Homework 7

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1. (a) Using our formulas to obtain the marginal PDFs, we write:

$$f_X(x) = \int_0^\infty f_{XY}(x,y)\,dy$$
$$f_Y(y) = \int_0^\infty f_{XY}(x,y)\,dx$$

This gives us:

$$f_X(x) = 8e^{-4x}\int_0^\infty e^{-2y}\,dy$$
$$f_Y(y) = 8e^{-2y}\int_0^\infty e^{-4x}\,dx$$

We continue to solve to get:

$$f_X(x) = 8e^{-4x}\int_0^\infty e^{-2y}\,dy$$
$$f_X(x) = -4e^{-4x}\left[e^{-2y}\right]\Big|_0^\infty$$
$$\boxed{f_X(x) = 4e^{-4x}, \quad x \geq 0}$$

$$f_Y(y) = 8e^{-2y}\int_0^\infty e^{-4x}\,dx$$
$$f_Y(y) = -2e^{-2y}\left[e^{-4x}\right]\Big|_0^\infty$$
$$\boxed{f_Y(y) = 2e^{-2y}, \quad y \geq 0}$$

We may observe that the two are independent random variables, since:

$$f_{XY}(x,y) = f_X(x)f_Y(y)$$

$$f_{XY}(x,y) = \left(4e^{-4x}\right)\left(2e^{-2y}\right)$$

$$\boxed{f_{XY}(x,y) = 8e^{-(4x+2y)} \quad \checkmark}$$

Furthermore, we may see that the individual PDFs follow an exponential form, with $\boxed{\lambda_x = 4}$ and $\boxed{\lambda_y = 2}$

(b) We may express this probability using the bounds defined by $y \geq 0$ and $x \geq y$, which gives us:

$$P[X > Y] = \int_0^\infty \int_y^\infty 8e^{-4x}e^{-2y}\,dx\,dy$$

We solve this to get:

$$P[X > Y] = \int_0^\infty -2e^{-2y}\left[e^{-4x}\right]\Big|_y^\infty\,dy$$

$$P[X > Y] = \int_0^\infty 2e^{-6y}\,dy$$

$$P[X > Y] = -\frac{1}{3}\left[e^{-6y}\right]\Big|_0^\infty$$

$$\boxed{P[X > Y] = \frac{1}{3}}$$

Similarly, we pay express $P[X+Y \leq 1]$ with bounds of $0 \leq x \leq 1$ and $0 \leq y \leq 1-x$, which gives us:

$$P[X+Y \leq 1] = \int_0^1 \int_0^{1-x} 8e^{-4x}e^{-2y}\,dy\,dx$$

We solve this to get:

$$P[X+Y \leq 1] = \int_0^1 -4e^{-4x}\left[e^{-2y}\right]\Big|_0^{1-x}\,dx$$

$$P[X+Y \leq 1] = \int_0^1 -4e^{-4x}\left[e^{-2+2x}-1\right]\,dx$$

$$P[X+Y \leq 1] = -4e^{-2}\int_0^1 e^{-2x}\,dx + 4\int_0^1 e^{-4x}\,dx$$

$$P[X+Y \leq 1] = 2e^{-2}\left[e^{-2x}\right]\Big|_0^1 - \left[e^{-4x}\right]\Big|_0^1$$

$$P[X+Y \leq 1] = 2e^{-4} - 2e^{-2} - e^{-4} + 1$$

$$\boxed{P[X+Y \leq 1] = .7476}$$

(c) Since X and Y are independent, we can expand this statement to write:

$$P[\min(X,Y) \geq .5] = P[X \geq .5, Y \geq .5] \rightarrow P[X \geq .5]P[Y \geq .5]$$

As such, we find each component as:

$$P[X \geq .5] = \int_{.5}^{\infty} 4e^{-4x}\, dx$$
$$P[Y \geq .5] = \int_{.5}^{\infty} 2e^{-2y}\, dy$$

We solve to find:

$$P[X \geq .5] = -\left[e^{-4x}\right]\Big|_{.5}^{\infty}$$
$$P[X \geq .5] = -\left[0 - e^{-2}\right]$$
$$P[X \geq .5] = .1353$$

$$P[Y \geq .5] = -\left[e^{-2y}\right]\Big|_{.5}^{\infty}$$
$$P[Y \geq .5] = -\left[0 - e^{-1}\right]$$
$$P[Y \geq .5] = .3679$$

We multiply the two to find:

$$P[\min(X,Y) \geq .5] = (.1353)(.3679)$$
$$\boxed{P[\min(X,Y) \geq .5] = .049787}$$

(d) Similar to part (c), we write:

$$P[\max(X,Y) \leq .5] = P[X \leq .5, Y \leq .5] \rightarrow P[X \leq .5]P[Y \leq .5]$$

This gives us:

$$P[X \leq .5] = \int_{0}^{.5} 4e^{-4x}\, dx$$
$$P[Y \leq .5] = \int_{0}^{.5} 2e^{-2y}\, dy$$

We solve to get:

$$P[X \leq .5] = \int_{0}^{.5} 4e^{-4x}\, dx$$
$$P[X \leq .5] = -\left[e^{-4x}\right]\Big|_{0}^{.5}$$

$$P[X \leq .5] = -\left[e^{-2} - 1\right]$$
$$P[X \leq .5] = .8647$$

$$P[Y \leq .5] = \int_0^{.5} 2e^{-2y}\, dy$$
$$P[Y \leq .5] = -\left[e^{-2y}\right] 0^{.5}$$

$$P[Y \leq .5] = -\left[e^{-1} - 1\right]$$
$$P[Y \leq .5] = .6321$$

We then multiply the two to find:

$$P[\max(X, Y) \leq .5] = (.8647)(.6321)$$
$$\boxed{P[\max(X, Y) \leq .5] = .5466}$$

2. (a)
   (b)
   (c)
   (d)

4. (a) To find $P[X \leq 1]$, we must first find the individual PDF of x. We begin by finding this:

$$f_X(x) = \frac{1}{24} \int_0^4 x + y\, dy$$

This gives us:

$$f_X(x) = \frac{1}{48} \left[2xy + y^2\right] \Big|_0^4$$
$$f_X(x) = \frac{x}{6} + \frac{1}{3}, \quad 0 \leq x \leq 2$$

From here, we get:

$$P[X \leq 1] = \int_0^1 f_X(x)\, dx$$
$$P[X \leq 1] = \frac{1}{6} \int_0^1 x + 2\, dx$$
$$P[X \leq 1] = \frac{1}{12} \left[x^2 + 4x\right] \Big|_0^1$$
$$\boxed{P[X \leq 1] = \frac{5}{12}}$$

(b)

(c)

6. (a)

   (b)

   (c)

   (d)

   (e)

7. (a)

   (b)

   (c)

   (d)

   (e)

8. (a)

   (b)

   (c)

   (d)

   (e)

9. (a)

   (b)

   (c)

   (d)

   (e)