Homework 5

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1. (a) To be a valid CDF, we know that the terms continuously build until they sum to 1. In this case, all of the terms become 1 at $v = 10$. Thus, we can differentiate to find the PDF:

$$f_V(v) = \frac{d}{dv}[F_V(v)]$$
$$f_V(v) = 2c(v-2), 2 \leq v < 10$$

We then know:

$$\int_2^{10} 2c(v-2)\,dv = 1$$

We can solve to get:

$$2c\left[\frac{v^2}{2} - 2v\Big|_2^{10}\right] = 1$$
$$2c\left[(50-2)-(20-4)\right] = 1$$
$$64c = 1$$

Which finally gets us:

$$\boxed{c = 64}$$

(b) We can then find the probability that the response time is greater than 5[ms] as:

$$P(v > 5) = 1 - F_V(v)$$
$$P(v > 5) = 1 - \frac{1}{64}(5-2)^2$$
$$\boxed{P(v > 5) = \frac{55}{64}}$$

(c) We can then find the response time probability for between 5 and 8 milliseconds:

$$P(5 \leq v < 8) = F_V(8) - F_V(5)$$
$$P(5 \leq v < 8) = \frac{1}{64}[(8-2)^2 - (5-2)^2]$$
$$\boxed{P(5 \leq v < 8) = \frac{27}{64}}$$

(d) We can find this to be:

$$P(v > 7 | 5 \leq v \leq 8) = \frac{P(7 < v \leq 8)}{P(5 \leq v \leq 8)}$$

We find the probability of the numerator:

$$P(7 < v \leq 8) = F_V(8) - F_V(7)$$
$$P(7 < v \leq 8) = \frac{1}{64}[(8-2)^2 - (7-2)^2]$$
$$P(7 < v \leq 8) = \frac{11}{64}$$

This gives us:

$$P(v > 7 | 5 \leq v \leq 8) = \frac{11/64}{27/64}$$
$$\boxed{P(v > 7 | 5 \leq v \leq 8) = \frac{11}{27}}$$

(e) To find the applicable value, we may write:

$$1 - F_V(a) = .36$$

We expand this to write:

$$1 - \frac{1}{64}(a-2)^2 = .36$$

We then solve:

$$a = \sqrt{64(.64)} + 2$$
$$a = \pm 6.4 + 2$$

Since the time has to be positive, we find:

$$\boxed{a = 8.4[\text{ms}]}$$

2. (a) To be a valid PDF, we know:

$$\int_{-\infty}^{\infty} ae^{-.2|x|}\,dx = 1$$

We expand:

$$\int_{-\infty}^{0} ae^{.2x}\,dx + \int_{0}^{\infty} ae^{-.2x}\,dx = 1$$

This gives us:

$$\left.\frac{ae^{.2x}}{.2}\right|_{-\infty}^{0} - \left.\frac{ae^{-.2x}}{.2}\right|_{0}^{\infty} = 1$$

We continue to solve:

$$(5a - 0) - (0 - 5a) = 1$$
$$10a = 1$$
$$\boxed{a = .1}$$

(b) We know that the expectation value can be expressed as:

$$E[x] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

This gives us:

$$E[x] = \frac{1}{10}\left[\int_{-\infty}^{0} xe^{.2x}\,dx + \int_{0}^{\infty} xe^{-.2x}\,dx\right]$$
$$\boxed{E[x] = 0[\mu\mathrm{V}]}$$

We may observe that, due to symmetry about the x axis, the expected value is zero.

(c) Once again, we break up the function to find the CDF in two regions. We begin with the first region:

$$F_X(x<0) = \int_{-\infty}^{x} .1e^{.2x}\,dx$$

This gives us:

$$F_X(x<0) = \left..5e^{.2x}\right|_{-\infty}^{x}$$
$$F_X(x<0) = .5e^{.2x}$$

We then find the second region:

$$F_X(x \geq 0) = \int_{-\infty}^{0} .1e^{.2x}\, dx + \int_{0}^{x} .1e^{-.2x}\, dx$$

