Homework 10

Michael Brodskiy

Professor: I. Salama

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1. We begin by writing an expression for $B_k = 0$, which gives us a waveform of:

$$x_0(t) = \cos\left(\frac{2\pi Nt}{T}\right), \quad t \in [(k-1)T, kT]$$

We then write, for $B_k = 1$:

$$x_0(t) = -\sin\left(\frac{2\pi Nt}{T}\right), \quad t \in [(k-1)T, kT]$$

We know that, given that f_o is a multiple of $1/T$, we know a full N cycles will be transmitted. We may observe that the sample space of X consists of 8 waveforms, as a result of the transmission of 3 bits (which means 2^3 possible combinations. These are given by:

$$\begin{gathered}
x_o(t), \quad t \in [0, 3T] \\
x_o(t), \quad t \in [0, 2T] \quad \text{and} \quad x_1(t), \quad t \in [2T, 3T] \\
x_o(t), \quad t \in [0, T] \cup [2T, 3T] \quad \text{and} \quad x_1(t), \quad t \in [T, 2T] \\
x_o(t), \quad t \in [T, 3T] \quad \text{and} \quad x_1(t), \quad t \in [0, T] \\
x_o(t), \quad t \in [0, T] \quad \text{and} \quad x_1(t), \quad t \in [T, 3T] \\
x_o(t), \quad t \in [T, 2T] \quad \text{and} \quad x_1(t), \quad t \in [0, T] \cup [2T, 3T] \\
x_o(t), \quad t \in [2T, 3T] \quad \text{and} \quad x_1(t), \quad t \in [0, 2T] \\
x_1(t), \quad t \in [0, 3T]
\end{gathered}$$

Accordingly, we find that the elements correspond to $(B_1, B_2, B - 3)$ as:

$$X_k(t) \to \begin{cases} 1, & (0,0,0) \\ 2, & (0,0,1) \\ 3, & (0,1,0) \\ 4, & (1,0,0) \\ 5, & (0,1,1) \\ 6, & (1,0,1) \\ 7, & (1,1,0) \\ 8, & (1,1,1) \end{cases}$$

To plot, we need to assume a value of N. Let us take this such that $N = T$ to simplify the functions. We proceed to plot each corresponding figure:

