Homework 7

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1. (a) First and foremost, we know:

$$Q_h = Q_l + W$$
$$W = Q_l - Q_h$$

Per reversible conditions:

$$\sigma_h = \sigma_l$$
$$\frac{Q_h}{\tau_h} = \frac{Q_l}{\tau_l}$$
$$Q_h = \frac{\tau_h Q_l}{\tau_l}$$

Thus, we can combine to write:

$$W = Q_h - \frac{\tau_l}{\tau_h} Q_h$$
$$W = Q_h \left(1 - \frac{\tau_l}{\tau_h}\right)$$
$$W = Q_h \left(\frac{\tau_h - \tau_l}{\tau_h}\right)$$

And finally:

$$\boxed{\frac{W}{Q_h} = \frac{\tau_h - \tau_l}{\tau_h}}$$

If the heat pump is not reversible, we know the efficiency is:

$$\frac{W}{Q_h} < \frac{\tau_h - \tau_l}{\tau_h}$$

(b) We can write:

$$W = (\tau_{hh} - \tau_l)\left(\frac{Q_{hh}}{\tau_{hh}} - \frac{Q_l}{\tau_l}\right)$$

We know that the work generated is used as Q_h in the heat pump:

$$Q_h = (\tau_{hh} - \tau_l)\left(\frac{Q_{hh}}{\tau_{hh}} - \frac{Q_l}{\tau_l}\right)$$

We can then rearrange to find the desired ratio:

$$\begin{aligned}\frac{Q_h}{\tau_{hh} - \tau_l} &= \frac{Q_{hh}\tau_l - Q_l\tau_{hh}}{\tau_{hh}\tau_l} \\ \frac{\tau_{hh}\tau_l}{\tau_{hh} - \tau_l} &= \frac{Q_{hh}\tau_l - Q_l\tau_{hh}}{Q_h}\end{aligned}$$

We know $Q_l = \frac{\tau_l}{\tau_h}Q_h$, which gives us:

$$\begin{aligned}\frac{\tau_{hh}\tau_l}{\tau_{hh} - \tau_l} &= \frac{Q_{hh}\tau_l}{Q_h} - \frac{\tau_l\tau_{hh}}{\tau_h} \\ \boxed{\frac{Q_{hh}}{Q_h} = \frac{\tau_{hh}}{\tau_{hh} - \tau_l} + \frac{\tau_{hh}}{\tau_h}}\end{aligned}$$

Using the given values, we obtain:

$$\begin{aligned}\frac{Q_{hh}}{Q_h} &= \frac{600}{600 - 270} + 2 \\ &\boxed{\frac{Q_{hh}}{Q_h} = 3.82}\end{aligned}$$

(c) For this problem, we essentially need to draw a "double Carnot engine," with the output work of one (W) being the Q_h of the second:

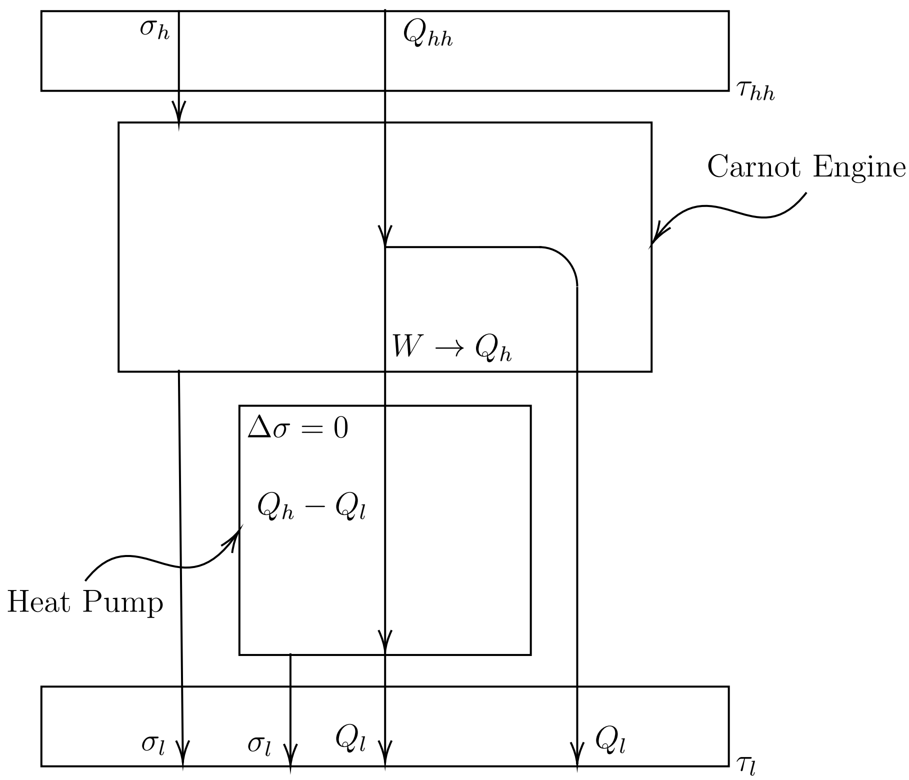

Figure 1: Carnot Engine Feeding a Heat Pump

2. (a) Similar to Figure 1, we must have the pilot flame feed into the refrigerator:

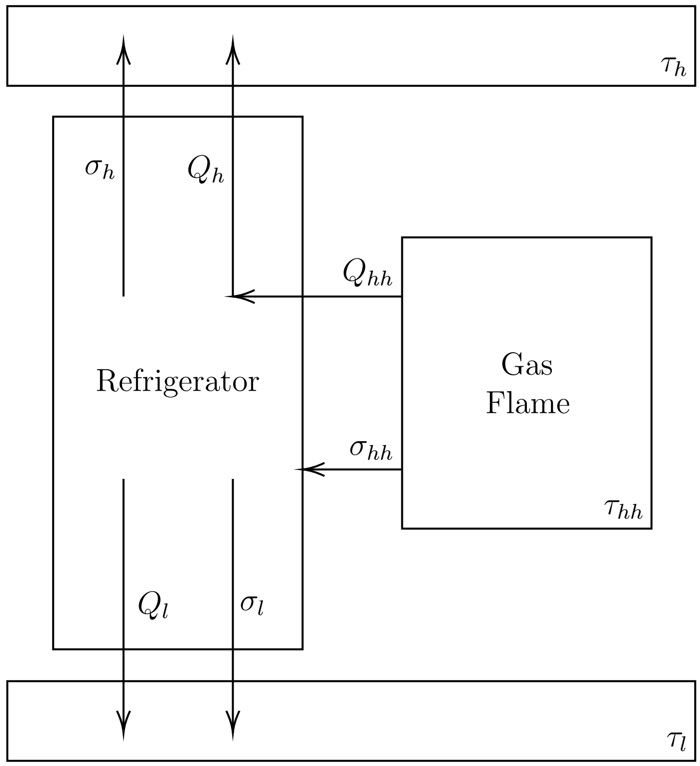

Figure 2: Pilot Flame Feeding a Refrigerator

(b) First, we assume that this is a reversible refrigerator. This gives us:

$$\frac{Q_{hh}}{\tau_{hh}} + \frac{Q_l}{\tau_l} - \frac{Q_h}{\tau_h} = 0$$

Furthermore, we know:

$$Q_h = Q_l + Q_{hh}$$

This gives us:

$$\frac{Q_{hh}}{\tau_{hh}} + \frac{Q_l}{\tau_l} = \frac{Q_l + Q_{hh}}{\tau_h}$$

We can rearrange:

$$\frac{\tau_l Q_{hh} + Q_l \tau_{hh}}{\tau_l \tau_{hh}} = \frac{Q_l + Q_{hh}}{\tau_h}$$

$$\tau_h(\tau_l Q_{hh} + Q_l \tau_{hh}) = (Q_l + Q_{hh})(\tau_l \tau_{hh})$$

$$\tau_h \tau_l Q_{hh} + Q_l \tau_{hh} \tau_h = \tau_l \tau_{hh} Q_l + \tau_l \tau_{hh} Q_{hh}$$

We then separate Q_{hh} and Q_l to different sides:

$$(\tau_h\tau_l - \tau_l\tau_{hh})Q_{hh} = (\tau_l\tau_{hh} - \tau_{hh}\tau_h)Q_l$$

This then gives us:

$$\boxed{\frac{Q_l}{Q_{hh}} = \frac{\tau_h\tau_l - \tau_l\tau_{hh}}{\tau_l\tau_{hh} - \tau_{hh}\tau_h}}$$

3. (a) The processes from stages $2 \to 3$ and $4 \to 1$ are both isentropic, indicating constant σ. This, in tandem with the fact that, for photons, $\sigma \propto V\tau^3$, allows us to write:

$$V_2\tau_h^3 = V_3\tau_l^3 \quad \text{and} \quad V_1\tau_h^3 = V_4\tau_l^3$$
$$\boxed{V_3 = \left(\frac{\tau_h}{\tau_l}\right)^3 V_2 \quad \text{and} \quad V_4 = \left(\frac{\tau_h}{\tau_l}\right)^3 V_1}$$

