

Lecture 1 — Basics of Probability Theory

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- What is a Set?

- A set is a collection of objects (elements) that make up the set
- We usually use upper case letters to describe a set and lower-case letters to refer to the elements
- A set can be defined using enumerations:

$$A = \{\text{Jane, Bill}, \dots\}$$

$$B = \{1, 2, 3, \dots\}$$

- A set can also be defined using a description method
- $A = \{x \mid x \text{ satisfies some property}\}$
- For example:

$$A = \{\text{Students} \mid \text{Students who earned an 'A'}\}$$

- A set can have a finite or infinite number of elements
- Useful notations:
 - * $x \in A \equiv$ element x is contained in A
 - * $x \notin A \equiv$ element x is not contained in A
 - * $C = \{\} = \emptyset$ — C is an empty or null set
 - * $D = S$ — Universal set including all elements in a given category
 - * $A \subset B$ — A is a subset of set B
 - * Simple Set: A set with a single element
 - * Set equality: $A = B$ only if $A \subset B$ and $B \subset A$
 - * $A^C \equiv$ complement of set A , which includes all elements in a given category that are not in set A

- The Inclusion-Exclusion Rule

- For two finite sets A and B , we have:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- This can be expanded to three sets (with a new set C):

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

- Types of Sets

- Collectively Exhaustive Sets — A collection of sets A_1, A_2, \dots, A_n are collectively exhaustive if (at least one of the events must occur):

$$A_1 \cup A_2 \cup \dots \cup A_n = S$$

- Mutually Exclusive Sets — Two sets are mutually exclusive if they have no elements in common:

$$A_i \cap A_j = \emptyset \quad i \neq j$$

- Partitions — A collection of sets A_1, A_2, \dots, A_n is a partition if they are both mutually exclusive and collectively exhaustive:

$$A_1 \cup A_2 \cup \dots \cup A_n = S \quad \text{and} \quad A_i \cap A_j = \emptyset \quad i \neq j$$

- Algebraic Rules of Manipulating Sets

- Commutative: $A \cap B = B \cap A$ and $A \cup B = B \cup A$
- Associative: $A \cap (B \cap C) = (A \cap B) \cap C$
- Distributive: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- De Morgan's Law: $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$

- Applying Set Theory to Probability Theory

- The Probability Function
 - * A probability measure $P[x]$ is a function that maps events in the sample space to real numbers
 - * The probability function assigns a value 0 to 1 to a certain event; for example, the probability function of rolling a single, standard die would be $P[1] = P[2] = \dots = P[6] = 1/6$
- Axioms of Probability
 - * For any event A , $P[A] \geq 0$
 - * $P[S] = 1$

- * For any countable collection of mutually exclusive events:

$$P[A_1 \cup A_2 \cup \dots \cup A_n] = P[A_1] + P[A_2] + \dots + P[A_n]$$

- Consequences of the Axioms

- * For mutually exclusive events, $P[A \cup B] = P[A] + P[B]$
- * For mutually exclusive events, A_1, A_2, \dots, A_N :

$$P[A_1 \cup A_2 \cup \dots \cup A_N] = P[A_1] + P[A_2] + \dots + P[A_N]$$

- Prior, Posterior, and Conditional Probabilities

- Prior Probability $P[A]$: The probability of an event A before any other information or evidence is considered
- Conditional Probability $P[A|B]$: The probability of A given that B has occurred, showing how B influences the likelihood of A

- * $P[A]$ = Prior Probability of Event $A \rightarrow P[A] = \frac{|A|}{|S|}$
- * $P[A|B]$ = Conditional Probability of A Given Event B Occurred
- * When event B occurs, the sample space is reduced to B

$$P[A|B] = \frac{|A \cap B|}{|B|} = \frac{P[A \cap B]}{P[B]}$$

- Posterior Probability $P[B|A]$: The probability of event B after observing event A . It represents a “revised belief” about B , incorporating the new information provided by A . Posterior probability adjusts the prior probability of B based on the evidence from A

- Bayes Rule and Statistical Independence

$$P[A \cap B] = P[AB] = P[B]P[A|B] = P[A]P[B|A]$$

- Bayes’ Rule states:

$$P[A|B] = \frac{P[AB]}{P[B]} \quad \text{and} \quad P[B|A] = \frac{P[AB]}{P[A]}$$

- A and B are statistically independent if:

$$P[AB] = P[A]P[B] \quad \text{or} \quad P[A|B] = P[A] \text{ and } P[B|A] = P[B]$$

- Probability Chain Rule of Three Events

$$P[ABC] = P[C]P[AB|C] = P[C]P[B|C]P[A|BC]$$

- A, B , and C are statistically independent if $P[ABC] = P[A]P[B]P[C]$ and each pair satisfies the condition of independence:

- * A and B are statistically independent
- * A and C are statistically independent
- * B and C are statistically independent

- Permutations

- The number of ways to arrange k distinguishable objects if a set of n distinguishable objects with repetition is:

$$\# = n^k$$

- Combinations

- Given a set of n distinct objects, any unordered subset k of the objects is called a combination of size k . The number of combinations of size k that can be formed from n distinct objects will be denoted by ${}_nC_k$ or $\binom{n}{k}$ and is given by:

$$\frac{n!}{k!(n-k)!}$$