Homework 1

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1. (a) We are given the following:

$$g(U) = CU^{\frac{3}{2}N}$$

We know, by definition, that $\sigma(U) = \ln(g(U))$. Thus, we may write:

$$\sigma(U) = \ln\left(CU^{\frac{3}{2}N}\right)$$

By properties of logarithms, we may expand this to:

$$\sigma(U) = \ln(C) + \frac{3N}{2}\ln(U)$$

Furthermore, by definition of the fundamental temperature, we know that $\dfrac{\partial\sigma(U)}{\partial U} = \dfrac{1}{\tau}$. As such, we obtain:

$$\frac{\partial\sigma(U)}{\partial U} = \frac{3N}{2U}$$

Substituting and rearranging, we get:

$$\frac{1}{\tau} = \frac{3N}{2U}$$

$$\boxed{U = \frac{3N}{2}\tau}$$

(b) If we return to the step before converting $\dfrac{\partial\sigma(U)}{\partial U}$ to $\frac{1}{\tau}$, we can differentiate once more:

$$\frac{\partial\sigma(U)}{\partial U} = \frac{3N}{2U}$$

$$\left(\frac{\partial^2\sigma(U)}{\partial U^2}\right)_N = \frac{3N}{2}\frac{\partial}{\partial U}\left(\frac{1}{U}\right)$$

$$\boxed{\left(\frac{\partial^2\sigma(U)}{\partial U^2}\right)_N = -\frac{3N}{U^2}}$$

Thus, we see this results in a function with a negative coefficient. Furthermore, because we know the quantity of particles can not be negative, and the square of a number can not be negative, the second order partial derivative must always be negative.

2. We know, by definition:

$$U = -(2s)mB$$

Rearranging, we see that:

$$s = -\frac{U}{2mB}$$

We also know that:

$$g(N,s) = 2^N\sqrt{\frac{2}{\pi N}}_{g(N,0)} e^{-\frac{2s^2}{N}}$$

Substituting the value of s above, we get:

$$g(N,s) = \underbrace{2^N\sqrt{\frac{2}{\pi N}}}_{g(N,0)} e^{-\frac{U^2}{2m^2B^2N}}$$

Finding the entropy, the resulting function looks as follows:

$$\sigma(N,s) = \left(-\frac{U^2}{2m^2B^2N}\right) + \underbrace{\ln\left(2^N\sqrt{\frac{2}{\pi N}}\right)}_{\sigma_o}$$

Now finding the fundamental temperature, we differentiate with respect to U:

$$\frac{\partial\sigma(N,s)}{\partial U} = \left(\left(-\frac{U^2}{2m^2B^2N}\right) + \underbrace{\ln\left(2^N\sqrt{\frac{2}{\pi N}}\right)}_{\sigma_o}\right)_{N,s} = -\frac{U}{m^2B^2N}$$

Now assuming that U is the average thermal energy, we get:

$$U = -2mB\langle s \rangle$$

We then return this to our function:

$$\frac{1}{\tau} = \frac{2\langle s \rangle}{mBN}$$
$$\tau = \frac{mBN}{2\langle s \rangle}$$

3. (a) From 1.55, we know:

$$g(N, n) = \frac{(N+n-1)!}{n!(N-1)!}$$

Thus, by definition of entropy, we know:

$$\sigma(N, n) = \ln\left(\frac{(N+n-1)!}{n!(N-1)!}\right) \to \ln((N+n-1)!) - \ln(n!) - \ln((N-1)!)$$

Replacing $N-1$ with N, we get:

$$\ln((N+n)!) - \ln(n!) - \ln(N!)$$

Through the Stirling approximation, we get:

$$(N+n)\ln(N+n) - (N+n) - n\ln(n) + n - N\ln(N) + N$$

This can be simplified to:

$$\boxed{\sigma(N, n) = (N+n)\ln(N+n) - n\ln(n) - N\ln(N)}$$

(b) We are given $U = n\hbar\omega$, which means $n = \frac{U}{\hbar\omega}$. Thus, we substitute to get:

$$\sigma\left(N, \frac{U}{\hbar\omega}\right) = \left(N + \frac{U}{\hbar\omega}\right)\ln\left(N + \frac{U}{\hbar\omega}\right) - \frac{U}{\hbar\omega}\ln\left(\frac{U}{\hbar\omega}\right) - N\ln(N)$$