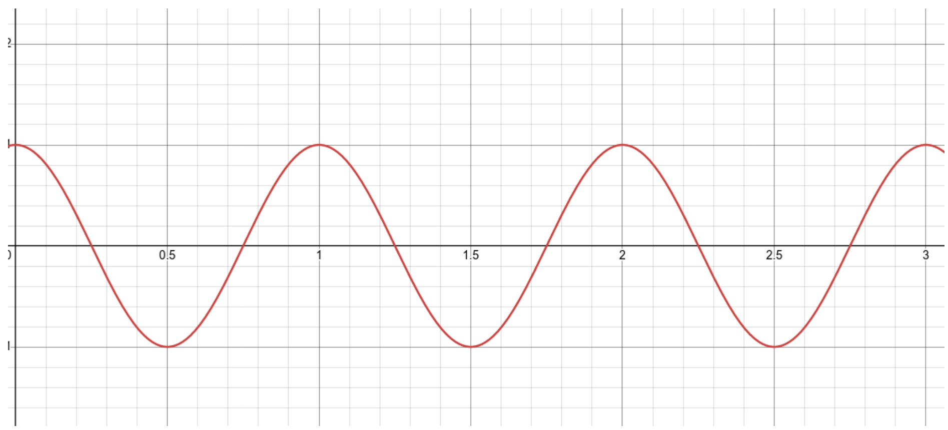

Figure 1: Plot for $(B_1, B_2, B_3) = (0, 0, 0)$

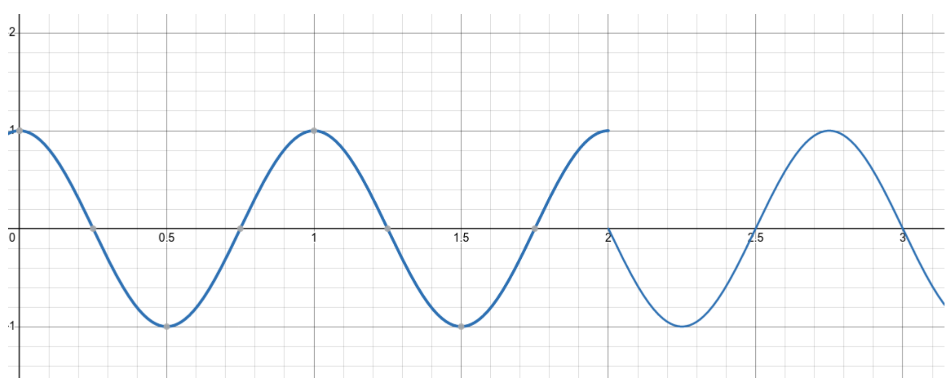

Figure 2: Plot for $(B_1, B_2, B_3) = (0, 0, 1)$

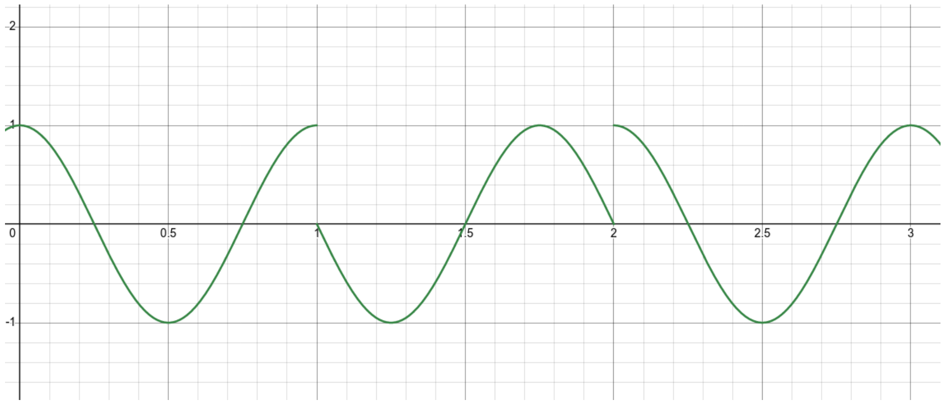

Figure 3: Plot for $(B_1, B_2, B_3) = (0, 1, 0)$

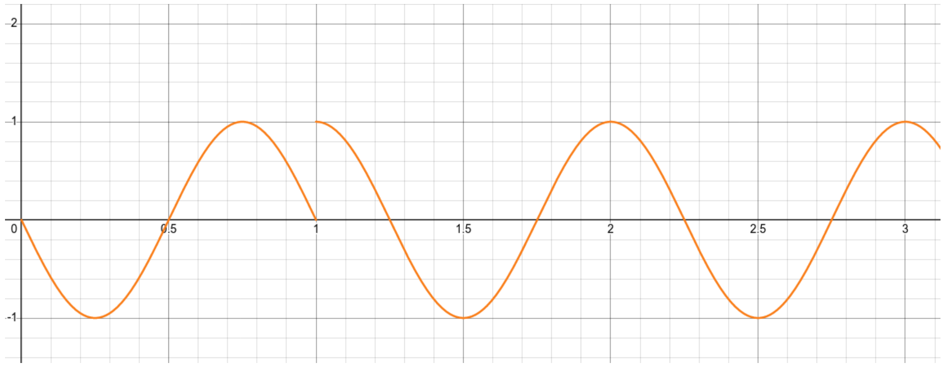

Figure 4: Plot for $(B_1, B_2, B_3) = (1, 0, 0)$

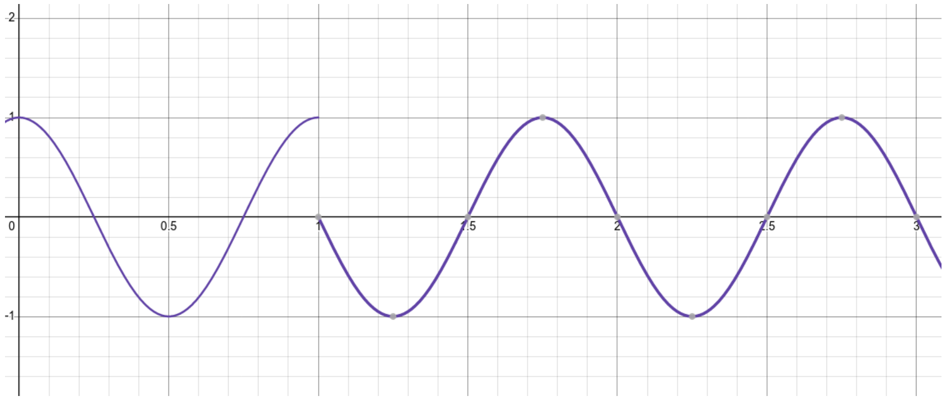

Figure 5: Plot for $(B_1, B_2, B_3) = (0, 1, 1)$

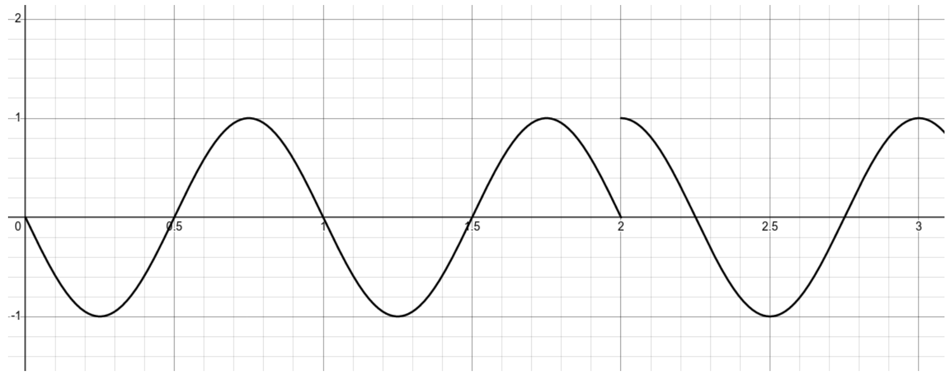

Figure 6: Plot for $(B_1, B_2, B_3) = (1, 1, 0)$

