Final

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On my honor, I pledge to uphold the values of honesty, integrity, and respect that are expected of me as a Northeastern student.

Northeastern University holds all students accountable for the honest completion of examinations, tests, papers, projects, and assignments. students are obligated to approach these tasks with unwavering integrity.

Students should complete this quiz independently, relying solely on their own efforts and understanding. The work submitted will be a true reflection of their individual endeavor. Please answer all components of the questions and provide comprehensive details of your work.

1. (a) First, we may write the autocorrelation function as:

$$R_{XX}[n,k] = E[X_n X_{n+k}]$$

Given their independence, for $k \neq 0$ we may write:

$$R_{XX}[n,k] = E[X_n]E[X_{n+k}]$$

We are given the expectation value of X, which allows use to write:

$$R_{XX}[n,k] = (\mu_x)^2$$
$$R_{XX}[n,k] = (2)^2$$

When we take $k \to 0$, we find that the variance component does not vanish such that:

$$R_{XX}[n,0] = E[X_n^2]$$
$$R_{XX}[n,0] = \mathrm{Var}(X_n) + (E[X_n])^2$$

$$R_{XX}[n,0] = 2^2 + (2)^2$$

$$\boxed{R_{XX}[n,k] = \begin{cases} 4, & k \neq 0 \\ 8, & k = 0 \end{cases}}$$

The autocovariance may be written as:

$$C_{XX}[n,k] = R_{XX}[n,k] - E[X_n]E[X_{n+k}]$$

We may notice that we simply subtract 4 from the autocorrelation to get:

$$\boxed{C_{XX}[n,k] = \begin{cases} 0, & k \neq 0 \\ 4, & k = 0 \end{cases}}$$

(b) We may express the expected power as:

$$E[X_n^2] = \text{Var}(X_n) + (E[X_n])^2$$

We may see that this is equivalent to the $k = 0$ value of the autocorrelation. Thus, we get:

$$\boxed{E[X_n^2] = 8}$$

(c) We can express the expected value of the output sequence as:

$$E[Y_n] = \frac{1}{2}(E[X_n] - E[X_{n-2}])$$

Since each X_n has the same mean, we find:

$$\boxed{E[Y_n] = 0}$$

(d) We may rewrite the expression as:

$$R_{YY}[n,k] = E[Y_n Y_{n+k}]$$

$$R_{YY}[n,k] = \frac{1}{4}E[(X_n - X_{n-2})(X_{n+k} - X_{n+k-2})]$$

We distribute to get:

$$R_{YY}[n,k] = \frac{1}{4}E[X_n X_{n+k} - X_{n-2}X_{n+k} - X_n X_{n+k-2} + X_{n-2}X_{n+k-2}]$$

We first take $k \to 0$ to get:

$$R_{YY}[n,0] = \frac{1}{4}E[X_n^2 - 2X_{n-2}X_n + X_{n-2}^2]$$

This gives us:

$$R_{YY}[n,0] = \frac{1}{4}(8 - 2(2)(2) + 8)$$
$$R_{YY}[n,0] = 2$$

Next, we take $k \to \pm 2$ to get:

$$R_{YY}[n,\pm 2] = \frac{1}{4}E[X_n X_{n\pm 2} - X_{n-2}X_{n\pm 2} - X_n X_{n\pm 2-2} + X_{n-2}X_{n\pm 2-2}]$$
$$R_{YY}[n,\pm] = \frac{1}{4}(4 - 4 - 8 + 4)$$
$$R_{YY}[n,\pm] = -1$$

When $k \neq 0$, we find:

$$R_{YY}[n,k] = \frac{1}{4}(2 - 2 - 2 + 2)$$
$$R_{YY}[n,k] = 0$$

As such, we conclude:

$$\boxed{R_{YY}[n,k] = \begin{cases} 0, & k \neq 0, \pm 2 \\ -1, & k = \pm 2 \\ 2, & k = 0 \end{cases}}$$

The autocovariance may be written as:

$$C_{YY}[n,k] = R_{YY}[n,k] - E[Y_n Y_{n+k}]$$

Since the expectation value of Y is zero, we get:

$$C_{YY}[n,k] = R_{YY}[n,k]$$

$$\boxed{C_{YY}[n,k] = \begin{cases} 0, & k \neq 0, \pm 2 \\ -1, & k = \pm 2 \\ 2, & k = 0 \end{cases}}$$

(e) The sequence Y_n is wide-sense stationary, since its mean is constant for all indices (it is always zero) and the autocorrelation is dependent solely on the index shift, k ($R_{YY}[n,k] = R_{YY}[n-k]$)

(f) The components of Y_n are not uncorrelated. We may conclude this since, as shown in part (d), the autocorrelation is nonzero for at least one value of $k \neq 0$ (when $k = \pm 2$ to be precise)

