

Homework 10

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- 1.
2. (a) Taking $t \rightarrow 100/3[\mu\text{s}]$, we obtain:

$$s_1 = 5 \cos(10^4 \pi (100/3) \cdot 10^{-6})$$

$$s_1 = 5 \cos(\pi/3)$$

$$s_1 = 2.5$$

We can then find $Y_1(t)$ by tacking on the noise term to get:

$$\boxed{Y_1(t) \rightarrow N(2.5, 1)}$$

Which means that this gives a normal distribution with a mean of 2.5 and standard deviation of 1.

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- (b) Similarly, we take $t \rightarrow 100[\mu\text{s}]$ to get:

$$s_2 = 5 \cos(10^4 \pi 200 \cdot 10^{-6})$$

$$s_2 = 5 \cos(2\pi)$$

$$s_2 = 5$$

This gives us:

$$\boxed{Y_2(t) \rightarrow N(5, 1)}$$

Or a normal distribution with mean 5 and standard deviation 1.

- (c) Given that $s(t)$ is static, we know that the covariance depends solely on the noise. This gives us:

$$\text{Cov}(Y_1, Y_2) = \text{Cov}(N_1, N_2)$$

Since N_1 and N_2 are normal functions independent of each other, we find:

$$\text{Cov}(Y_1, Y_2) = \text{Cov}(N_1, N_2) = 0$$

Which means that Y_1 and Y_2 are independent

- (d) Summing the two distributions
4. (a)
- (b)
5. (a) From the fact that $C(\tau) \neq C(-\tau)$, we may observe that this is not a valid autocovariance function for a WSS random process
- (b) Although the given function is even, it does not satisfy the positive-definite requirement. That is, it does not follow:

$$\sum_{i=1}^n \sum_{j=1}^n c_i c_j \mathcal{R}_x(\tau_i, \tau_j) \geq 0$$

If we take $\tau_1 = -1$ and $\tau_2 = -2$, then we get:

$$-c_1^2 - 2c_2^2$$

We observe that this is less than zero, and, therefore, this is not a valid autocovariance function for a WSS random process

- (c) The function is even and non-negative for all of τ , and it is positive definite. Therefore, we conclude that this is a valid autocovariance function for a WSS random process. Note: we may verify that it is positive definite by showing that the sum would yield:

$$c_i c_j \rightarrow \frac{1}{|\tau_1|} \frac{1}{|\tau_2|}$$

Which must be positive.

- (d) We may observe that the function is even, non-negative for all of τ , and is positive definite. Therefore, we conclude that this is a valid autocovariance function for a WSS random process
6. (a) We may observe that $X(t)$ is not periodic, since the autocorrelation function does not contain a periodic term

- (b) The function is not periodic, so we use the autocorrelation. We want to take $\tau \rightarrow \infty$; however, since τ is bounded, in this case, we take $\tau \rightarrow 6$:

$$E[X] = 3 - \frac{1}{2}(6)$$

$$\boxed{E[X] = 0}$$

- (c) We may write the expected power as:

$$E[X^2(t)] = \mathcal{R}_{XX}(0)$$

This gives us:

$$E[X^2(t)] = 3 - \frac{1}{2}(0)$$

$$\boxed{E[X^2(t)] = 3}$$

- (d) We can express this using the value of the uniform distribution to get:

$$P[X(t=1) > 1] = \int_1^3 \frac{1}{3 - (-3)} dx$$

$$P[X(t=1) > 1] = \int_1^3 \frac{1}{6} dx$$

$$P[X(t=1) > 1] = \left. \frac{x}{6} \right|_1^3$$

$$P[X(t=1) > 1] = \frac{3-1}{6}$$

$$\boxed{P[X(t=1) > 1] = \frac{1}{3}}$$

- (e) We can break this up to write:

$$\begin{aligned} E[(X(1) + X(2) + X(3))^2] &= E[X^2(1)] + E[X^2(2)] + E[X^2(3)] + \\ &2E[X(1)X(2)] + 2E[X(2)X(3)] + 2E[X(1)X(3)] \end{aligned}$$

This is equivalent to:

$$E[(X(1) + X(2) + X(3))^2] = 3\mathcal{R}_{XX}(0) + 4\mathcal{R}_{XX}(1) + 2\mathcal{R}_{XX}(2)$$

We evaluate to get:

$$E[(X(1) + X(2) + X(3))^2] = 3[3] + 4\left[\frac{5}{2}\right] + 2[2]$$

$$\boxed{E[(X(1) + X(2) + X(3))^2] = 23}$$

(f) Expanding, we get:

$$E[Y] = 2E[X] + 3$$

$$\boxed{E[Y] = 3}$$

We can then find:

$$\mathcal{R}_{YY}(t) = E[(2X(t) + 3)(2X(t + \tau) + 3)]$$

We expand this to get:

$$\mathcal{R}_{YY}(t) = 4E[X(t)2X(t + \tau)] + 6E[X(t + \tau)] + 6E[X(t)] + 9$$

$$\boxed{\mathcal{R}_{YY}(t) = 4\mathcal{R}_{XX}(t) + 9}$$

7. (a)

