Fermi and Bose Gases

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- Gibbs Distribution

$$P(N,\varepsilon_S)=\frac{e^{\frac{N\mu-\varepsilon_S}{\tau}}}{\mathfrak{z}}$$

  This is the probability to find S in a quantum state $S(N)$ of N particles and energy ε_S

$$\mathfrak{z}(\tau,\mu,V)=\sum_N\sum_{\varepsilon_S}e^{\frac{N\mu-\varepsilon_S}{\tau}}$$

- Activity

  - We can define the activity as:

$$e^{\frac{\mu}{\tau}}=\lambda$$

  - This allows us to rewrite:

$$P(N,\varepsilon_S)=\frac{\lambda^N e^{-\frac{\varepsilon_S}{\tau}}}{\mathfrak{z}}$$

$$\mathfrak{z}(\tau,\mu,V)=\sum_N\sum_{\varepsilon_S}\lambda^N e^{-\frac{\varepsilon_S}{\tau}}$$

  - Some important averages that follow from this are:

$$\langle N\rangle=\sum_N\sum_{\varepsilon_S}N\cdot P(N,\varepsilon_S)=\tau\left(\frac{\partial\ln(\mathfrak{z})}{\partial\mu}\right)_{\tau,V}$$

$$\langle\varepsilon_S\rangle=\sum_N\sum_{\varepsilon_S}\varepsilon_S\cdot P(N,\varepsilon_S)=\tau^2\left(\frac{\partial\ln(\mathfrak{z})}{\partial\tau}\right)_{\mu,V}+\tau\mu\left(\frac{\partial\ln(\mathfrak{z})}{\partial\mu}\right)_{\tau,V}$$

- Fermi and Bose Gases

- Such gases are non-interacting (mono-atomic particles of spin = 0)
- Fermions: Half-integer spin particles, like protons, neutrons, electrons, positrons, and hydrogen
- Bosons: Integer spin particles, like photons, phonons
- Pauli Exclusion Principle: An orbital can only be occupied by 0 or 1 fermions of the same species

- Fermi-Dirac Distribution
  - A system, S, inside a reservoir, R, is filled with a gas of non-interacting fermions, and is in thermal and diffusive equilibrium
  - Orbital, ε_n, is in thermal and diffusive equilibrium with other orbitals

- Gibbs Sum for Orbital ε_n

$$\varepsilon_n = \lambda^0 e^{-0/\tau} + \lambda^1 e^{-\frac{\varepsilon_n}{\tau}} = 1 + \lambda e^{-\frac{\varepsilon_n}{\tau}}$$

The average quantity of gas particles, will be written as:

$$\langle N \rangle \equiv f(\varepsilon_n) = \frac{\lambda e^{-\frac{\varepsilon_n}{\tau}}}{\mathfrak{z}_n} = \frac{1}{\frac{1}{\lambda} e^{\frac{\varepsilon_n}{\tau}} + 1}$$

Inserting λ in, we get the Fermi-Dirac Distribution:

$$f(\varepsilon_n) = \frac{1}{e^{\frac{\varepsilon_n - \mu}{\tau}} + 1}$$

This gives us the average occupancy of an orbital ε_n, also known as the average number of particles, or the probability of occupancy Since it is a probability, we know:

$$0 \leq f(\varepsilon_n) \geq 1$$

- Bose-Einstein Distribution
  - For this distribution, we obtain a geometric series for the grand partition function:

$$\mathfrak{z}_n = 1 + \lambda e^{-\frac{\varepsilon_n}{\tau}} + \left(\lambda e^{-\frac{\varepsilon_n}{\tau}}\right) + \ldots = \frac{1}{1 - \lambda e^{-\frac{\varepsilon_n}{\tau}}}$$

  - Thus, we assume $\lambda e^{-\frac{\varepsilon_n}{\tau}} < 1 \, \forall \, n$
  - Finally, we get:

$$f(\varepsilon_n) = \frac{1}{e^{\frac{\varepsilon_n - \mu}{\tau}} - 1}$$

  - In the classical regime, $e^{\frac{\varepsilon_n - \mu}{\tau}} >> 1 \, \forall \, n$ and there is no difference between a Bose-Einstein and Fermi-Dirac distribution. The classical distribution function is:

$$f(\varepsilon_n) = e^{\frac{\mu - \varepsilon_n}{\tau}} << 1$$