Homework 6

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1. We begin by sketching the CDF:

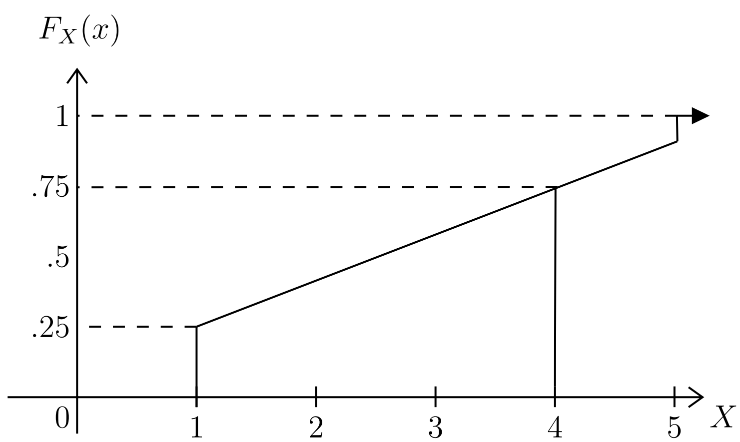

Figure 1: CDF of Given Function

(a) From the CDF, we may conclude:

$$P[X < 1] = F_X(X^- = 1) - F_X(X = 0)$$
$$\boxed{P[X < 1] = 0}$$

And then:

$$P[X \leq 1] = F_X[X = 1] - F_X[X = 0]$$
$$P[X \leq 1] = .25 - 0$$
$$\boxed{P[X \leq 1] = .25}$$

(b) Since the given probability occurs in the continuous portion, we can find that:

$$P[X < 2] = P[X \leq 2]$$

Furthermore, we can write:

$$P[X \leq 2] = F_X[X = 2] - F_X[X = 0]$$
$$P[X \leq 2] = \frac{25}{60} - 0$$
$$\boxed{P[X \leq 2] = P[X < 2] = .416\bar{6}}$$

(c) We continue to find:

$$P[X > 5] = 1 - F_X(X^- = 5)$$
$$P[X > 5] = 1 - .916\bar{6}$$
$$\boxed{P[X > 5] = .083\bar{3}}$$

Since the right side is not in the continuous portion, we get:

$$P[X \geq 5] = 1 - F_X(X^+ = 5)$$
$$P[X \geq 5] = 1 - 1$$
$$\boxed{P[X \geq 5] = 0}$$

(d) With the given range, we may find:

$$P[1 < X < 2] = F_X(X^- = 2) - F_X(X^+ = 1)$$
$$P[1 < X < 2] = .416\bar{6} - .25$$
$$\boxed{P[1 < X < 2] = .16\bar{6}}$$

2. (a) We may begin by finding a PDF for the continuous region:

$$f_{W,2}(w) = (1 - f_{W,1})\lambda e^{-\lambda w}$$

Furthermore, we know that:

$$f_{W,1}(w) = .3$$

Given the expectation value given successful communications, we may know:

$$E[W] = \frac{1}{\lambda}$$

This gives us:

$$\lambda = \frac{1}{3}[\text{min}]$$

Thus, the pdf becomes:

$$\boxed{f_W(w) = \begin{cases} \dfrac{3}{10}, & w = 0 \\ \\ \dfrac{7}{30}e^{-\frac{1}{3}w} & w > 0 \end{cases}}$$

(b) We may find the CDF by integrating to get:

$$F_W(w) = \begin{cases} \dfrac{3}{10}, & w = 0 \\ \\ 1 - .7e^{-\frac{1}{3}w} & w > 0 \end{cases}$$

We may note that this is continuous, so we may simply write:

$$\boxed{F_W(w) = 1 - .7e^{-\frac{1}{3}w} \quad w \geq 0}$$

(c) We can write the expectation value as:

$$E[W] = \int_0^\infty w f_W(w)\, dw$$

This gives us:

$$E[W] = .3(w \to 0) + \int_{0^+}^\infty \frac{7w}{30}e^{-\frac{1}{3}w}\, dw$$

We evaluate to get:

$$E[W] = .3(0) + 2.1$$

$$\boxed{E[W] = 2.1[\text{min}]}$$

We can then find the variance using:

$$\text{Var}[W] = E[W^2] - E[W]^2$$

From here, we can get:

$$\text{Var}[W] = \int_0^\infty \frac{7w^2}{30}e^{-\frac{1}{3}w}\, dw - 2.1^2$$

This gives us:

$$\text{Var}[W] = 12.6 - 4.41$$

$$\boxed{\text{Var}[W] = 8.19[\text{min}^2]}$$

(d) To define this probability, we simply use the CDF to write:

$$P[W < 1] = F_W(w = 1^-)$$

We can find this using:

$$P[W < 1] = 1 - .7e^{-1/3}$$

$$\boxed{P[W < 1] = .4984}$$

3. (a) We know that the expression for the joint PMF relies on X, which can be 0 to 3 unsuccessful attempts, and Y, which can be 0 to 2 attempts to success after the second attempt. Thus, we may express the joint PMF as a matrix:

$$P_{XY}(x, y) = \begin{Bmatrix} .512 & .384 & .096 & .008 \\ 0 & .096 & .096 & .024 \\ 0 & 0 & 0 & .008 \end{Bmatrix}$$

