Homework 1

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1. (a) We know that union is a set containing all elements from both A and B. Thus, we get:

$$\boxed{A \cup B = \{\mathbf{4GB}, \mathbf{8GB}, \mathbf{12GB}, \mathbf{32GB}\}}$$

On the other hand, an intersection is all elements common to A and B. Thus, we get:

$$\boxed{A \cap B = \{\mathbf{32GB}\}}$$

(b) Complements contain all elements from the sample space, less the elements present in the original set, thus, we get:

$$\boxed{A^c = \{\mathbf{4GB}, \mathbf{12GB}, \mathbf{16GB}, \mathbf{64GB}\}}$$
$$\boxed{B^c = \{\mathbf{8GB}, \mathbf{16GB}, \mathbf{64GB}\}}$$

(c) Set subtraction removes elements common to both sets. Thus, we get:

$$\boxed{B - A = \{\mathbf{4GB}, \mathbf{12GB}\}}$$

A and B^c from above, we take the intersection to find:

$$\boxed{A \cap B^c = \{\mathbf{8GB}\}}$$

(d) We know that $B \cap A = A \cap B$, and, therefore, we perform the following operation:

$$\{\mathbf{4GB}, \mathbf{12GB}\} \cup \{\mathbf{32GB}\}$$

This gives us:

$$\boxed{(B - A) \cup (B \cap A) = \{\mathbf{4GB}, \mathbf{12GB}, \mathbf{32GB}\}}$$

We can note that this simply returns our original B.

(e) First, we find the complement of the union. From (a) we take the complement of the union to get:

$$\boxed{(A \cup B)^c = \{\mathbf{16GB}, \mathbf{64GB}\}}$$

Alternatively, we find the intersection of the A and B complements from part (b) to see:

$$\boxed{A^c \cap B^c = \{\mathbf{16GB}, \mathbf{64GB}\}}$$

As De Morgan's Laws state, the two should be equivalent.

2. (a) The sample space is made up of permutations of the possible modes for three processors. This gives us:

$$\boxed{S = \{\mathbf{AAA}, \mathbf{AAB}, \mathbf{ABA}, \mathbf{ABB}, \mathbf{BAA}, \mathbf{BAB}, \mathbf{BBA}, \mathbf{BBB}\}}$$

Note that we may confirm that this is an exhaustive list, as there are 3 binary options meaning that there are $2^3 = 8$ possible permutations of these modes, as shown above.

(b) Event 1 constrains the sample space to all cases in which the first processor operates in **Mode** '**A**' which gives us:

$$\boxed{\mathbf{E_1} = \{\mathbf{AAA}, \mathbf{AAB}, \mathbf{ABA}, \mathbf{ABB}\}}$$

(c) Event 2 constrains the sample space to all cases in which the first and third processors operate in **Mode** '**B**' which gives us:

$$\boxed{\mathbf{E_2} = \{\mathbf{BAB}, \mathbf{BBB}\}}$$

(d) We can take the intersection of the two events to find:

$$\mathbf{E_1} \cap \mathbf{E_2} = \emptyset$$

<u>Therefore, the two are mutually exclusive</u>

(e) Taking the union of the two events, we find:

$$\mathbf{E_1} \cap \mathbf{E_2} \neq S$$

<u>Therefore, the two are **not** collectively exhaustive</u>

(f) Subtracting $\mathbf{E_2}$ from $\mathbf{E_1}$ gives us:

$$\mathbf{E_1} - \mathbf{E_2}$$

Given that the two share no elements, we simply get $\mathbf{E_1}$ such that:

$$\mathbf{E_1} - \mathbf{E_2} = \mathbf{E_1} = \{\mathbf{AAA}, \mathbf{AAB}, \mathbf{ABA}, \mathbf{ABB}\}$$

4. We can express the event A as:

$$A = \left\{ (i,j) \,\middle|\, |i-j| = 1,\, i+j > 4 \right\}$$

First, let us define pairs such that $|i-j| = 1$. This gives us:

$$E_{|i-j|=1} = \{(1,0),(1,2),(3,2),(3,4)\}$$

Then, we define pairs such that $i+j > 4$:

$$E_{i+j>4} = \{(3,2),(3,4),(4,2),(4,4)\}$$

We then take the intersection of the two sets to find:

$$\boxed{A = \{(3,4),(3,2)\}}$$

We know that the total amount of outcomes (since each is equally likely) is the quantity of possibilities of i time the quantity of possibilities of j, which gives us the quantity of elements in S:

