gCore: Exploring Cross-layer Cohesiveness in Multi-layer Graphs

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ABSTRACT

As multi-layer graphs can give a more accurate and reliable picture of the complex relationships between entities, cohesive subgraph mining, a fundamental task in graph analysis, has been studied on multi-layer graphs in the literature. However, existing cohesive subgraph models are designated for special multi-layer graphs, such as multiplex networks and heterogeneous information networks. In this paper, we propose generalized core (gCore), a new notion of cohesive subgraph on general multi-layer graphs without any predefined constraints on inter-connections between vertices. The gCore model considers both intra-layer and cross-layer cohesiveness of vertices. Three related problems are studied in this paper including gCore search (GCS), gCore decomposition (GCD) and gCore indexing (GCI). A polynomial-time algorithm based on the peeling paradigm is proposed to solve the GCS problem. By considering the containment between gCores, a "tree of trees" data structure called KP-tree is designed for efficiently solving the GCD problem and serving as a compact storage and index of all gCores. Several advanced lossless compaction techniques including node/subtree-elimination, subtree transplant and subtree merge are proposed to help reduce the storage overhead of the KP-tree and speed up the process of solving GCD and GCI. Besides, a KP-tree-based GCS algorithm is designed, which can retrieve any gCore in linear time in the size of the gCore and the height of the KP-tree. The experiments on 10 real-world graphs verify the effectiveness of the gCore model and the efficiency of the proposed algorithms.

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The source code, data, and/or other artifacts have been made available at https://github.com/SSSuperDan/gCore.

1 INTRODUCTION

Motivation. Graphs have been widely used to model complex relationships between entities. However, the most comprehensively studied single-layer graph model is limited in characterizing multiple types of relationships like various types of social relationships [8]. It will become an inevitable trend to use *multi-layer graphs* [20] to fully characterize multi-faceted relationships between entities. A multi-layer graph is modeled as a set of inter-related layered graphs (*layers* for short), each of which represents one type of relationship.

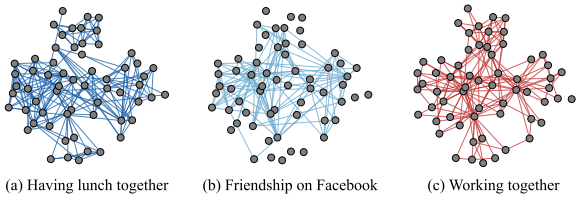

Figure 1: An example of pillar multi-layer graph extracted from the AUCS dataset [8] which consists of three layers. Each layer depicts a type of relationship between the employees in a university.

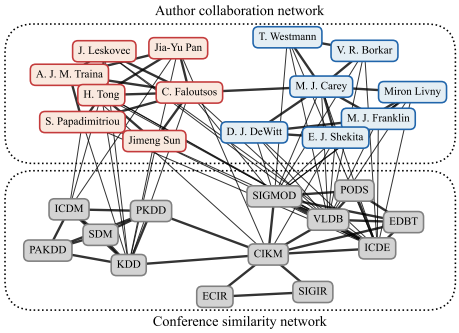

Figure 2: An example of general multi-layer graph extracted from the DBLP dataset [32]. It consists of two layers representing the collaboration between authors and the similarity between some top venues in the fields of databases and data mining, respectively. Each author is connected by cross-layer links to the top-3 venues where he/she has published the most papers.

According to how vertices across layers are inter-connected, multi-layer graphs can be categorized into *pillar multi-layer graphs* and *general multi-layer graphs* [19]. As shown in Figure 1, a pillar multi-layer graph has the same set of vertices (entities) on all layers, and every vertex in a layer has exactly one cross-layer link to its copy (mirror) on every other layer. A general multi-layer graph, as illustrated in Figure 2, may have different vertex sets on different layers, and any vertex in a layer has zero to many cross-layer links to the vertices on every other layer.

