APPENDIX

A. Proof of Property 1

Proof. We prove by contradiction. Suppose there are two distinct $(\mathcal{F},\lambda,k)$-FoCores C and C'. According to Definition 2, for all $u \in C$ ($u \in C'$ resp.), we have $deg_C(u,l) \geq k$ ($deg_{C'}(u,l) \geq k$ resp.) on at least λ layers including $\mathcal{F}$. Therefore, for all $v \in C \cup C'$, we have $deg_{C\cup C'}(v,l) \geq \max(deg_C(v,l), deg_{C'}(v,l)) \geq k$ on at least λ layers including $\mathcal{F}$, so $C \cup C'$ is also a $(\mathcal{F},\lambda,k)$-FoCore. It contradicts to the maximality of C and C'. Thus, the property holds. □

B. Proof of Property 2

Proof. Let v be an arbitrary vertex in $C^{k+1}_{\mathcal{F},\lambda}$. By Definition 2, v has at least λ support layers w.r.t. the minimum degree threshold $k+1$. Certainly, these layers support v w.r.t. the minimum degree threshold k. Moreover, in the induced subgraph $G[C^k_{\mathcal{F},\lambda}]$, the degree of v on each focus layer $l \in \mathcal{F}$ is at least $k+1$, which is certainly greater than k. Therefore, all vertices in $C^{k+1}_{\mathcal{F},\lambda}$ satisfy the constraints specified in Definition 2 w.r.t. the parameters $\mathcal{F}$, λ and k. Due to the maximality and the uniqueness of $C^k_{\mathcal{F},\lambda}$ (Property 1), we have $C^{k+1}_{\mathcal{F},\lambda} \subseteq C^k_{\mathcal{F},\lambda}$. □

C. Proof of Property 3

Proof. Let $C_1 = C^k_{\mathcal{F}_1,\lambda_1}$ and $C_2 = C^k_{\mathcal{F}_2,\lambda_2}$. By Definition 2, every vertex $v \in C_2$ has at least k neighbors on at least λ_2 layers (of course, on at least λ_1 layers), so C_2 satisfies the background constraint of the $(\mathcal{F}_1,\lambda_1,k)$-FoCore. By Definition 2, every vertex $v \in C_2$ satisfies $deg_{C_2}(v,l) \geq k$ for all $l \in \mathcal{F}_2$ (certainly, for all $l \in \mathcal{F}_1$), so C_2 also satisfies the focus constraint of the $(\mathcal{F}_1,\lambda_1,k)$-FoCore. Due to the maximality of C_1, we have $C_2 \subseteq C_1$. Thus, the property holds. □

D. Proof of Theorem 1

Proof. As will be presented in Section III, there exists algorithms that solve the FoCore decomposition problem in $f(|L|) \cdot |E|^{O(1)}$ time, where f is a function of $|L|$ that is independent of $|E|$. Thus, the theorem holds. □

E. Proof of Lemma 1

Proof. Let $k_1 = cn_{\mathcal{F}_1,\lambda_1}(v)$ and $k_2 = cn_{\mathcal{F}_2,\lambda_2}(v)$. We have $v \in C^{k_1}_{\mathcal{F}_1,\lambda_1}(v)$ and $v \in C^{k_2}_{\mathcal{F}_2,\lambda_2}(v)$. By Definition 4, k_1 is the maximum k such that $v \in C^k_{\mathcal{F}_1,\lambda_1}$. Since $(\mathcal{F}_1,\lambda_1) \sqsubseteq (\mathcal{F}_2,\lambda_2)$, we have $C^{k_2}_{\mathcal{F}_2,\lambda_2}(v) \subseteq C^{k_2}_{\mathcal{F}_1,\lambda_1}(v)$ by Property 3, so $v \in C^{k_2}_{\mathcal{F}_1,\lambda_1}(v)$. Thus, we have $k_1 \geq k_2$. □

F. Proof of Lemma 2

Proof. We prove the lemma by considering the above 3 cases:

Case 1: When $\mathcal{F} = \emptyset$, $\lambda = 1$ and $k = 1$, we set $R = V$, so $C^k_{\mathcal{F},\lambda} \subseteq V = R$.

Case 2: When $(\mathcal{F},\lambda) \neq (\emptyset, 1)$ and $k = 1$, we have $R = \bigcap_{(\mathcal{F}',\lambda')\sqsubseteq(\mathcal{F},\lambda)} C^k_{\mathcal{F}',\lambda'}$. According to Property 3, we have $C^k_{\mathcal{F},\lambda} \subseteq C^k_{\mathcal{F}',\lambda'}$, so $C^k_{\mathcal{F},\lambda} \subseteq R$.

Case 3: When $k > 1$, we have $R = C^{k-1}_{\mathcal{F},\lambda}$. According to Property 2, we have $C^k_{\mathcal{F},\lambda} \subseteq R$. □

G. Proof of Theorem 2

Proof. (Necessity) If $cn_{\mathcal{F},\lambda}(v) = k$, we have $v \in C^k_{\mathcal{F},\lambda}$ and $|N(v) \cap C^k_{\mathcal{F},\lambda}| \geq k$. For all $w \in N(v) \cap C^k_{\mathcal{F},\lambda}$, we have $cn_{\mathcal{F},\lambda}(w) \geq k$, so the condition is met. Next, we prove by contradiction that k is the largest integer that satisfies the condition. Suppose $k' > k$ also satisfies the condition. Let $U \subseteq N(v)$ be the set of vertices that satisfy the condition w.r.t. k'. For $w \in U$, let C_w be the $(\mathcal{F},\lambda,k')$-FoCore containing w. We have that $\bigcup_{w\in U} C_w \cup \{v\}$ forms a $(\mathcal{F},\lambda,k')$-FoCore, so $cn_{\mathcal{F},\lambda}(v) = k'$, which contradicts with the fact that $cn_{\mathcal{F},\lambda}(v) = k$.

(Sufficiency) For each vertex $u \in N(v)$ that satisfies the condition, let C_u be the $(\mathcal{F},\lambda,k)$-FoCore containing u. Consider $U = \bigcup C_u \cup \{v\}$. For all vertex $w \in U$, we have $deg_{G[U]}(w,l) \geq k$ for all $l \in \mathcal{F}$, and w has at least λ support layers in $G[U]$, so $v \in U \subseteq C^k_{\mathcal{F},\lambda}$. Since there is no $k' > k$ that satisfies the condition, we have $v \notin C^{k'}_{\mathcal{F},\lambda}$. Thus, $cn_{\mathcal{F},\lambda}(v) = k$. □

H. Proof of Lemma 3

Proof. By the definition of C^μ, C^μ is a $(\mathcal{F}_\mu,\lambda_\mu,k_\mu)$-FoCore, where $\mathcal{F}_\mu = \{l^\mu\}$, $\lambda_\mu = 1$, and k_μ is the maximum k such that the $(\mathcal{F}_\mu,\lambda_\mu,k)$-FoCore is not empty. Since C^* is the densest FoCore, we have $\rho(C^*) \geq \rho(C^\mu)$. □