This gives us:

$$F_X(x \geq 0) = .5e^{.2x} + \left[-.5e^{-.2x}\right]\Big|_0^x$$

$$F_X(x \geq 0) = 1 - .5e^{-.2x}$$

Finally, we may express this as:

$$\boxed{F_X(x) = \begin{cases} .5e^{.2x}, & x < 0 \\ 1 - .5e^{-.2x}, & x \geq 0 \end{cases}}$$

(d) We can calculate the former as:

$$P[X > 0] = 1 - P[X \leq 0]$$

$$P[X > 0] = 1 - \int_{-\infty}^{0} f_X(x)\, dx$$

This gives us:

$$P[X > 0] = 1 - \left[.5e^{.2x}\right]\Big|_{-\infty}^{0}$$

$$\boxed{P[X > 0] = .5}$$

Similarly, we may find:

$$P[X > 2] = 1 - P[X \leq 2]$$

This gives us:

$$P[X > 2] = 1 - P[X > 0] - \int_{0}^{2} .1e^{-.2x}\, dx$$

We continue to evaluate:

$$P[X > 2] = 1 - .5 - \left[-.5e^{-.2x}\right] 0^2$$

$$P[X > 2] = 1 - .5 - [-.5e^{-.4} - (-.5)]$$

$$P[X > 2] = .5e^{-.4}$$

$$\boxed{P[X > 2] = .3352}$$

(e) Given that the PDF is symmetrical, we may simply write:

$$P[|X| > 2] = 2P[X > 2]$$
$$P[|X| > 2] = 2(.3352)$$
$$\boxed{P[|X| > 2] = .6704}$$

3. (a) Given that the PDF can be found as the differential of the CDF, we get:

$$f_X(x) = \frac{d}{dx}\left[F_X(x)\right]$$
$$f_X(x) = \frac{1}{16}\frac{d}{dx}\left[(x-2)^2\right], \quad 2 \le x < 6$$

This gives us:

$$\boxed{f_X(x) = \frac{x-2}{8}, \quad 2 \le x < 6}$$

(b) We can find the expected value using the formula:

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

This gives us:

$$E[X] = \int_{-\infty}^{\infty} x\left(\frac{x-2}{8}\right)\, dx$$
$$E[X] = \frac{1}{8}\int_{2}^{6} \left(x^2 - 2x\right)\, dx$$

We evaluate to find:

$$E[X] = \frac{1}{8}\left[\frac{x^3}{3} - x^2\right]\Big|_2^6$$
$$E[X] = \frac{1}{8}\left[(72-36) - \left(\frac{8}{3} - 4\right)\right]$$
$$\boxed{E[X] = \frac{112}{24} = 4.6\bar{6}[\text{ms}]}$$

Using this mean value, we can find the variance as:

$$\text{Var}[X] = \int_2^6 (x - 4.6\bar{6})^2 f_X(x)\, dx$$

We evaluate to get:

$$\boxed{\text{Var}[X] = .88\bar{8}[\text{ms}^2]}$$

(c) Using the PDF, we may find $P[X > 4] = 1 - P[X \leq 4]$. This gives us:

$$P[X > 4] = 1 - \int_2^4 f_X(x)\,dx$$
$$P[X > 4] = 1 - \int_2^4 \frac{x-2}{8}\,dx$$

We evaluate to get:

$$P[X > 4] = 1 - \frac{1}{16}(x^2 - 2x)\Big|_2^4$$
$$P[X > 4] = 1 - \frac{1}{16}[(16-8) - (4-4)]$$
$$\boxed{P[X > 4] = \frac{1}{2} = .5}$$

From here, we may determine the conditional probability of:

$$P[X > 5|X > 4] = \frac{P[X > 5]}{P[X > 4]}$$

From here, we find:

$$P[X > 5] = 1 - \frac{1}{16}(x^2 - 2x)\Big|_2^5$$
$$P[X > 5] = 1 - \frac{1}{16}[(25-10) - (4-4)]$$
$$\boxed{P[X > 5] = \frac{1}{16} = .0625}$$

