

# Lecture XII Notes

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## 1 The Multivariable Chain Rule – 14.5

In three dimensions, the formula for the chain rule looks as follows:

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

Conditions  $z = f(x, y); x = x(t), y = y(t)$

*Example:*

$$z = x^2y + 3xy^4; x = \sin 2t, y = \cos t$$

$$\frac{dz}{dt} = (2xy + 3y^4)(2 \cos 2t) + (x^2 + 12xy^3)(-\sin t)$$

Another case looks as follows:

If  $z = f(x, y); x = g(s, t), y = h(s, t)$ , then:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

The general case of the chain rule, where  $u$  is a multivariable function of  $n$  variables  $x_1, x_2, \dots, x_n$ , and each  $x$  is a function of  $m$  variables  $t_1, \dots, t_m$ , then:

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2} \frac{\partial x_2}{\partial t_i} + \dots + \frac{\partial u}{\partial x_n} \frac{\partial x_n}{\partial t_i}$$

## 1.1 Implicit Differentiation

If  $\frac{\partial f}{\partial y} \neq 0$ :

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

In addition to this:

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}}$$

and

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}}$$

## 2 The Directional Derivative – 14.6

The definition of a directional derivative is:

$$D_{\hat{u}} = \lim_{h \rightarrow 0} \frac{f(x_o + ha, y_o + hb) - f(x_o, y_o)}{h}$$

In more simplified terms:

$$D_{\hat{u}} = \frac{\partial f}{\partial x}a + \frac{\partial f}{\partial y}b$$

### 2.1 The Gradient Vector

The gradient vector is written  $\vec{\nabla} f$ , and:

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}}$$

In the case that  $u = f(x, y, z)$ :

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}$$

and

$$D_{\hat{u}} = \vec{\nabla} f(x, y, z) \cdot \hat{u}$$

The greatest value of the gradient, written as  $|\vec{\nabla} f|$ , occurs in the same direction for  $\hat{\mathbf{u}}$  and  $\vec{\nabla} f$