z is not shifted and Y becomes:

$$\boxed{f_{Y|S_o}(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}}}$$

(b) The MAP decision rule states:

$$P[S_1|Y] \geq P[S_o|Y]$$

We apply Bayes' Theorem to obtain:

$$P[S|Y] = \frac{P[Y|S]P[S]}{P[Y]}$$

Plugging this into the equation we want to find:

$$P[Y|S_1]P[S_1] \geq P[Y|S_o]P[S_o]$$

We substitute our expressions from (a), as well as the given probability values to get:

$$pf_{Y|S_1}(y) \geq (1-p)f_{Y|S_o}(y)$$

This gives us:

$$\frac{p}{\sqrt{2\pi\sigma^2}}e^{-\frac{(y-A)^2}{2\sigma^2}} \geq \frac{1-p}{\sqrt{2\pi\sigma^2}}e^{-\frac{y^2}{2\sigma^2}}$$
$$-\frac{(y-A)^2}{2\sigma^2} + \ln(p) \geq -\frac{y^2}{2\sigma^2} + \ln(1-p)$$

We continue to simplify and solve for Y:

$$\frac{-(y-A)^2 + y^2}{2\sigma^2} \geq \ln\left(\frac{1-p}{p}\right)$$
$$\frac{2Ay - A^2}{2\sigma^2} \geq \ln\left(\frac{1-p}{p}\right)$$
$$\frac{2Ay}{2\sigma^2} \geq \frac{A^2}{2\sigma^2} + \ln\left(\frac{1-p}{p}\right)$$
$$y \geq \frac{A}{2} + \frac{\sigma^2}{A}\ln\left(\frac{1-p}{p}\right)$$

Accordingly, we may state that the threshold voltage is:

$$\boxed{V_{th} = \frac{A}{2} + \frac{\sigma^2}{A}\ln\left(\frac{1-p}{p}\right)}$$

If we take $p \rightarrow .5$, then we obtain:

$$V_{th} = \frac{A}{2} + \frac{\sigma^2}{A}\ln(1)$$
$$\boxed{V_{th} = \frac{A}{2}}$$

8. We may observe that the random variable is given by:

$$h(y) = f_x(g^{-1}(y))\left[\frac{d}{dy}g^{-1}(y)\right]$$

Using this, we take:

$$Y = 2 + 5X^2 \rightarrow X = \sqrt{\frac{Y-2}{5}}$$

Which gives us:

$$g^{-1}(y) = \sqrt{\frac{Y-2}{5}}$$

Taking the differential gives us:

$$\frac{d}{dy}[g^{-1}(y)] = \frac{1}{10}\sqrt{\frac{5}{Y-2}}$$

Since f_X is uniform, we write:

$$f_X = \frac{1}{4}, \quad -2 \leq x \leq 2$$

Accordingly, we may find that $h(y)$ is:

$$h(y) = \frac{1}{40}\sqrt{\frac{5}{Y-2}}$$

Furthermore, we know that: $-2 \leq X \leq 2$, which gives us:

$$0 < \sqrt{\frac{Y-2}{5}} \leq 2 \rightarrow 2 < Y \leq 22$$

Thus, the PDF is:

$$\boxed{f_Y(y) = \frac{1}{40}\sqrt{\frac{5}{y-2}}, \quad 2 < y \leq 22}$$

We integrate to find the CDF:

$$F_Y(y) = \frac{1}{40}\int_2^y \left(\frac{5}{y-2}\right)^{\frac{1}{2}} dy$$

$$\boxed{F_Y(y) = \begin{cases} 0, & y < 2 \\ \frac{\sqrt{5y-10}}{20}, & 2 \leq y \leq 22 \\ 1, & \text{otherwise} \end{cases}}$$

9. We can find the probability within the given range by writing:

$$P[W=0] = P[|V| < .4] = \int_{-.4}^{.4} \frac{1}{20}\, dv$$

$$\boxed{P[W=0] = P[|V| < .4] = \frac{.8}{20} = .04}$$

From here, we may write the PDF of W as:

$$\boxed{f_W(w) = \begin{cases} .05, & -10 \leq w < -.04 \\ .04, & w = 0 \\ .05, & .04 < w \leq 10 \\ 0, & \text{otherwise} \end{cases}}$$