(b) We proceed to find the marginal PMFs as:

$$P_X(x) = \sum_y P_{XY}(x, y)$$

This gives us:

$$P_X(x) = \begin{Bmatrix} .512 + 0 + 0 + 0 \\ .384 + .096 \\ .096 \cdot 2 \\ .024 + 2 \cdot .008 \end{Bmatrix}$$

$$\boxed{P_X(x) = \begin{Bmatrix} .512 \\ .48 \\ .192 \\ .04 \end{Bmatrix}}$$

Similarly, we may find the marginal PMF of Y as:

$$P_Y(y) = \sum_x P_{XY}(x, y)$$

This gives us:

$$P_Y(y) = \begin{Bmatrix} .512 + .384 + .096 + .008 \\ .024 + 2 \cdot .096 \\ .008 \end{Bmatrix}$$

$$\boxed{P_Y(y) = \begin{Bmatrix} 1 \\ .216 \\ .008 \end{Bmatrix}}$$

(c) We can find this using the formula:

$$P_{X|Y}(x|y = 1) = \frac{P_{XY}(x, 1)}{P_Y(1)}$$

Using the results from (b), we obtain:

$$P_{X|Y}(x|y = 1) = \frac{\begin{Bmatrix} 0 \\ .096 \\ .096 \\ .024 \end{Bmatrix}}{.216}$$

We then get:

$$\boxed{P_{X|Y}(x|y = 1) = \begin{Bmatrix} 0 \\ .444\bar{4} \\ .444\bar{4} \\ .111\bar{1} \end{Bmatrix}}$$

(d) Using the result from (c), we get:

$$E[X|Y = 1] = 0 \cdot 0 + 1 \cdot .44\bar{4} + 2 \cdot .44\bar{4} + 3 \cdot .11\bar{1}$$

This becomes:

$$\boxed{E[X|Y = 1] = 1.66\bar{6}}$$

4. (a) Per the given joint PMF, we may find:

$$\boxed{P_{X_1X_2}(x_1 = 2, x_2 = 2) = .1}$$

(b) We may sum along the diagonal to get:

$$P(A \Rightarrow X_1 = X_2) = .07 + .15 + .1 + .07$$

$$\boxed{P(A \Rightarrow X_1 = X_2) = .39}$$

(c) First, we may express this as:

$$P[B] = P(|X_1 - X_2| \geq 3)$$

This gets us the combinations of:

$$(X_1, X_2) \Rightarrow (0,3),(0,4),(1,4),(3,0)$$

We then sum each probability to get:

$$P(|X_1 - X_2| \geq 3) = .01 + .02 + .01 + 0$$
$$\boxed{P(|X_1 - X_2| \geq 3) = .04}$$

(d) First, we express this as:

$$P[\text{tasks} = 4] = P(X_1 + X_2 = 4)$$

Which gives us:

$$P(X_1 + X_2 = 4) = .1 + .03 + .04 + .01$$
$$\boxed{P(X_1 + X_2 = 4) = .18}$$

We can then find:

$$P[\text{tasks} \geq 4] = P(X_1 + X_2 = 4) + P(X_1 + X_2 > 4)$$

As such, we get:

$$P(X_1 + X_2 \geq 4) = .18 + .06 + .02 + .04 + .04 + .07 + .05$$
$$\boxed{P(X_1 + X_2 \geq 4) = .46}$$

(e) We can find the marginal PMF of X_1 by summing across the rows of X_2 to get:

$$P_{X_1}(x_1) = \begin{Bmatrix} .07 + .06 + .04 + 0 + .1 \\ .08 + .15 + .05 + .03 + .02 \\ .03 + .05 + .1 + .04 + .04 \\ .01 + .04 + .06 + .07 + .05 \end{Bmatrix}$$

We evaluate to find:

$$\boxed{P_{X_1}(x_1) = \begin{Bmatrix} .27 \\ .33 \\ .26 \\ .23 \end{Bmatrix}}$$

(f) Using the results from part (e), we may write the conditional marginal PMF as:

$$P_{X_2|X_1=2}(x_2) = \frac{P_{X_2}(X_1 = 2, x_2)}{P_{X_1}(2)}$$

This gives us:

$$P_{X_2|X_1=2}(x_2) = \frac{\begin{Bmatrix} .03 \\ .05 \\ .1 \\ .04 \\ .04 \end{Bmatrix}}{.26}$$

$$\boxed{P_{X_2|X_1=2}(x_2) = \begin{Bmatrix} .1154 \\ .1923 \\ .3846 \\ .1538 \\ .1538 \end{Bmatrix}}$$

(g) First, we find each value:

$$P(X_1 = 0) = .18$$
$$P(X_2 = 3) = .22$$
$$P(X_1 = 0, X_2 = 3) = 0$$

Since $\boxed{P(X_1 = 0, X_2 = 3) \neq P(X_1 = 0)P(X_2 = 3)}$, the two are dependent.

