Homework 3

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1. (a) We want to find the probability that both units failed. This can be expressed as event A AND event B, indicating failure of the processors. Per our formulas, we know:

$$P[A|B] = \frac{P[A \cap B]}{P[B]}$$

Per the given information, we know that $P[A|B] = .3$ and $P[B] = .02$. Thus, we may calculate to get:

$$\boxed{P[A \cap B] = .006}$$

(b) We may calculate the probability of no good units in five systems by simply raising part (a) to the power of 5 to get:

$$\boxed{P_5[A \cap B] = 7.776 \cdot 10^{-12}}$$

(c) The probability that there is a working unit in a single computer can be given by the complement of no working units. Thus, we see:

$$P[(A \cap B)^c] = 1 - .006 = .994$$

We can then sum the possible combinations for the other two computers (at least one working or none working) to get:

$$P_3[(A \cap B)^c] = .994(.994^2 + (.994)(.006) + .006^2)$$
$$\boxed{P_3[(A \cap B)^c] = .9881}$$

2. (a) The probability that the system does not fail may be expressed as:

$$P[F^c] = (1 - p)^{20}$$

We can thus take the complement and say that failure will occur for:

$$P[F] = 1 - (1-p)^{20}$$

From the problem, the system failure rate is 20%, which gives us:

$$1 - (1-p)^{20} = .2$$

From here, we solve:

$$\begin{aligned} (1-p)^{20} &= .8 \\ 1-p &= \sqrt[20]{.8} \\ p &= 1 - \sqrt[20]{.8} \\ &\boxed{p = .011095} \end{aligned}$$

(b) Similar to part (a), we now solve for a failure rate of .1:

$$\begin{aligned} 1 - (1-p)^{20} &= .1 \\ (1-p)^{20} &= .9 \\ p &= 1 - \sqrt[20]{.9} \\ &\boxed{p = .0052542} \end{aligned}$$

4. (a) We can calculate this by multiplying the success rates together:

$$P_3 = (.9)^2(.8)$$

$$\boxed{P_3 = .648}$$

(b) We want to calculate the probability of at least one of the components succeeding. This may be expressed as:

$$P_{1+} = \underbrace{2[(.9)(.1)(.2)]}_{\text{One router}} + \overbrace{[(.1)^2(.8)]}^{\text{Only switch}} + \underbrace{[(.9)^2(.2)]}_{\text{Both routers}} + \overbrace{2[(.9)(.8)(.1)]}^{\text{One router/one switch}} + .648$$

$$\boxed{P_{1+} = .998}$$

(c) To find the odds of exactly one of the components passing, we simply subtract cases from above in which multiple devices pass. This gives us:

$$P_1 = .998 - 2[(.9)(.8)(.1)] - [(.9)^2(.2)] - .648$$

$$\boxed{P_1 = .044}$$

(d) Since we know the probability that no components fail, we simply take the complement to find the probability that a component does fail:

$$P_3^c = 1 - .648$$
$$\boxed{P_3^c = .352}$$

(e) Per our formulas, we can find $P[\text{all}|\text{one}]$ as:

$$P[\text{all}|\text{one}] = \frac{.648}{.998}$$
$$\boxed{P[\text{all}|\text{one}] = .6493}$$

5. (a) Given that n can be the range of $[2, 5]$, we know that the PMF must contain:

$$\left\{\frac{c}{4}, \frac{c}{8}, \frac{c}{16}, \frac{c}{32}\right\}$$

We know that the probabilities within a PMF must sum to 1, so we get:

$$\frac{c}{4} + \frac{c}{8} + \frac{c}{16} + \frac{c}{32} = 1$$

We simplify to find:

$$8c + 4c + 2c + c = 32$$
$$\boxed{c = \frac{32}{17}}$$

(b) We can take the complement of the probability that two packets are transmitted to find:

$$P[3+] = 1 - \frac{1}{4}\left(\frac{32}{17}\right)$$
$$\boxed{P[3+] = \frac{7}{17} = .4118}$$

(c) We can find the probability of an odd packet to be:

$$P[3 + \cap Odd] = \left(\frac{1}{8} + \frac{1}{32}\right)\frac{32}{17}$$
$$P[3 + \cap Odd] = \frac{5}{17}$$

We then divide by the probability of three or more to find:

$$\boxed{P[Odd|3+] = \frac{5}{7} = .7143}$$

(d) We can express the probability for a given n as:

$$P[C] = \sum_{n=2}^{5} P_N(n)(.95)^n$$

This gives us the following values:

n	$P_n[C]$
2	.4247
3	.2017
4	.095824
5	.045517
Total	.7677

6. (a) Based on the given information, we may express the probability of 2 or 3 servers as c. This gives us:

$$\left\{\frac{c}{4}, \frac{c}{2}, c, c, \frac{c}{4}\right\}$$

We know the values need to sum to 1, which gives us:

$$\frac{2c}{4} + \frac{c}{2} + 2c = 1$$
$$\frac{c}{2} + \frac{c}{2} + 2c = 1$$
$$3c = 1$$
$$\boxed{c = \frac{1}{3}}$$

