

Lecture 3 — Multiple Random Variables

Michael Brodskiy

Professor: I. Salama

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- The Law of Total Expectation

$$E[Y] = E_X[E[Y|X]]$$

- The Law of Total Variance

$$\text{Var}[Y] = E_X[\text{Var}[Y|X]] + \text{Var}_X[E[Y|X]]$$

- For joint random variables, we know:

- $F_{XY}(x, y) = \int_{-\infty}^y \int_{-\infty}^x f_{XY}(x, y) dx dy$
- $F_{XY}(\infty, \infty) = 1$
- $F_{XY}(x, \infty)$ represents the marginal CDF of X , $F_X(x)$
- $F_{XY}(\infty, y)$ represents the marginal CDF of Y , $F_Y(y)$

- Sum of Two Random Variables

- $Z = X + Y$
- $E[Z] = E[X] + E[Y]$ ($\mu_Z = \mu_X + \mu_Y$)
- $\text{Var}[Z] = \text{Var}[X + Y] = E[((X + Y) - (\mu_X + \mu_Y))^2]$
- $\text{Var}[Z] = E[((X - \mu_X)^2 + 2(X - \mu_X)(Y - \mu_Y) + (Y - \mu_Y)^2)]$
- $\text{Var}[Z] = \text{Var}[X] + \text{Var}[Y] + 2E[(X - \mu_X)(Y - \mu_Y)]$
- $\text{Var}[Z] = \text{Var}[X] + \text{Var}[Y] + 2\text{Cov}(X, Y)$
- $\text{Var}[Z] = E[((X - \mu_X)^2 + 2(X - \mu_X)(Y - \mu_Y) + (Y - \mu_Y)^2)]$
- Covariance:

$$\text{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$$

- The Correlation Coefficient

$$\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

- Standardizes covariance to range from -1 to 1, making it unitless and easier to interpret
- When the correlation coefficient magnitude is close to unity, it implies a high level of correlation

- Properties of Covariance

- $\text{Cov}(X, X) = \text{Var}(X)$
- If X and Y are independent, $\text{Cov}(X, Y) = 0$
- $\text{Cov}(X, Y) = \text{Cov}(Y, X)$
- $\text{Cov}(aX, bY) = ab\text{Cov}(X, Y)$
- $\text{Cov}(X + c, Y) = \text{Cov}(X, Y)$
- Distributive Property: $\text{Cov}(aX + bY + c, Z) = a\text{Cov}(X, Z) + b\text{Cov}(Y, Z)$