By definition of the fundamental temperature, we get:

$$\frac{1}{\tau} = \frac{\partial}{\partial U}\left(\left(N + \frac{U}{\hbar\omega}\right)\ln\left(N + \frac{U}{\hbar\omega}\right) - \frac{U}{\hbar\omega}\ln\left(\frac{U}{\hbar\omega}\right) - N\ln(N)\right)$$
$$\frac{1}{\tau} = \left(\frac{1}{\hbar\omega}\right)\ln\left(N + \frac{U}{\hbar\omega}\right) - \frac{1}{\hbar\omega}\ln\left(\frac{U}{\hbar\omega}\right)$$
$$\frac{1}{\tau} = \frac{1}{\hbar\omega}\ln\left(\frac{N\hbar\omega}{U} + 1\right)$$

Rearranging for U, we get:

$$e^{\frac{\hbar\omega}{\tau}} - 1 = \frac{N\hbar\omega}{U}$$

And finally:

$$\boxed{U = \frac{N\hbar\omega}{e^{\frac{\hbar\omega}{\tau}} - 1}}$$

4. (a) Given that there is only one correct key out of the 44 possibilities per press, we know that the possibility of a single key being correct is:

$$\frac{1}{44} \approx .0227$$

Upon repeating this for a sequence of 10^5 characters, this probability becomes:

$$\boxed{\left(\frac{1}{44}\right)^{100{,}000} \approx 10^{-164{,}345}}$$

(b) From the given values, we know the typing speed is 10 keys per second, and the age of the universe is 10^{18} seconds. Thus, we get:

$$10^{18}(10) = 10^{19}[\text{keys}]$$

Given that there are 10^{10} monkeys, we can multiply the numbers out to produce the total keys pressed:

$$\left(10^{19}\right)\left(10^{10}\right) = 10^{29}[\text{keys total}]$$

Then, we also know that, per one hamlet, there are 10^5 characters. This yields:

$$10^{29}\left(\frac{1}{10^5}\right) = 10^{24}[\text{monkey-Hamlets}]$$

Finally, given are probability from above, we obtain:

$$\boxed{\left(10^{24}\right)\left(10^{-164{,}345}\right) = 10^{-164{,}316}[\text{probability of Hamlet in given time}]}$$

5. (a) From equation (17) we know:

$$g_1(N_1, \hat{s}_1 + \delta)g_2(N_2, \hat{s}_1 - \delta) = (g_1, g_2)_{max}e^{-\left(\frac{2\delta^2}{N_1} + \frac{2\delta^2}{N_2}\right)}$$

We can rearrange this formula to write:

$$\frac{g_1(N_1, \hat{s}_1 + \delta)g_2(N_2, \hat{s}_1 - \delta)}{(g_1, g_2)_{max}} = e^{-\left(\frac{2\delta^2}{N_1} + \frac{2\delta^2}{N_2}\right)}$$

Thus, we find that:

$$\frac{g_1 g_2}{(g_1, g_2)_{max}} = e^{-\left(\frac{4(10^{11})^2}{10^{22}}\right)} \Rightarrow$$

$$\boxed{\frac{g_1 g_2}{(g_1, g_2)_{max}} = e^{-4} \approx .01832}$$

(b) From the problem above, we know:

$$g_1(N_1, \hat{s}_1 + \delta) g_2(N_2, \hat{s}_1 - \delta) = (g_1, g_2)_{max} e^{-\left(\frac{2\delta^2}{N_1} + \frac{2\delta^2}{N_2}\right)}$$

This also gives us:

$$\sum_{s_1} g_1(N_1, s_1) g_2(N_2, s - s_1) = (g_1, g_2)_{max} \sum_{\delta} e^{-\left(\frac{2\delta^2}{N_1} + \frac{2\delta^2}{N_2}\right)}$$

