States of a Model System

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- Consider a system of N quantum particles in a stationary quantum state (U, V, $N\dots$ are independent of time)
- Multiplicity of degeneracy of an energy level ε_n is the number of quantum states, g_n, corresponding to ε_n
- Hydrogen Atom:
  - One proton, one electron
  - $\varepsilon_n = -\dfrac{13.6[\text{eV}]}{n^2}$
  - $\psi_{n,l,m,s} = R_{n,l}(r)\Theta_{l,m}(\theta)\Phi_m(\phi)(s)$
    - n, l, m, s are quantum numbers
    - n is called the principle quantum number
    - l is the angular momentum quantum number
      - $0 \leq l \leq n-1$
    - m is the magnetic quantum number
      - $-l \leq m \leq l$
    - s is the spin-magnetic quantum number
      - $s = \pm\frac{1}{2}$
    - $g_n = 2n^2$
- Quantum Particle in a Box ($L \times L \times L$)
  - We find $\varepsilon_{n_x,n_y,n_z} = \frac{\hbar^2\pi^2}{2mL^2}(n_x^2 + n_y^2 + n_z^2)$
  - $\psi_{n_x,n_y,n_z}(x, y, z) = \dots$
  - $1 \leq n_x, n_y, n_z \leq \infty$

n_x	n_y	n_z	$\varepsilon_{n_x,n_y,n_z}/A$
1	1	1	3
2	1	1	6
1	2	1	6
1	1	2	6
2	2	1	9
2	1	2	9
1	2	2	9

- We can see that the "6" energy level is degenerate, with a multiplicity of 3, just like "9"

- Binary Model System
  - Energy of the system, $\varepsilon = -MB$
    - M is total magnetic moment: $M =$ (spins up - spins down)$m \to M = (N_\uparrow - N_\downarrow)\, m$
    - $(N_\uparrow - N_\downarrow) = 2s$ — spin excess
    - Thus, $\varepsilon = -2mBs$, meaning it is dependent on spin excess
    - $N = 3$ example:
      - $\uparrow\uparrow\uparrow$ — $2s = N_\uparrow - N_\downarrow = 3 \Rightarrow g = 1$
      - $\uparrow\uparrow\downarrow$, $\uparrow\downarrow\uparrow$, $\downarrow\uparrow\uparrow$ — $2s = N_\uparrow - N_\downarrow = 1 \Rightarrow g = 3$
      - $\uparrow\downarrow\downarrow$, $\downarrow\downarrow\uparrow$, $\downarrow\uparrow\downarrow$ — $2s = N_\uparrow - N_\downarrow = -1 \Rightarrow g = 3$
      - $\downarrow\downarrow\downarrow$ — $2s = N_\uparrow - N_\downarrow = -3 \Rightarrow g = 1$
    - In general, there are $N + 1$ values of $2s$ (or M or ε) and 2^N states of the system in total $\Rightarrow$ same energy levels have very high multiplicity
  - Calculation of $g(N, s)$

$$\begin{cases} N_\uparrow - N_\downarrow = 2s \\ N_\uparrow + N_\downarrow = N \end{cases} \qquad \begin{cases} N_\uparrow = \frac{N}{2} + s \\ N_\downarrow = \frac{N}{2} - s \end{cases}$$

  - $g(N, s) = \dfrac{N!}{N_\uparrow! N_\downarrow!}$
  - Drawing from combinatorics below, an approximation of $g(N, s)$ for $N >> 1$ and $s << N$, we can use the Stirling formula:

$$N! \approx \sqrt{2\pi N}\left(\frac{N}{e}\right)^N$$

  - $\ln(N!) \approx N\ln(N) - N + \frac{1}{2}\ln(N) + \frac{1}{2}\ln(2\pi) + 0\left(\frac{1}{N}\right)$
  - $\ln(1 + x) \approx x - \frac{1}{2}x^2 + 0x^3, \quad -1 \le x \le 1$
  - Thus, $g(N, s) \approx 2^N\sqrt{\frac{2}{\pi N}}e^{-\frac{2s^2}{N}}$

- Important Combinatorics
  - $(x + y^n) = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$ — Binomial Expansion
  - $\binom{n}{k} = \frac{n!}{k!(n-k!)}$ — Binomial Coefficient
- Gaussian Probability Density Function (PDF)

$$G(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

  - σ represents the standard deviation of G
- Macroscopic properties of a large system are well defined (*i.e.* fluctuations about the mean values are small $\approx O(\sqrt{N})$)