Calculus III Formula Sheet

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1 Chapter 12

1.1 Unit 1

1. The distance formula in three dimensions:
$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

2. A sphere with center (h, k, l):
$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

1.2 Unit 2

3. Vector given two points, $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_1)$
$$\overrightarrow{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

4. Magnitude of vector in:
   (a) Two dimensions ($a = \langle a_1, a_2 \rangle$):
   $$|\overrightarrow{a}| = \sqrt{(a_1)^2 + (a_2)^2}$$
   (b) Three dimensions ($a = \langle a_1, a_2, a_3 \rangle$):
   $$|\overrightarrow{a}| = \sqrt{(a_1)^2 + (a_2)^2 + (a_3)^2}$$

5. Standard Basis Vectors:
$$\hat{\mathbf{i}} = \langle 1, 0, 0 \rangle \quad \hat{\mathbf{j}} = \langle 0, 1, 0 \rangle \quad \hat{\mathbf{k}} = \langle 0, 0, 1 \rangle$$

6. Unit Vectors (Any vector with magnitude 1):
$$\overrightarrow{u}_a = \frac{\overrightarrow{a}}{|\overrightarrow{a}|}$$

1.3 Unit 3

6. Dot Product[1] (Also Known As Scalar Product):

$$\overrightarrow{a} \cdot \overrightarrow{b} = a_1b_1 + a_2b_2 + a_3b_3$$

7. Dot Product Angle Formula:

$$\overrightarrow{a} \cdot \overrightarrow{b} = |\overrightarrow{a}||\overrightarrow{b}|\cos\theta$$

8. Direction Angles:[2]

$$\cos\alpha = \frac{\overrightarrow{a}\hat{\mathbf{i}}}{|\overrightarrow{a}|}$$
$$\cos\beta = \frac{\overrightarrow{a}\hat{\mathbf{j}}}{|\overrightarrow{a}|}$$
$$\cos\gamma = \frac{\overrightarrow{a}\hat{\mathbf{k}}}{|\overrightarrow{a}|}$$

9. Projections:

   (a) Scalar projection of $\overrightarrow{b}$ onto $\overrightarrow{a}$:

$$comp_{\overrightarrow{a}}\overrightarrow{b} = \frac{\overrightarrow{a} \cdot \overrightarrow{b}}{|\overrightarrow{a}|}$$

   (b) Vector projection of $\overrightarrow{b}$ onto $\overrightarrow{a}$:

$$proj_{\overrightarrow{a}}\overrightarrow{b} = \left(\frac{\overrightarrow{a} \cdot \overrightarrow{b}}{|\overrightarrow{a}|}\right)\frac{\overrightarrow{a}}{|\overrightarrow{a}|}$$

1.4 Unit 4

10. Cross Product:[3]

$$\overrightarrow{a} \text{ x } \overrightarrow{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

11. Cross Product Angle Formula:[4]

$$|\overrightarrow{a} \text{ x } \overrightarrow{b}| = |\overrightarrow{a}||\overrightarrow{b}|\sin\theta$$

12. Volume of the Parallelepiped created by vectors $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$:

$$V = |\overrightarrow{a} \cdot (\overrightarrow{b} \text{ x } \overrightarrow{c})|$$

[1] Two vectors are orthogonal (perpendicular) if $\overrightarrow{a} \cdot \overrightarrow{b} = 0$

[2] α corresponds to the x axis, β to the y axis, and γ to the z axis

[3] The vector created by $\overrightarrow{a}$ x $\overrightarrow{b}$ is orthogonal to both $\overrightarrow{a}$ and $\overrightarrow{b}$

[4] If the cross product equals zero, the two vectors are parallel

1.5 Unit 5

13. Parametric line equations, given parallel vector $\langle a, b, c\rangle$, through point (x_o, y_o, z_o):

$$x = x_o + at \quad y = y_o + bt \quad z = z_o + ct$$

14. Symmetric Equations:

$$t = \frac{x - x_o}{a} = \frac{y - y_o}{b} = \frac{z - z_o}{c}$$

15. Equation of a plane:

$$a(x - x_o) + b(y - y_o) + c(z - z_o) = 0$$

16. Distance from plane to point $(x_1 + y_1 + z_1)$:

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

1.6 Unit 6

17. Quadric Surface Formulas:

Figure	Equation
Ellipsoid: A Figure in Which All Traces are Ellipses	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
Cone: A Figure in Which Horizontal Traces are Ellipses and Vertical Traces in x and y are Hyperbolas	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$
Elliptic Paraboloid: Horizontal Traces are Ellipses and Vertical Traces are Parabolas	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$
Hyperboloid of One Sheet: Horizontal Traces are Ellipses and Vertical Traces are Hyperbolas	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
Hyperbolic Paraboloid: Horizontal Traces are Hyperbolas and Vertical Traces are Parabolas	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$
Hyperboloid of Two Sheets: Horizontal Traces are Ellipses in z and Vertical Traces are Hyperbolas	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

