Homework 4

Michael Brodskiy

Professor: A. Stepanyants

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1. First, we can describe the external chemical potential as:

$$\mu_{ext} = -\frac{1}{2}Mr^2\omega^2$$

Note, the sign is negative so at larger r values, the chemical potential is lower. We can add the internal potential to find the total:

$$\mu_{tot} = \tau \ln \left(\frac{n(r)}{n_Q} \right) - \frac{1}{2}Mr^2\omega^2$$

For diffusion equilibrium, we know:

$$\mu_{tot}(r) = \mu_{tot}(0)$$

This lets us write:

$$\tau \ln \left(\frac{n(0)}{n_Q} \right) = \tau \ln \left(\frac{n(r)}{n_Q} \right) - \frac{1}{2}Mr^2\omega^2$$

We can subtract the ln term over:

$$\tau \ln \left(\frac{n(r)}{n_Q} \right) - \tau \ln \left(\frac{n(0)}{n_Q} \right) = \frac{1}{2}Mr^2\omega^2$$

Then we continue to simplify:

$$\tau \ln \left(\frac{n(r)}{n(0)} \right) = \frac{1}{2}Mr^2\omega^2$$

And finally, we get:

$$\boxed{n(r) = n(0)e^{\frac{Mr^2\omega^2}{2\tau}}}$$

3. From (1), we know we can write:

$$n(h) = n(0)e^{-\beta Mgh}$$

We can calculate the total number of atoms as:

$$n(0) \int_0^\infty e^{-\beta Mgh}\, dh$$

The total potential energy can be calculated as:

$$Mgn(0) \int_0^\infty h e^{-\beta Mgh}\, dh$$

We can then find the potential energy per atom:

$$U = \frac{Mg \displaystyle\int_0^\infty h e^{-\beta Mgh}\, dh}{\displaystyle\int_0^\infty e^{-\beta Mgh}\, dh}$$

After calculating, we get:

$$U = k_B T$$

The heat capacity can be found as:

$$C = k_B T + \frac{3}{2}k_B T = \frac{5}{2}k_B T$$

Thus, the heat capacity is given by:

$$\boxed{C = \frac{5}{2}k_B T}$$

4. We can find the difference in chemical potential as:

$$\mu_{cell} - \mu_{pond} = \tau \ln\left(\frac{n_{cell}}{n_{pond}}\right)$$

Thus, this gives us the potential drop:

$$\boxed{\Delta\mu = (8.617 \cdot 10^{-5})(300)\ln\left(10^4\right) \approx .24[\text{eV}]}$$

6. (a) The grand partition function may be written as:

$$\mathfrak{z}(\tau, \mu, V) = \sum_{N} \sum_{\varepsilon_S} \lambda^N e^{-\frac{\varepsilon_S}{\tau}}$$

We can see that there are three states to account for:

N	ε_S
0	0
1	0
1	ε

This gives us:

$$\mathfrak{z}(\tau, \mu, V) = \lambda^0 e^0 + \lambda^1 e^0 + \lambda^1 e^{-\frac{\varepsilon}{\tau}}$$

This evaluates to:

$$\boxed{\mathfrak{z} = 1 + \lambda + \lambda e^{-\frac{\varepsilon}{\tau}}}$$

(b) From our formulas, we know:

$$\langle N \rangle = \sum_{N} \sum_{\varepsilon_S} N \cdot P(N, \varepsilon_S)$$

Evaluating each combination, we get:

$$\langle N \rangle = \frac{(0)\lambda^0 e^0 + (1)\lambda^1 e^0 + (1)\lambda^1 e^{-\frac{\varepsilon}{\tau}}}{\mathfrak{z}}$$

This finally evaluates to:

$$\boxed{\langle N \rangle = \frac{\lambda + \lambda e^{-\frac{\varepsilon}{\tau}}}{1 + \lambda + \lambda e^{-\frac{\varepsilon}{\tau}}}}$$

(c) For $\langle N(\varepsilon) \rangle$, we can write the formula as:

$$\langle N(\varepsilon) \rangle = \sum_{N} N \cdot \frac{\lambda^N e^{-\frac{\varepsilon}{\tau}}}{\mathfrak{z}}$$

This only occurs at $N = 1$, which gives us:

$$\langle N(\varepsilon) \rangle = \frac{\lambda e^{-\frac{\varepsilon}{\tau}}}{\mathfrak{z}}$$

This can be expanded to:

$$\boxed{\frac{\lambda e^{-\frac{\varepsilon}{\tau}}}{1 + \lambda + \lambda e^{-\frac{\varepsilon}{\tau}}}}$$

