Lecture 2 — Random Variables

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- A random variable is a function that maps the outcomes of a random experiment into a set of real numbers
- The Probability Mass Function (PMF)
  - The PMF is a probability measure that gives us probabilities of the possible values for a random variable
  - The PMF may be defined as:

$$P_x(x) = \begin{cases} P(X = x), & \text{if } x \in S_x \\ 0, & \text{Otherwise} \end{cases}$$

  - The probability mass function can be obtained using the probabilities of the corresponding sample space outcomes
- The Bernoulli Random Variable
  - We may write this as $X = \text{Bernoulli}(p)$
  - This can be expressed as:

$$P_x(x) = \begin{cases} 1 - p, & x = 0 \\ p, & x = 1 \\ 0, & \text{otherwise} \end{cases}$$

  - Expectation Value: $E[X] = p$
  - $\text{Var}[X] = p(1 - p)$
- The Binomial Random Variable
  - We may write this as $X = \text{Binomial}(n, p)$

- This can be expressed as:

$$P_x(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k}, & 0 \le k \le n \\ 0, & \text{otherwise} \end{cases}$$

- Expectation Value: $E[X] = np$
- $\text{Var}[X] = np(1-p)$

- The Geometric Random Variable
  - We may write this as: $X = \text{Geometric}(p)$
  - This can be expressed as:

$$P_x(x) = \begin{cases} p(1-p)^{x-1}, & x = 1, 2, 3 \cdots \\ 0, & \text{otherwise} \end{cases}$$

  - Expectation Value: $E[X] = \frac{1}{p}$
  - $\text{Var}[X] = \frac{1-p}{p^2}$

- The Poisson Random Variable
  - We may write this as: $X = \text{Poisson}(a)$
  - This can be expressed as:

$$P_K(k) = \begin{cases} \frac{a^k}{k!} e^{-a}, & k = 0, 1, 2, 3 \cdots \\ 0, & \text{otherwise} \end{cases}$$

  - The Poisson random variable can be described using the average arrival rate $a = \lambda T$
  - Expectation Value: $E[X] = \alpha$
  - $\text{Var}[X] = \alpha$

- The Pascal or Negative Binomial Random Variable
  - We may write this as: $X = \text{Pascal}(k, p)$
  - This can be expressed as:

$$P_X(x) = \begin{cases} \binom{x-1}{k-1} p^k (1-p)^{x-k}, & x = k, k+1, \cdots \\ 0, & \text{otherwise} \end{cases}$$

  - Expectation Value: $E[X] = \frac{k}{p}$
  - $\text{Var}[X] = \frac{k(1-p)}{p^2}$

- The Discrete Uniform Random Variable
  - We may write this as: $N = \text{Uniform}(k, l)$
  - This can be expressed as:

$$P_N(n) = \begin{cases} \frac{1}{l-k-1}, & l-k = 2, 3, 4, \cdots \\ 0, & \text{otherwise} \end{cases}$$

  - Expectation Value: $E[X] = \frac{l+k}{2}$
  - $\text{Var}[X] = \frac{l-k}{12}(l-k+2)$
- The Cumulative Distribution Function (CDF)
  - The CDF of a discrete random variable is defined for all values of the random variable as:

$$F_x(x) = P[X \leq x] = \sum_{j:x_j \leq x} P_x(x_j)$$

  - For any value x, $F_x(x)$ is the probability that the observed value of X will be at most x
- Variance

$$\text{Var}[X] = E[(X - \mu_x)^2]$$

  - $\text{Var}[X_1 + X_2 + \cdots + X_n] = \text{Var}[X_1] + \text{Var}[X_2] + \cdots + \text{Var}[X_n]$
- The Exponential Random Variable
  - We may write this as: $X = \text{exponential}(\lambda)$
  - This can be expressed as:

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

  - Expectation value: $\dfrac{1}{\lambda}$
  - $\text{Var}[X] = \dfrac{1}{\lambda^2}$
- The Gaussian or Normal Random Variable
  - $Z = \text{Normal}(\mu, \sigma)$

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

- The Standard Normal CDF

$$\Phi(z) = F_Z(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\, du$$

  - Q is the inverse function, such that $Q(z) = 1 - \Phi(z)$