We then find:

$$P[X > 5|X > 4] = \frac{.0625}{.5}$$
$$\boxed{P[X > 5|X > 4] = .125}$$

(d) We know that the PDF will scale such that:

$$f_{X|A}(x) = \frac{f_X(x)}{P[A]}$$

Thus, we can find:

$$P[A] = 1 - \frac{1}{16}(x^2 - 2x)\Big|_2^3$$
$$P[A] = 1 - \frac{1}{16}[(9-6) - (4-4)]\Big|_2^3$$

$$P[A] = \frac{13}{16} = .8125$$

Thus, we scale the initial pdf to get:

$$f_{X|A}(x) = \frac{16}{13}\left[\frac{x-2}{8}\right]$$

$$\boxed{f_{X|A}(x) = \frac{2x-4}{13}, \quad 3 \leq x < 6}$$

4. (a) We know that the PDF is continuous and equivalent over the whole interval, and, thus, we may write:

$$f_X(x) = \frac{1}{a-(-a)}, \quad -a \leq x \leq a$$

$$\boxed{f_X(x) = \frac{1}{2a}, \quad -a \leq x \leq a}$$

(b) Given that we know the mean will be zero, we may write:

$$\text{Var}[X] = \int_{-a}^{a} (x-0)^2 \frac{1}{2a}\, dx$$

$$\text{Var}[X] = \int_{-a}^{a} \frac{x^2}{2a}\, dx$$

We then evaluate:

$$\text{Var}[X] = \frac{x^3}{6a} {-a}^{a}$$

$$\text{Var}[X] = \left[\frac{a^3}{6a} - \left(-\frac{a^3}{6a}\right)\right]$$

$$\text{Var}[X] = \left[\frac{a^2}{6} + \frac{a^2}{6}\right]$$

$$\boxed{\text{Var}[X] = \frac{a^2}{3}}$$

(c) We can write:

$$P\left[X > \frac{a}{2} | X > 0\right] = \frac{P\left[X > \frac{a}{2}\right]}{P[X > 0]}$$

We begin by finding:

$$P[X > 0] = 1 - \frac{0-(-a)}{2a}$$

$$P[X > 0] = \frac{1}{2}$$

We then find:

$$P\left[X > \frac{a}{2}\right] = 1 - \frac{\frac{a}{2} - (-a)}{a - (-a)}$$

$$P\left[X > \frac{a}{2}\right] = \frac{1}{4}$$

We then divide to get:

$$P\left[X > \frac{a}{2}|X > 0\right] = \frac{.25}{.5}$$

$$\boxed{P\left[X > \frac{a}{2}|X > 0\right] = \frac{1}{2}}$$

(d) Given the uniform probability, we may simply double our value of $P\left[X > \frac{a}{2}\right]$ to find:

$$P\left[|X| > \frac{a}{2}\right] = 2P\left[X > \frac{a}{2}\right]$$

This then gives us the probability of a fault trigger as:

$$\boxed{P\left[|X| > \frac{a}{2}\right] = \frac{1}{2}}$$

(e) We begin by setting the equation appropriately to get:

$$2X^2 > \frac{a^2}{3}$$

Solving this gives us:

$$|X| > \frac{a}{\sqrt{6}}$$

Thus, we want to find:

$$P\left[|X| > \frac{a}{\sqrt{6}}\right] = 2P\left[X > \frac{a}{\sqrt{6}}\right]$$

This gives us:

$$P\left[|X| > \frac{a}{\sqrt{6}}\right] = 2\left[1 - \frac{\frac{a}{\sqrt{6}} - (-a)}{2a}\right]$$

$$\boxed{P\left[|X| > \frac{a}{\sqrt{6}}\right] = .5918}$$

6. (a) For an exponential random variable, we know:

$$\lambda = \frac{1}{\mu}$$

Which gives us:

$$\lambda = \frac{1}{2}$$

Thus, we may write the PDF as:

$$\boxed{f_X(x) = .5e^{-.5x}, \quad x > 0}$$

(b) First, we want to find the probability of the request taking less than two seconds:

$$P[X < 2] = \int_0^2 .5e^{-.5x}\,dx$$
$$P[X < 2] = -e^{-.5x}\Big|_0^2$$
$$P[X < 2] = [-e^{-1} - (-1)]$$
$$\boxed{P[X < 2] = .6321}$$

