Homework 6

Michael Brodskiy
Professor: A. Stepanyants

November 4, 2023

2. (a) We can find the density of states $d\varepsilon$ as:

$$D(\varepsilon)\,d\varepsilon = \frac{V\varepsilon^2}{\pi^2\hbar^3c^3}\,d\varepsilon$$

We can then find:

$$N = \int_0^{\varepsilon_F} \frac{V\varepsilon^2}{\pi^2\hbar^3c^3}\,d\varepsilon$$
$$N = \frac{V\varepsilon_F^3}{3\pi^2\hbar^3c^3}$$
$$N(3\pi^2\hbar^3c^3) = V\varepsilon_F^3$$
$$\varepsilon_F^3 = \frac{N}{V}(3\pi^2\hbar^3c^3)$$
$$\varepsilon_F^3 = 3n(\pi^2\hbar^3c^3)$$
$$\varepsilon_F^3 = \frac{3n}{\pi}(\pi\hbar c)^3$$

And finally:

$$\boxed{\varepsilon_F = \left(\frac{3n}{\pi}\right)^{\frac{1}{3}}(\pi\hbar c)}$$

(b) The energy can be defined in a similar manner:

$$U = \int_0^{\varepsilon_F} \varepsilon D(\varepsilon)\,d\varepsilon$$

Which gives us:

$$U = \int_0^{\varepsilon_F} \frac{V\varepsilon^3}{\pi^2\hbar^3c^3}\,d\varepsilon$$

$$U = \frac{V\varepsilon_F^4}{4\pi^2\hbar^3c^3}$$

$$U = \left(\frac{V\varepsilon_F^3}{3\pi^2\hbar^3c^3}\right)\frac{3}{4}\varepsilon_F$$

And finally:

$$\boxed{U_o = \frac{3}{4}N\varepsilon_F}$$

3. (a) We know that the ground state energy may be expressed as:

$$U_o = \frac{3N\hbar^2}{10M}\left(\frac{3\pi^2 N}{V}\right)^{\frac{2}{3}}$$

We also know the relation:

$$P = -\frac{\partial U}{\partial V}$$

This gives us:

$$P = -\left(-\frac{2}{3}\frac{3\hbar^2(3\pi^2)^{\frac{2}{3}}}{10M}\left(\frac{N}{V}\right)^{\frac{5}{3}}\right)$$

$$P = \frac{\hbar^2(3\pi^2)^{\frac{2}{3}}}{5M}\left(\frac{N}{V}\right)^{\frac{5}{3}}$$

$$\boxed{P = \frac{(3\pi^2)^{\frac{2}{3}}}{5}\frac{\hbar^2}{M}\left(\frac{N}{V}\right)^{\frac{5}{3}}}$$

(b) We can define:

$$\delta\sigma(\tau) = \int \frac{1}{\tau}\,dU$$

We know the heat capacity of an electron gas as:

$$C_V = \frac{1}{2}\pi^2 N\left(\frac{\tau}{\tau_F}\right)$$

We know the heat capacity is equal to $\frac{\partial U}{\partial \tau}$, which allows us to write:

$$\sigma(\tau) = \int_0^\tau \frac{1}{\tau}C_v\,d\tau$$

$$\sigma(\tau) = \int_0^\tau \frac{1}{2}\pi^2 N\frac{1}{\tau_F}\,d\tau$$

$$\boxed{\sigma(\tau) = \frac{1}{2}\pi^2 N \frac{\tau}{\tau_F}}$$

Notice: $\sigma = C_V$, as is correct for a degenerate gas

5. (a) We can describe $n = \frac{N}{V}$ as:

$$n = \frac{\rho}{M} = \frac{.081}{3 \cdot 1.67 \cdot 10^{-24}}$$
$$n = 1.617 \cdot 10^{22} \left[\mathrm{cm}^{-3}\right]$$
$$n = 1.617 \cdot 10^{28} \left[\mathrm{m}^{-3}\right]$$

