Homework 7

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1. (a) Using our formulas to obtain the marginal PDFs, we write:

$$f_X(x) = \int_0^\infty f_{XY}(x,y)\,dy$$
$$f_Y(y) = \int_0^\infty f_{XY}(x,y)\,dx$$

This gives us:

$$f_X(x) = 8e^{-4x}\int_0^\infty e^{-2y}\,dy$$
$$f_Y(y) = 8e^{-2y}\int_0^\infty e^{-4x}\,dx$$

We continue to solve to get:

$$f_X(x) = 8e^{-4x}\int_0^\infty e^{-2y}\,dy$$
$$f_X(x) = -4e^{-4x}\left[e^{-2y}\right]\Big|_0^\infty$$
$$\boxed{f_X(x) = 4e^{-4x}, \quad x \geq 0}$$

$$f_Y(y) = 8e^{-2y}\int_0^\infty e^{-4x}\,dx$$
$$f_Y(y) = -2e^{-2y}\left[e^{-4x}\right]\Big|_0^\infty$$
$$\boxed{f_Y(y) = 2e^{-2y}, \quad y \geq 0}$$

We may observe that the two are independent random variables, since:

$$f_{XY}(x,y) = f_X(x)f_Y(y)$$

$$f_{XY}(x,y) = \left(4e^{-4x}\right)\left(2e^{-2y}\right)$$

$$\boxed{f_{XY}(x,y) = 8e^{-(4x+2y)} \quad \checkmark}$$

Furthermore, we may see that the individual PDFs follow an exponential form, with $\boxed{\lambda_x = 4}$ and $\boxed{\lambda_y = 2}$

(b) We may express this probability using the bounds defined by $y \geq 0$ and $x \geq y$, which gives us:

$$P[X > Y] = \int_0^\infty \int_y^\infty 8e^{-4x}e^{-2y}\,dx\,dy$$

We solve this to get:

$$P[X > Y] = \int_0^\infty -2e^{-2y}\left[e^{-4x}\right]\Big|_y^\infty dy$$

$$P[X > Y] = \int_0^\infty 2e^{-6y}\,dy$$

$$P[X > Y] = -\frac{1}{3}\left[e^{-6y}\right]\Big|_0^\infty$$

$$\boxed{P[X > Y] = \frac{1}{3}}$$

Similarly, we pay express $P[X + Y \leq 1]$ with bounds of $0 \leq x \leq 1$ and $0 \leq y \leq 1 - x$, which gives us:

$$P[X + Y \leq 1] = \int_0^1 \int_0^{1-x} 8e^{-4x}e^{-2y}\,dy\,dx$$

We solve this to get:

$$P[X + Y \leq 1] = \int_0^1 -4e^{-4x}\left[e^{-2y}\right]\Big|_0^{1-x} dx$$

$$P[X + Y \leq 1] = \int_0^1 -4e^{-4x}\left[e^{-2+2x} - 1\right]\,dx$$

$$P[X + Y \leq 1] = -4e^{-2}\int_0^1 e^{-2x}\,dx + 4\int_0^1 e^{-4x}\,dx$$

$$P[X + Y \leq 1] = 2e^{-2}\left[e^{-2x}\right]\Big|_0^1 - \left[e^{-4x}\right]\Big|_0^1$$

$$P[X + Y \leq 1] = 2e^{-4} - 2e^{-2} - e^{-4} + 1$$

$$\boxed{P[X + Y \leq 1] = .7476}$$

(c) Since X and Y are independent, we can expand this statement to write:

$$P[\min(X,Y) \geq .5] = P[X \geq .5, Y \geq .5] \rightarrow P[X \geq .5]P[Y \geq .5]$$

As such, we find each component as:

$$P[X \geq .5] = \int_{.5}^{\infty} 4e^{-4x}\,dx$$
$$P[Y \geq .5] = \int_{.5}^{\infty} 2e^{-2y}\,dy$$