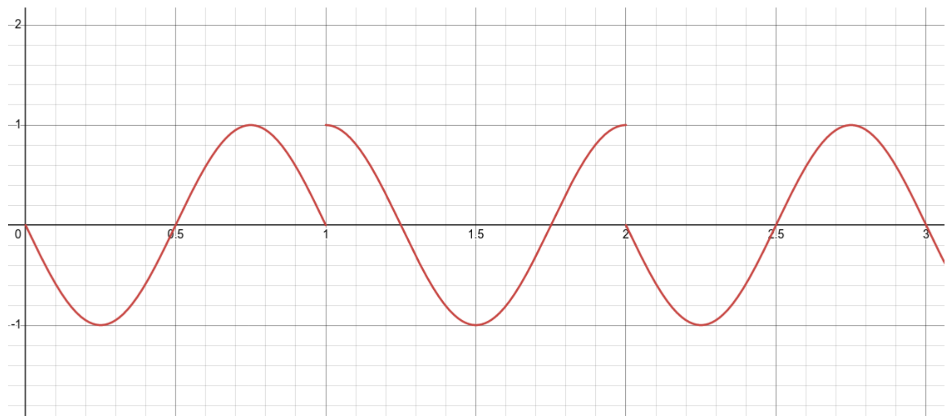

Figure 7: Plot for $(B_1, B_2, B_3) = (1, 0, 1)$

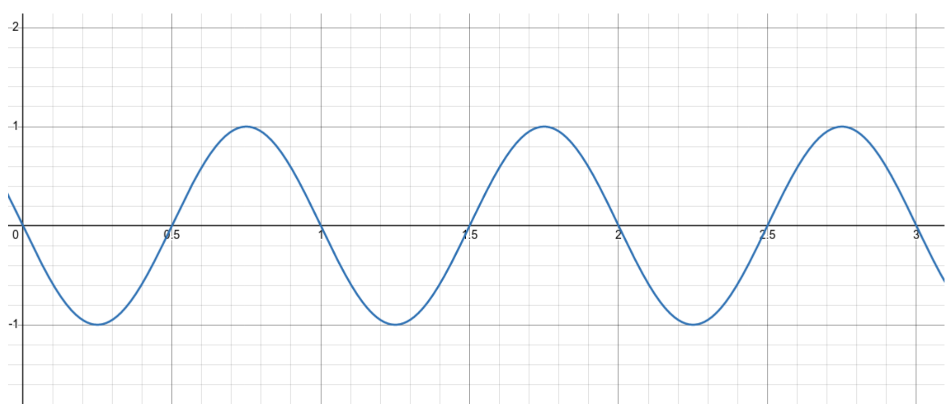

Figure 8: Plot for $(B_1, B_2, B_3) = (1, 1, 1)$

2. (a) Taking $t \to 100/3[\mu s]$, we obtain:

$$s_1 = 5\cos\big(10^4\pi(100/3)\cdot 10^{-6}\big)$$

$$s_1 = 5\cos(\pi/3)$$
$$s_1 = 2.5$$

We can then find $Y_1(t)$ by tacking on the noise term to get:

$$\boxed{Y_1(t) \to N(2.5, 1)}$$

Which means that this gives a normal distribution with a mean of 2.5 and standard deviation of 1.

