Homework 2

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1. (a) First, we know that:

$$z(\varepsilon) = e^{-\frac{\varepsilon}{\tau}}$$

Plugging in the respective values of $\varepsilon = 0$ and $\varepsilon = \varepsilon_o$:

$$z = z(0) + z(\varepsilon_o) = 1 + e^{-\frac{\varepsilon_o}{\tau}}$$

Free energy can also be expressed in terms of z:

$$\boxed{F = -\tau \ln(z) \to -\tau \ln\left(1 + e^{\frac{-\varepsilon_o}{\tau}}\right)}$$

(b) We know the energy may be expressed as:

$$U = -\tau^2 \frac{\partial}{\partial \tau}\left(\frac{F}{\tau}\right)$$
$$= \tau^2 \frac{\partial}{\partial \tau}\left(\ln\left(1 + e^{-\frac{\varepsilon_o}{\tau}}\right)\right)$$
$$= \tau^2 \left(\frac{ae^{-\frac{\varepsilon_o}{\tau}}}{\tau^2\left(1 + e^{-\frac{\varepsilon_o}{\tau}}\right)}\right)$$
$$= \frac{\varepsilon_o e^{-\frac{\varepsilon_o}{\tau}}}{\left(1 + e^{-\frac{\varepsilon_o}{\tau}}\right)}$$

Furthermore, entropy may be expressed as:

$$\sigma = -\frac{\partial F}{\partial \tau}$$
$$= \frac{\partial}{\partial \tau}\left(\tau \ln\left(1 + e^{-\frac{\varepsilon_o}{\tau}}\right)\right)$$
$$= \ln\left(1 + e^{-\frac{\varepsilon_o}{\tau}}\right) + \frac{\varepsilon_o e^{-\frac{\varepsilon_o}{\tau}}}{\tau\left(1 + e^{-\frac{\varepsilon_o}{\tau}}\right)}$$

2. (a)

   (b)

   (c)

3. (a) We want to find the free energy over all values of s. Thus, we set up a sum:

$$z = \sum_{0}^{\infty} e^{-\frac{s\hbar\omega}{\tau}}$$

   We can find by reversing the expansion:

$$\sum_{0}^{\infty} e^{-\frac{s\hbar\omega}{\tau}} = \frac{1}{1 - e^{-\frac{\hbar\omega}{\tau}}}$$

   Using the formula for free energy, we obtain:

$$\boxed{F = -\tau \ln(z) = \tau \ln\left(1 - e^{-\frac{\hbar\omega}{\tau}}\right)}$$

   (b) We know the following:

$$\sigma = -\frac{\partial F}{\partial \tau}$$

   Taking the partial derivative with respect to τ, we find:

$$-\frac{\partial F}{\partial \tau} = -\frac{\partial}{\partial \tau}\left(\tau \ln\left(1 - e^{-\frac{\hbar\omega}{\tau}}\right)\right) \Rightarrow$$

$$-\left(\ln\left(1 - e^{-\frac{\hbar\omega}{\tau}}\right) + \tau \frac{\partial}{\partial \tau}\left[\ln\left(1 - e^{-\frac{\hbar\omega}{\tau}}\right)\right]\right) = -\ln\left(1 - e^{-\frac{\hbar\omega}{\tau}}\right) - \tau \left[-\frac{\hbar\omega e^{-\frac{\hbar\omega}{\tau}}}{\tau^2\left(1 - e^{-\frac{\hbar\omega}{\tau}}\right)}\right] \Rightarrow$$

$$\boxed{\sigma = \frac{\hbar\omega}{\tau\left(e^{\frac{\hbar\omega}{\tau}} + 1\right)} - \ln\left(1 - e^{-\frac{\hbar\omega}{\tau}}\right)}$$

4.

5.

6. From quantum mechanics, we know the energy of such a particle, with a single n (because it is unidimensional), may be expressed as:

$$\varepsilon = \frac{\hbar^2 \pi^2 n^2}{2mL^2}$$

   Thus, we know:

$$z = \sum_{n=1}^{\infty} e^{-\frac{\varepsilon}{\tau}}$$

If we define some value $x = \dfrac{\hbar\pi}{L\sqrt{2m\tau}}$, we can write this as a Gaussian integral:

$$z = \int_0^{\infty} e^{-x^2 n^2}\, dn = \frac{\sqrt{\pi}}{2x}$$

Using the formula for free energy, we can find:

$$-\tau \ln(z) \to N\tau \ln\left(\frac{2x}{\sqrt{\pi}}\right) = N\tau \ln\left(\frac{2\hbar\pi}{L\sqrt{2m\tau\pi}}\right)$$

To simplify our calculations, we can take out the square root:

$$N\tau \ln\left(\frac{2\hbar\pi}{L\sqrt{2m\tau\pi}}\right) = \frac{1}{2} N\tau \ln\left(\frac{2\hbar^2\pi}{L^2 m\tau}\right)$$

Now we take the partial derivative with respect to τ to find the entropy, using $y = \dfrac{2\hbar^2\pi}{L^2 m}$ to simplify:

$$\sigma = -\frac{\partial}{\partial\tau}\left(\frac{1}{2} N\tau \ln\left(\frac{y}{\tau}\right)\right)$$

$$-\frac{N}{2}\ln\left(\frac{y}{\tau}\right) - \frac{N\tau}{2}\frac{\partial}{\partial\tau}\left[\ln\left(\frac{y}{\tau}\right)\right] = -\frac{N}{2}\ln\left(\frac{y}{\tau}\right) - \frac{N\tau^2}{2y}\frac{\partial}{\partial\tau}\left[\frac{y}{\tau}\right]$$

$$-\frac{N}{2}\ln\left(\frac{y}{\tau}\right) - \frac{N\tau^2}{2y}\left(-\frac{y}{\tau^2}\right) = \frac{N}{2}\left(1 - \ln\left(\frac{y}{\tau}\right)\right)$$

$$\boxed{\sigma = \frac{N}{2}\left(1 - \ln\left(\frac{2\hbar^2\pi}{L^2 m\tau}\right)\right)}$$