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(d) The thermal average energy can be evaluated as:

$$\langle \varepsilon_S \rangle = \sum_N \sum_{\varepsilon_S} \varepsilon_S \cdot P(N, \varepsilon_S)$$

Since $\varepsilon_S \neq 0$ only for the last term, the last term is the only one remaining:

$$\boxed{\frac{\lambda \varepsilon e^{-\frac{\varepsilon}{\tau}}}{1 + \lambda + \lambda e^{-\frac{\varepsilon}{\tau}}}}$$

(e) In the given case, we can write the combinations as:

N	ε_S
0	0
1	0
1	ε
2	ε

Thus, we can add one term to our initial partition function:

$$\mathfrak{z}_{new} = \mathfrak{z}_{init} + \lambda^2 e^{-\frac{\varepsilon}{\tau}}$$

This gives us:

$$\boxed{\mathfrak{z} = 1 + \lambda + \lambda e^{-\frac{\varepsilon}{\tau}} + \lambda^2 e^{-\frac{\varepsilon}{\tau}} = (1 + \lambda)\left(1 + \lambda e^{-\frac{\varepsilon}{\tau}}\right)}$$

8. (a) First, we must find the partition function:

$$\mathfrak{z} = 1 + \lambda e^{-\frac{\varepsilon_A}{\tau}}$$

From here, we can find the probability as:

$$P(O_2) = \frac{\lambda e^{-\frac{\varepsilon_A}{\tau}}}{1 + \lambda e^{-\frac{\varepsilon_A}{\tau}}}$$

To simplify, we can write:

$$P(O_2) = \frac{1}{\frac{1}{\lambda} e^{\frac{\varepsilon_A}{\tau}} + 1}$$

We can then substitute the values we know:

$$1 = .9\left(\frac{1}{1 \cdot 10^{-5}} e^{\frac{\varepsilon_A}{\tau}} + 1\right)$$

$$.1 = \frac{9}{1 \cdot 10^{-4}} e^{\frac{\varepsilon_A}{\tau}}$$

Then, we get:

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$$\frac{\varepsilon_A}{\tau} = \ln\left(\frac{10^{-5}}{9}\right)$$

$$\varepsilon_A = k_B T \ln\left(\frac{10^{-5}}{9}\right)$$

$$\varepsilon_A = \left(8.617 \cdot 10^{-5}\right)(312)\ln\left(\frac{10^{-5}}{9}\right)$$

Finally, we find:

$$\boxed{\varepsilon_A = -.3686 \,[\text{eV per } O_2]}$$

(b) Now with another particle, we can find the partition function is:

$$з = 1 + \lambda_{O_2} e^{-\frac{\varepsilon_A}{\tau}} + \lambda_{CO} e^{-\frac{\varepsilon_B}{\tau}}$$

We can then write the probability of O_2 as:

$$P(O_2) = \frac{\lambda_{O_2} e^{-\frac{\varepsilon_A}{\tau}}}{1 + \lambda_{O_2} e^{-\frac{\varepsilon_A}{\tau}} + \lambda_{CO} e^{-\frac{\varepsilon_B}{\tau}}}$$

Plugging in the values we know:

$$.1 = \frac{\left(10^{-5}\right) e^{-\frac{\varepsilon_A}{\tau}}}{1 + \left(10^{-5}\right) e^{-\frac{\varepsilon_A}{\tau}} + \left(10^{-7}\right) e^{-\frac{\varepsilon_B}{\tau}}}$$

$$.9\left(10^{-5}\right) e^{-\frac{\varepsilon_A}{\tau}} = .1 + .1\left(10^{-7}\right) e^{-\frac{\varepsilon_B}{\tau}}$$

$$9\left(10^{-5}\right) e^{-\frac{\varepsilon_A}{\tau}} + 1 = \left(10^{-7}\right) e^{-\frac{\varepsilon_B}{\tau}}$$

$$900 e^{-\frac{\varepsilon_A}{\tau}} + 10^7 = e^{-\frac{\varepsilon_B}{\tau}}$$

$$-\frac{\varepsilon_B}{\tau} = \ln\left(900 e^{-\frac{\varepsilon_A}{\tau}} + 10^7\right)$$

$$\varepsilon_B = -k_B T \ln\left(900 e^{-\frac{\varepsilon_A}{\tau}} + 10^7\right)$$

$$\varepsilon_B = -\left(8.617 \cdot 10^{-5}\right)(312)\ln\left(900 e^{-\frac{-.3686}{.026885}} + 10^7\right)$$

Finally, this yields:

$$\boxed{\varepsilon_B = -.5518 [\text{eV per } CO]}$$