(b) Similarly, we take $t \to 100[\mu\text{s}]$ to get:

$$s_2 = 5\cos\left(10^4\pi 200 \cdot 10^{-6}\right)$$
$$s_2 = 5\cos(2\pi)$$
$$s_2 = 5$$

This gives us:

$$\boxed{Y_2(t) \to N(5, 1)}$$

Or a normal distribution with mean 5 and standard deviation 1.

(c) Given that $s(t)$ is static, we know that the covariance depends solely on the noise. This gives us:

$$\text{Cov}(Y_1, Y_2) = \text{Cov}(N_1, N_2)$$

Since N_1 and N_2 are normal functions independent of each other, we find:

$$\text{Cov}(Y_1, Y_2) = \text{Cov}(N_1, N_2) = 0$$

Which means that Y_1 and Y_2 are independent

(d) Summing the two distributions and averaging their means and standard deviations, we find:

$$E\left[\frac{Y_1 + Y_2}{2}\right] = \frac{1}{2}(5 + 2.5)$$
$$E\left[\frac{Y_1 + Y_2}{2}\right] = 3.75$$

And then:

$$\sigma_{(Y_1+Y_2)/2} = \left(\frac{1}{2}\right)^2 (1 + 1)$$
$$\sigma_{(Y_1+Y_2)/2} = .5$$

Thus, we get:

$$\boxed{\frac{Y_1(t)+Y_2(t)}{2} \rightarrow N(3.75, .5)}$$

A normal distribution, with a mean of 3.75 and standard deviation of .5

4. (a) We know that the mean of $X(t)$ may be expressed as:

$$\mu_X = E[X(t)] = E[A_o \cos(\omega_o t + \theta)]$$

Given that this is over a uniform function from $-\pi$ to π, we find:

$$\boxed{\mu_X = 0}$$

We proceed to find the autocorrelation function as:

$$\mathcal{R}_{XX}(t,\tau) = E[X(t)X(t+\tau)]$$

We expand to get:

$$\mathcal{R}_{XX}(t,\tau) = E[A_o^2 \cos(\omega_o t + \theta)\cos(\omega_o(t+\tau)+\theta)]$$

Applying our cosine properties gives us:

$$\mathcal{R}_{XX}(t,\tau) = E\left[\frac{A_o^2}{2}(\cos(\omega_o\tau) + \cos(2\omega_o t + \omega_o\tau))\right]$$

This simplifies to:

$$\boxed{\mathcal{R}_{XX}(t,\tau) = \frac{A_o^2}{2}\cos(\omega_o\tau)}$$

As such, since the autocorrelation depends on the time difference, and the mean is constant, we may conclude that this <u>is a wide-sense stationary process</u>

(b) We may write the mean as:

$$\mu_y = E[V + X(t)] = E[V] + E[X(t)]$$

This gives us:

$$\boxed{\mu_y = \mu_V}$$

We then write the autocorrelation function as:

$$\mathcal{R}_{YY}(t,\tau) = E[Y(t)Y(t+\tau)]$$

Expanding gives us:

$$\mathcal{R}_{YY}(t,\tau) = E[(V + X(t))(V + X(t+\tau))]$$
$$\mathcal{R}_{YY}(t,\tau) = E[V^2 + VX(t+\tau) + VX(t) + X(t)X(t+\tau)]$$

We then simplify to:

$$\mathcal{R}_{YY}(t,\tau) = E[V^2 + X(t)X(t+\tau)]$$
$$\mathcal{R}_{YY}(t,\tau) = E[V^2] + E[X(t)X(t+\tau)]$$
$$\boxed{\mathcal{R}_{YY}(t,\tau) = \sigma_V^2 + \frac{A_o^2}{2}\cos(\omega_o\tau)}$$

Since the mean is constant due to its independence from θ, and the autocorrelation depends on the time difference, this is a wide-sense stationary function