We solve to find:

$$P[X \geq .5] = -\left[e^{-4x}\right]\Big|_{.5}^{\infty}$$
$$P[X \geq .5] = -\left[0 - e^{-2}\right]$$
$$P[X \geq .5] = .1353$$

$$P[Y \geq .5] = -\left[e^{-2y}\right]\Big|_{.5}^{\infty}$$
$$P[Y \geq .5] = -\left[0 - e^{-1}\right]$$
$$P[Y \geq .5] = .3679$$

We multiply the two to find:

$$P[\min(X,Y) \geq .5] = (.1353)(.3679)$$
$$\boxed{P[\min(X,Y) \geq .5] = .049787}$$

(d) Similar to part (c), we write:

$$P[\max(X,Y) \leq .5] = P[X \leq .5, Y \leq .5] \rightarrow P[X \leq .5]P[Y \leq .5]$$

This gives us:

$$P[X \leq .5] = \int_{0}^{.5} 4e^{-4x}\,dx$$
$$P[Y \leq .5] = \int_{0}^{.5} 2e^{-2y}\,dy$$

We solve to get:

$$P[X \leq .5] = \int_{0}^{.5} 4e^{-4x}\,dx$$
$$P[X \leq .5] = -\left[e^{-4x}\right]\Big|_{0}^{.5}$$

$$P[X \leq .5] = -\left[e^{-2} - 1\right]$$
$$P[X \leq .5] = .8647$$

$$P[Y \leq .5] = \int_0^{.5} 2e^{-2y}\, dy$$
$$P[Y \leq .5] = -\left[e^{-2y}\right] 0^{.5}$$

$$P[Y \leq .5] = -\left[e^{-1} - 1\right]$$
$$P[Y \leq .5] = .6321$$

We then multiply the two to find:

$$P[\max(X, Y) \leq .5] = (.8647)(.6321)$$
$$\boxed{P[\max(X, Y) \leq .5] = .5466}$$

2. (a) We may find the CDF as:

$$F_X(x) = \int_0^x f_X(x)\, dx$$

This gives us:

$$F_X(x) = \int_0^x \frac{x}{50}\, dx$$

We evaluate to get:

$$F_X(x) = \left[\frac{x^2}{100}\right]\Big|_0^x$$

$$\boxed{F_X(x) = \begin{cases} \frac{x^2}{100}, & 0 \leq x \leq 10 \\ 1, & x > 10 \\ 0, & \text{otherwise} \end{cases}}$$

(b) Given the independence of X_1 and X_2, we may express this probability as:

$$P[X_1 \leq 5, X_2 \leq 5] = (F_X(5))^2$$

This gives us:

$$P[X_1 \leq 5, X_2 \leq 5] = \left(\frac{1}{4}\right)^2$$

$$\boxed{P[X_1 \leq 5, X_2 \leq 5] = \frac{1}{16}}$$

(c) Once again, due to the independence, we may write:

$$F_W[w] = P[W \leq w] = (P[X \leq w])^2$$

Since we are given $w = 5$, we simply use the answer from (b):

$$F_W[5] = (P[X \leq 5])^2$$
$$\boxed{F_W[5] = \frac{1}{16}}$$

(d) We may observe that the CDF may be written as the product of the two individual CDFs; however, because they are independent, identically distributed systems, we obtain:

$$F_W(w) = F_{X_1}(w)F_{X_2}(w)$$
$$F_W(w) = F_X(w)F_X(w) \quad (X_1 = X_2)$$
$$F_W(w) = [F_X(w)]^2$$

This gives us:

$$\boxed{F_W(w) = \begin{cases} \frac{w^4}{10000}, & 0 \leq w \leq 10 \\ 1, & w > 10 \\ 0, & \text{otherwise} \end{cases}}$$