Thus, we get:

$$\left\{\frac{1}{12}, \frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{12}\right\}$$

(b) Based on the PMF above, we sum to get:

$$P[2+] = 1 - \frac{1}{12} - \frac{1}{6}$$
$$\boxed{P[2+] = .75}$$

7. We may express the PMF of A as a binomial distribution, since the pulled resistor is either A or not. This gives us $n = 10$ and a probability of $p = 1/3$, since each resistor is equally likely. This gives us the terms of the PMF as:

$$\binom{10}{n}(1/3)^p(2/3)^{p-n}$$

We iterate to get:

$$P_n(x) = \{.017342, .086708, .19509, .26012, .22761, .13656, .056902,$$
$$.016258, .0030483, .00033870, .000016935\}$$

From here, we find the probability of 2 or more A resistors as:

$$P[2+] = 1 - .017342 - .086708$$
$$\boxed{P[2+] = .896}$$

8. We begin by finding the probability that a generation-ending pair is combined. We may iterate through possible values of S_M and S_L and find that the pairs with absolute differences greater than three, in form (l, m) are:

$$(1, 5), (1, 8), (2, 8), \text{ and } (3, 8)$$

We know that the total quantity of possible pairs can be found by multiplying together the quantity of possible values of l and m, which gives:

$$N = (4)(3) = 12$$

Thus, we find the probability that a generation-ending pair is found is:

$$p = \frac{4}{12} = .33\bar{3}$$

We can write the PMF as:

$$\boxed{P_N(x) = \left(\frac{2}{3}\right)^{1-N}\left(\frac{1}{3}\right)}$$

We can find the probability that $N = 5$ as:

$$P_5(x) = \left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)$$
$$\boxed{P_5(x) = .065844}$$

We may define the probability that $N > 5$ as:

$$P_{>5}(x) = 1 - \sum_{n=1}^{5} P_N(x)$$
$$\boxed{P_{>5}(x) = .1317}$$

9. (a) We can define the PMF as:

$$\boxed{P_N(x) = \binom{20}{N} p^N (1-p)^{20-N}}$$

(b) Using the above, we take that $N \leq 15$ and $p = .8$ to calculate the probability of rejection as:

$$P(\text{rejection}|p = .8) = \sum_{N=0}^{15} \binom{20}{N} (.8)^N (.2)^{20-N}$$

We iterate to find:

$$\boxed{P(\text{rejection}|p = .8) = .3704}$$

(c) Similar to part (b), we can find the probability that the claim is <u>not</u> rejected given $p = .7$ by writing:

$$P(\text{not rejected}|p = .7) = 1 - \sum_{N=0}^{15} \binom{20}{N} (.7)^N (.3)^{20-N}$$

This gives us:

$$\boxed{P(\text{not rejected}|p = .7) = .2375}$$

(d) We can see that (b) would decrease, since one less value (when $N = 15$) is added to the probability, while the probability for (c) would increase, since this value is no longer subtracted. Thus, we find:

$$P_{b,\text{new}} = .3704 - \binom{20}{15} (.8)^{15} (.2)^5$$

$$\boxed{P_{b,\text{new}} = .1958}$$

$$P_{c,\text{new}} = .2375 + \binom{20}{15} (.7)^{15} (.3)^5$$

$$\boxed{P_{c,\text{new}} = .4164}$$

10. (a) We may observe that the PMF is simply a geometric distribution and can be written as ($n \geq 1$):

$$\boxed{P_N(x) = p^n (1-p)^{n-1}}$$

(b) We can calculate the probability as:

$$P_3(x) = \sum_{n=1}^{3} p^n (1-p)^{n-1}$$

We can expand this to get:

$$p + p(1-p) + p(1-p)^2 \geq .95$$
$$-.95 + p + p(1-p) + p(1-p)^2 \geq 0$$

Solving, we find that the minimal value of p is:

$$\boxed{p \geq .6316}$$

(c) Although the formula for the PMF remains the same, it is different in that n is now capped, such that $1 \leq n \leq 5$. With $p = .7$ we may write this as:

$$\boxed{P_N(x) = (.7)^n (.3)^{n-1}, \quad 1 \leq n \leq 5}$$

We can calculate the probability that delivery fails as:

$$P_{>5}(x) = 1 - \sum_{n=1}^{5} (.7)^n (.3)^{n-1}$$

This gives us:

$$\boxed{P_{>5}(x) = .1143}$$