5. (a) From the fact that $C(\tau) \neq C(-\tau)$, we may observe that this is not a valid autocovariance function for a WSS random process

   (b) Although the given function is even, it does not satisfy the positive-definite requirement. That is, is does not follow:

   $$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \mathcal{R}_x(\tau_i, \tau_j) \geq 0$$

   If we take $\tau_1 = -1$ and $\tau_2 = -2$, then we get:

   $$-c_1^2 - 2c_2^2$$

   We observe that this is less than zero, and, therefore, this is not a valid autocovariance function for a WSS random process

   (c) The function is even and non-negative for all of τ, and it is positive definite. Therefore, we conclude that this is a valid autocovariance function for a WSS random process. Note: we may verify that it is positive definite by showing that the sum would yield:

   $$c_i c_j \rightarrow \frac{1}{|\tau_1|}\frac{1}{|\tau_2|}$$

   Which must be positive.

   (d) We may observe that the function is even, non-negative for all of τ, and is positive definite. Therefore, we conclude that this is a valid autocovariance function for a WSS random process

6. (a) We may observe that $X(t)$ is not periodic, since the autocorrelation function does not contain a periodic term

(b) The function is not periodic, so we use the autocorrelation. We want to take $\tau \to \infty$; however, since τ is bounded, in this case, we take $\tau \to 6$:

$$E[X] = 3 - \frac{1}{2}(6)$$
$$\boxed{E[X] = 0}$$

(c) We may write the expected power as:

$$E[X^2(t)] = \mathcal{R}_{XX}(0)$$

This gives us:

$$E[X^2(t)] = 3 - \frac{1}{2}(0)$$
$$\boxed{E[X^2(t)] = 3}$$

(d) We can express this using the value of the uniform distribution to get:

$$P[X(t=1) > 1] = \int_1^3 \frac{1}{3-(-3)}\,dx$$
$$P[X(t=1) > 1] = \int_1^3 \frac{1}{6}\,dx$$
$$P[X(t=1) > 1] = \frac{x}{6}\bigg|_1^3$$
$$P[X(t=1) > 1] = \frac{3-1}{6}$$
$$\boxed{P[X(t=1) > 1] = \frac{1}{3}}$$

(e) We can break this up to write:

$$E[(X(1)+X(2)+X(3))^2] = E[X^2(1)] + E[X^2(2)] + EX^2(3)] +$$
$$2E[X(1)X(2)] + 2E[X(2)X(3)] + 2E[X(1)X(3)]$$

This is equivalent to:

$$E[(X(1)+X(2)+X(3))^2] = 3\mathcal{R}_{XX}(0) + 4\mathcal{R}_{XX}(1) + 2\mathcal{R}_{XX}(2)$$

We evaluate to get:

$$E[(X(1)+X(2)+X(3))^2] = 3\,[3] + 4\left[\frac{5}{2}\right] + 2[2]$$
$$\boxed{E[(X(1)+X(2)+X(3))^2] = 23}$$

(f) Expanding, we get:

$$E[Y] = 2E[X] + 3$$

$$\boxed{E[Y] = 3}$$

We can then find:

$$\mathcal{R}_{YY}(t) = E[(2X(t) + 3)(2X(t + \tau) + 3)]$$

We expand this to get:

$$\mathcal{R}_{YY}(t) = 4E[X(t)2X(t + \tau)] + 6E[X(t + \tau)] + 6E[X(t)] + 9$$

$$\boxed{\mathcal{R}_{YY}(t) = 4\mathcal{R}_{XX}(t) + 9}$$

7. (a) We may begin by computing the autocovariance as:

$$C_{XX}(n, k) = \text{Cov}(X_n, X_k)$$

But because all X_n are i.i.d, we get:

$$\boxed{C_{XX}(n, k) = \begin{cases} \text{Var}(X_n), & n = k \\ 0, & n \neq k \end{cases}}$$

We then compute the autocorrelation as:

$$R_{XX}(n, k) = E[X_n X_k]$$

Because of independence, we write:

$$R_{XX}(n, k) = E[X_n]E[X_k]$$

Accordingly, we get:

$$\boxed{R_{XX}(n, k) = \begin{cases} p, & n = k \\ p^2, & n \neq k \end{cases}}$$

