Lecture XVIII Notes

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1 Triple Integrals – 15.6

Given a region B where:

$$B = \{(x, y, z)|a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}$$

If this region is filled with many smaller boxes, or sub-boxes, the triple Riemann sum would look as such:

$$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\,\Delta V$$

To form an integral, this triple Riemann must approach infiinty:

$$\iiint_B f(x, y, z)\,dV = \lim_{(l,m,n)\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\,\Delta V$$

Furthermore, Fubini's Theorem, which was applied to double integrals, may be applied to triple integrals, if f is continuous on $B = [a, b]$ x $[c, d]$ x $[r, s]$:

$$\iiint_B f(x, y, z)\,dV = \int_a^b \int_r^s \int_c^d f(x, y, z)\,dy\,dz\,dx$$

Example: Evaluate the following triple integral along
$B = \{(x, y, z)|0 \leq x \leq 1, -1 \leq y \leq 2, 0 \leq z \leq 3\}$:

$$\iiint_B xyz^2\,dV$$

$$\int_0^1 \int_{-1}^2 \int_0^3 xyz^2\,dz\,dy\,dx$$

$$\int_0^1 \int_{-1}^2 3xy\,dy\,dx$$

$$\int_0^1 \frac{27x}{2}\, dx$$

$$\frac{27x^2}{4}\Big|_0^1 = \frac{27}{4}$$

1.1 Triple Integrals over General Regions

A Type I general region is defined by E, where:

$$E = \{(x, y, z)|(x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

These boundaries result in the following Integral:

$$\iiint_E f(x, y, z)\, dV$$

This may be simplified, using Fubini's Theorem, to form the following:

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\right] dA$$

A more complicated region is formed when D and E are Type I regions, which means:
$E = \{(x, y, z)|a \leq x \leq b, g_1(x) \leq y \leq g_2(x), u_1(x, y) \leq z \leq u_2(x, y)\}$
This yields the triple integral:

$$\iiint_E f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\, dy\, dx$$

If D is a Type II region, while E is Type I, this results in:

$$\iiint_E f(x, y, z)\, dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\, dx\, dy$$

Region E is a Type II region when $E = \{(x, y, z)|(y, z) \in D, u_1(y, z) \leq x \leq u_2(y, z)$

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\, dx\right] dA$$

Finally, region E is Type III if $E = \{(x, y, z)|(x, z) \in D, u_1(x, z) \leq y \leq u_2(x, z)$
This forms the integral:

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\, dy\right] dA$$

Much like in double integrals, a function's mass, moments, and center of mass may be determined through triple integration. The formulas are as follows: For mass:

$$m = \iiint_E \rho(x, y, z)\, dV$$

For the moments about the three coordinate planes:

$$M_{yz} = \iiint_E x\rho(x,y,z)\,dV \quad M_{xz} = \iiint_E y\rho(x,y,z)\,dV$$

$$M_{xy} = \iiint_E z\rho(x,y,z)\,dV$$

The center of mass at $(\overline{x}, \overline{y}, \overline{z})$

$$\overline{x} = \frac{M_{yz}}{m} \quad \overline{y} = \frac{M_{xz}}{m} \quad \overline{z} = \frac{M_{xy}}{m}$$

Finally, the moment of inertia formulas are:

$$I_x = \iiint_E (y^2 + z^2)\rho(x,y,z)\,dV$$

$$I_y = \iiint_E (x^2 + z^2)\rho(x,y,z)\,dV$$

$$I_z = \iiint_E (x^2 + y^2)\rho(x,y,z)\,dV$$

1.2 Change of Variables in Triple Integrals

In three dimensions, a change of variables revolves around the Jacobian. The Jacobian of a transformation where $x = g(u,v)$ and $y = h(u,v)$ looks as follows:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

In a double integral, this change of variables would result in the integrand being multiplied by the Jacobian:

$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$