This means that the multiplication factor, $\mathbb{F} = \sum_{\delta} e^{-\left(\frac{2\delta^2}{N_1} + \frac{2\delta^2}{N_2}\right)}$

Using $N = N_1 = N_2$, and converting to an integral expression, we obtain:

$$\int_{-\infty}^{\infty} e^{-\frac{4\delta^2}{N}} \, d\delta$$

As this is a Gaussian integral, we can approximate it to:

$$\boxed{\mathbb{F} = \sqrt{\frac{\pi N}{4}} = 8.862 \cdot 10^{10}}$$

(c) The definition of such error may be written as:

$$\frac{\sigma_f - \sigma_i}{\sigma_f}$$

where σ_f is factorless and σ_i includes the factor. This gives us two expressions for the entropy, defined as:

$$\begin{cases} \sigma_f &= \ln\left((g_1, g_2)_{max} \mathbb{F}\right) \\ \sigma_i &= \ln\left(g_1, g_2\right)_{max} \end{cases}$$

By the properties of logarithms, combined with the above expression for error, we get:

$$\frac{\ln(\mathbb{F})}{\ln\left((g_1, g_2)_{max} \mathbb{F}\right)}$$

We know that $(g_1, g_2)_{max}$ may be expressed as follows:

$$(g_1, g_2)_{max} = g_1(N,0)g_2(N,0)e^{-\frac{s^2}{N}}$$

This manipulates the logarithm in the denominator into:

$$\frac{\ln(\mathbb{F})}{-\frac{s^2}{N} + \ln\left(g_1(N,0)g_2(N,0)\mathbb{F}\right)}$$

Finally, we also know that:

$$g_1(N,0) = g_2(N,0) = \sqrt{\frac{2}{\pi N}}2^N$$

Using the information that $N = 10^{22}$ and $s = 10^{20}$, we can finally solve:

$$\boxed{\text{error} = \frac{\ln(8.862 \cdot 10^{10})}{\ln\Big((8.862 \cdot 10^{10})(\frac{2}{\pi(10^{22})}2^{10^{23}})\Big) - 10^{18}} = 1.818 \cdot 10^{-21}}$$

Thus, the factor doesn't carry much influence.

6. From equation (17) we know:

$$g_1(N_1, \hat{s}_1 + \delta)g_2(N_2, \hat{s}_1 - \delta) = (g_1, g_2)_{max}e^{-\left(\frac{2\delta^2}{N_1} + \frac{2\delta^2}{N_2}\right)}$$

Since $N_1 = N_2$, we replace both with N:

$$g_1(N, \hat{s}_1 + \delta)g_2(N, \hat{s}_1 - \delta) = (g_1, g_2)_{max}e^{-\frac{4\delta^2}{N}}$$

The total number of states, given $x = \frac{\delta}{N}$ can be expressed as:

$$\text{states} = (g_1, g_2)_{max}N\int_{-\infty}^{\infty} e^{-4Nx^2}\,dx$$

For the case larger than or equal to 10^{-10}:

$$\text{states} = (g_1, g_2)_{max}N\int_{10^{-10}}^{\infty} e^{-4Nx^2}\,dx$$

To rewrite this in an easily calculable form, we can take $a = 2\sqrt{N}x$:

$$da = 2\sqrt{N}dx$$
$$a = 2\sqrt{N}x = \frac{2\delta}{\sqrt{N}} \to \frac{\delta}{N} \geq 10^{-10} \to a \geq 20$$
$$\text{states} = (g_1, g_2)_{max}\frac{\sqrt{N}}{2}\int_{20}^{\infty} e^{-a^2}\,da$$

Then, using the complementary error function, we get:

$$\frac{\sqrt{N}}{2}\frac{e^{-400}}{40}(1+e)$$

Finding a ratio, we obtain:

$$r = \frac{\frac{\sqrt{N}}{2}\frac{e^{-400}}{40}(1+e)}{\frac{\sqrt{\pi N}}{2}} = \frac{e^{-400}}{40\sqrt{\pi}}(1+e)$$

$$\boxed{r = 2.7 \cdot 10^{-176}}$$

Thus, it is quite improbable.