(b) Since the distributions X_n are i.i.d, we may obtain the probability as simply the sum of individual probabilites. Since the probability of each is the same, we get:

$$\boxed{E[Y_n] = np}$$

Similarly, we can sum the variances by writing:

$$\text{Var}(Y_n) = n\text{Var}(X_n)$$

$$\boxed{\text{Var}(Y_n) = np(1 - p)}$$

(c) We may observe that each X_n represents a Bernoulli trial. accordingly, since Y_n is the sum of Bernoulli trials, it represents a Binomial distribution such that $\boxed{Y_n = \text{Binom}(n, p)}$

(d) We know that, to be a wide-sense stationary process, two conditions must be met:

i. $E[Y_n]$ is constant

ii. $\text{Cov}(Y_n, Y_k)$ is dependent solely on the difference $|n - k|$

From (b), we see that $E[Y_n] = np$ is not constant, and, therefore, the process is not wide-sense stationary.

(e) We begin by writing:

$$C_{YY}(n, k) = \text{Cov}(Y_n, Y_k)$$

This gives us:

$$C_{YY}(n, k) = \sum_{i=1}^{n} \sum_{j=1}^{k} \text{Cov}(X_i, X_j)$$

Given the independence of the distributions, we see that only diagonal terms remain, which means that the only non-zero covariances occur when:

$$i = j \to \text{Cov}(X_i, X_j) = p(1 - p)$$

Therefore, we rewrite the above to get:

$$C_{YY}(n, k) = \sum_{i=1}^{\min(n,k)} p(1 - p)$$

As such, we finally get:

$$\boxed{C_{YY}(n, k) = \min(n, k)p(1 - p)}$$

8. (a) Given that X and Y are wide sense stationary processes, we may write:

$$V(t) = 2X(t) + Y(t) \to E[V] = 2E[X] + E[Y]$$
$$\mathcal{R}_V(t, \tau) = E[\bar{V}(t)V(t + \tau)]$$

We expand this to get:

$$\mathcal{R}_V(t, \tau) = E[(2X(t) + Y(t))(2X(t + \tau) + Y(t + \tau))]$$
$$\mathcal{R}_V(t, \tau) = E[(2X(t)2X(t + \tau) + Y(t)Y(t + \tau)) + 2X(t)Y(t + \tau) + 2Y(t)X(t + \tau)]$$

And thus we conclude:

$$\boxed{\mathcal{R}_V(t,\tau) \neq \mathcal{R}_X(t,\tau) + \mathcal{R}_Y(t,\tau)}$$

Therefore, it is <u>not wide sense stationary</u>

(b) Similar to the above, we write:

$$E[W] = E[XY]$$

Since Y and X are independent, we get:

$$E[W] = E[X]E[Y] = \mu_X \mu_Y$$

We then write:

$$\mathcal{R}_W(t,\tau) = E[\bar{W}(t)W(t+\tau)]$$

We expand:

$$\mathcal{R}_W(t,\tau) = E[X(t)Y(t)X(t+\tau)Y(t+\tau)]$$

Once again, because X and Y are independent, we get:

$$\boxed{\mathcal{R}_W(t,\tau) = E[X(t)X(t+\tau)][EY(t)Y(t+\tau)] = \mathcal{R}_X(t,\tau)\mathcal{R}_Y(t,\tau)}$$

And, therefore, W <u>is independent and wide sense stationary</u>

9. We begin by writing the autocorrelation as:

$$\mathcal{R}_{WW}(t,\tau) = E[W(t)W(t+\tau)]$$

We expand to get:

$$\mathcal{R}_{WW}(t,\tau) = E[(X\cos\big(10^8\pi t\big)+Y\sin\big(10^8\pi t\big))(X\cos\big(10^8\pi(t+\tau)\big)+Y\sin\big(10^8\pi(t+\tau)\big))]$$

Since X and Y are uncorrelated, we know that any expectation value involving both will cancel, so we simplify to:

$$\mathcal{R}_{WW}(t,\tau) = E[X^2\cos\big(10^8\pi t\big)\cos\big(10^8\pi(t+\tau)\big) + Y^2\sin\big(10^8\pi t\big)\sin\big(10^8\pi(t+\tau)\big)]$$

We use the identity that $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ to get:

$$\mathcal{R}_{WW}(t,\tau) = E[X^2\cos\big(10^8\pi\tau\big)\cos\big(10^8\pi t\big) + Y^2\cos\big(10^8\pi\tau\big)\sin\big(10^8\pi t\big)]$$

Since X and Y both have mean 0 and variance σ^2, we find that, as long as t and $t+\tau$ are within the range of integration:

$$\boxed{\mathcal{R}_{WW}(t,\tau) = \sigma^2 \cos\big(10^8\pi(t-\tau)\big)}$$

Otherwise, it is zero.

