Lecture 1 — Basics of Probability Theory

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- What is a Set?
  - A set is a collection of objects (elements) that make up the set
  - We usually use upper case letters to describe a set and lower-case letters to refer to the elements
  - A set can be defined using enumerations:

$$A = \{\text{Jane}, \text{Bill}, \cdots\}$$
$$B = \{1, 2, 3, \cdots\}$$

  - A set can also be defined using a description method
  - $A = \{x \,|\, x \text{ satisfies some property}\}$
  - For example:

$$A = \{\text{Students} \mid \text{Students who earned an 'A'}\}$$

  - A set can have a finite or infinite number of elements
  - Useful notations:
    * $x \in A \equiv$ element x is contained in A
    * $x \notin A \equiv$ element x is not contained in A
    * $C = \{\} = \emptyset$ — C is an empty or null set
    * $D = S$ — Universal set including all elements in a given category
    * $A \subset B$ — A is a subset of set B
    * Simple Set: A set with a single element
    * Set equality: $A = B$ only if $A \subset B$ and $B \subset A$
    * $A^C \equiv$ complement of set A, which includes all elements in a given category that are not in set A
- The Inclusion-Exclusion Rule

- For two finite sets A and B, we have:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- This can be expanded to three sets (with a new set C):

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

- Types of Sets
  - Collectively Exhaustive Sets — A collection of sets $A_1, A_2, \cdots, A_n$ are collectively exhaustive if (at least one of the events must occur):

$$A_1 \cup A_2 \cup \cdots \cup A_n = S$$

  - Mutually Exclusive Sets — Two sets are mutually exclusive if they have no elements in common:

$$A_i \cap A_f = \emptyset \quad i \neq j$$

  - Partitions — A collection of sets $A_1, A_2, \cdots, A_n$ is a partition if they are both mutually exclusive and collectively exhaustive:

$$A_1 \cup A_2 \cup \cdots \cup A_n = S \quad \text{and} \quad A_i \cap A_j = \emptyset \quad i \neq j$$

- Algebraic Rules of Manipulating Sets
  - Commutative: $A \cap B = B \cap A$ and $A \cup B = B \cup A$
  - Associative: $A \cap (B \cap C) = (A \cap B) \cap C$
  - Distributive: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  - De Morgan's Law: $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$

- Applying Set Theory to Probability Theory
  - The Probability Function
    * A probability measure $P[x]$ is a function that maps events in the sample space to real numbers
    * The probability function assigns a value 0 to 1 to a certain event; for example, the probability function of rolling a single, standard die would be $P[1] = P[2] = \cdots = P[6] = 1/6$
  - Axioms of Probability
    * For any event A, $P[A] \geq 0$
    * $P[S] = 1$

- For any countable collection of mutually exclusive events:

$$P[A_1 \cup A_2 \cup \cdots \cup A_n] = P[A_1] + P[A_2] + \cdots + P[A_n]$$

- Consequences of the Axioms
  - For mutually exclusive events, $P[A \cup B] = P[A] + P[B]$
  - For mutually exclusive events, $A_1, A_2, \cdots, A_N$:

$$P[A_1 \cup A_2 \cup \cdots \cup A_N] = P[A_1] + P[A_2] + \cdots + P[A_N]$$

- Prior, Posterior, and Conditional Probabilities
  - Prior Probability $P[A]$: The probability of an event A before any other information or evidence is considered
  - Conditional Probability $P[A|B]$: The probability of A given that B has occurred, showing how B influences the likelihood of A
    - $P[A]$ = Prior Probability of Event $A \to P[A] = \dfrac{|A|}{|S|}$
    - $P[A|B]$ = Conditional Probability of A Given Event B Occurred
    - When event B occurs, the sample space is reduced to B

$$P[A|B] = \frac{|A \cap B|}{|B|} = \frac{P[A \cap B]}{P[B]}$$

  - Posterior Probability $P[B|A]$: The probability of event B after observing event A. It represents a "revised belief" about B, incorporating the new information provided by A. Posterior probability adjusts the prior probability of B based on the evidence from A

- Bayes Rule and Statistical Independence

$$P[A \cap B] = P[AB] = P[B]P[A|B] = P[A]P[B|A]$$

  - Bayes' Rule states:

$$P[A|B] = \frac{P[AB]}{P[B]} \quad \text{and} \quad P[B|A] = \frac{P[AB]}{P[A]}$$

  - A and B are statistically independent if:

$$P[AB] = P[A]P[B] \quad \text{or} \quad P[A|B] = P[A] \text{ and } P[B|A] = P[B]$$

- Probability Chain Rule of Three Events

$$P[ABC] = P[C]P[AB|C] = P[C]P[B|C]P[A|BC]$$

- A, B, and C are statistically independent if $P[ABC] = P[A]P[B]P[C]$ and each pair satisfies the condition of independence:
    * A and B are statistically independent
    * A and C are statistically independent
    * B and C are statistically independent

- Permutations

  - The number of ways to arrange k distinguishable objects if a set of n distinguishable objects with repetition is:

$$\# = n^k$$

- Combinations

  - Given a set of n distinct objects, any unordered subset k of the objects is called a combination of size k. The number of combinations of size k that can be formed from n distinct objects will be denoted by ${}_nC_k$ or $\binom{n}{k}$ and is given by:

$$\frac{n!}{k!(n-k)!}$$

- Binomial Probability

  - The probability of having exactly n_1 successes and n_o failures in n independent trials is given by:

$$p_b = \binom{n}{n_1} p^{n_1}(1-p)^{n-n_1}$$

  - $n_o + n_1 = n$
  - p is the probability of success in any trial independent of any other trial

- Multinomial Probability

  - In n independent repetitions of an experiment with sample space $[s_1, s_2, \cdots, s_n]$ and the corresponding probabilities $[p_1, p_2, \cdots, p_n]$, the probability of having exactly n_i outcomes of type s_i is given by:

$$p = \frac{n!}{n_1!n_2!\cdots!n_m!} p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m} \qquad \text{where } n = n_1 + n_2 + \cdots + n_m$$