Homework 9

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1. (a) Since it is given that $X, Y \in [0,1]$, we find that the range of $W = \text{Max}(X, Y) \rightarrow \boxed{W \in [0,1]}$. Therefore, we write:

$$\boxed{S_W = [0,1]}$$

(b) Given that w is the upper bound of x and/or y, we may write:

$$F_W(w) = \int_0^w \int_0^w 4xy\,dx\,dy$$

We evaluate this to get:

$$F_W(w) = 4\int_0^w y \int_0^w x\,dx\,dy$$

$$F_W(w) = 4\left[\frac{y^2}{2}\right]\Big|_0^w \left[\frac{x^2}{2}\right]\Big|_0^w$$

$$\boxed{F_W(w) = \begin{cases} w^4, & 0 \leq w < 1 \\ 1, & w \geq 1 \\ 0, & \text{otherwise} \end{cases}}$$

From here, we know that:

$$f_W(w) = \frac{d}{dw}[F_W(w)]$$

This gives us:

$$\boxed{f_W(w) = 4w^3, \quad 0 \leq w < 1}$$

2. (a) Given that $y \geq x$, we know that the lowest point for W will occur when $y = x$, or $W = 0$. Similarly, the biggest difference occurs when $y = 1$ and $x = 0$, which produces $W = 1$. Thus, we write:

$$\boxed{S_W = [0,1]}$$

(b) From here, we may write:

$$F_W(w) = \int_0^1 \int_0^{y-w} 6x \, dx \, dy$$

We evaluate to get:

$$F_W(w) = 6 \int_0^1 [x^2/2]\Big|_0^{y-w} dy$$

$$F_W(w) = 3 \int_0^1 (y-w)^2 \, dy$$

$$F_W(w) = 3 \int_0^1 y^2 - 2yw + w^2 \, dy$$

$$F_W(w) = y^3 - 3y^2 w + 3w^2 y\Big|_0^1$$

$$\boxed{F_W(w) = \begin{cases} 1 - 3w + 3w^2, & 0 \le w \le 1 \\ 1, & w > 1 \\ 0, & \text{otherwise} \end{cases}}$$

We then differentiate to get:

$$\boxed{f_W(w) = 6w - 3, \quad 0 \le w \le 1}$$

3. (a) We can express the PDF of Y using the law of total probability as:

$$f_Y(y) = P[Z = 1] f_x(y) + P[Z = -1] f_x(-y)$$

Substituting our values, since we know that X is normal with $\mu = 0$ and $\sigma^2 = 1$, we get:

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} \left[p e^{-\frac{y^2}{2}} + (1-p) e^{-\frac{y^2}{2}} \right]$$

Distributing and simplifying gives us:

$$\boxed{f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}}$$

Thus, we see that this returns to simply being a normal distribution.

(b) When we take $\mu \to 10$, this gives us:

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} \left[p e^{-\frac{(y-10)^2}{2}} + (1-p) e^{-\frac{(-y-10)^2}{2}} \right]$$

We simplify to get:

$$\boxed{f_Y(y) = \frac{1}{\sqrt{2\pi}}\left[pe^{-\frac{(y-10)^2}{2}} + (1-p)e^{-\frac{(y+10)^2}{2}}\right]}$$

5. We begin by finding the moment generating function of X. We can write this as:

$$\phi_x(s) = E[e^{sX}]$$
$$E[e^{sX}] = \sum_{x=0}^{\infty} e^{sx}\left(\frac{e^{-\alpha_1}\alpha_1^x}{x!}\right)$$

   We rearrange to get:

$$E[e^{sX}] = e^{-\alpha_1}\sum_{x=0}^{\infty}\left(\frac{(e^s\alpha_1)^x}{x!}\right)$$

   We know that the series expansion for an exponential is:

$$e^x \to \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

   As such, we get:

$$\boxed{\phi_x(s) = e^{\alpha_1(e^s-1)}}$$

   Similarly, we find the MGF of Y:

$$\boxed{\phi_y(s) = e^{\alpha_2(e^s-1)}}$$

   Since X and Y are independent, the MGF of $Z = X + Y$ will simply be the product of the individual MGFs:

$$\boxed{\phi_z(s) = e^{(\alpha_1+\alpha_2)(e^s-1)}}$$

   Note that, if we were to apply the same process to find the MGF of Z as we did for X and Y, we would still get the same answer.

