Homework 7

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1. (a) First and foremost, we know:

$$Q_h = Q_l + W$$
$$W = Q_l - Q_h$$

Per reversible conditions:

$$\sigma_h = \sigma_l$$
$$\frac{Q_h}{\tau_h} = \frac{Q_l}{\tau_l}$$
$$Q_h = \frac{\tau_h Q_l}{\tau_l}$$

Thus, we can combine to write:

$$W = Q_h - \frac{\tau_l}{\tau_h} Q_h$$
$$W = Q_h \left(1 - \frac{\tau_l}{\tau_h}\right)$$
$$W = Q_h \left(\frac{\tau_h - \tau_l}{\tau_h}\right)$$

And finally:

$$\boxed{\frac{W}{Q_h} = \frac{\tau_h - \tau_l}{\tau_h}}$$

If the heat pump is not reversible, we know the efficiency is:

$$\frac{W}{Q_h} < \frac{\tau_h - \tau_l}{\tau_h}$$

(b) We can write:

$$W = (\tau_{hh} - \tau_l)\left(\frac{Q_{hh}}{\tau_{hh}} - \frac{Q_l}{\tau_l}\right)$$

We know that the work generated is used as Q_h in the heat pump:

$$Q_h = (\tau_{hh} - \tau_l)\left(\frac{Q_{hh}}{\tau_{hh}} - \frac{Q_l}{\tau_l}\right)$$

We can then rearrange to find the desired ratio:

$$\begin{aligned}\frac{Q_h}{\tau_{hh} - \tau_l} &= \frac{Q_{hh}\tau_l - Q_l\tau_{hh}}{\tau_{hh}\tau_l}\\ \frac{\tau_{hh}\tau_l}{\tau_{hh} - \tau_l} &= \frac{Q_{hh}\tau_l - Q_l\tau_{hh}}{Q_h}\end{aligned}$$

We know $Q_l = \frac{\tau_l}{\tau_h}Q_h$, which gives us:

$$\frac{\tau_{hh}\tau_l}{\tau_{hh} - \tau_l} = \frac{Q_{hh}\tau_l}{Q_h} - \frac{\tau_l\tau_{hh}}{\tau_h}$$

$$\boxed{\frac{Q_{hh}}{Q_h} = \frac{\tau_{hh}}{\tau_{hh} - \tau_l} + \frac{\tau_{hh}}{\tau_h}}$$

Using the given values, we obtain:

$$\frac{Q_{hh}}{Q_h} = \frac{600}{600 - 270} + 2$$

$$\boxed{\frac{Q_{hh}}{Q_h} = 3.82}$$

(c) For this problem, we essentially need to draw a "double Carnot engine," with the output work of one (W) being the Q_h of the second:

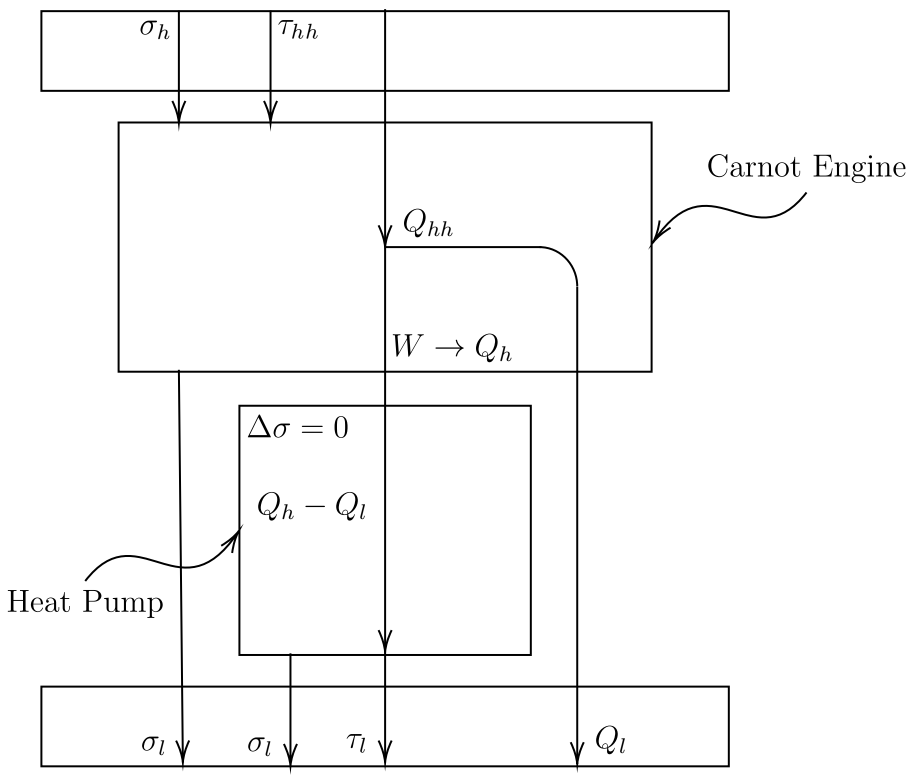

Figure 1: Carnot Engine Feeding a Heat Pump

2. (a)
   (b)
3. (a)
   (b)
   (c)
   (d)
5. First and foremost, we may write:

$$W = (\tau_h - \tau_l)\sigma_l$$

Given the assumption that the process is reversible, we may write:

$$\sigma_l = \frac{Q_l}{\tau_l}$$

This gives us:

$$W = \left(\frac{\tau_h}{\tau_l} - 1\right) Q_l$$

We can apply the given parameters to find:

$$W = \left(\frac{500}{20} - 1\right) 1500$$

$$\boxed{W = 36[\text{GW}]}$$

Given improvements, we can write:

$$W = \left(\frac{600}{20} - 1\right) 1500$$

$$\boxed{W = 43.5[\text{GW}]}$$

Clearly, there is nearly a 20% increase in the ability to produce energy

6. (a) First and foremost, we may write:

$$W = \left(1 - \frac{\tau_l}{\tau_h}\right) Q_h = \left(\frac{\tau_h}{\tau_l} - 1\right) Q_l$$

Given that:

$$\frac{dW}{dt} = P$$

We may obtain:

$$P = \left(\frac{\tau_h}{\tau_l} - 1\right) \frac{dQ_l}{dt}$$

From the problem, we may substitute:

$$P = A\left(\frac{\tau_h}{\tau_l} - 1\right)(\tau_h - \tau_l)$$

Multiplying both sides by τ_l and dividing by A, we get:

$$\frac{P\tau_l}{A} = (\tau_h - \tau_l)^2$$

$$\frac{P\tau_l}{A} = \tau_h^2 - 2\tau_h\tau_l + \tau_l^2$$

We can then move all terms to one side:

$$\frac{P\tau_l}{A} + 2\tau_h\tau_l - \tau_l^2 - \tau_h^2 = 0$$

$$\left(\frac{P}{A}+2\tau_h\right)\tau_l-\tau_l^2-\tau_h^2=0$$

Employing the quadratic equation, we obtain:

$$\tau_l^2-\left(\frac{P}{A}+2\tau_h\right)\tau_l+\tau_h=0\rightarrow\begin{cases}a=1\\ b=-\left(\frac{P}{A}+2\tau_h\right)\\ c=\tau_h^2\end{cases}$$

$$\tau_l=\frac{1}{2}\left(\frac{P}{A}+2\tau_h-\sqrt{\left(\frac{P}{A}+2\tau_h\right)^2-4\tau_h^2}\right)$$ [1]

Distributing, we finally obtain:

$$\tau_l=\frac{P}{2A}+\tau_h-\left[\left(\frac{P}{2A}+\tau_h\right)^2-\tau_h^2\right]^{\frac{1}{2}}$$

The Boltzmann factor cancels out, leaving us with:

$$\boxed{T_l=\left(T_h+\frac{P}{2A}\right)-\left[\left(T_h+\frac{P}{2A}\right)^2-T_h^2\right]^{\frac{1}{2}}}$$

(b) We can rearrange one of the formulas obtained in (a) to write:

$$A=\frac{PT_l}{(T_h-T_l)^2}$$

This gives us:

$$A=\frac{(2\cdot10^3)(290)}{(20)^2}$$

$$A=\frac{(580\cdot10^3)}{400}$$

$$\boxed{A=1450\left[\frac{\mathrm{W}}{\mathrm{K}}\right]}$$

7. We know from (1) that:

$$\frac{W}{Q_l}=\left(\frac{\tau_h}{\tau_l}-1\right)$$

And also that:

$$W+Q_l=Q_h$$

We can express the work done by the light bulb as $W = Q_l$. The total work of the two processes needs to equal zero, which allows us to write:

$$\left(\frac{\tau_h}{\tau_l} - 1\right) Q_l - Q_l = 0$$

This simplifies to:

$$\tau_h = 2\tau_l$$

Room temperature implies $27[°\mathrm{C}] \rightarrow 300[\mathrm{K}]$

Thus, the Carnot refrigerator, at peak efficiency, may cool to:

$$\boxed{\tau_l = \frac{300}{2} = 150[\mathrm{K}]}$$

Thus, the refrigerator should be able to cool below room temperature.