4. (a) To find $P[X \leq 1]$, we must first find the individual PDF of x. We begin by finding this:

$$f_X(x) = \frac{1}{24} \int_0^4 x + y \, dy$$

This gives us:

$$f_X(x) = \frac{1}{48} \left[2xy + y^2\right] \Big|_0^4$$
$$f_X(x) = \frac{x}{6} + \frac{1}{3}, \quad 0 \leq x \leq 2$$

From here, we get:

$$P[X \leq 1] = \int_0^1 f_X(x) \, dx$$
$$P[X \leq 1] = \frac{1}{6} \int_0^1 x + 2 \, dx$$
$$P[X \leq 1] = \frac{1}{12} \left[x^2 + 4x\right] \Big|_0^1$$

$$\boxed{P[X \leq 1] = \frac{5}{12}}$$

(b) We may write the conditional PDF as:

$$f_{XY|A}(x,y) = \frac{f_{XY}(x,y)}{f(A)}$$

As determined in part (a), this gives us:

$$f_{XY|A}(x,y) = \frac{(x+y)/24}{5/12}, \quad 0 \leq x \leq 1,\ 0 \leq y \leq 4$$

We simplify to get:

$$\boxed{f_{XY|A}(x,y) = \frac{x+y}{10}, \quad 0 \leq x \leq 1,\ 0 \leq y \leq 4}$$

(c) Using the result from part (b), we may write the conditional marginal PDFs as:

$$f_{X|A}(x) = \int_0^4 f_{XY|A}(x,y)\,dy$$

$$f_{Y|A}(y) = \int_0^1 f_{XY|A}(x,y)\,dx$$

We expand this to get:

$$f_{X|A}(x) = \int_0^4 \frac{x+y}{10}\,dy$$

$$f_{Y|A}(y) = \int_0^1 \frac{x+y}{10}\,dx$$

We then solve:

$$f_{X|A}(x) = \frac{2xy+y^2}{20}\Big|_0^4$$

$$\boxed{f_{X|A}(x) = \frac{2x+4}{5}, \quad 0 \leq x \leq 1}$$

$$f_{Y|A}(y) = \frac{x^2+2xy}{20}\Big|_0^1$$

$$\boxed{f_{Y|A}(y) = \frac{1+2y}{20}, \quad 0 \leq y \leq 4}$$

We can then use the first result to find:

$$E[X|A] = \int_0^1 x\left(\frac{2x+4}{5}\right) dx$$
$$E[X|A] = \int_0^1 \frac{2x^2+4x}{5} dx$$
$$E[X|A] = \frac{2x^3+6x^2}{15}\Big|_0^1$$
$$\boxed{E[X|A] = \frac{8}{15}}$$

6. (a) We can find $f_Y(y)$ as:

$$f_Y(y) = \int_{-2}^{y} \frac{1}{8} dx$$
$$f_Y(y) = \left[\frac{x}{8}\right]\Big|_{-2}^{y}$$
$$\boxed{f_Y(y) = \frac{y}{8} + \frac{1}{4}, \quad -2 \leq y \leq 2}$$

(b) We may apply the conditional PDF formula to get:

$$f_{X|Y}(x,y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

Substituting in our known values, we get:

$$f_{X|Y}(x,y) = \frac{1}{8} \cdot \frac{8}{y+2}$$
$$\boxed{f_{X|Y}(x,y) = \frac{1}{y+2}, \quad x \leq y \leq 2}$$

(c) We may see that the above conditional PDF is simply a uniform distribution with $b = y$ and $a = -2$. As such, we may state that:

$$\boxed{E[X|Y=y] = \frac{y-2}{2}}$$

(d) We may write the covariance as:

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y]$$

We find the expectation values, first finding the marginal PDF of x:

$$f_X(x) = \int_x^2 \frac{1}{8} dy$$

$$f_X(x) = \left[\frac{y}{8}\right]\Big|_x^2$$

This gives us:

$$f_X(x) = \frac{1}{4} - \frac{x}{8}$$

We then compute the expectation values of each. First, we write:

$$E[X] = \int_{-2}^{2} x\left[\frac{1}{4} - \frac{x}{8}\right]\, dx$$

$$E[Y] = \int_{-2}^{2} y\left[\frac{y}{8} + \frac{1}{4}\right]\, dy$$

We solve:

$$E[X] = \frac{1}{8}\int_{-2}^{2} 2x - x^2\, dx$$

$$E[X] = \frac{1}{8}\left[x^2 - \frac{x^3}{3}\right]\Big|_{-2}^{2}$$

$$\boxed{E[X] = -\frac{2}{3}}$$

$$E[Y] = \frac{1}{8}\int_{-2}^{2} y(y+2)\, dy$$

$$E[Y] = \frac{1}{8}\left[\frac{y^3}{3} + 2y\right]\Big|_{-2}^{2}$$

$$\boxed{E[Y] = \frac{2}{3}}$$

We then find:

$$E[XY] = \int_{-2}^{2}\int_{x}^{2} \frac{xy}{8}\, dy\, dx$$

$$\boxed{E[XY] = 0}$$

Thus, we this gives us a covariance of:

$$\boxed{\mathrm{Cov}(X, Y) = \frac{4}{9}}$$

(e) We can break apart this covariance:

$$\text{Cov}(4Y, 2X + 2Y + .1) \rightarrow 4\text{Cov}(Y, 2X + 2Y + .1)$$
$$4\text{Cov}(Y, 2X + 2Y + .1) \rightarrow 8\text{Cov}(Y, X + Y + .05)$$
$$8\text{Cov}(Y, X + Y + .05) \rightarrow 8\text{Cov}(Y, X + Y)$$
$$8\text{Cov}(Y, X + Y) \rightarrow 8[\text{Cov}(Y, X) + \text{Cov}(Y, Y)]$$

And finally, we find:

$$8[\text{Cov}(Y, X) + \text{Cov}(Y, Y)] \rightarrow 8\text{Cov}(X, Y) + 8\text{Var}(Y)$$

Thus, we find the variance of Y as:

$$\text{Var}(Y) = \int_{-2}^{2} \left(y - \frac{2}{3}\right)\left(\frac{y}{8} + \frac{1}{4}\right) dy$$
$$\boxed{\text{Var}(Y) = \frac{8}{9}}$$

This gives us:

$$\boxed{\text{Cov}(4Y, 2X + 2Y + .1) = \frac{96}{9}}$$

7. (a) Given that the distribution is uniform, we can write the PDF as:

$$f_{Y|X}(y|x) = \frac{1}{x}, \quad 0 \leq y \leq x$$

Thus, we can apply our expectation value formula to get:

$$\boxed{E[Y|X = x] = \frac{x + 0}{2} = \frac{x}{2}}$$

We apply the variance formula in a similar manner to say:

$$\text{Var}(Y|X) = \frac{(x - 0)^2}{12}$$
$$\boxed{\text{Var}(Y|X = x) = \frac{x^2}{12}}$$

(b) Now, we find the joint PDF. First and foremost, given its uniform nature, we may state that:

$$f_X(x) = 1, \quad 0 \leq x \leq 1$$

Using our formula for $f_{Y|X}(y|x)$ from (a), and combining it with the joint PDF formula:

$$f_{XY}(x,y) = f_{Y|X}(y|x)f_X(x)$$

We multiply and combine to get:

$$\boxed{f_{XY}(x,y) = \frac{1}{x}, \quad 0 \leq y \leq x \leq 1}$$

(c) We find the marginal PDF of y by integrating over x:

$$f_Y(y) = \int_y^1 f_{XY}(x,y)\,dx$$

We substitute our formula and solve:

$$f_Y(y) = \int_y^1 \frac{1}{x}\,dx$$

$$f_Y(y) = \ln(x)\Big|_y^1$$

$$\boxed{f_Y(y) = \ln\left(\frac{1}{y}\right), \quad 0 \leq y \leq 1}$$

(d) We then apply the marginal PDF to find the expectation value:

$$E[Y] = \int_0^1 y\ln\left(\frac{1}{y}\right)\,dy$$

Using integration by parts, we find:

$$\boxed{E[Y] = \frac{1}{4}[\mathrm{MW}]}$$

We then find the variance as:

$$\mathrm{Var}(Y) = \int_0^1 (y-.25)^2\ln\left(\frac{1}{y}\right)\,dy$$

$$\boxed{\mathrm{Var}(Y) = .04861[\mathrm{MW}^2]}$$

8. (a) No actual problem, only problem statement

(b) We know that:

$$\rho_{XY}(x,y) = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}$$

As such, we may write:

$$\mathrm{Cov}(U,V) = \mathrm{Cov}(aX,bY) = ab\mathrm{Cov}(X,Y)$$

We solve the first equation to get:

$$\mathrm{Cov}(X,Y) = \left(\frac{1}{4}\right)\sqrt{16^2}$$
$$\mathrm{Cov}(X,Y) = 4$$

As such, we conclude:

$$\boxed{\mathrm{Cov}(U,V) = 4ab}$$

(c) We know variances shift according to:

$$\mathrm{Var}(aX) = a^2\mathrm{Var}(X)$$

Accordingly, we may write:

$$\rho_{UV}(U,V) = \frac{\mathrm{Cov}(U,V)}{\sqrt{\mathrm{Var}(aX)\mathrm{Var}(bY)}}$$

Thus, we incorporate known values to get:

$$\rho_{UV}(U,V) = \frac{4ab}{ab\sqrt{16}}$$
$$\boxed{\rho_{UV}(U,V) = 1}$$

(d) We may rewrite W as:

$$W = aX - bY$$

We want W and X to be uncorrelated, or:

$$\mathrm{Cov}(W,X) = 0$$

We expand this to get:

$$\mathrm{Cov}(aX - bY, X) = 0$$
$$a\mathrm{Cov}(X,X) - b\mathrm{Cov}(Y,X) = 0$$

As such, we see that we want:

$$a\text{Var}(X) = b\text{Cov}(X, Y)$$

We know both values, so we write:

$$16a = 4b$$

We conclude that, for W and X to be uncorrelated, we want:

$$\boxed{\frac{b}{a} = 4}$$

(e) We may expand the expectation value of Z as:

$$E[Z] = E[cX + dY]$$
$$E[cX + dY] = cE[X] + dE[Y]$$

As given, we take $c \to 1$ and plug in our known values to get:

$$E[Z] = 2 + d$$

We then write the variance as:

$$\text{Var}(Z) = \text{Var}(cX + dY)$$
$$\text{Var}(Z) = \text{Var}(cX) + \text{Var}(dY) + 2\text{Cov}(cX, dY)$$
$$\text{Var}(cX) + \text{Var}(dY) + 2\text{Cov}(cX, dY) = c^2\text{Var}(X) + d^2\text{Var}(Y) + 2cd\text{Cov}(X, Y)$$

As such, we enter known values to get:

$$\text{Var}(Z) = 16 + 16d^2 + 8d$$

Substituting into the SNR equation, we get:

$$\text{SNR} = \frac{4 + 4d + d^2}{16 + 8d + 16d^2}$$

Plotting the signal-to-noise ratio against d, we see:

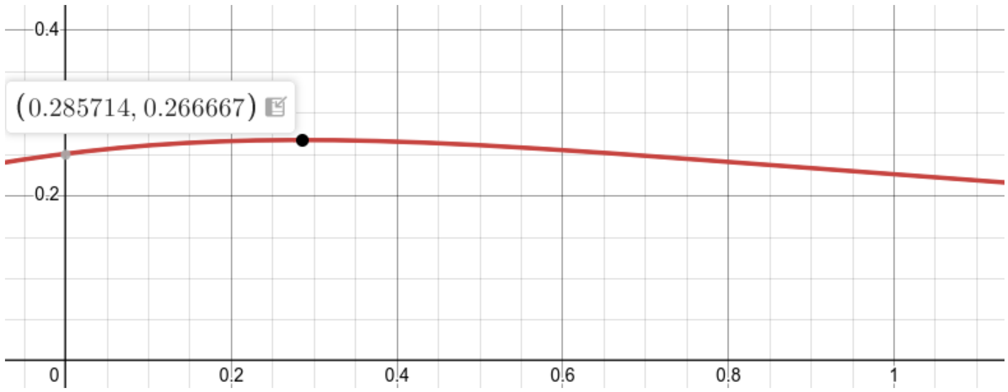

Figure 1: SNR versus d Plot

As such, the SNR is maximized when $\boxed{d = .285714}$.

9. (a) We may begin by expressing the joint PMF as a matrix:

$$P_{XY}(x, y) = \begin{bmatrix} 0 & 1/8 & 3/8 & 1/4 \\ 0 & 0 & 1/8 & 0 \\ 0 & 0 & 0 & 1/8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Summing the columns, we see that $X = \{2, 3, 4\}$ with probabilities $\left\{\frac{1}{8}, \frac{1}{2}, \frac{3}{8}\right\}$, respectively. Similarly, summing the rows shows us that $Y = \{1, 2, 3\}$ with probabilities $\left\{\frac{3}{4}, \frac{1}{8}, \frac{1}{8}\right\}$, respectively. Using this gives us:

$$E[X] = \frac{1}{8}(2) + \frac{1}{2}(3) + \frac{3}{8}(4)$$
$$E[Y] = \frac{3}{4}(1) + \frac{1}{8}(2) + \frac{1}{8}(3)$$

We solve to get:

$$\boxed{E[X] = \frac{13}{4}}$$
$$\boxed{E[Y] = \frac{11}{8}}$$

(b) To find the variances, we begin by writing:

$$E[X^2] = \frac{1}{8}(2)^2 + \frac{1}{2}(3)^2 + \frac{3}{8}(4)^2$$
$$E[Y^2] = \frac{3}{4}(1)^2 + \frac{1}{8}(2)^2 + \frac{1}{8}(3)^2$$

This gives us:

$$E[X^2] = 11$$
$$E[Y^2] = \frac{19}{8}$$

We then find the variance as:

$$\text{Var}(X) = E[X^2] - (E[X])^2$$
$$\text{Var}(Y) = E[Y^2] - (E[Y])^2$$

This gives us:

$$\text{Var}(X) = 11 - \frac{169}{16}$$
$$\text{Var}(Y) = \frac{19}{8} - \frac{121}{64}$$

And finally:

$$\boxed{\text{Var}(X) = .4375}$$
$$\boxed{\text{Var}(Y) = .4844}$$

(c) We can find the correlation as:

$$r_{X,Y} = E[XY] = \sum\sum xyP(x,y)$$

We expand this to get:

$$r_{X,Y} = \frac{1}{8}(2)(1) + \frac{3}{8}(3)(1) + \frac{1}{4}(4)(1) + \frac{1}{8}(3)(2) + \frac{1}{8}(4)(3)$$

Solving gives us:

$$\boxed{r_{X,Y} = \frac{37}{8} = 4.625}$$

(d) The covariance may be written as:

$$\text{Cov}(X,Y) = E[XY] - E[X]E[Y]$$

Using our obtained values gives us:

$$\text{Cov}(X,Y) = 4.625 - (3.25)(1.375)$$
$$\boxed{\text{Cov}(X,Y) = .1562}$$

(e) We can then write the correlation coefficient as:

$$\rho_{X,Y} = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

This gives us:

$$\rho_{X,Y} = \frac{.1562}{\sqrt{(.4375)(.4844)}}$$

$$\boxed{\rho_{X,Y} = .3394}$$