(b) As with our ideal gas, we can derive the heat and work in a similar manner. First and foremost:

$$Q_h = \tau_h\left(\sigma_2 - \sigma_1\right)$$

For a photon gas, we know:

$$\sigma = \frac{4\pi^2 V\tau^3}{45h^3c^3}$$

This gives us:

$$Q_h = \frac{4\pi^2\tau_h^4}{45h^3c^3}(V_2 - V_1)$$

Then, we need to employ the equation $Q_h = \Delta U + W$. We begin by finding ΔU:

$$U = \frac{\pi^2 V\tau^4}{15h^3c^3}$$
$$\Delta U = \frac{\pi^2\tau_h^4}{15h^3c^3}(V_2 - V_1)$$

We can see that the dependency on the volume makes it so that $\Delta U \neq 0$, unlike an ideal gas. As such we may say that $\boxed{Q_h \neq W}$. We can also find the actual value of W as:

$$Q_h - \Delta U = \frac{\pi^2\tau_h^4}{45h^3c^3}(V_2 - V_1)$$

Thus, we can see:

$$\boxed{\frac{4\pi^2\tau_h^4}{45h^3c^3}(V_2 - V_1) \neq \frac{\pi^2\tau_h^4}{45h^3c^3}(V_2 - V_1)}$$

(c) For the isentropic processes, we know:

$$Q = 0$$

This allows us to write:

$$\Delta U + W = 0$$
$$W = -\Delta U$$

For process $2 \rightarrow 3$, we may write:

$$W = -(U_3 - U_2)$$
$$W = -\frac{\pi^2}{15h^3c^3}(V_3\tau_l^4 - V_2\tau_h^4)$$

For process $4 \rightarrow 1$, we may write:

$$W = -(U_1 - U_4)$$
$$W = -\frac{\pi^2}{15h^3c^3}(V_1\tau_h^4 - V_4\tau_l^4)$$

Employing our results from part (a), we may write:

$$W_{23} = -\frac{\pi^2}{15h^3c^3}(V_2\tau_l\tau_h^3 - V_2\tau_h^4)$$
$$W_{41} = -\frac{\pi^2}{15h^3c^3}(V_1\tau_h^4 - V_1\tau_h^3\tau_l)$$

These are equivalent to:

$$W_{23} = \frac{\pi^2 V_2\tau_h^3}{15h^3c^3}(\tau_h - \tau_l)$$
$$W_{41} = \frac{\pi^2 V_1\tau_h^3}{15h^3c^3}(\tau_l - \tau_h)$$

Thus, we can see that, though similar, unlike an ideal gas, $\boxed{W_{23} \neq W_{41}}$

(d) Using our results from (b) and (c), we can write:

$$W_{12} = \frac{\pi^2\tau_h^4}{45h^3c^3}(V_2 - V_1)$$
$$W_{23} = \frac{\pi^2 V_2\tau_h^3}{15h^3c^3}(\tau_h - \tau_l)$$
$$W_{34} = \frac{\pi^2\tau_l^4}{45h^3c^3}(V_4 - V_3)$$

$$W_{41} = \frac{\pi^2 V_1 \tau_h^3}{15h^3c^3}(\tau_l - \tau_h)$$

For W_{34}, we can employ the identities found in (a):

$$W_{34} = \frac{\pi^2 \tau_l \tau_h^3}{45h^3c^3}(V_1 - V_2)$$

Now, we can sum:

$$W_{tot} = \frac{\pi^2 \tau_h^4}{45h^3c^3}(V_2 - V_1) + \frac{\pi^2 V_2 \tau_h^3}{15h^3c^3}(\tau_h - \tau_l) + \frac{\pi^2 \tau_l \tau_h^3}{45h^3c^3}(V_1 - V_2) + \frac{\pi^2 V_1 \tau_h^3}{15h^3c^3}(\tau_l - \tau_h)$$

We can factor out the coefficients:

$$W_{tot} = \frac{\pi^2 \tau_h^3}{45h^3c^3}\left(\tau_h(V_2 - V_1) + 3V_2(\tau_h - \tau_l) + \tau_l(V_1 - V_2) + 3V_1(\tau_l - \tau_h)\right)$$

We may continue by summing similar terms:

$$W_{tot} = \frac{\pi^2 \tau_h^3}{45h^3c^3}\left((\tau_h - \tau_l)(V_2 - V_1) + 3(V_2 - V_1)(\tau_h - \tau_l)\right)$$

Continuing simplification, we get:

$$\boxed{W_{tot} = \frac{4\pi^2 \tau_h^3}{45h^3c^3}(\tau_h - \tau_l)(V_2 - V_1)}$$