7. (a) Using the moment generating function, we know that:

$$E[Y] = \phi_y(0)'$$

   As such, we differentiate to get:

$$\phi_y'(s) = \frac{2}{(1-s)^3}$$

We evaluate at $s = 0$, which gives us:

$$\boxed{E[Y] = 2}$$

Similarly, we may write:

$$E[Y^2] = \phi_y(0)''$$

We differentiate a second time to get:

$$\phi_y''(s) = \frac{6}{(1-s)^4}$$

Evaluating at $s = 0$, we find:

$$\boxed{E[Y^2] = 6}$$

(b) Similar to (a), we find:

$$E[V] = \phi_w(0)'$$
$$E[V] = \frac{3}{(1-0)^4}$$
$$\boxed{E[V] = 3}$$

This gives us:

$$E[W] = E[Y] + E[V]$$
$$E[W] = 2 + 3$$
$$\boxed{E[W] = 5}$$

We then find:

$$E[V^2] = \phi_w(0)''$$
$$E[V^2] = \frac{12}{(1-0)^5}$$
$$\boxed{E[V^2] = 12}$$

From here, we may write:

$$\text{Var}[W] = \text{Var}[Y] + \text{Var}[V]$$

This gives us:

$$\text{Var}[W] = (6 - 2^2) + (12 - 3^2)$$
$$\boxed{\text{Var}[W] = 5}$$

(c) Given that Y and V are independent, we may take their product to find:

$$\phi_w(s) = \frac{1}{(1-s)^5}$$

We may observe that, accordingly, $W = \text{Erlang}(\lambda = 1, k = 5)$, which makes sense, given that $Y = \text{Erlang}(1, 2)$ and $V = \text{Erlang}(1, 3)$. As such, we get:

$$\boxed{f_W(w) = \frac{w^4 e^{-w}}{4!}, \quad w \geq 0}$$

(d) We may re-express this as:

$$W_1 = \frac{1}{2}(Y + V) \to W_1 = \frac{1}{2}W$$

Accordingly, we get:

$$E[W_1] = \frac{1}{2}E[W]$$

$$\boxed{E[W_1] = 2.5}$$

And then:

$$\text{Var}[W_1] = \frac{1}{4}\text{Var}[W]$$

$$\boxed{\text{Var}[W_1] = 1.25}$$

And finally:

$$\boxed{f_{W_1}(w_1) = \frac{2(2w_1)^4 e^{-2w_1}}{4!}, \quad w_1 \geq 0}$$

8. (a) Knowing our results for a standard normal distribution, may write the moment generating function as:

$$\phi_x(s) = E[e^{sX}]$$

$$\boxed{\phi_x(s) = e^{\frac{s^2}{2}}}$$

(b) Given that each X_n is independent of the others, we can multiply the MGFs k times to write:

$$E[e^{sV} | K = k] = e^{\frac{ks^2}{2}}$$

From here, we use the given probabilities to write:

$$\phi_v(s) = \sum_{k=1}^{\infty} P_K(k) e^{\frac{ks^2}{2}}$$

This gives us:

$$\boxed{\phi_v(s) = \frac{.8e^{\frac{s^2}{2}}}{1 - .2e^{\frac{s^2}{2}}}}$$

Before evaluating the expected value with the MGF, we may find the value to be:

$$E[V] = E[E[V|K]]$$

From above, we know:

$$E[V|K] = kE[X]$$

And since each X is normally distributed, $E[X] = 0$. Therefore, we may conclude:

$$\boxed{E[V] = 0}$$

We confirm using the MGF:

$$\phi_v(s)' = \frac{.8se^{\frac{s^2}{2}}}{\left(1 - .2e^{\frac{s^2}{2}}\right)^2}$$

At $s = 0$ this gives:

$$\phi_v(0)' = 0$$

And therefore, we confirm:

$$\boxed{E[V] = \phi_v'(0) = 0}$$

9. (a) We may observe that the given distribution is binomial, such that we may write:

$$K_{50} = \binom{50}{k} .7^k .3^{1-k}$$

Accordingly, we may find:

$$E[K_{50}] = np$$

$$E[K_{50}] = 50(.7)$$

$$\boxed{E[K_{50}] = 35 \text{ video packets}}$$

(b) Again, given this is a binomial distribution, we write:

$$\sigma_{50} = \sqrt{np(1-p)}$$
$$\sigma_{50} = \sqrt{35(.3)}$$
$$\boxed{\sigma_{50} = 3.2404 \text{ video packets}}$$