2 Chapter 13

2.1 Unit 1

18. Limit of a vector function:

$$\lim_{t\to a} \overrightarrow{r}(t) = \langle \lim_{t\to a} x(t), \lim_{t\to a} y(t), \lim_{t\to a} z(t) \rangle$$

2.2 Unit 2

19. Derivative of a vector function:

$$\frac{d}{dt}[\overrightarrow{r}(t)] = \langle x'(t), y'(t), z'(t) \rangle$$

20. Derivative of cross and dot products:

    (a) Dot Product:

$$\frac{d}{dt}[\overrightarrow{u}(t) \cdot \overrightarrow{v}(t)] = \overrightarrow{u}'(t) \cdot \overrightarrow{v}(t) + \overrightarrow{u}(t) \cdot \overrightarrow{v}'(t)$$

    (b) Cross Product:

$$\frac{d}{dt}[\overrightarrow{u}(t) \text{ x } \overrightarrow{v}(t)] = \overrightarrow{u}'(t) \text{ x } \overrightarrow{v}(t) + \overrightarrow{u}(t) \text{ x } \overrightarrow{v}'(t)$$

21. Integral of a vector function:

$$\int_a^b \overrightarrow{r}(t)\, dt = \left(\int_a^b x(t)\, dt \right) \hat{\mathbf{i}} + \left(\int_a^b y(t)\, dt \right) \hat{\mathbf{j}} + \left(\int_a^b z(t)\, dt \right) \hat{\mathbf{k}}$$

2.3 Unit 3

22. Arc Length of a parametric vector function:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_a^b |\overrightarrow{r}'(t)|\, dt$$

23. Unit Tangent Vector:

$$\overrightarrow{T}(t) = \frac{\overrightarrow{r}'(t)}{|\overrightarrow{r}'(t)|}$$

24. Curvature:

    (a) Using the unit tangent vector:

$$\kappa(t) = \frac{|\overrightarrow{T}'(t)|}{|\overrightarrow{r}'(t)|}$$

(b) Using first and second order derivatives:

$$\kappa(t) = \frac{|\overrightarrow{r}'(t) \text{ x } \overrightarrow{r}''(t)|}{|\overrightarrow{r}'(t)|^3}$$

(c) For single variable functions:

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{\frac{3}{2}}}$$

25. Unit Normal Vector:

$$\overrightarrow{N}(t) = \frac{\overrightarrow{T}'(t)}{|\overrightarrow{T}'(t)|}$$

26. Binormal Vector:

$$\overrightarrow{B}(t) = \overrightarrow{T}(t) \text{ x } \overrightarrow{N}(t)$$

2.4 Unit 4

27. Velocity:

$$\overrightarrow{v}(t) = \overrightarrow{r}'(t)$$

28. Speed:

$$|\overrightarrow{v}(t)| = |\overrightarrow{r}'(t)|$$

29. Acceleration:

$$\overrightarrow{a}(t) = \overrightarrow{v}'(t) = \overrightarrow{r}''(t)$$

3 Chapter 14

3.1 Unit 1

30. Level curves are used to demonstrate the height of a function, by drawing a line where $f(x, y, z) = k$, where k is any constant in the domain of f

3.2 Unit 2

31. To evaluate a multivariable limit, one must evaluate it along different paths: *Example*

$$\lim_{(x,y)\to(0,0)} \frac{x}{y}$$

Evaluate along $y = mx$, which is any line through the origin

$$\lim_{(x,y)\to(0,0)} \frac{\cancel{x}}{m\cancel{x}} \Rightarrow \frac{1}{m}$$

Therefore, this limit does not exist because, for different slopes, the value is different

3.3 Unit 3

32. To find a partial derivative, hold all variables aside from the one being differentiated with respect to to find a partial derivative.

3.4 Unit 4

33. If f has continuous partial derivatives, the following equation may be used to find a tangent plane:
$$z - z_o = f_x(x_o, y_o)(x - x_o) + f_y(x_o, y_o)(y - y_o)$$

34. Total differential:
$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$
Or, with a multivariable function:
$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

3.5 Unit 5

35. The Chain Rule (Where x and y are differentiable functions of t, $x(t)$ and $y(t)$ and $z = f(x(t), y(t))$:
$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

36. The Chain Rule (Where x and y are differentiable functions of (s, t), $x(s, t)$, and $y(s, t)$ and $z = f(x(s, t), y(s, t))$:
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

37. Implicit differentiation:
$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

38. Implicit Function Theorem:
$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

3.6 Unit 6

39. Directional derivative of function $f(x, y)$ in the direction of unit vector $u = \langle a, b\rangle$:
$$D_u f(x, y) = f_x(x, y)a + f_y(x, y)b$$
Or, in three dimensions:
$$D_u f(x, y, z) = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c$$

