# Homework 3

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October 7, 2023

1. We can find the sum of all photons to be:

$$\sum \langle s_n \rangle = \sum \frac{1}{e^{\frac{\hbar\omega_n}{\tau}} - 1}$$

We know that in a cavity of volume $V$, the edge length is $L$, which means $V = L^3$. From here, we know $\omega_n = \frac{n\pi c}{L}$. Now, if we were to assume that $n_x$, $n_y$, and $n_z$ are in a positive octant, we can get:

$$\sum_n \frac{1}{e^{\frac{\hbar n\pi c}{L\tau}} - 1} = \frac{1}{4}\int_0^\infty \left(\frac{4\pi n^2}{e^{\frac{\hbar n\pi c}{L\tau}} - 1}\right) dn$$
$$= \pi \int_0^\infty \left(\frac{4\pi n^2}{e^{\frac{\hbar n\pi c}{L\tau}} - 1}\right) dn$$

If we perform a $u$-substitution, we obtain:

$$u = \frac{\hbar n\pi c}{L\tau}$$
$$du = \frac{\hbar\pi c}{L\tau} dn$$

Substituting this in, we get:

$$\pi \left(\frac{L\tau}{\hbar\pi c}\right)^3 \int_0^\infty \frac{u^2}{e^u - 1}\, du$$

Using a numerical solver, we find:

$$\int_0^\infty \frac{u^2}{e^u - 1}\, du = 2.404$$

Thus, we get:

$$N = \frac{L^3 \tau^3}{\pi^2 \hbar^3 c^3}(2.404)$$

We can simplify the $L^3$ to get:

$$\boxed{N = \frac{V \tau^3}{\pi^2 \hbar^3 c^3}(2.404)}$$

Finding $\frac{\sigma}{N}$, we find:

$$\begin{aligned}
\frac{\sigma}{N} &= \left(\frac{4\pi^2 V \tau^3}{45\hbar^3 c^3}\right)\left(\frac{\pi^2 \hbar^3 c^3}{V\tau^3(2.404)}\right) \\
&= \frac{4\pi^4}{45} \\
&= 3.6017
\end{aligned}$$

2. (a) To find the surface energy production rate, we need to simply multiply the density by the surface area. This gives us:

$$\begin{aligned}
4\pi R^2 \cdot C_s &= 4\pi(1.49 \cdot 10^{11})^2 \cdot .136 \\
&= 3.794 \cdot 10^{22}
\end{aligned}$$

We then need to convert the radius, which was in meters, to centimeters:

$$\begin{aligned}
\text{Energy rate } &= 3.794 \cdot 10^{22} \cdot \left(10^2\right)^2 \\
\text{Energy rate } &= 3.794 \cdot 10^{26} \left[\frac{\text{J}}{\text{s}}\right]
\end{aligned}$$

$$\boxed{\text{Energy rate } \approx 4 \cdot 10^{26}\, [\text{W}]}$$

(b) We first find the energy emission per unit are of the sun itself, using the radius provided:

$$\frac{4 \cdot 10^{26}}{4\pi(7 \cdot 10^{10})^2} = 6496.1 \left[\frac{\text{W}}{\text{cm}^2}\right]$$

We know this value is equal to the Stefan-Boltzmann constant multiplied by the temperature to the power of four. From here, we can solve:

$$\begin{aligned}
\frac{6496.1}{5.67 \cdot 10^{-12}} &= T^4 \\
T^4 &= 1.1457 \cdot 10^{15} [\text{K}^4] \\
T &= 5,817.9 [\text{K}]
\end{aligned}$$

$$\boxed{T \approx 6,000 [\text{K}]}$$

4. (a) First we start off by finding the atomic mass difference when converting hydrogen to helium:

$$4(1.0078) - 4.0026 = .0286[\text{u}]$$

We then convert this value to kilograms:

$$.0286 \cdot \left(\frac{1.6726 \cdot 10^{26}}{1.00727647}\right) = 4.79 \cdot 10^{-29}[\text{kg}]$$

