# Homework 1

Michael Brodskiy
Professor: A. Stepanyants

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1. (a) We are given the following:

$$g(U) = CU^{\frac{3}{2}N}$$

We know, by definition, that $\sigma(U) = \ln(g(U))$. Thus, we may write:

$$\sigma(U) = \ln\left(CU^{\frac{3}{2}N}\right)$$

By properties of logarithms, we may expand this to:

$$\sigma(U) = \ln(C) + \frac{3N}{2}\ln(U)$$

Furthermore, by definition of the fundamental temperature, we know that $\dfrac{\partial\sigma(U)}{\partial U} = \dfrac{1}{\tau}$. As such, we obtain:

$$\frac{\partial\sigma(U)}{\partial U} = \frac{3N}{2U}$$

Substituting and rearranging, we get:

$$\frac{1}{\tau} = \frac{3N}{2U}$$

$$\boxed{U = \frac{3N}{2}\tau}$$

(b) If we return to the step before converting $\dfrac{\partial\sigma(U)}{\partial U}$ to $\frac{1}{\tau}$, we can differentiate once more:

$$\frac{\partial\sigma(U)}{\partial U} = \frac{3N}{2U}$$

$$\left(\frac{\partial^2\sigma(U)}{\partial U^2}\right)_N = \frac{3N}{2}\frac{\partial}{\partial U}\left(\frac{1}{U}\right)$$

$$\boxed{\left(\frac{\partial^2\sigma(U)}{\partial U^2}\right)_N = -\frac{3N}{U^2}}$$

Thus, we see this results in a function with a negative coefficient. Furthermore, because we know the quantity of particles can not be negative, and the square of a number can not be negative, the second order partial derivative must always be negative.

2. We know, by definition:

$$U = -(2s)mB$$

Rearranging, we see that:

$$s = -\frac{U}{2mB}$$

We also know that:

$$g(N,s) = 2^N\sqrt{\frac{2}{\pi N}}_{g(N,0)} e^{-\frac{2s^2}{N}}$$

Substituting the value of $s$ above, we get:

$$g(N,s) = \underbrace{2^N\sqrt{\frac{2}{\pi N}}}_{g(N,0)} e^{-\frac{U^2}{2m^2B^2N}}$$

Finding the entropy, the resulting function looks as follows:

$$\sigma(N,s) = \left(-\frac{U^2}{2m^2B^2N}\right) + \underbrace{\ln\left(2^N\sqrt{\frac{2}{\pi N}}\right)}_{\sigma_o}$$

Now finding the fundamental temperature, we differentiate with respect to $U$:

$$\frac{\partial\sigma(N,s)}{\partial U} = \left(\left(-\frac{U^2}{2m^2B^2N}\right) + \underbrace{\ln\left(2^N\sqrt{\frac{2}{\pi N}}\right)}_{\sigma_o}\right)_{N,s} = -\frac{U}{m^2B^2N}$$

Now assuming that $U$ is the average thermal energy, we get:

$$U = -2mB\langle s\rangle$$

We then return this to our function:

$$\frac{1}{\tau} = \frac{2\langle s\rangle}{mBN}$$
$$\tau = \frac{mBN}{2\langle s\rangle}$$

3. (a) From 1.55, we know:

$$g(N,n) = \frac{(N+n-1)!}{n!(N-1)!}$$

Thus, by definition of entropy, we know:

$$\sigma(N,n) = \ln\left(\frac{(N+n-1)!}{n!(N-1)!}\right) \to \ln((N+n-1)!) - \ln(n!) - \ln((N-1)!)$$

Replacing $N-1$ with $N$, we get:

$$\ln((N+n)!) - \ln(n!) - \ln(N!)$$

Through the Stirling approximation, we get:

$$(N+n)\ln(N+n) - (N+n) - n\ln(n) + n - N\ln(N) + N$$

This can be simplified to:

$$\boxed{\sigma(N,n) = (N+n)\ln(N+n) - n\ln(n) - N\ln(N)}$$

(b) We are given $U = n\hbar\omega$, which means $n = \frac{U}{\hbar\omega}$. Thus, we substitute to get:

$$\sigma\left(N, \frac{U}{\hbar\omega}\right) = \left(N + \frac{U}{\hbar\omega}\right)\ln\left(N + \frac{U}{\hbar\omega}\right) - \frac{U}{\hbar\omega}\ln\left(\frac{U}{\hbar\omega}\right) - N\ln(N)$$