We know that the Carnot Energy efficiency formula may be expressed as:

$$\eta_c = \frac{W_{tot}}{Q_{rec}}$$

Dividing the above answer by the heat from part (b), we get:

$$\boxed{\eta_c = \frac{(\tau_h - \tau_l)}{\tau_h} = \left(1 - \frac{\tau_l}{\tau_h}\right)}$$

As can be seen, the efficiency formula remains the same

5. First and foremost, we may write:

$$W = (\tau_h - \tau_l)\sigma_l$$

Given the assumption that the process is reversible, we may write:

$$\sigma_l = \frac{Q_l}{\tau_l}$$

This gives us:

$$W = \left(\frac{\tau_h}{\tau_l} - 1\right) Q_l$$

We can apply the given parameters to find:

$$W = \left(\frac{500}{20} - 1\right) 1500$$

$$\boxed{W = 36[\text{GW}]}$$

Given improvements, we can write:

$$W = \left(\frac{600}{20} - 1\right) 1500$$

$$\boxed{W = 43.5[\text{GW}]}$$

Clearly, there is nearly a 20% increase in the ability to produce energy

6. (a) First and foremost, we may write:

$$W = \left(1 - \frac{\tau_l}{\tau_h}\right) Q_h = \left(\frac{\tau_h}{\tau_l} - 1\right) Q_l$$

Given that:

$$\frac{dW}{dt} = P$$

We may obtain:

$$P = \left(\frac{\tau_h}{\tau_l} - 1\right) \frac{dQ_l}{dt}$$

From the problem, we may substitute:

$$P = A\left(\frac{\tau_h}{\tau_l} - 1\right)(\tau_h - \tau_l)$$

Multiplying both sides by τ_l and dividing by A, we get:

$$\frac{P\tau_l}{A} = (\tau_h - \tau_l)^2$$

$$\frac{P\tau_l}{A} = \tau_h^2 - 2\tau_h\tau_l + \tau_l^2$$

We can then move all terms to one side:

$$\frac{P\tau_l}{A} + 2\tau_h\tau_l - \tau_l^2 - \tau_h^2 = 0$$
$$\left(\frac{P}{A} + 2\tau_h\right)\tau_l - \tau_l^2 - \tau_h^2 = 0$$

Employing the quadratic equation, we obtain:

$$\tau_l^2 - \left(\frac{P}{A} + 2\tau_h\right)\tau_l + \tau_h = 0 \rightarrow \begin{cases} a = 1 \\ b = -\left(\frac{P}{A} + 2\tau_h\right) \\ c = \tau_h^2 \end{cases}$$
$$\tau_l = \frac{1}{2}\left(\frac{P}{A} + 2\tau_h - \sqrt{\left(\frac{P}{A} + 2\tau_h\right)^2 - 4\tau_h^2}\right)$$

Note: we can forego the solution with a positive determinant, as that would make $\tau_l > \tau_h$. Distributing, we finally obtain:

$$\tau_l = \frac{P}{2A} + \tau_h - \left[\left(\frac{P}{2A} + \tau_h\right)^2 - \tau_h^2\right]^{\frac{1}{2}}$$

The Boltzmann factor cancels out, leaving us with:

$$\boxed{T_l = \left(T_h + \frac{P}{2A}\right) - \left[\left(T_h + \frac{P}{2A}\right)^2 - T_h^2\right]^{\frac{1}{2}}}$$

(b) We can rearrange one of the formulas obtained in (a) to write:

$$A = \frac{PT_l}{(T_h - T_l)^2}$$

This gives us:

$$A = \frac{(2 \cdot 10^3)(290)}{(20)^2}$$
$$A = \frac{(580 \cdot 10^3)}{400}$$
$$\boxed{A = 1450 \left[\frac{\mathrm{W}}{\mathrm{K}}\right]}$$

7. We know from (1) that:

$$\frac{W}{Q_l} = \left(\frac{\tau_h}{\tau_l} - 1\right)$$

And also that:

$$W + Q_l = Q_h$$

We can express the work done by the light bulb as $W = Q_l$. The total work of the two processes needs to equal zero, which allows us to write:

$$\left(\frac{\tau_h}{\tau_l} - 1\right) Q_l - Q_l = 0$$

This simplifies to:

$$\tau_h = 2\tau_l$$

Room temperature implies $27[°\mathrm{C}] \to 300[\mathrm{K}]$

Thus, the Carnot refrigerator, at peak efficiency, may cool to:

$$\boxed{\tau_l = \frac{300}{2} = 150[\mathrm{K}]}$$

Thus, the refrigerator should be able to cool below room temperature.