$$N_S = (3)(4) = 12$$

Given that there are two outcomes possible for A, we can find the probability of A occurring as:

$$\boxed{P[A] = \frac{2}{12} = .1667}$$

5. (a) First and foremost, we may rewrite $P[A \cup B]$ to get:

$$P[A] + P[B] - P[A \cap B] \geq P[A]$$

This is equivalent to:

$$P[A] + P[A^c \cap B] \geq P[A]$$

Subtracting over, we get:

$$P[A^c \cap B] \geq 0$$

Since probability must always be greater than or equal to zero, we know that the above statement will always be true, and, therefore, the following must be true as well:

$$\boxed{\therefore P[A \cup B] \geq P[A]}$$

(b) Similarly to part A, we may rewrite $P[A \cap B]$ to get:

$$P[A] + P[B] - P[A \cup B] \leq P[A]$$

We subtract over to get:

$$P[B] - P[A \cup B] \leq 0$$
$$P[B] \leq P[A \cup B]$$

Since a union will always have either more or an equal amount of elements as the original set, the above statement is always true, and, therefore, we may write:

$$\boxed{\therefore P[A \cap B] \leq P[A]}$$

(c) Since subtraction removes common elements, we know that there may be two cases: A and B have common elements, or A and B have no common elements. In the case of the latter, we know that the two are mutually exclusive, and therefore: $P[A - B] = P[A] - P[B]$, which gives us:

$$P[A] - P[B] \leq P[A]$$
$$P[B] \geq 0$$

We know that the above statement is always true, since probability can not be negative. Now, in the case of the former, set $A - B$ would contain less elements than A. Therefore, we can state that:

$$\boxed{\therefore P[A - B] \leq P[A]}$$

6. (a) The first event (among A, B, and C only B is applied) is shown below:

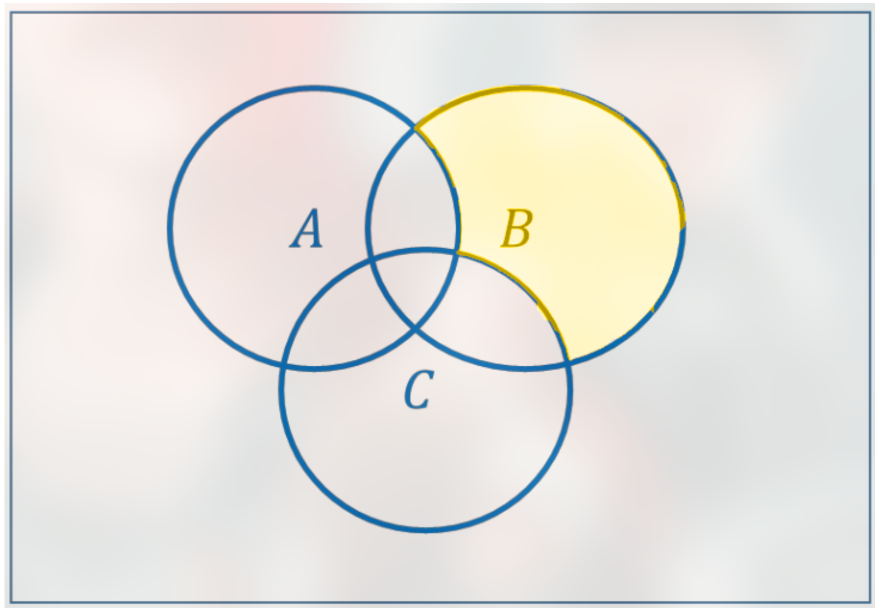

Figure 1: Event 1 Highlighted in Gold

This set can be expressed as:

$$\boxed{\mathbf{E_1} = B - (A \cup C)}$$

(b) The second event (at least one technique, A, B, or C is applied) is shown below:

