Homework 1

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1. (a) We are given the following:

$$g(U) = CU^{\frac{3}{2}N}$$

We know, by definition, that $\sigma(U) = \ln(g(U))$. Thus, we may write:

$$\sigma(U) = \ln\left(CU^{\frac{3}{2}N}\right)$$

By properties of logarithms, we may expand this to:

$$\sigma(U) = \ln(C) + \frac{3N}{2}\ln(U)$$

Furthermore, by definition of the fundamental temperature, we know that $\dfrac{\partial\sigma(U)}{\partial U} = \dfrac{1}{\tau}$. As such, we obtain:

$$\frac{\partial\sigma(U)}{\partial U} = \frac{3N}{2U}$$

Substituting and rearranging, we get:

$$\frac{1}{\tau} = \frac{3N}{2U}$$

$$\boxed{U = \frac{3N}{2}\tau}$$

(b) If we return to the step before converting $\dfrac{\partial\sigma(U)}{\partial U}$ to $\frac{1}{\tau}$, we can differentiate once more:

$$\frac{\partial\sigma(U)}{\partial U} = \frac{3N}{2U}$$

$$\left(\frac{\partial^2 \sigma(U)}{\partial U^2}\right)_N = \frac{3N}{2} \frac{\partial}{\partial U}\left(\frac{1}{U}\right)$$

$$\boxed{\left(\frac{\partial^2 \sigma(U)}{\partial U^2}\right)_N = -\frac{3N}{U^2}}$$

Thus, we see this results in a function with a negative coefficient. Furthermore, because we know the quantity of particles can not be negative, and the square of a number can not be negative, the second order partial derivative must always be negative.

2.

3. (a) From 1.55, we know:

$$g(N,n) = \frac{(N+n-1)!}{n!(N-1)!}$$

Thus, by definition of entropy, we know:

$$\sigma(N,n) = \ln\left(\frac{(N+n-1)!}{n!(N-1)!}\right) \to \ln((N+n-1)!) - \ln(n!) - \ln((N-1)!)$$

Replacing $N-1$ with N, we get:

$$\ln((N+n)!) - \ln(n!) - \ln(N!)$$

Through the Stirling approximation, we get:

$$(N+n)\ln(N+n) - (N+n) - n\ln(n) + n - N\ln(N) + N$$

This can be simplified to:

$$\boxed{\sigma(N,n) = (N+n)\ln(N+n) - n\ln(n) - N\ln(N)}$$

(b) We are given $U = n\hbar\omega$, which means $n = \frac{U}{\hbar\omega}$. Thus, we substitute to get:

$$\sigma\left(N, \frac{U}{\hbar\omega}\right) = \left(N + \frac{U}{\hbar\omega}\right)\ln\left(N + \frac{U}{\hbar\omega}\right) - \frac{U}{\hbar\omega}\ln\left(\frac{U}{\hbar\omega}\right) - N\ln(N)$$

By definition of the fundamental temperature, we get:

$$\frac{1}{\tau} = \frac{\partial}{\partial U}\left(\left(N + \frac{U}{\hbar\omega}\right)\ln\left(N + \frac{U}{\hbar\omega}\right) - \frac{U}{\hbar\omega}\ln\left(\frac{U}{\hbar\omega}\right) - N\ln(N)\right)$$

$$\frac{1}{\tau} = \left(\frac{1}{\hbar\omega}\right)\ln\left(N + \frac{U}{\hbar\omega}\right) - \frac{1}{\hbar\omega}\ln\left(\frac{U}{\hbar\omega}\right)$$

$$\frac{1}{\tau} = \frac{1}{\hbar\omega}\ln\left(\frac{N\hbar\omega}{U} + 1\right)$$

Rearranging for U, we get:

$$e^{\frac{\hbar\omega}{\tau}} - 1 = \frac{N\hbar\omega}{U}$$

And finally:

$$\boxed{U = \frac{N\hbar\omega}{e^{\frac{\hbar\omega}{\tau}} - 1}}$$

4. (a) Given that there is only one correct key out of the 44 possibilities per press, we know that the possibility of a single key being correct is:

$$\frac{1}{44} \approx .0227$$

Upon repeating this for a sequence of 10^5 characters, this probability becomes:

$$\left(\frac{1}{44}\right)^{100,000} \approx 10^{-164,345}$$

(b) From the given values, we know the typing speed is 10 keys per second, and the age of the universe is 10^{18} seconds. Thus, we get:

$$10^{18}(10) = 10^{19}[\text{keys}]$$

Given that there are 10^{10} monkeys, we can multiply the numbers out to produce the total keys pressed:

$$\left(10^{19}\right)\left(10^{10}\right) = 10^{29}[\text{keys total}]$$

Then, we also know that, per one hamlet, there are 10^5 characters. This yields:

$$10^{29}\left(\frac{1}{10^5}\right) = 10^{24}[\text{monkey-Hamlets}]$$

Finally, given are probability from above, we obtain:

$$\left(10^{24}\right)\left(10^{-164,345}\right) = 10^{-164,316}[\text{probability of Hamlet in given time}]$$

5. (a)

 (b)

 (c)

6.