We then plug this into Einstein's equation to find the energy:

$$4.79 \cdot 10^{-29} \cdot (3 \cdot 10^{8})^{2} = 4.27 \cdot 10^{-12}[\text{J}]$$

We then multiply this by the mass of the sun, divided by the quantity of 10% of the hydrogen to find the available energy:

$$\boxed{E = (.1)(2 \cdot 10^{30})\left(\frac{1}{4 \cdot 1.0078}\right)\left(\frac{1.00727647}{1.6726 \cdot 10^{-27}}\right)(4.27 \cdot 10^{-12}) = 1.277 \cdot 10^{44}[\text{J}]}$$

(b) We need to divide the amount from part (a), and divide it by the energy radiation rate:

$$\frac{1.277 \cdot 10^{44}}{4 \cdot 10^{26}} = 3.1925 \cdot 10^{17}[\text{s}]$$

We then convert seconds to years to find:

$$\boxed{2.875 \cdot 10^{18} \cdot (3600)^{-1} \cdot (24)^{-1} \cdot (365)^{-1} = 1.012 \cdot 10^{10}[\text{yr}]}$$

5. Setting up a ratio, we can find that:

$$R_{sun}^{2} T_{sun}^{4} = R_{earth}^{2} T_{earth}^{4}$$

This gives us:

$$T_{earth} = T_{sun}\sqrt{\frac{R_{sun}}{R_{earth}}}$$

Plugging in our values, we get:

$$T_{earth} = 5800\sqrt{\frac{7 \cdot 10^{10}}{1.5 \cdot 10^{13}}}$$

$$\boxed{T_{earth} = 396.22[\text{K}]}$$

6. (a) We know that in a "$j$-th" mode, the total energy can be described as:

$$U = \sum_j \varepsilon_j$$

For a harmonic, we can find that $\varepsilon_j = s_j \hbar \omega_j$:

$$U = \sum_j s_j \hbar \omega_j$$

Then, we know:

$$P = -\frac{\partial U}{\partial V}$$

Combining the two, we see:

$$\boxed{P = -\sum_j s_j \hbar \frac{\partial \omega_j}{\partial V}}$$

(b) Since we know:

$$\omega_j = \frac{j\pi c}{L} \text{ and } V = L^3$$

We can combine the two to find:

$$\omega_j = \frac{j\pi c}{\sqrt[3]{V}}$$

Then, we differentiate:

$$\frac{\partial \omega_j}{\partial V} = -\frac{1}{3} j\pi c V^{-\frac{4}{3}}$$

This can be simplified to find:

$$\boxed{\frac{\partial \omega_j}{\partial V} = -\frac{\omega_j}{3V}}$$

(c) Combining (a) and (b), we find:

$$P = \sum_j \frac{s_j \hbar \omega_j}{3V}$$

Which becomes:

$$\boxed{P = \frac{U}{3V}}$$

(d) We may use the formula:

$$P = \frac{Nk_BT}{V}$$

This would yield:

$$P = \underbrace{\left(6.022 \cdot 10^{23}\right)\left(10^2\right)^3}_{\text{molecules per volume}} \underbrace{\left(1.381 \cdot 10^{-23}\right)}_{\text{Boltzmann constant}} \underbrace{\left(2 \cdot 10^7\right)}_{\text{temperature}} = 1.6633 \cdot 10^{14}\,[\text{Pa}]$$

Now, using the formula from (c)

$$P = \frac{3U}{V}$$

$$P = \frac{\pi^2 \tau^4}{(3)(15)\hbar^3 c^3}$$

To convert to temperatures, we use the multiply both sides by the Boltzmann constant:

$$\frac{\pi^2 k_B^4 T^4}{45\hbar^3 c^3} = \frac{Nk_BT}{V}$$

Simplifying, we find:

$$T = \sqrt[3]{\frac{45\hbar^3 c^3 N}{\pi^2 k_B^3 V}}$$

$$\boxed{T = 3.2 \cdot 10^7 [\text{K}]}$$