From here, since each request is independent, we simply find:

$$P[X_3 < 2] = (P[X < 2])^3$$
$$P[X_3 < 2] = (.6321)^3$$
$$\boxed{P[X_3 < 2] = .2526}$$

(c) i. Although the λ value stays the same as the above, we define n as the quantity of requests, or 3, such that:

$$\boxed{Y = \text{Erlang}(3, .5)}$$

ii. Using the Erlang distribution, we write:

$$\boxed{F_Y(y) = .0625y^2e^{-.5y}}$$

iii. We can find the mean value as:

$$E[Y] = \int_0^\infty .0625y^2e^{-.5y}\,dy$$

We solve to get:

$$\boxed{E[Y] = 6}$$

Then, we find the variance:

$$\text{Var}[Y] = \int_0^\infty .0625(y-6)^2 y^2 e^{-.5y}\,dy$$

Solving gives us:

$$\boxed{\text{Var}[Y] = 12}$$

iv. We can find the probability as:

$$P[Y < 5] = \int_0^5 .0625 y^2 e^{-.5y}\,dy$$

Solving gives us:

$$\boxed{P[Y < 5] = .45618}$$

8. (a) Given that we want 95% of all readings to be within 1^o of μ, we can use the inverse normal function to find a z-score of:

$$\frac{x-\mu}{\sigma} = \pm 1.96$$

Furthermore, we know that:

$$x - \mu = \pm 1$$

We combine the two to get:

$$\pm 1.96\sigma = \pm 1$$
$$\boxed{\sigma = .5102}$$

(b) Using our z-score formula, we may write:

$$z = \frac{25-\mu}{1} = 25 - \mu$$

We know that:

$$P[Y \leq 25^o] = P[Z \leq 25 - \mu]$$

Given the probability for $Y \leq 25^o$, we can use the inverse normal function to write:

$$z = 1.4985$$

This gives us:

$$25 - \mu = 1.4985$$
$$\boxed{\mu = 23.501^o}$$

(c) Similarly, we may find:

$$z = -1.9954$$

From this, we write:

$$-\mu = -1.9954$$
$$\boxed{\mu = 1.9954^o}$$

(d) If the probability above or below a certain point is equal to 1/2, then we are at the mean. Therefore, we may write:

$$\boxed{\mu = 25^o}$$

9. (a) We want to find a value for which 90% are less than or equal to, in terms of storage. We may begin by taking the inverse of the normal to find the z score:

$$\text{invnorm}(.9) = 1.2816$$

Now that we know the z score, we may write:

$$\frac{X - 500}{100} = 1.2816$$

Solving, we find:

$$X = 100(1.2816) + 500$$

Given that we want 90% of users to be 30 GB below the set threshold, we may find:

$$\boxed{W_{Th} = 658.16[\text{GB}]}$$

(b) We can calculate this by transferring to a z score:

$$z = \frac{600 - 500}{100}$$
$$z = 1$$

And then taking the normal function to get:

$$\%_{\text{users}} = 1 - .8413$$
$$\boxed{\%_{\text{users}} = 15.87\%}$$

(c) Given that we can write the initial PDF as:

$$f_W(w) = \text{NormalPDF}(500, 100)$$

We can write the conditional probability as:

$$f_{W|A}(w) = \frac{P[A]}{\text{NormalPDF}(500, 100)}$$

As such, we use our probability from above to write:

$$f_{W|A}(w) = \left[\frac{6.303}{100\sqrt{2\pi}} e^{-\frac{(w-500)^2}{20000}}\right]^{-1}$$

This can be simplified to:

$$\boxed{f_{W|A}(w) = 39.769 e^{\frac{(w-500)^2}{20000}}}$$

(d) We can shift the PDF such that:

$$E[(W - 600)|A] = E[W|W > 600] - 600$$

For a normal distribution, this gives us:

$$E[W|W > 600] = \mu + \sigma \left[\frac{\Phi(z)}{1 - \Phi(z)}\right]$$

We use the previously-obtained z-score to get:

$$E[W|W > 600] = 500 + 100 \left[\frac{\phi(1)}{1 - \Phi(1)}\right]$$

We can find:

$$\phi(1) = \frac{1}{\sqrt{2\pi}} e^{-.5}$$
$$\phi(1) = .242$$

We know from earlier that:

$$1 - \Phi(1) = .1587$$

Thus, we get:

$$E[W|W > 600] = 500 + 100 \left[\frac{.242}{.1587}\right]$$
$$E[W|W > 600] = 652.47[\text{GB}]$$