Cohesive **s**ubgraph **m**ining (CSM) is the task of finding densely connected vertices, which has been extensively studied on single-layer graphs in the literature [21]. Recently, CSM on multi-layer graphs (ML-CSM for short) has attracted increasing research attention to detect more reliable cohesive subgraphs based on multiple types of relationships [2, 13, 19, 22, 39]. Existing work investigates ML-CSM on special multi-layer graphs like **m**ulti**p**lex **n**etworks (MPN) [11] and **h**eterogeneous **i**nformation **n**etworks (HIN) [9],

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where an MPN is a typical pillar multi-layer graph, and a HIN can be translated to a general multi-layer graph by grouping each type of vertices into a layer [19].

Studying ML-CSM on general multi-layer graphs is able to enhance the discovered cohesive subgraphs by investigating the interconnections between cohesive subgraphs across layers. For example, the red vertices in Figure 2 form a cohesive subgraph in the author layer. Many neighbors of the red vertices on the conference layer also form a cohesive subgraph, thereby enhancing the cohesiveness of the red vertices. However, ML-CSM has been so far rarely studied on general multi-layer graphs.

Prior Work. Although many cohesive subgraph models have been proposed for single-layer graphs, e.g. clique [23], quasi-clique [25], k-core [30], k-truss [7], k-edge-connected subgraph [38], and nucleus [29], they are obviously inapplicable to ML-CSM.

Two well-known cohesive subgraph models for ML-CSM have been proposed for multiplex networks (MPN), namely *cross-layer quasi-clique* [5, 28, 36] and *multi-layer core* [11, 22, 39]. They can be regarded as extensions from γ-quasi-clique [25] and k-core [30], respectively. Specifically, a cross-layer quasi-clique (resp. multi-layer k-core) is defined as a subset of vertices that form a γ-quasi-clique (resp. k-core) in every layer. Certainly, such concepts are not suitable for general multi-layer graphs because different layers may have different vertex sets.

Pei et al. [28] and Jiang et al. [18] have discussed extending cross-layer quasi-clique to multi-layer graphs with heterogeneous vertices by associating vertices across layers through surjective mappings f. A subset Q of vertices forms a cohesive subgraph in a multi-layer graph if 1) Q is a γ-quasi-clique in a certain layer, and 2) in each other layer, the vertex subset $Q' = \{f(v)|v \in Q\}$ is a γ'-quasi-clique. However, this extension does not work if any vertex in Q has more than one neighbor in some other layers, which is very likely to happen in practice. Zhu et al. [39] has shown that such extension has several intrinsic limitations inherited from the γ-quasi-clique model, including high computational cost and low flexibility in characterizing large cohesive subgraphs caused by the stringent constraint imposed on the degrees of vertices.

The typical ML-CSM models proposed for heterogeneous information networks (HIN) are also extended from the classical models like k-core, for example, $(k, \mathcal{P})$-core [10] and relational community (r-com) [17]. However, a HIN focuses on relationships between heterogeneous types of entities [19], making it hard to distinguish different types of relationships between vertices of the same type.

Solution. In this paper, a new cohesive subgraph model called *generalized core (gCore)* is proposed for general multi-layer graphs, which overcomes the shortcomings pointed out by Zhu et al. [39]. To obtain reliable and robust cohesive subgraphs, we certainly expect that the vertices show cohesiveness in each layer [18, 39]. A thorny problem is to define the cohesiveness of vertices on different layers based on the many-to-many cross-layer mappings between vertices. Our solution is to extend the extension scheme proposed by Pei et al. [28] and Jiang et al. [18] with the k-core model as described below. Suppose there are l layers in a general multi-layer graph and the l-th layer is of users' interest. Let $k_1, k_2, \ldots, k_l \in \mathbb{N}$ and $p_1, p_2, \ldots, p_{l-1} \in [0, 1]$. A subset Q of vertices on the l-th layer forms a gCore if 1) Q is a k_l-core on the l-th layer; 2) in each other i-th layer, there exists a k_i-core $Q' \subseteq N(Q)$ such that every vertex in Q has at least a fraction p_i of neighbors on the i-th layer participating Q', where $N(Q)$ is the set of all vertices in the i-th layer that link to vertices in Q; and 3) Q is maximal. The properties of the gCore model are studied in this paper. Most importantly, there is a unique gCore in a general l-layer graph given vectors $\mathbf{k} = (k_1, k_2, \ldots, k_l)$ and $\mathbf{p} = (p_1, p_2, \ldots, p_{l-1})$, so the gCore is also called the $(\mathbf{k}, \mathbf{p})$-core.

