Homework 8

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1. We may write the correlation coefficient as:

$$\rho_{XY} = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

Given that $Y = X + 2Z$, we obtain:

$$\text{Cov}(X,Y) = \text{Cov}(X, X + 2Z)$$

We break this apart to get:

$$\text{Cov}(X, X + 2Z) = \text{Cov}(X, X) + 2\text{Cov}(X, Z)$$
$$\text{Cov}(X, X + 2Z) = \text{Var}(X) + 2\text{Cov}(X, Z)$$

Additionally, we get:

$$\text{Var}(Y) = \text{Var}(X + 2Z)$$
$$\text{Var}(X + 2Z) = 3\text{Var}(X) + 4\text{Var}(Z) + 8\text{Cov}(X, Z)$$

Thus, the expression becomes:

$$\rho_{XY} = \frac{\text{Var}(X) + 2\text{Cov}(X, Z)}{\sqrt{\text{Var}(X)[3\text{Var}(X) + 4\text{Var}(Z) + 8\text{Cov}(X, Z)]}}$$

The final step is to calculate the covariance between X and Z. Since it is stated that the two are independent, we arrive at:

$$\text{Cov}(X, Z) = 0$$

And thus:

$$\rho_{XY} = \frac{\text{Var}(X)}{\sqrt{\text{Var}(X)[3\text{Var}(X) + 4\text{Var}(Z)]}}$$

We plug in our known values to get:

$$\rho_{XY} = \frac{16}{\sqrt{16[3(16) + 4(4)]}}$$

$$\boxed{\rho_{XY} = \frac{1}{2}}$$

Since the correlation coefficient is not zero, $\underline{X \text{ and } Y \text{ are not independent}}$.

Now, given $W = 2X - Z$, we want to find:

$$E[W] = E[2X - Z]$$

$$\text{Var}(W) = \text{Var}(2X - Z)$$

$$\text{Cov}(W, Y) = \text{Cov}(2X - Z, X + 2Z)$$

For the expectation value, we simply decompose to get:

$$E[W] = 2E[X] - E[Z]$$

$$E[W] = 2(2) - 1$$

$$\boxed{E[W] = 3}$$

The variance can be expanded to get:

$$\text{Var}(W) = 4\text{Var}(X) + \text{Var}(Z) - \cancel{4\text{Cov}(X, Z)}$$

$$\text{Var}(W) = 4(16) + 4$$

$$\boxed{\text{Var}(W) = 68}$$

Finally, we find the covariance as:

$$\text{Cov}(W, Y) = \text{Cov}(2X, X + 2Z) - \text{Cov}(Z, X + 2Z)$$

$$\text{Cov}(W, Y) = 2\text{Var}(X) + 3\text{Cov}(X, Z) - 2\text{Var}(Z)$$

Plugging in values, we get:

$$\text{Cov}(W, Y) = 2(16) + \cancel{3\text{Cov}(X, Z)} - 2(4)$$

$$\boxed{\text{Cov}(W, Y) = 24}$$

2. (a)
   (b)
   (c)
3. (a)
   (b)
   (c)
4.
5. (a)
   (b)
   (c)
   (d)
   (e)
   (f)
   (g)

5. **Extra Credit**

   (a)
   (b)
   (c)
   (d)
   (e)
   (f)
   (g)

6. (a)
   (b)
   (c)
   (d)
   (e)
   (f)
7. (a)
   (b)
8.
9.