By definition of the fundamental temperature, we get:

$$\frac{1}{\tau} = \frac{\partial}{\partial U}\left(\left(N + \frac{U}{\hbar\omega}\right)\ln\left(N + \frac{U}{\hbar\omega}\right) - \frac{U}{\hbar\omega}\ln\left(\frac{U}{\hbar\omega}\right) - N\ln(N)\right)$$
$$\frac{1}{\tau} = \left(\frac{1}{\hbar\omega}\right)\ln\left(N + \frac{U}{\hbar\omega}\right) - \frac{1}{\hbar\omega}\ln\left(\frac{U}{\hbar\omega}\right)$$
$$\frac{1}{\tau} = \frac{1}{\hbar\omega}\ln\left(\frac{N\hbar\omega}{U} + 1\right)$$

Rearranging for $U$, we get:

$$e^{\frac{\hbar\omega}{\tau}} - 1 = \frac{N\hbar\omega}{U}$$

And finally:

$$\boxed{U = \frac{N\hbar\omega}{e^{\frac{\hbar\omega}{\tau}} - 1}}$$

4. (a) Given that there is only one correct key out of the 44 possibilities per press, we know that the possibility of a single key being correct is:

$$\frac{1}{44} \approx .0227$$

Upon repeating this for a sequence of $10^5$ characters, this probability becomes:

$$\left(\frac{1}{44}\right)^{100{,}000} \approx 10^{-164{,}345}$$

(b) From the given values, we know the typing speed is 10 keys per second, and the age of the universe is $10^{18}$ seconds. Thus, we get:

$$10^{18}(10) = 10^{19}[\text{keys}]$$

Given that there are $10^{10}$ monkeys, we can multiply the numbers out to produce the total keys pressed:

$$\left(10^{19}\right)\left(10^{10}\right) = 10^{29}[\text{keys total}]$$

Then, we also know that, per one hamlet, there are $10^5$ characters. This yields:

$$10^{29}\left(\frac{1}{10^5}\right) = 10^{24}[\text{monkey-Hamlets}]$$

Finally, given are probability from above, we obtain:

$$\left(10^{24}\right)\left(10^{-164{,}345}\right) = 10^{-164{,}316}[\text{probability of Hamlet in given time}]$$

5. (a) From equation (17) we know:

$$g_1(N_1, \hat{s}_1 + \delta)g_2(N_2, \hat{s}_1 - \delta) = (g_1, g_2)_{max}e^{-\left(\frac{2\delta^2}{N_1} + \frac{2\delta^2}{N_2}\right)}$$

We can rearrange this formula to write:

$$\frac{g_1(N_1, \hat{s}_1 + \delta)g_2(N_2, \hat{s}_1 - \delta)}{(g_1, g_2)_{max}} = e^{-\left(\frac{2\delta^2}{N_1} + \frac{2\delta^2}{N_2}\right)}$$

Thus, we find that:

$$\frac{g_1 g_2}{(g_1, g_2)_{max}} = e^{-\left(\frac{4(10^{11})^2}{10^{22}}\right)} \Rightarrow$$
$$\frac{g_1 g_2}{(g_1, g_2)_{max}} = e^{-4} \approx .01832$$

(b)

(c)

6. From equation (17) we know:

$$g_1(N_1, \hat{s}_1 + \delta) g_2(N_2, \hat{s}_1 - \delta) = (g_1, g_2)_{max} e^{-\left(\frac{2\delta^2}{N_1} + \frac{2\delta^2}{N_2}\right)}$$

Since $N_1 = N_2$, we replace both with $N$:

$$g_1(N, \hat{s}_1 + \delta) g_2(N, \hat{s}_1 - \delta) = (g_1, g_2)_{max} e^{-\frac{4\delta^2}{N}}$$

The total number of states, given $x = \frac{\delta}{N}$ can be expressed as:

$$\text{states} = (g_1, g_2)_{max} N \int_{-\infty}^{\infty} e^{-4Nx^2}\, dx$$

For the case larger than or equal to $10^{-10}$:

$$\text{states} = (g_1, g_2)_{max} N \int_{10^{-10}}^{\infty} e^{-4Nx^2}\, dx$$

To rewrite this in an easily calculable form, we can take $a = 2\sqrt{N}x$:

$$da = 2\sqrt{N} dx$$
$$a = 2\sqrt{N}x = \frac{2\delta}{\sqrt{N}} \to \frac{\delta}{N} \geq 10^{-10} \to a \geq 20$$
$$\text{states} = (g_1, g_2)_{max} \frac{\sqrt{N}}{2} \int_{20}^{\infty} e^{-a^2}\, da$$

Then, using the complementary error function, we get:

$$\frac{\sqrt{N}}{2} \frac{e^{-400}}{40} (1 + e)$$

Finding a ratio, we obtain:

$$r = \frac{\frac{\sqrt{N}}{2}\frac{e^{-400}}{40}(1+e)}{\frac{\sqrt{\pi N}}{2}} = \frac{e^{-400}}{40\sqrt{\pi}}(1+e)$$

$$r = 2.7 \cdot 10^{-176}$$

Thus, it is quite improbable.