Based on the gCore model, we study three related problems in this paper, namely *gCore search (GCS), gCore decomposition (GCD)* and *gCore indexing (GCI)*. GCS finds the $(\mathbf{k}, \mathbf{p})$-core in a general l-layer graph for given vectors $\mathbf{k}$ and $\mathbf{p}$. This problem is key to the retrieval of a cohesive subgraph with particular features. GCD finds all distinct nonempty gCores in a general multi-layer graph. This problem is essential for exploring the structure of a general multi-layer graph. GCI constructs an index to speed up GCS.

To solve the GCS problem, we propose a polynomial-time algorithm that repeatedly removes vertices violating the constraint imposed on degree or neighbor coverage.

To solve the GCD problem, we first propose a naïve algorithm that consists of two separate phases to enumerate $(\mathbf{k}, \mathbf{p})$ pairs and retrieve $(\mathbf{k}, \mathbf{p})$-cores, respectively. To reduce redundant computations in the naïve algorithm, a "tree of trees" data structure called *KP-tree* is proposed to organize all gCores in a systematic way by leveraging their containment relationships. The KP-tree determines a good order to find all gCores that avoids computing gCores from scratch and enables fast identification of empty gCores.

Moreover, the KP-tree derives an index structure to support fast retrieval of any $(\mathbf{k}, \mathbf{p})$-core. Based on this index structure, a more efficient algorithm is proposed to solve the GCS problem, which runs in linear time in the size of the queried $(\mathbf{k}, \mathbf{p})$-core and the height of the KP-tree. Considering that the information in the KP-tree index is redundant, we design a series of lossless compaction schemes, including node/subtree elimination, subtree transplant, and subtree merge, to further save storage and improve index construction efficiency. It is proved that the compacted index guarantees the correctness of the index-based GCS algorithm.

Extensive experiments have been performed to evaluate the gCore model and the proposed algorithms. It shows the following results: (1) Vertices in the $(\mathbf{k}, \mathbf{p})$-core attain more closeness with each other compared with those in the k-core; (2) The time cost of our algorithm for solving the GCS problem is comparable to that of other algorithms for searching core-based cohesive subgraphs in multi-layer graphs. Based on the KP-tree index, the $(\mathbf{k}, \mathbf{p})$-core search efficiency is improved by up to 1–4 orders of magnitude; (3) The index compaction techniques significantly reduce both the KP-tree construction time, which includes the time to solve the GCD problem, and the memory cost of the KP-tree.

Contributions. The contributions of the paper are as follows:

(1) A new cohesive subgraph model called generalized core (gCore) is proposed for general multi-layer graphs. The uniqueness of gCores and the containment among gCores are studied.
(2) Three related problems concerning the gCore model are formulated, namely gCore search (GCS), gCore decomposition (GCD) and gCore indexing (GCI).
(3) A polynomial-time algorithm is proposed to solve the GCS problem based on the peeling paradigm.

(4) A data structure called KP-tree is designed to systematically organize all gCores, and a KP-tree-based algorithm is proposed to efficiently solve the GCD problem.
(5) Based on the KP-tree, an index structure is designed to support fast online GCS. A series of compaction schemes are developed to eliminate redundant information from the KP-tree index, which further reduce storage overhead and improve index construction efficiency.
(6) Extensive experiments have been carried out to evaluate the gCore model and the proposed algorithms.

