Lecture XXI Notes

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July 20, 2020

1 Fundamental Theorem of Line Integrals – 16.3

The Fundamental Theorem of Line Integrals states that, if C is a smooth, continuous curve, and $\overrightarrow{r}'(t)$, $a \leq t \leq b$, then:

$$\int_C \nabla f \, dr = f(\overrightarrow{r}(b)) - f(\overrightarrow{r}(a))$$

Example: Given $f(x, y, z) = \cos(\pi x) + \sin(\pi y) - xyz$, and C is any path from $(1, \frac{1}{2}, 2)$ to $(2, 1, -1)$, Find:

$$\int_C \nabla f \, dr$$

$$\int_C \nabla f \, dr = f(2, 1, -1) - f(1, \frac{1}{2}, 2)$$

$$(\cos(2\pi) + \sin(\pi) + 2) - (\cos(\pi) + \sin(\frac{\pi}{2}) - 1)$$

$$\int_C \nabla f \, dr = 4$$

Recalling conservative vector fields, where $\overrightarrow{F} = \nabla f \Leftarrow$ potential function

The Fundamental Theorem of Line Integrals also states that conservative vector fields are independent of path. That is, as long as any path, say C_1 and C_2, start and end at the same point, then:

$$\int_{C_1} \nabla f \, dr = \int_{C_2} \nabla f \, dr$$

Any path, C, that starts at the same point at which it terminates is called closed.

There are two types of closed paths:

1. Simple

2. Non-Simple

A simple path is a shape such as a circle or ellipse. A non-simple path crosses over itself, and example of such a shape being the "∞" symbol.

A region, D is called open when it does not contain any of its own boundaries.

Any region, D, in which one may connect any two points without exiting the boundaries is known as connected.

Furthermore, a region, D may be called simply connected when there are no holes.

1. If C is independent of path:

$$\int_{C_1} \nabla f \, dr = \int_{C_2} \nabla f \, dr$$

2. $\overrightarrow{F}$ is conservative if $\overrightarrow{F} = \nabla f$

3. $\overrightarrow{F}$ is a conservative vector field if D is an open and connected region, where $\int_C \overrightarrow{F} \, dr$ is independent of path (in D).

4. $\int_C \overrightarrow{F} \, dr$ is independent of path in a region, D, if $\int_C \overrightarrow{F} \, dr = 0$ on every closed path C in D.

Given a function, $\overrightarrow{F}(x, y) = P(x, y)\hat{\mathbf{i}} + Q(x, y)\hat{\mathbf{j}}, \overrightarrow{F}$ is conservative if:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

1.1 Solving for ∇f

Given $\overrightarrow{F}(x, y) = (2x^3y^4 + x)\hat{\mathbf{i}} + (2x^4y^3 + y)\hat{\mathbf{j}}$, determine whether $\overrightarrow{F}$ is a conservative vector field, and, if so, find ∇f

$$\frac{\partial P}{\partial y} = 8x^3y^3$$
$$\frac{\partial Q}{\partial x} = 8x^3y^3$$

Therefore, it is conservative

$$\frac{\partial f}{\partial x} = 2x^3y^4 + x$$
$$\frac{\partial f}{\partial y} = 2x^4y^3 + y$$
$$f(x, y) = \int_C 2x^3y^4 + x \, dx \Rightarrow \frac{1}{2}(x^4y^4 + x^2) + h(y)$$
$$\frac{\partial f}{\partial y} = 2x^4y^3 + h'(y) = 2x^4y^3 + y$$

$$2x^4y^3 + h'(y) = \cancel{2x^4y^3} + y$$

$$h'(y) = y \Rightarrow h(y) = \frac{1}{2}y^2$$

$$f(x,y) = \frac{1}{2}(x^4y^4 + x^2 + y^2)$$