Lecture 3 — Multiple Random Variables

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March 17, 2025

- The Law of Total Expectation

$$E[Y] = E_X[E[Y|X]]$$

- The Law of Total Variance

$$\text{Var}[Y] = E_X[\text{Var}[Y|X]] + \text{Var}_X[E[Y|X]]$$

- For joint random variables, we know:
  - $F_{XY}(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{XY}(x, y)\, dx\, dy$
  - $F_{XY}(\infty, \infty) = 1$
  - $F_{XY}(x, \infty)$ represents the marginal CDF of X, $F_X(x)$
  - $F_{XY}(\infty, y)$ represents the marginal CDF of Y, $F_Y(y)$

- Sum of Two Random Variables
  - $Z = X + Y$
  - $E[Z] = E[X] + E[Y]$ $(\mu_Z = \mu_X + \mu_Y)$
  - $\text{Var}[Z] = \text{Var}[X + Y] = E[((X + Y) - (\mu_X + \mu_Y)^2)]$
  - $\text{Var}[Z] = E[((X - \mu_X)^2 + 2(X - \mu_X)(Y - \mu_Y) + (Y - \mu_Y)^2)]$
  - $\text{Var}[Z] = \text{Var}[X] + \text{Var}[Y] + 2E[(X - \mu_X)(Y - \mu_Y)]$
  - $\text{Var}[Z] = \text{Var}[X] + \text{Var}[Y] + 2\text{Cov}(X, Y)$
  - $\text{Var}[Z] = E[((X - \mu_X)^2 + 2(X - \mu_X)(Y - \mu_Y) + (Y - \mu_Y)^2)]$
  - Covariance:

$$\text{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$$

- The Correlation Coefficient

$$\rho(X,Y) = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y}$$

  - Standardizes covariance to range from -1 to 1, making it unitless and easier to interpret
  - When the correlation coefficient magnitude is close to unity, it implies a high level of correlation

- Properties of Covariance
  - $\text{Cov}(X,X) = \text{Var}(X)$
  - If X and Y are independent, $\text{Cov}(X,Y) = 0$
  - $\text{Cov}(X,Y) = \text{Cov}(Y,X)$
  - $\text{Cov}(aX,bY) = ab\text{Cov}(X,Y)$
  - $\text{Cov}(X+c,Y) = \text{Cov}(X,Y)$
  - Distributive Property: $\text{Cov}(aX+bY+c,Z) = a\text{Cov}(X,Z) + b\text{Cov}(Y,Z)$

- Standard Bivariate Normal Distribution

$$f_{XY}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)}$$

  - Where $-1 < \rho < 1$