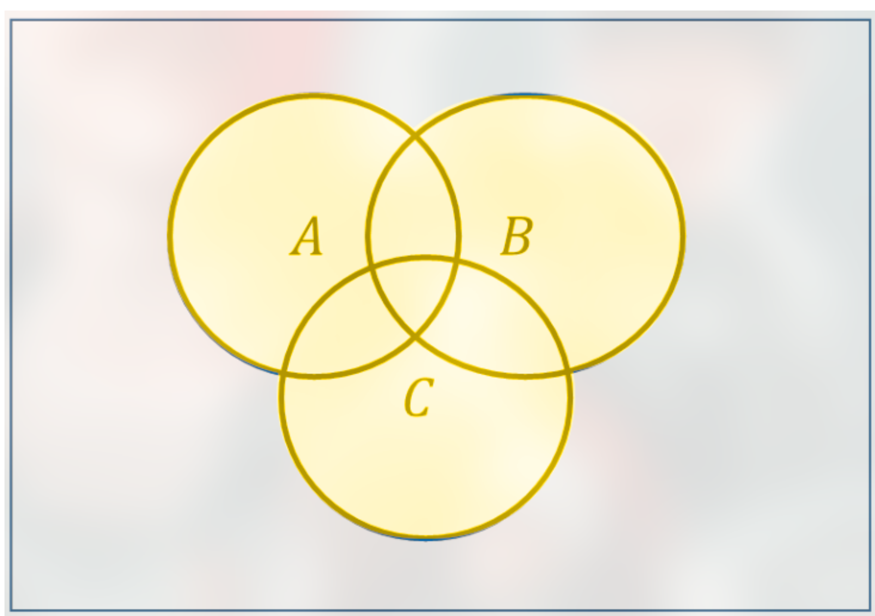

Figure 2: Event 2 Highlighted in Gold

This set can be expressed as:

$$\boxed{\mathbf{E_2} = A \cup B \cup C}$$

(c) The third event (A and C, but not B are applied) is shown below:

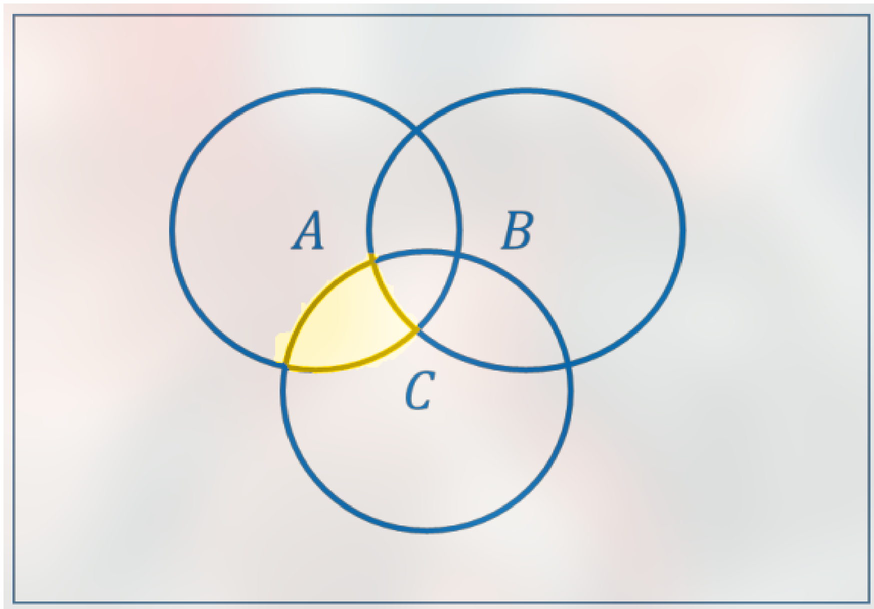

Figure 3: Event 3 Highlighted in Gold

This can be expressed as:

$$\boxed{\mathbf{E_3} = A \cap C - B}$$

(d) The fourth event (at most two of the techniques are applied) is shown below:

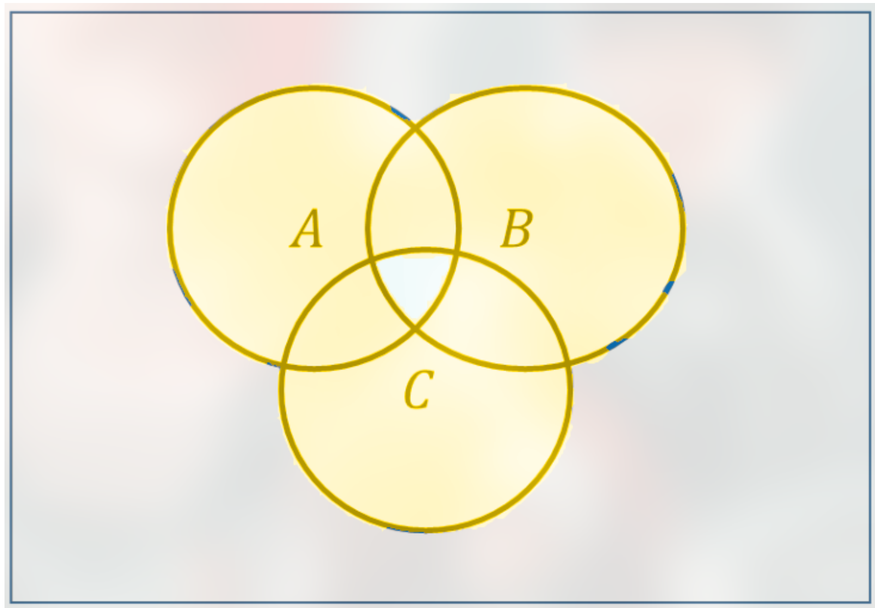

Figure 4: Event 4

This can be expressed as:

