

Homework 1

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1. (a) We are given the following:

$$g(U) = CU^{\frac{3}{2}N}$$

We know, by definition, that $\sigma(U) = \ln(g(U))$. Thus, we may write:

$$\sigma(U) = \ln(CU^{\frac{3}{2}N})$$

By properties of logarithms, we may expand this to:

$$\sigma(U) = \ln(C) + \frac{3N}{2} \ln(U)$$

Furthermore, by definition of the fundamental temperature, we know that $\frac{\partial \sigma(U)}{\partial U} = \frac{1}{\tau}$. As such, we obtain:

$$\frac{\partial \sigma(U)}{\partial U} = \frac{3N}{2U}$$

Substituting and rearranging, we get:

$$\frac{1}{\tau} = \frac{3N}{2U}$$

$$U = \frac{3N}{2}\tau$$

- (b) If we return to the step before converting $\frac{\partial \sigma(U)}{\partial U}$ to $\frac{1}{\tau}$, we can differentiate once more:

$$\frac{\partial \sigma(U)}{\partial U} = \frac{3N}{2U}$$

$$\left(\frac{\partial^2 \sigma(U)}{\partial U^2}\right)_N = \frac{3N}{2} \frac{\partial}{\partial U} \left(\frac{1}{U}\right)$$

$$\boxed{\left(\frac{\partial^2 \sigma(U)}{\partial U^2}\right)_N = -\frac{3N}{U^2}}$$

Thus, we see this results in a function with a negative coefficient. Furthermore, because we know the quantity of particles can not be negative, and the square of a number can not be negative, the second order partial derivative must always be negative.

2. We know, by definition:

$$U = -(2s)mB$$

Rearranging, we see that:

$$s = -\frac{U}{2mB}$$

We also know that:

$$g(N, s) = 2^N \sqrt{\frac{2}{\pi N}} e^{-\frac{2s^2}{N}}$$

Substituting the value of s above, we get:

$$g(N, s) = \underbrace{2^N \sqrt{\frac{2}{\pi N}}}_{g(N,0)} e^{-\frac{U^2}{2m^2 B^2 N}}$$

Finding the entropy, the resulting function looks as follows:

$$\sigma(N, s) = \left(-\frac{U^2}{2m^2 B^2 N}\right) + \underbrace{\ln \left(2^N \sqrt{\frac{2}{\pi N}}\right)}_{\sigma_o}$$

Now finding the fundamental temperature, we differentiate with respect to U :

$$\frac{\partial \sigma(N, s)}{\partial U} = \left(\left(-\frac{U^2}{2m^2 B^2 N}\right) + \underbrace{\ln \left(2^N \sqrt{\frac{2}{\pi N}}\right)}_{\sigma_o} \right)_{N,s} = -\frac{U}{m^2 B^2 N}$$

Now assuming that U is the average thermal energy, we get:

$$U = -2mB\langle s \rangle$$

We then return this to our function:

$$\begin{aligned}\frac{1}{\tau} &= \frac{2\langle s \rangle}{mBN} \\ \tau &= \frac{mBN}{2\langle s \rangle}\end{aligned}$$

3. (a) From 1.55, we know:

$$g(N, n) = \frac{(N + n - 1)!}{n!(N - 1)!}$$

Thus, by definition of entropy, we know:

$$\sigma(N, n) = \ln \left(\frac{(N + n - 1)!}{n!(N - 1)!} \right) \rightarrow \ln((N + n - 1)!) - \ln(n!) - \ln((N - 1)!)$$

Replacing $N - 1$ with N , we get:

$$\ln((N + n)!) - \ln(n!) - \ln(N!)$$

Through the Stirling approximation, we get:

$$(N + n) \ln(N + n) - (N + n) - n \ln(n) + n - N \ln(N) + N$$

This can be simplified to:

$$\boxed{\sigma(N, n) = (N + n) \ln(N + n) - n \ln(n) - N \ln(N)}$$

(b) We are given $U = n\hbar\omega$, which means $n = \frac{U}{\hbar\omega}$. Thus, we substitute to get:

$$\sigma \left(N, \frac{U}{\hbar\omega} \right) = \left(N + \frac{U}{\hbar\omega} \right) \ln \left(N + \frac{U}{\hbar\omega} \right) - \frac{U}{\hbar\omega} \ln \left(\frac{U}{\hbar\omega} \right) - N \ln(N)$$

By definition of the fundamental temperature, we get:

$$\begin{aligned}\frac{1}{\tau} &= \frac{\partial}{\partial U} \left(\left(N + \frac{U}{\hbar\omega} \right) \ln \left(N + \frac{U}{\hbar\omega} \right) - \frac{U}{\hbar\omega} \ln \left(\frac{U}{\hbar\omega} \right) - N \ln(N) \right) \\ \frac{1}{\tau} &= \left(\frac{1}{\hbar\omega} \right) \ln \left(N + \frac{U}{\hbar\omega} \right) - \frac{1}{\hbar\omega} \ln \left(\frac{U}{\hbar\omega} \right) \\ \frac{1}{\tau} &= \frac{1}{\hbar\omega} \ln \left(\frac{N\hbar\omega}{U} + 1 \right)\end{aligned}$$

Rearranging for U , we get:

$$e^{\frac{\hbar\omega}{\tau}} - 1 = \frac{N\hbar\omega}{U}$$

And finally:

$$\boxed{U = \frac{N\hbar\omega}{e^{\frac{\hbar\omega}{\tau}} - 1}}$$

4. (a) Given that there is only one correct key out of the 44 possibilities per press, we know that the possibility of a single key being correct is:

$$\frac{1}{44} \approx .0227$$

Upon repeating this for a sequence of 10^5 characters, this probability becomes:

$$\left(\frac{1}{44}\right)^{100,000} \approx 10^{-164,345}$$

- (b) From the given values, we know the typing speed is 10 keys per second, and the age of the universe is 10^{18} seconds. Thus, we get:

$$10^{18}(10) = 10^{19}[\text{keys}]$$

Given that there are 10^{10} monkeys, we can multiply the numbers out to produce the total keys pressed:

$$(10^{19})(10^{10}) = 10^{29}[\text{keys total}]$$

Then, we also know that, per one hamlet, there are 10^5 characters. This yields:

$$10^{29} \left(\frac{1}{10^5}\right) = 10^{24}[\text{monkey-Hamlets}]$$

Finally, given are probability from above, we obtain:

$$(10^{24})(10^{-164,345}) = 10^{-164,316}[\text{probability of Hamlet in given time}]$$

5. (a)
(b)
(c)
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