We can then use the formula:

$$\varepsilon_F = \frac{\hbar^2}{2M}\left(3\pi^2 n\right)\right)^{\frac{2}{3}}$$
$$\varepsilon_F = \frac{(1.055 \cdot 10^{-34})^2}{2 \cdot 3 \cdot 1.67 \cdot 10^{-27}}\left(3\pi^2(1.617 \cdot 10^{28})\right)^{\frac{2}{3}}$$
$$\varepsilon_F = 6.79 \cdot 10^{-23}[\mathrm{J}]$$
$$\boxed{\varepsilon_F = 4.24 \cdot 10^{-4}[\mathrm{eV}]}$$

Now, we can find v_F as:

$$\varepsilon_F = \frac{1}{2} M v_F^2$$
$$v_F = \sqrt{\frac{2\varepsilon_F}{M}}$$
$$v_F = \sqrt{\frac{2(6.79 \cdot 10^{-23})}{3 \cdot 1.67 \cdot 10^{-27}}}$$
$$\boxed{v_F = 164.638 \left[\frac{\mathrm{m}}{\mathrm{s}}\right]}$$

And, finally, T_F:

$$T_F = \frac{\tau_F}{k_B} = \frac{\varepsilon_F}{k_B}$$
$$T_F = \frac{6.79 \cdot 10^{-23}}{1.381 \cdot 10^{-23}}$$
$$T_F = \frac{6.79}{1.381}$$
$$\boxed{T_F = 4.92[\mathrm{K}]}$$

(b) We may begin by using:

$$C_V = \frac{\pi^2 N}{2}\frac{\tau}{\tau_F}$$

$$C_V = \frac{\pi^2}{2\tau_F}(N\tau)$$

$$C_V = \frac{2(\pi^2)}{(4.92)}(Nk_BT)$$

$$\boxed{C_V = 1.003Nk_BT}$$

The coefficient $1.003 < 2.89$, the experimental value.

6. (a) The energy may be defined as:

$$U = -G\int_0^R \frac{\rho^2(4\pi r^2)\left(\frac{4}{3}\pi r^3\right)}{r}\,dr$$

$$U = -\rho^2 G\int_0^R \frac{(4\pi)^2 r^5}{3r}\,dr$$

$$U = -\frac{16\pi^2\rho^2 G}{3}\int_0^R r^4\,dr$$

$$U = -\frac{16\pi^2\rho^2 G}{3}\frac{R^5}{5}$$

$$U = -\frac{16\pi^2\rho^2 G R^5}{15}$$

We know the density becomes $\rightarrow \dfrac{3M}{4\pi R^3}$, which gives:

$$\boxed{U = -\frac{3M^2G}{5R}}$$

Thus, we can see this is on the order of $\dfrac{GM^2}{R}$

(b) The Fermi energy may be defined as:

$$\varepsilon_F = \frac{\hbar^2}{2M}(3\pi^2 n)^{\frac{2}{3}}$$

And the volume is:

$$V = \frac{4}{3}\pi R^3$$

We can write the total energy as:

$$NK_e = N\varepsilon_F$$

$$K_{tot} = N\frac{\hbar^2}{2m}\left(3\pi^2\frac{N}{V}\right)^{\frac{2}{3}}$$

$$K_{tot} = \frac{\hbar^2}{2m}\left(\frac{9\pi^2}{4\pi R^3}\right)^{\frac{2}{3}} N^{\frac{5}{3}}$$

$$K_{tot} = \frac{\hbar^2}{2mR^2}\left(\frac{9}{4}\pi\right)^{\frac{2}{3}} N^{\frac{5}{3}}$$

$$K_{tot} = \frac{\hbar^2}{mR^2}\left(\frac{81}{128}\pi^2\right)^{\frac{2}{3}} N^{\frac{5}{3}}$$

This can then be written:

$$K_{tot} = c\frac{\hbar^2}{mR^2}N^{\frac{5}{3}}$$

with $c = \left(\frac{81}{128}\pi^2\right)^{\frac{2}{3}}$. Now, since $M_H << M$, we can write $N = \frac{M}{M_H}$, which gives us:

$$\boxed{K_{tot} = c\frac{\hbar^2 M^{\frac{5}{3}}}{mM_H^{\frac{5}{3}}R^2}}$$

Thus, we can see the momentum is on the order of $\frac{\hbar^2 M^{\frac{5}{3}}}{mM_H^{\frac{5}{3}}R^2}$