$$\boxed{\mathbf{E_4} = (A \cup B \cup C) - (A \cap B \cap C)}$$

(e) The fifth event (only one technique is applied) is shown below:

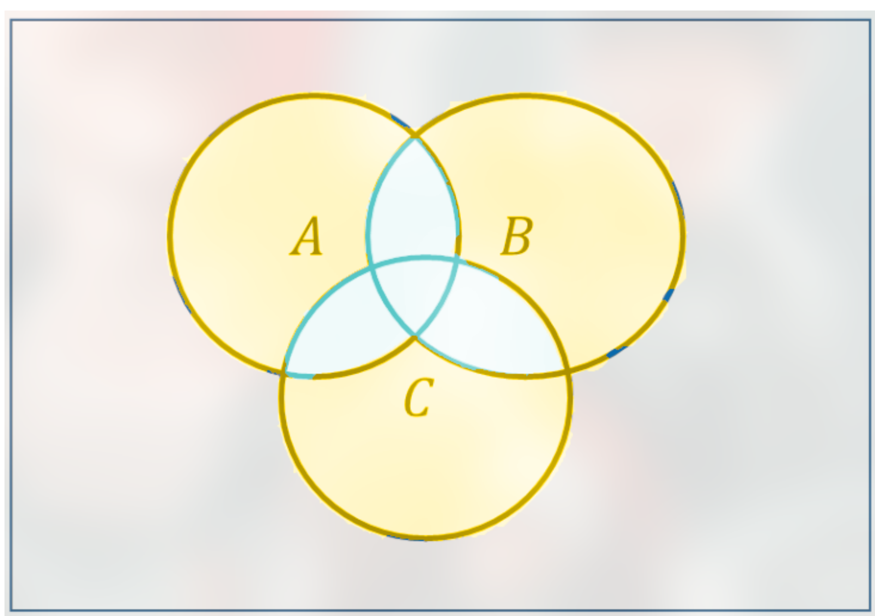

Figure 5: Event 5

This can be expressed as:

$$\boxed{\mathbf{E_5} = (A \cup B \cup C) - (A \cap B) - (A \cap C) - (B \cap C) - (A \cap B \cap C)}$$

7. (a) By our equations, we know:

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

Alternatively, we may write:

$$P[A \cap B] = P[A] + P[B] - P[A \cup B]$$

This gives us:

$$P[A \cap B] = \frac{1}{2} + \frac{1}{3} - \frac{1}{2}$$

$$\boxed{P[A \cap B] = \frac{1}{3} = .33\bar{3}}$$

(b) Two events are independent if:

$$P[A \cap B] = P[A]P[B]$$

We can thus calculate:

$$P[A]P[B] = \frac{1}{6}$$

Since:

$$\frac{1}{3} \neq \frac{1}{6}$$

The two events are **not** independent

(c) The conditions for a partition are that the events are both mutually exclusive and collectively exhaustive. Since A and B are not independent (that is, $P[A \cap B] \neq 0$), at least one of these conditions is not met, and A, B, and C are therefore not partitions.

(d) We can break down the expression to:

$$P[C \cap (A \cup B)] = P[C] + P[A \cup B] - P[C \cup (A \cup B)]$$

By the properties of sets, we know that unions may be expressed in any order, and, thus, the last term is 1. This gives us:

$$P[C \cap (A \cup B)] = P[C] + P[A \cup B] - 1$$

We plug in known values to write:

$$P[C \cap (A \cup B)] = \frac{3}{4} + \frac{1}{2} - 1$$

$$\boxed{P[C \cap (A \cup B)] = \frac{1}{4}}$$

(e) We can simplify this as:

$$P[(A \cup B) - C] = P[A \cup B] - P[(A \cup B) \cap C]$$

Per our previous calculation we can find this as:

$$\boxed{P[(A \cup B) - C] = \frac{1}{2} - \frac{1}{4} = .25}$$

8. (a)
   - We can find the probability of A by dividing the total range of numbers over the range of A. This gives us:

   $$\boxed{P[A] = \frac{1}{2} = .5}$$

   - Similarly, we can find $P[B]$:
     The first range is defined by:

     $$x \geq -.5$$

     The second range is defined by:

     $$x \leq 1.5$$

     Thus, we see that the valid range for x, provided the constraints in x values is:

     $$-.5 \leq x \leq 1$$

     This gives us:

     $$\boxed{P[B] = \frac{1.5}{2} = .75}$$

   - We continue to find:

   $$\boxed{P[C] = \frac{.25}{2} = .125}$$

   - We find the joint probability. This means:

     $$x \geq 0 \quad \text{and} \quad |x - .5| \leq 1$$

     This means that the range of x is decreased to:

$$0 \leq x \leq 1$$

This gives us:

$$\boxed{P[A \cap B] = \frac{1}{2} = .5}$$

- We find the second joint probability, which constrains x with:

$$x \geq 0 \quad \text{and} \quad x \leq -.75$$

Since these events are mutually exclusive, we get:

$$\boxed{P[A \cap C] = 0}$$

(b)

- First we find $P[A \cup B]$. This means that:

$$x \geq 0 \quad \text{or} \quad |x - .5| \leq 1$$

Thus, this means that x is either:

$$x \geq 0$$

Or:

$$-.5 \leq x \leq 1$$

We take the possible range of values to get:

$$\boxed{P[A \cup B] = \frac{1.5}{2} = .75}$$

Using our theorems, we know that:

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

We apply this to get:

$$\boxed{P[A \cup B] = .5 + .75 - .5 = .75}$$

- Similarly, we find the relationship between A and C:

$$x \geq 0 \quad \text{and} \quad x \leq -.75$$

This gives us a range of 1.25, such that:

$$\boxed{P[A \cup C] = \frac{1.25}{2} = .625}$$

Alternatively, we may use our theorems to write:

$$P[A \cup C] = P[A] + P[C] - P[A \cap C]$$

This gives us:

$$\boxed{P[A \cup C] = .5 + .125 - 0 = .625}$$

9. (a) The sample space consists of all combinations of the features $\{2, 5, 4\}$, such that each feature appears only once. This gives us:

$$\boxed{S = \{245, 254, 425, 452, 524, 542\}}$$

(b) E_1 implies that the first value is even. We can take the subset of the sample space where this is true to write:

$$\boxed{E_1 = \{245, 254, 425, 452\}}$$

Similarly, we find E_2:

$$\boxed{E_2 = \{425, 524, 542, 245\}}$$

Finally, we can find O_1:

$$\boxed{O_1 = \{524, 542\}}$$

(c) We may look at the subset of values that meet E_2 constraints within E_1 and divide the quantity by the total quantity within E_1 to get:

$$P[E_2|E_1] = \frac{P[E_1 \cap E_2]}{P[E_1]}$$

This gives us:

$$\boxed{P[E_2|E_1] = \frac{2}{4} = .5}$$

(d) Similarly, we may find:

$$P[E_1|E_2] = \frac{P[E_1 \cap E_2]}{P[E_2]}$$

This gives us:

$$\boxed{P[E_1|E_2] = \frac{2}{4} = .5}$$

(e) We use the same technique to write:

$$P[E_2|O_1] = \frac{P[O_1 \cap E_2]}{P[O_1]}$$

This gives us:

$$\boxed{P[E_2|O_1] = \frac{2}{2} = 1}$$

(f) We can get:

$$P[O_1|O_2] = \frac{P[O_1 \cap O_2]}{P[O_2]}$$

This gives us:

$$\boxed{P[O_1|O_2] = \frac{0}{2} = 0}$$

(g) Since the numbers must be unique, and there are only two numbers that are even, the probability that all three numbers are even is 0. Thus, we write:

$$\boxed{P[(E_1 \cap E_2)|E_3] = 0}$$

(h) Assuming the first number is even, the number can be either 2 or 4, which means that there is a 1 in 2 chance. Thus, we write:

$$\boxed{P[C_1 = 2|E_1] = .5}$$

10. (a) We can find this probability as:

$$\boxed{P[H_0|B] = \frac{.4}{.6} = .66\bar{6}}$$

(b) We can find this probability as:

$$\boxed{P[L|H_1] = \frac{.1}{.2} = .5}$$

(c) We can find this probability as:

$$\boxed{P[H_1 \cap H_2|L] = \frac{.3}{.4} = .75}$$