Lecture XI Notes

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1 The Multivariable Chain Rule – 14.5

In three dimensions, the formula for the chain rule looks as follows:

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Conditions $z = f(x, y); x = x(t), y = y(t)$

Example:

$$z = x^2y + 3xy^4; x = \sin 2t, y = \cos t$$
$$\frac{dz}{dt} = (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)$$

Another case looks as follows:
If $z = f(x, y); x = g(s, t), y = h(s, t)$, then:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

The general case of the chain rule, where u is a multivariable function of n variables x_1, x_2,… x_n, and each x is a function of m variables t_1,… t_m, then:

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

1.1 Implicit Differentiation

If $\frac{\partial f}{\partial y} \neq 0$:

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

In addition to this:

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}}$$

and

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}}$$

2 The Directional Derivative – 14.6

The definition of a directional derivative is:

$$D_{\hat{u}} = \lim_{h \to 0} \frac{f(x_o + ha, y_o + hb) - f(x_o, y_o)}{h}$$

In more simplified terms:

$$D_{\hat{u}} = \frac{\partial f}{\partial x} a + \frac{\partial f}{\partial y} b$$

2.1 The Gradient Vector

The gradient vector is written $\overrightarrow{\nabla} f$, and:

$$\overrightarrow{\nabla} f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}}$$

In the case that $u = f(x, y, z)$:

$$\overrightarrow{\nabla} f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}$$

and

$$D_{\hat{u}} = \overrightarrow{\nabla} f(x, y, z) \cdot \hat{u}$$

The greatest value of the gradient, written as $|\overrightarrow{\nabla} f|$, occurs in the same direction for $\hat{\mathbf{u}}$ and $\overrightarrow{\nabla} f$