5. (a) Since k ranges from 0 to n, we simply drop the k-dependent terms to get:

$$\boxed{P_N(n) =} \frac{5^n}{n!} e^{-5}$$

We may observe that the expected rate is $\boxed{\lambda = 5}$ packets per minute.

(b) We may write the conditional PMF as:

$$P_{K|N}(k|n) = \frac{P_{NK}(n, k)}{P_N(n)}$$

We plug both functions in to get:

$$\boxed{P_{K|N}(k|n) = \binom{n}{k} p^k (1 - p)^{n-k}}$$

(c) We know that an event is independent if:

$$P_{NK}(n,k) = P_N(n)P_K(k)$$

Since the n choose k function can not be separated into a function of n and a function of k, we see that the events are dependent

(d) We can calculate this as:

$$P[N < 3, K = 1] = P_{NK}(0,1) + P_{NK}(1,1) + P_{NK}(2,1)$$

We take $p = .8$ to write:

$$P[N < 3, K = 1] = \sum_{n=0}^{2} \frac{5^n}{n!} e^{-5} \binom{n}{1} (.8)(.2)^{n-1}$$

This gives us:

$$\boxed{P[N < 3, K = 1] = .053904}$$

(e) Using the conditional PMF obtained in (b), we get:

$$E[K|N = 10] = \sum_{k=0}^{10} \binom{10}{k} (.8)^k (.2)^{10-k}$$

$$\boxed{E[K|N = 10] = 8[\text{packets}]}$$

6. (a) We can find the range by finding the minimum and maximum amount of files to be uploaded. First, we know that it is possible that there are no users waiting to upload a file. In such a case, we have $Y = 0$. On the other hand, it is possible that there are 3 users waiting to upload 3 files each, which gives us: $Y = 3^2 = 9$. Thus, we find the range to be:

$$\boxed{Y = [0, 9]}$$

(b) To find this, we may write:

$$P(X = 3, Y = 3) = P(X = 3) \cdot P(Y = 3|X = 3)$$

The following is given:

$$P(X = 3) = .3$$

We then find:

$$P(Y = 3|X = 3) = (.5)^3$$

$$P(Y=3|X=3) = .125$$

This gives us:

$$P(X=3, Y=3) = (.3)(.125)$$
$$\boxed{P(X=3, Y=3) = .0375}$$

(c) Similarly, we find:

$$P(Y=7|X=3) = 3[(.1)^2(.5) + (.1)(.4)^2]$$
$$P(Y=7|X=3) = .063$$

From here, we get:

$$P(X=3, Y=7) = (.3)(.063)$$
$$\boxed{P(X=3, Y=7) = .0189}$$

(d) Since there are now exactly 2 users, this means there are, at most, 3 files each in queue, and at least 1, which gives us:

$$\boxed{Y = [2, 6]}$$

Given the two users, we can iterate through to calculate the PMF as:

$$P_Y(y) = \begin{Bmatrix} (.5)^2 \\ (2)(.5)(.4) \\ (.4)^2 + 2(.1)(.5) \\ (2)(.4)(.1) \\ (.1)^2 \end{Bmatrix}$$

Evaluating, this gives us:

$$\boxed{P_Y(y) = \begin{Bmatrix} .25 \\ .4 \\ .26 \\ .08 \\ .01 \end{Bmatrix}}$$

We then compute the expectation value as:

$$E[Y|X=2] = 2(.25) + 3(.4) + 4(.26) + 5(.08) + 6(.01)$$
$$\boxed{E[Y|X=2] = 3.2}$$

7. (a) We can integrate over the interval to write:

$$\int_{-1}^{1}\int_{0}^{x^2} cx^2\, dy\, dx = 1$$

We evaluate to get:

$$\int_{-1}^{1} cx^4\, dx = 1$$

$$\frac{cx^5}{5} - 1^1 = 1$$

$$\frac{2c}{5} = 1$$

$$\boxed{c = \frac{5}{2}}$$

Thus, we may write:

$$f_{X,Y}(x,y) = \begin{cases} 2.5x^2, & -1 \le x \le 1,\quad 0 \le y \le x^2 \\ 0, & \text{otherwise} \end{cases}$$

(b) We can find the marginal PDF with respect to X as:

$$f_X(x) = \int_{0}^{x^2} cx^2\, dy$$

This gives us:

$$\boxed{f_X(x) = \frac{5}{2}x^4, \quad -1 \le x \le 1}$$

(c) Similarly, we may find the marginal PDF with respect to Y as:

$$f_Y(y) = \int_{-\sqrt{y}}^{\sqrt{y}} cx^2\, dx$$

$$f_Y(y) = \frac{c}{3}\left[x^3\right] - \sqrt{y}^{\sqrt{y}}$$

$$\boxed{f_Y(y) = \frac{5}{3}y^{3/2}, \quad 0 \le y \le 1}$$