(b)

(c)

(d)

(e)

8. (a) Given that X and Y are wide sense stationary processes, we may write:

$$V(t) = 2X(t) + Y(t) \rightarrow E[V] = 2E[X] + E[Y]$$

$$\mathcal{R}_V(t, \tau) = E[\bar{V}(t)V(t + \tau)]$$

We expand this to get:

$$\mathcal{R}_V(t, \tau) = E[(2X(t) + Y(t))(2X(t + \tau) + Y(t + \tau))]$$

$$\mathcal{R}_V(t, \tau) = E[(2X(t)2X(t + \tau) + Y(t)Y(t + \tau)) + 2X(t)Y(t + \tau) + 2Y(t)X(t + \tau)]$$

And thus we conclude:

$$\boxed{\mathcal{R}_V(t, \tau) \neq \mathcal{R}_X(t, \tau) + \mathcal{R}_Y(t, \tau)}$$

Therefore, it is not wide sense stationary

(b) Similar to the above, we write:

$$E[W] = E[XY]$$

Since Y and X are independent, we get:

$$E[W] = E[X]E[Y] = \mu_X\mu_Y$$

We then write:

$$\mathcal{R}_W(t, \tau) = E[\bar{W}(t)W(t + \tau)]$$

We expand:

$$\mathcal{R}_W(t, \tau) = E[X(t)Y(t)X(t + \tau)Y(t + \tau)]$$

Once again, because X and Y are independent, we get:

$$\boxed{\mathcal{R}_W(t, \tau) = E[X(t)X(t + \tau)][EY(t)Y(t + \tau)] = \mathcal{R}_X(t, \tau)\mathcal{R}_Y(t, \tau)}$$

And, therefore, W is independent and wide sense stationary

9. We begin by writing the autocorrelation as:

$$\mathcal{R}_{WW}(t, \tau) = E[W(t)W(t + \tau)]$$

We expand to get:

$$\mathcal{R}_{WW}(t, \tau) = E[(X \cos(10^8 \pi t) + Y \sin(10^8 \pi t))(X \cos(10^8 \pi(t + \tau)) + Y \sin(10^8 \pi(t + \tau)))]$$

Since X and Y are uncorrelated, we know that any expectation value involving both will cancel, so we simplify to:

$$\mathcal{R}_{WW}(t, \tau) = E[X^2 \cos(10^8 \pi t) \cos(10^8 \pi(t + \tau)) + Y^2 \sin(10^8 \pi t) \sin(10^8 \pi(t + \tau))]$$

We use the identity that $\cos(x + y) = \cos(x) \cos(y) - \sin(x) \sin(y)$ to get:

$$\mathcal{R}_{WW}(t, \tau) = E[X^2 \cos(10^8 \pi \tau) \cos(10^8 \pi t) + Y^2 \cos(10^8 \pi \tau) \sin(10^8 \pi t)]$$

Since X and Y both have mean 0 and variance σ^2 , we find that, as long as t and $t + \tau$ are within the range of integration:

$$\boxed{\mathcal{R}_{WW}(t, \tau) = \sigma^2 \cos(10^8 \pi(t - \tau))}$$

Otherwise, it is zero.

Accordingly, we observe that the process $W(t)$ is wide sense stationary, since the mean is zero and the autocorrelation only depends on the time difference, $t - \tau$

We can find the autocovariance as:

$$C_{WW}(t, \tau) = R_{WW}(t, \tau) - E[W(t)]E[W(t + \tau)]$$

Since we know the terms that go to zero, we find:

$$C_{WW}(t, \tau) = R_{WW}(t, \tau)$$

$$\boxed{C_{WW}(t, \tau) = \sigma^2 \cos(10^8 \pi(t - \tau))}$$

We use the autocovariance to find:

$$C_{WW}(0, .001) = \sigma^2 \cos(-10^8 \pi \cdot 10^{-3})$$

$$C_{WW}(0, .001) = \sigma^2 \cos(-10^5 \pi)$$

This simplifies to:

$$\boxed{C_{WW}(0, .001) = \sigma^2}$$

We can find the mean signal power by taking $\mathcal{R}_{WW}(0)$:

$$E[W^2(t)] = \sigma^2 \cos(0)$$

$$\boxed{E[W^2(t)] = \sigma^2}$$

Since the mean is zero, nothing is subtracted from the power, which gives us a variance of:

$$\boxed{\text{Var}(W(t)) = \sigma^2}$$

10.

12. (a)

(b)