(g) i. We begin by finding the mean. We do this by writing:

$$E[W_n] = E[B_n X_n]$$

Since B_n and X_n are independent, we get:

$$E[W_n] = E[B_n]E[X_n]$$

The mean for X_n is given and, by the properties of Bernoulli random variables, we know $E[B_n] = p$. Thus, we get:

$$\mu_W = E[W_n] = (p)(2)$$

$$\boxed{\mu_W = (p)(2)}$$

We can find the autocorrelation by writing:

$$R_{WW}[n,k] = E[W_n W_{n+k}]$$

This gives us:

$$R_{WW}[n,k] = E[B_n X_n B_{n+k} X_{n+k}]$$

Taking $k \to 0$ gives us:

$$R_{WW}[n,k] = E[B_n^2 X_n^2]$$

Given their independence, we get:

$$R_{WW}[n,0] = E[B_n^2]E[X_n^2]$$

$$R_{WW}[n,0] = (p)(8)$$

$$R_{WW}[n,0] = 8p$$

We then find, that for $k \neq 0$, we have:

$$R_{WW}[n,k] = E[B_n]E[X_n]E[X_{n+k}]E[B_{n+k}]$$

$$R_{WW}[n,k] = (p)(2)(2)(p)$$

$$R_{WW}[n,k] = 4p^2$$

Thus, we write:

$$\boxed{R_{WW}[n,k] = \begin{cases} 4p^2, & k \neq 0 \\ 8p, & k = 0 \end{cases}}$$

We may write the autocovariance as:

$$C_{WW}[k] = R_{WW}[k] - (E[W_k])^2$$

If we take $k \to 0$, we get:

$$C_{WW}[0] = R_{WW}[0] - (E[W_0])^2$$
$$C_{WW}[0] = 8p - (2p)^2$$

For $k \neq 0$, we find:

$$C_{WW}[k] = R_{WW}[k] - 4p^2$$
$$C_{WW}[k] = 4p^2 - 4p^2$$
$$C_{WW}[k] = 0$$

Thus, we write:

$$\boxed{C_{WW}[k] = \begin{cases} 8p - 4p^2, & k = 0 \\ 0, & k \neq 0 \end{cases}}$$

Since the autocorrelation is zero for all values when $k \neq 0$, we may conclude that the components of W_n are uncorrelated

ii. Similar to how we did it previously, we get:

$$R_{WX}[n,k] = E[W_n X_{n+k}]$$

We expand this to get:

$$R_{WX}[n,k] = E[B_n X_n X_{n+k}]$$

Taking $k \to 0$, we get:

$$R_{WX}[n,0] = E[B_n X_n^2]$$
$$R_{WX}[n,0] = 8p$$

And then for $k \neq 0$, we get:

$$R_{WX}[n,k] = E[B_n X_n X_{n+k}]$$
$$R_{WX}[n,k] = 4p$$

Thus, we write:

$$\boxed{R_{WX}[n,k] = \begin{cases} 4p, & k \neq 0 \\ 8p, & k = 0 \end{cases}}$$

We continue to get:

$$C_{WX}[k] = R_{WX}[k] - E[W_k]E[X_{n+k}]$$

This gives us (for $k \to 0$):

$$C_{WX}[0] = R_{WX}[0] - (2p)(2)$$

$$C_{WX}[0] = 8p - (4p)$$
$$C_{WX}[0] = 4p$$

And for $k \neq 0$:

$$C_{WX}[k] = R_{WX}[k] - E[W_k]E[X_k]$$
$$C_{WX}[k] = 4p - 4p$$
$$C_{WX}[k] = 0$$

Thus, we conclude:

$$\boxed{C_{WX}[k] = \begin{cases} 4p, & k = 0 \\ 0, & k \neq 0 \end{cases}}$$

iii. Finally, we compare Y and W by writing:

$$R_{YW}[n,k] = E[Y_n W_{n+k}]$$

We expand to get:

$$R_{YW}[n,k] = \frac{1}{2}E[(X_n - X_{n-2})B_{n+k}X_{n+k}]$$
$$R_{YW}[n,k] = \frac{1}{2}E[X_n B_{n+k}X_{n+k} - B_{n+k}X_{n+k}X_{n-2}]$$

Taking $k \to 0$, we see:

$$R_{YW}[n,0] = \frac{1}{2}E[B_n X_n^2 - B_n X_n X_{n-2}]$$
$$R_{YW}[n,0] = \frac{1}{2}[(8)(p) - (p)(2)(2)]$$
$$R_{YW}[n,0] = 2p$$