(c) From the two results above, we may write:

$$\frac{\hbar^2 M^{\frac{5}{3}}}{mM_H^{\frac{5}{3}}R^2} \approx \frac{GM^2}{R}$$

Rearranging, we get:

$$\frac{\hbar^2}{GmM_H^{\frac{5}{3}}} \approx M^{\frac{1}{3}}R$$

Now we can calculate:

$$M^{\frac{1}{3}}R \approx \frac{\hbar^2}{GmM_H^{\frac{5}{3}}}$$

We will have two different values for $\hbar$, one in electron-volts, and one in joules to verify the correct dimensionality with respect to G and m:

$$M^{\frac{1}{3}}R \approx \frac{(6.582 \cdot 10^{-22})(1.055 \cdot 10^{-34})c^2}{(6.67 \cdot 10^{-11})[.511](1.67 \cdot 10^{-24})^{\frac{5}{3}}}$$

$$M^{\frac{1}{3}}R \approx 7.8 \cdot 10^{11}$$

Note: this result does not have the correct units. Correcting this, we find:

$$7.8 \cdot 10^{11}(10^8) = 7.8 \cdot 10^{19}$$

Then multiplying by the constants/coefficients removed in (a) and (b), we get:

$$7.8 \cdot 10^{19} \cdot \left(\frac{81\pi^2}{128}\right)^{\frac{2}{3}} \frac{5}{3} = 4.4 \cdot 10^{20}$$

Thus, we can see that the value is on the order of

$$\boxed{M^{\frac{1}{3}}R \to 10^{20}[\mathrm{g}^{\frac{1}{3}}\mathrm{cm}]}$$

(d) We know:

$$\rho = \frac{3M}{4\pi R^3}$$

$$\rho = \frac{3M}{4\pi\left(\frac{\cdot 10^{20}}{M}\right)}$$

$$\rho = \frac{3M^2}{4\pi\,(10^{60})}$$

$$\rho = \frac{3(2 \cdot 10^{66})}{4\pi\,(10^{60})}$$

$$\boxed{\rho = 4.77465 \cdot 10^5 \left[\frac{\mathrm{g}}{\mathrm{cm}^3}\right]}$$

(e) The only difference between a proton and electron gas would be the energy per c^2, which, for neutrons, is on the order of 10^9 (giga) instead of 10^6 (mega). Thus, $M^{\frac{1}{3}}R$ would be divided by a factor of 10^3:

$$10^{20}(10^{-3}) = 10^{17}[\mathrm{g}^{\frac{1}{3}}\mathrm{cm}]$$

We can then write:

$$R = \frac{10^{17}}{(2 \cdot 10^{33})^{\frac{1}{3}}} = 793701[\mathrm{cm}]$$

$$\boxed{R = 7.937[\mathrm{km}]}$$

7. Since we want to find the critical temperature at which $N_e < N$, we can define:

$$N = N_e = \frac{2.404 V \tau^3}{\pi^2 \hbar^3 c^3}$$

Which can be rearranged to give:

$$\tau = \left(\frac{N \pi^2 \hbar^3 c^3}{2.404 V} \right)^{\frac{1}{3}}$$

We can convert:

$$10^{20}[\mathrm{cm}^{-3}] \rightarrow 10^{26}[\mathrm{m}^{-3}]$$

We can then calculate:

$$k_B T = \left(\frac{\pi^2 (1.055 \cdot 10^{-34})^3 (3 \cdot 10^8)^3}{2.404} 10^{26} \right)^{\frac{1}{3}}$$

$$k_B T = 2.35 \cdot 10^{-17}$$

$$\boxed{T = 1.7 \cdot 10^6 [\mathrm{K}]}$$

Below this temperature, $N_e < N$