40. Gradient Vector:

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}}$$

Or, in three dimensions:

$$\nabla f(x, y, z) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}$$

41. Tangent Multivariable Planes:

$$F_x(x_o, y_o, z_o)(x - x_o) + F_y(x_o, y_o, z_o)(y - y_o) + F_z(x_o, y_o, z_o)(z - z_o) = 0$$

3.7 Unit 7

42. Second Derivative Test:

$$D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum
(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum
(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum

4 Chapter 15

4.1 Unit 1

43. Double Integral over Rectangles: Given rectangle $R = \{(x, y) \big| a \leq x \leq b, c \leq y \leq d\}$ The double integral of the function $f(x, y)$ is:

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx$$

This yields the volume of the shape under the function $f(x, y)$ and above rectangle, R

44. Midpoint Rule for Double integrals:

$$\int_R f(x, y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_i)\, \Delta A$$

4.2 Unit 2

45. Type I Region ($D = \{(x, y) \big| a \leq x \leq b, g_1(x) \leq y \leq g_2(x)$):

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

46. Type II Region ($D = \{(x, y) \big| h_1(y) \leq x \leq h_2(y), c \leq y \leq d$):

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

4.3 Unit 3

47. Change to Polar Coordinates:

$$\int_{\alpha}^{\beta}\int_{a}^{b} f(r\cos\theta, r\sin\theta) r\, dr\, d\theta$$

48. Polar bounded by function(s) ($D = \{(r,\theta) \big| \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$):

$$\int_{\alpha}^{\beta}\int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) r\, dr\, d\theta$$

4.4 Unit 4

49. Mass from density function:

$$m = \iint_D \rho(x,y)\, dA$$

50. Moment about the:

    (a) x axis:

    $$M_x = \iint_D y\rho(x,y)\, dA$$

    (b) y axis:

    $$M_y = \iint_D x\rho(x,y)\, dA$$

51. Center of mass:

$$\bar{x} = \frac{M_y}{m} \qquad \bar{y} = \frac{M_x}{m}$$

52. Moment of Inertia about the:

    (a) x axis:

    $$I_x = \iint_D y^2\rho(x,y)\, dA$$

    (b) y axis:

    $$I_y = \iint_D x^2\rho(x,y)\, dA$$

    (c) Origin (Polar):

    $$I_o = \iint_D (x^2+y^2)\rho(x,y)\, dA$$

4.5 Unit 5

53. Surface Area:
$$\iint_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\, dA$$

4.6 Unit 6

54. Triple Integral on Box ($B = \{(x,y,z) \big| a \le x \le b, c \le y \le d, r \le z \le s\}$):
$$\int_a^b \int_c^d \int_r^s f(x,y,z)\, dx\, dy\, dz$$

55. Type I E ($E = \{(x,y,z) \big| (x,y) \in D, u_1(x,y) \le z \le u_2(x,y)\}$):
$$\iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz \right] dA$$

56. Type I D and E ($E = \{(x,y,z) \big| a \le x \le b, g_1(x) \le y \le g_2(x), u_1(x,y) \le z \le u_2(x,y)\}$):
$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz\, dy\, dx$$

57. Type II D and Type I E ($E = \{(x,y,z) \big| h_1(y) \le x \le h_2(y), c \le y \le d, u_1(x,y) \le z \le u_2(x,y)\}$):
$$\int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz\, dx\, dy$$

58. Type II E ($E = \{(x,y,z) \big| u_1(y,z) \le x \le u_2(y,z), c \le y \le d, r \le z \le s\}$)
$$\iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\, dx \right] dA$$

59. Type III E ($E = \{(x,y,z) \big| a \le x \le b, u_1(x,z) \le y \le u_2(x,z), r \le z \le s\}$)
$$\iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\, dy \right] dA$$

4.7 Unit 7 (Skip)

4.8 Unit 8 (Skip)

4.9 Unit 9

60. The Jacobian Transformation:
$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

61. Change of Variables in Double Integrals

$$\iint_S f(x(u,v), y(u,v))J \, du \, dv$$

5 Chapter 16

5.1 Unit 1

62. Gradient vector fields:

$$\nabla f(x,y,z) = f_x(x,y,z)\hat{\mathbf{i}} + f_y(x,y,z)\hat{\mathbf{j}} + f_z(x,y,z)\hat{\mathbf{k}}$$

5.2 Unit 2

63. Line Integrals:

$$\int_C f(x,y,z)\, ds = \int_C f(x,y,z)\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}\, dt$$

5.3 Unit 3

64. Fundamental Theorem of Line Integrals

$$\int_C \mathbf{F}\, d\mathbf{r} = \int_C \nabla f \, d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

65. Conservative Vector Field If:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

5.4 Unit 4

66. Green's Theorem:

$$\int_C P\, dx + Q\, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$