(c) We begin by standardizing to a normal variable, Z:

$$Z = \frac{X - 35}{3.2404}$$

We take the limits as $30 \leq X \leq 40$, such that:

$$\frac{-5}{3.2404} \leq Z \leq \frac{5}{3.2404}$$
$$-1.543 \leq Z \leq 1.543$$

We can find this probability by first finding:

$$P[Z \geq -1.543] = .9386$$

We then subtract the upper bound:

$$P[30 \leq X \leq 40] = .9386 - P[Z \geq 1.543]$$
$$P[30 \leq X \leq 40] = .9386 - .061415$$
$$P[30 \leq X \leq 40] = .8772$$

(d) We begin by using continuity correction to write:

$$P[a \leq K_n \leq b] \approx P\left[\frac{a-\mu}{\sigma} \leq Z \leq \frac{b-\mu}{\sigma}\right]$$

This gives us:

$$P[30 \leq K_n \leq 50] \approx P\left[29.5 \leq K_n \leq 40.5\right]$$
$$P[30 \leq K_n \leq 50] \approx P\left[-1.6973 \leq Z \leq 1.6973\right]$$

We can use the inverse normal operation to find:

$$P[30 \leq K_n \leq 50] \approx P[Z \leq 1.6973] - P[Z \leq -1.6973]$$
$$P[30 \leq K_n \leq 50] \approx .955 - .045$$
$$\boxed{P[30 \leq K_n \leq 50] \approx .91}$$

11. (a) We know that the expectation value for a single test is given by the inverse of its probability, or:

$$E[X_n] = \frac{1}{p}$$

Given that we sum together a series consisting of n of these geometric distributions, we can find:

$$E[K_n] = E[X_1] + E[X_2] + \cdots + E[X_n] = \frac{n}{p}$$

This gives us:

$$E[K_n] = \frac{900}{.9}$$

$$\boxed{E[K_n] = 1000 \text{ tests}}$$

(b) Similarly, the variance of K_n may be written as:

$$\text{Var}[K_n] = n\text{Var}[X_n]$$

We may find:

$$\text{Var}[X_n] = \frac{1-p}{p^2}$$

$$\text{Var}[X_n] = \frac{.1}{.9^2}$$

$$\boxed{\text{Var}[X_n] = .1235 \text{ tests}^2}$$

We then multiply by n to find:

$$\boxed{\text{Var}[K_n] = 111.11 \text{ tests}^2}$$

(c) Using the CLT for <u>at least</u> 1000 tests, we write:

$$Z = \frac{1000-\mu}{\sigma} = \frac{1000-1000}{\sqrt{111.11}}$$

$$Z = 0$$

From here, we apply a normal distribution to write:

$$\boxed{P[Z \geq 0] = .5}$$

12. We can begin by finding:

$$E[V] = 20 - 10E[W^3]$$

We can find this by using:

$$E[W^3] = \int_{-1}^{1} \frac{w^3}{2}\,dw$$
$$E[W^3] = \frac{w^4}{8}\Big|_{-1}^{1}$$
$$E[W^3] = 0$$

This gives us:

$$E[V] = 20$$

Similarly, we write:

$$\text{Var}[V] = 100\text{Var}[W^3]$$
$$\text{Var}[V] = 100\int_{-1}^{1} (w^3)^2 \cdot \frac{1}{2}\,dw$$
$$\text{Var}[V] = 100\int_{-1}^{1} \frac{w^6}{2}\,dw$$
$$\text{Var}[V] = 100\left[\frac{w^7}{14}\right]\Big|_{-1}^{1}$$
$$\text{Var}[V] = 100/7 \approx 14.285$$

As such, we may write expressions for X as:

$$E[X] = 5 \cdot 30 \cdot E[V]$$
$$\sigma_X = 5 \cdot \sqrt{30 \cdot \text{Var}[V]}$$

These give us:

$$E[X] = 3000 \text{ Mb}$$
$$\sigma_X = 103.51 \text{ Mb}$$
$$\text{Var}[X] = 5 \cdot 30 \cdot \text{Var}[V]$$

We then want to find:

$$P\left[X \geq 95\right] \rightarrow P\left[Z \geq \frac{X-\mu}{\sigma}\right]$$

We enter our values to find:

$$Z = \frac{95 - 3000}{103.51}$$
$$Z = -28.065$$

From here, we find:

$$P\left[X \geq 95\right] = 1 - P[Z \leq -28.065]$$
$$\boxed{P\left[X \geq 95\right] \approx 1}$$