From here, we can obtain the value we want:

$$E[(W-600)|A] = 652.47 - 600$$
$$\boxed{E[(W-600)|A] = 52.47[\text{GB}]}$$

The expected revenue per user is then:

$$R = 52.47 \cdot .05$$
$$R = 2.6235\$ \left[\frac{\text{Dollars}}{\text{Users}}\right]$$

With 10,000 users, we get:

$$\boxed{R_{10k} = 26,235\$ \,[\text{Dollars}]}$$

10. (a) The event $X \geq t$ is the same as $A_1 \cap A_2 \cap A_3$

    (b) Given that the events are independent, along with the statement from part (a), we may write that an individual event can be expressed as:

    $$P[A_i] = e^{-.01t}$$

    And therefore $P[X \geq t]$ can be expressed as:

    $$P[X \geq t] = \prod_{i=1}^{3} P[A_i]$$
    $$P[X \geq t] = \left(e^{-.01t}\right)^3$$
    $$\boxed{P[X \geq t] = e^{-.03t}}$$

    Since the above is the probability that X is greater than or equal to a certain time, we can find $F_X(t) = P[X < t]$ as:

    $$\boxed{F_X(t) = 1 - e^{-.03t}}$$

    We can then differentiate to find:

    $$\boxed{f_X(t) = .03e^{-.03t}}$$

    We may observe that this is an exponential (Laplace) random distribution.

    (c) Per our distribution rules we know:

    $$\lambda = .03$$

    And also:

$$E[X] = \lambda^{-1}$$

This gives us:

$$\boxed{E[X] = 33.33\bar{3}}$$

Furthermore, we know:

$$\text{Var}[X] = \frac{1}{\lambda^2}$$

This gives us:

$$\boxed{\text{Var}[X] = 1111.11}$$

11. For the first plan, we may simply write:

$$\boxed{Y_A = .05X}$$

For the second plan, we have a two-step system, for which we get:

$$\boxed{Y_B = \begin{cases} .2, & X \leq 10 \\ .05X - .3, & X > 10 \end{cases}}$$

Given the scaling factor, we may write the expected value of Y_A as:

$$E[Y_A] = E[.05X]$$
$$E[Y_A] = .05E[X]$$
$$\boxed{E[Y_A] = .05\mu}$$

For the expectation value of the second plan, the process is a bit more complex. Given that the expectation value is an exponential random variable, we can begin by writing the PDF:

$$f_X(x) = \frac{1}{\mu}e^{-\frac{x}{\mu}}$$

From here, we can write out the expectation values as:

$$Y_B = .2P[X \leq 10] + E[.05X - .3|X > 10]P[X > 10]$$

From the PDF, we may integrate to get the PDF as:

$$F_X(X \leq x) = 1 - e^{-\frac{x}{\mu}}$$

As such, we may find:

$$P[X \leq 10] = 1 - e^{-\frac{10}{\mu}} \quad \text{and} \quad P[X > 10] = e^{-\frac{10}{\mu}}$$

We plug this into our expectation value formula above to get:

$$Y_B = .2\left(1 - e^{-\frac{10}{\mu}}\right) + (.2 + .05\mu)e^{-\frac{10}{\mu}}$$

We simplify to get:

$$\boxed{Y_B = .2 + .05\mu e^{-\frac{10}{\mu}}}$$

We then use the above formulas at $\mu = 5, 8$[GB] to get:

$$\$_{A,5} = .05(5) = .25[\text{Dollars}]$$
$$\$_{A,8} = .05(8) = .4[\text{Dollars}]$$
$$\$_{B,5} = .2 + .05(5)e^{-2} = .2338[\text{Dollars}]$$
$$\$_{B,8} = .2 + .05(8)e^{-(10/8)} = .3146[\text{Dollars}]$$

<u>As such, Plan B is better in both cases</u>