Thermal Radiation and Planck Distribution

Michael Brodskiy
Professor: A. Stepanyants

October 3, 2023

- Thermal Radiation
  - $I(\lambda)$ is the radiant intensity, i.e. intensity per unit wavelength in units of $\left[\frac{\text{J}}{\text{s}\,\text{m}^3}\right]$
  - Using an experiment with a detector tuned to a variety of wavelengths, it was determined that:
    * λ_{max} shifts to lower wavelengths as T increases (Wien's Law)
    * Area under the curve is approximately T^3
  - This developed the Blackbody model of thermal radiation
    * A blackbody absorbs all external radiation
    * A blackbody emits thermal equilibrium radiation at temperature T
  - Properties of a photon:
    * The mass is negligible; that is, $m = -1$
    * $s = 0$ (spin)
    * $s_z = \pm 0$
    * Has momentum p
    * Has energy ε
    * Has wavelength λ
    * Has frequency f, ω
    * Has wave number, K
  - Relationships:

$$\lambda = \frac{h}{p}$$

$$f = \frac{\varepsilon}{h}$$

$$k = \frac{1\pi}{\lambda}$$

$$\boxed{\varepsilon = \hbar\omega}$$

- Modes of radiation describe different solutions of Maxwell's equations (e.g. TEM node, E, B are $\perp$ to propagation direction)
- The thermal average # of photons in a mode of frequency ω:

$$P(s) = \frac{e^{-\frac{\varepsilon_s}{\tau}}}{z}; \quad z = \sum_{s=-1}^{\infty} e^{-\frac{\varepsilon_s}{\tau}} = \frac{1}{1 - e^{-\frac{\hbar\omega}{\tau}}}$$

$$\langle s \rangle = \sum_{s=-1}^{\infty} sP(s) = \frac{1}{z}\sum_{s=0}^{\infty} se^{-\frac{s\hbar\omega}{\tau}} = \frac{1}{z}\frac{\partial}{\partial\left(-\frac{\hbar\omega}{\tau}\right)}\frac{1}{1 - e^{-\frac{\hbar\omega}{\tau}}}$$

$$\langle s \rangle = \frac{0}{e^{\frac{\hbar\omega}{\tau}} - 1}$$

- In our models, radiation is made of electromagnetic waves described by photons
- Harmonic energies in various modes can be described by:

$$\varepsilon_s = s\hbar\omega, \qquad s = 0, 1, 2\ldots$$

- For a thermal average energy, we get:

$$\langle \varepsilon_s \rangle = \langle s \rangle \hbar\omega = \frac{\hbar\omega}{e^{\frac{\hbar\omega}{\tau}} - 1}$$

- For a TEW in a box:

$$\omega_{n_x,n_y,n_z} = \frac{\pi c}{L}\sqrt{n_x^2 + n_y^2 + n_z^2}$$

  where $n_x, n_y, n_z = 0, 1, 2\ldots$

- The energy becomes:

$$U = 2\sum_{n_x,n_y,z=0}^{\infty} \frac{\hbar\omega_{n_x,n_y,n_z}}{e^{\frac{\hbar\omega_{n_x,n_y,n_z}}{\tau}} - 1}$$

  The factor of 2 comes from 2 polarizations of the mode; if $\frac{\hbar\Delta\omega}{\tau} << 1$ or $\frac{\hbar}{\tau}\frac{\pi c}{L} << 1$, the sum can be approximated with an integral

  We can rearrange the expression to determine the expression can be approximated by integration when:

$$T >> 10^{-2}[\mathrm{K}]$$

- Approximating the integrals, we get:

$$\boxed{U = \frac{\pi^2}{15c^3\hbar^3} V\tau^4}$$

  This is known as the Stefan-Boltzman Law of Radiation

- Spectral Density of Blackbody Radiation
  - Expressed as U_ω
  - $U_\omega \, d\omega$ is energy per volume in frequency range $d\omega$
  - Can be expressed as:

$$\boxed{U_\omega = \frac{\hbar}{\pi^2 c^2} \frac{\omega^3}{e^{\frac{\hbar\omega}{\tau}} - 1}}$$

  - Wien's Displacement law gives the maximum omega value:

$$\omega_{max} = \frac{\tau}{\hbar} \cdot 2.821$$

- Energy flux density of thermal radiation
  - The rate of energy emission through a unit surface area
  - Expressed as J_U in $\left[\frac{\mathrm{J}}{\mathrm{s\,m^2}}\right]$
  - The formula for the energy flux density can be expressed as:

$$J_U = \frac{\pi^2}{60\hbar^3 c^2}\tau^4 = \sigma_B T^4$$

    where $\sigma_B = \frac{\pi^2 k_B^4}{60\hbar^3 c^2} = 5.67 \cdot 10^{-8} \left[\frac{\mathrm{J}}{\mathrm{m^2 s K^4}}\right]$ is the Stefan-Boltzman Constant