

Lecture XXI Notes

Michael Brodskiy

Professor: V. Cherkassky

July 20, 2020

1 Fundamental Theorem of Line Integrals – 16.3

The Fundamental Theorem of Line Integrals states that, if C is a smooth, continuous curve, and $\vec{r}'(t)$, $a \leq t \leq b$, then:

$$\int_a^b \nabla f dr = f(\vec{r}(b)) - f(\vec{r}(a))$$

Example: Given $f(x, y, z) = \cos(\pi x) + \sin(\pi y) - xyz$, and C is any path from $(1, \frac{1}{2}, 2)$ to $(2, 1, -1)$, Find:

$$\int_C \nabla f dr$$

$$\int_C \nabla f dr = f(2, 1, -1) - f\left(1, \frac{1}{2}, 2\right)$$

$$(\cos(2\pi) + \sin(\pi) + 2) - (\cos(\pi) + \sin\left(\frac{\pi}{2}\right) - 1)$$

$$\int_C \nabla f dr = 4$$

Recalling conservative vector fields, where $\vec{F} = \nabla f \Leftarrow$ potential function

The Fundamental Theorem of Line Integrals also states that conservative vector fields are independent of path. That is, as long as any path, say C_1 and C_2 , start and end at the same point, then:

$$\int_{C_1} \nabla f dr = \int_{C_2} \nabla f dr$$

Any path, C , that starts at the same point at which it terminates is called closed.

There are two types of closed paths:

1. Simple

2. Non-Simple

A simple path is a shape such as a circle or ellipse. A non-simple path crosses over itself, and an example of such a shape being the “ ∞ ” symbol.

A region, D is called open when it does not contain any of its own boundaries.

Any region, D , in which one may connect any two points without exiting the boundaries is known as connected.

Furthermore, a region, D may be called simply connected when there are no holes.

1. If C is independent of path:

$$\int_{C_1} \nabla f \, dr = \int_{C_2} \nabla f \, dr$$

2. \vec{F} is conservative if $\vec{F} = \nabla f$
3. \vec{F} is a conservative vector field if D is an open and connected region, where $\int_C \vec{F} \, dr$ is independent of path (in D).
4. $\int_C \vec{F} \, dr$ is independent of path in a region, D , if $\int_C \vec{F} \, dr = 0$ on every closed path C in D .

Given a function, $\vec{F}(x, y) = P(x, y)\hat{\mathbf{i}} + Q(x, y)\hat{\mathbf{j}}$, \vec{F} is conservative if:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

1.1 Solving for ∇f

Given $\vec{F}(x, y) = (2x^3y^4 + x)\hat{\mathbf{i}} + (2x^4y^3 + y)\hat{\mathbf{j}}$, determine whether \vec{F} is a conservative vector field, and, if so, find ∇f

$$\frac{\partial P}{\partial y} = 8x^3y^3$$

$$\frac{\partial Q}{\partial x} = 8x^3y^3$$

Therefore, it is conservative

$$\frac{\partial f}{\partial x} = 2x^3y^4 + x$$

$$\frac{\partial f}{\partial y} = 2x^4y^3 + y$$

$$f(x, y) = \int_C 2x^3y^4 + x \, dx \Rightarrow \frac{1}{2}(x^4y^4 + x^2) + h(y)$$

$$\frac{\partial f}{\partial y} = 2x^4y^3 + h'(y) = 2x^4y^3 + y$$

$$\cancel{2x^4y^3} + h'(y) = \cancel{2x^4y^3} + y$$

$$h'(y) = y \Rightarrow h(y) = \frac{1}{2}y^2$$

$$f(x, y) = \frac{1}{2}(x^4y^4 + x^2 + y^2)$$