Accordingly, we observe that the process $W(t)$ is wide sense stationary, since the mean is zero and the autocorrelation only depends on the time difference, $t-\tau$

We can find the autocovariance as:

$$C_{WW}(t,\tau) = R_{WW}(t,\tau) - E[W(t)]E[W(t+\tau)]$$

Since we know the terms that go to zero, we find:

$$C_{WW}(t,\tau) = R_{WW}(t,\tau)$$
$$\boxed{C_{WW}(t,\tau) = \sigma^2 \cos\big(10^8\pi(t-\tau)\big)}$$

We use the autocovariance to find:

$$C_{WW}(0,.001) = \sigma^2 \cos\left(-10^8\pi \cdot 10^{-3}\right)$$
$$C_{WW}(0,.001) = \sigma^2 \cos\left(-10^5\pi\right)$$

This simplifies to:

$$\boxed{C_{WW}(0,.001) = \sigma^2}$$

We can find the mean signal power by taking $\mathcal{R}_{WW}(0)$:

$$E[W^2(t)] = \sigma^2 \cos(0)$$
$$\boxed{E[W^2(t)] = \sigma^2}$$

Since the mean is zero, nothing is subtracted from the power, which gives us a variance of:

$$\boxed{\mathrm{Var}(W(t)) = \sigma^2}$$

10. We are given that the weather on each day is normally distributed with $E[W] = 20[^\circ\mathrm{C}]$ and $\sigma = 5[^\circ\mathrm{C}]$. Let us then express the two day-averaged distribution as:

$$W_n = \frac{2X_n + X_{n-1}}{3} \quad \text{and} \quad W_{n+1} = \frac{2X_{n+1} + X_n}{3}$$

We find the Covariance between daily "steps" to get:

$$\text{Cov}(W_n, W_{n+1}) = \text{Cov}\left(\frac{2X_n + X_{n-1}}{3}, \frac{2X_{n+1} + X_n}{3}\right)$$

We can break this apart to get:

$$\text{Cov}(W_n, W_{n+1}) = \frac{1}{9}\left[2\text{Cov}\left(X_n, X_n\right) + 4\text{Cov}\left(X_n, X_{n+1}\right) + 2\text{Cov}\left(X_{n-1}.X_{n+1}\right) + \text{Cov}(X_{n-1}, X_n)\right]$$

$$\text{Cov}(W_n, W_{n+1}) = \frac{1}{9}\left[2\text{Var}\left(X_n\right)\right]$$

$$\boxed{\text{Cov}(W_n, W_{n+1}) = \frac{50}{9}\left[{}^\circ\text{C}^2\right]}$$

Thus, since the covariance is not zero, the W_n distributions are not independent, and, therefore it is not an i.i.d random sequence.

12. (a) We may begin by constructing our distribution by using the formula for a Poisson distribution for each individual train line:

$$P[n] = \frac{(\lambda T)e^{-\lambda T}}{n!}$$

We combine the individual PMFs to get a general one for train arrivals as:

$$\lambda = \lambda_R + \lambda_G + \lambda_B$$

Entering the given information, we write:

$$\lambda = .1 + .05 + .15$$

$$\lambda = .3[\text{min}^{-1}]$$

Accordingly, we find the quantity of trains in an hour as:

$$\lambda T = 60(.03)$$

$$\lambda T = 18[\text{trains/hour}]$$

Applying this to our Poisson distribution formula, we find:

$$\boxed{P_N[n] = \text{Poisson}(18) = \frac{(18)^n e^{-18}}{n!}}$$

(b) We want to find the conditional PMF, which we can do by constructing a binomial distribution

$$P[R|N = 10] = \binom{10}{R}\left(\frac{.1}{.3}\right)^R\left(1 - \frac{.1}{.3}\right)^{10-R}$$

We simplify to find:

$$\boxed{P[R|N = 10] = \binom{10}{R}\left(\frac{1}{3}\right)^R\left(\frac{2}{3}\right)^{10-R}}$$

As such, we see that this follows a binomial distrbution with $n = 10$ and $p = 1/3$