For $k = 2$, we see:

$$R_{YW}[n,2] = \frac{1}{2}E[X_n B_{n+2}X_{n+2} - B_{n+2}X_{n+2}X_{n-2}]$$
$$R_{YW}[n,2] = \frac{1}{2}[(2)(p)(2) - (p)(2)(2)]$$
$$R_{YW}[n,2] = 0$$

Furthermore, for $k = -2$, we find:

$$R_{YW}[n,-2] = \frac{1}{2}E[X_n B_{n-2}X_{n-2} - B_{n-2}X_{n-2}^2]$$
$$R_{YW}[n,-2] = \frac{1}{2}[(2)(p)(2) - (p)(8)]$$
$$R_{YW}[n,-2] = -2p$$

For $k \neq 0, -2, 2$, this is:

$$R_{YW}[n,k] = \frac{1}{2}E[X_n B_{n+k} X_{n+k} - B_{n+k} X_{n+k} X_{n-2}]$$
$$R_{YW}[n,k] = \frac{1}{2}[(2)(2)(p) - (p)(2)(2)]$$
$$R_{YW}[n,k] = 0$$

As such, we write:

$$\boxed{R_{YW}[n,k] = \begin{cases} 0, & k \neq 0, -2 \\ 2p, & k = 0 \\ -2p, & k = -2 \end{cases}}$$

Given that the expectation value of Y is zero, we get:

$$C_{YW}[k] = R_{YW}[k] - E[Y_n]E[W_{n+k}]$$
$$C_{YW}[k] = R_{YW}[k]$$
$$\boxed{C_{YW}[k] = \begin{cases} 0, & k \neq 0, -2 \\ 2p, & k = 0 \\ -2p, & k = -2 \end{cases}}$$

2. (a) First and foremost, we know that, for a Gaussian distribution, we have:

$$\boxed{E[X(t)] = 0}$$

Now, we may find the power by using the autocorrelation and taking $\tau \to 0$:

$$E[X^2(t)] = 25e^{-20000\tau^2}$$
$$E[X^2(t)] = 25e^{-20000(0)^2}$$
$$\boxed{E[X^2(t)] = 25}$$

(b) We can find this probability by converting to a Z-score:

$$Z = \frac{X(1[\text{ms}])}{\sigma}$$

This gives us:

$$P\left[Z < \frac{2}{\sqrt{25}}\right] = P\left[\frac{X(1[\text{ms}])}{5} < .4\right]$$
$$\boxed{P\left[Z < .4\right] = .6554}$$

(c) We may begin by expanding

(d)

(e)

(f)

3. (a) We may begin by writing:

$$E[X(t)] = AE[\cos(\omega t + \theta)]$$

We use the cosine identity:

$$\cos(\omega t + \theta) = \cos(\omega t)\cos(\theta) - \sin(\omega t)\sin(\theta)$$

To get:

$$E[X(t)] = A\left(E[\cos(\omega t)]E[\cos(\theta)] - E[\sin(\omega t)]E[\sin(\theta)]\right)$$

Since θ is a uniform distribution from $-\pi$ to π, the θ-dependent terms become zero, which gives us:

$$E[X(t)] = A(0 - 0)$$

$$\boxed{E[X(t)] = 0}$$

(b) We can write the autocorrelation function as:

$$R_{XX}(t, \tau) = E[X(t)X(t + \tau)]$$

We substitute our values to get:

$$R_{XX}(t, \tau) = A^2 E[\cos(\omega t + \theta)\cos(\omega(t + \tau) + \theta)]$$

Using the identity:

$$\cos(a)\cos(b) = \frac{1}{2}\left[\cos(a - b) + \cos(a + b)\right]$$

This gives us:

$$E[X(t)X(t + \tau)] = \frac{A^2}{2} E\left[\cos(\omega \tau) + \cos(2\omega t + \omega \tau + 2\theta)\right]$$

We may observe that, since the second term is dependent on θ, it cancels out, which leaves us with:

$$E[X(t)X(t + \tau)] = \frac{A^2}{2} E\left[\cos(\omega \tau)\right]$$

We proceed to calculate the expectation value as:

$$E[X(t)X(t+\tau)] = \frac{A^2}{2}\int_{-\omega_o}^{\omega_o} \cos(\omega\tau)\,d\omega$$

We see that this results in the sinc function, which lets us write:

$$\boxed{R_{XX}(t,\tau) = \frac{A^2}{2}\text{sinc}\left(\frac{\omega_o\tau}{\pi}\right)}$$

(c) We may conclude that $X(t)$ is wide-sense stationary since the mean is constant (0, per part a), and the autocorrelation depends only on the time difference, τ (as per part b)

(d) We find the variance as:

$$\text{Var}(X(t)) = R_{XX}(X(0))$$

$$\text{Var}(X(t)) = \frac{A^2}{2}\text{sinc}(0)$$

$$\boxed{\text{Var}(X(t)) = \frac{A^2}{2}}$$