2 RELATED WORK

Multi-layer graphs. Multi-layer graphs have been widely studied in the fields of complex systems and big data analysis. For simplicity, existing work mostly studies special multi-layer graphs with specific constraints imposed on inter-connections between vertices, such as multiplex networks (MPN) [11], interdependent networks [6], heterogeneous information networks (HIN) [9], and multidimensional networks [3]. Refs. [4, 20] comprehensively surveyed the existing multi-layer graph models. Among them, MPNs and HINs are the most popularly used in big data analysis to model the complex relationships between entities in domains like sociology [8], biology [12, 14], and e-commerce[15]. However, as stated in Section 1, the analysis tools developed for MPNs and HINs cannot be applied to each other and to general multi-layer graphs.

CSM on MPNs. To characterize cohesive subgraphs on an MPN, existing work always uses a unified classic cohesive subgraph model to restrict the cohesiveness of vertices in some or all layers. Wang et al. [34] and Zeng et al. [36] proposed to mine maximal frequent cliques and γ-quasi-cliques in a series of graphs. Pei et al. [28] and Jiang et al. [18] extended the notion of quasi-clique and introduced the cross-layer quasi-clique model and the frequent cross-layer quasi-clique model. Due to the #P-complexity of enumerating all (frequent) cross-layer quasi-cliques, exact branch and bound approaches with several pruning methods are proposed. Boden et al. [5] introduced an MLCS cluster model for MPNs with edge labels, which is a variant of cross-layer quasi-clique with supplementary consideration of similarities between edge labels. A best-first search approach is presented in [5] to find qualified MLCS clusters with low redundancy.

To overcome the intrinsic limitations inherited from the γ-quasi-clique model in the above models, Zhu et al. [39] proposed a notion of d-coherent core (d-CC), which requires a vertex subset to form a d-core in a given subset of layers. Three approximation algorithms with provable guarantees are proposed in [39] to extract a fixed number of d-CCs that attain the largest diversity. Liu et al. [22] studied the d-coherent core decomposition problem. Galimberti et al. [11] studied the multi-layer core model, another extension of k-core. It allows using varying k in different layers to characterize the different levels of cohesiveness. Three decomposition algorithms relying on different search orders are given in [11]. Hashemi et al. [13] introduced a relaxed version of the d-CC model called (k, λ)-FirmCore, and studied the FirmCore decomposition problem. Huang et al. [16] and Behrouz et al. [2] further combined the extension scheme of the d-coherent core and the (k, λ)-FirmCore with the k-truss model, respectively. All the above models are tailored to fit multiplex networks with all layers sharing the same set of vertices. Therefore, they cannot be used in general multi-layer graphs.

CSM on HINs. Fang et al. [9] surveyed the existing cohesive subgraph models and the CSM algorithms for HINs. Here, we review several representative works. As HINs are usually represented by single-layer graphs with vertices labeled by types, meta-paths (sometimes meta-subgraphs) [9] are specified to designate the type of vertices and edges forming the cohesive structures. Fang et al. [10] defined a $(k, \mathcal{P})$-core model given a symmetric meta-path $\mathcal{P}$, which requires each vertex in a $(k, \mathcal{P})$-core is connected to at least k vertices through instances of $\mathcal{P}$. Two variants of the $(k, \mathcal{P})$-core are also studied in [10] that require the meta-path instances contributing to k are vertex-disjoint and edge-disjoint. Yang et al. [35] combined the extension scheme of the $(k, \mathcal{P})$-core and the vertex-disjoint $(k, \mathcal{P})$-core with the k-truss model. Jian et al. [17] proposed another extension of k-core called relational community (r-com). It uses the same way as the $(k, \mathcal{P})$-core to constrain cohesiveness, but allowing a series of meta-paths $\mathcal{P}$ of fixed length 2 and integers k specific to each $\mathcal{P}$. An r-com may contain heterogeneous vertices. Hu et al. [15] designed an extension of the clique model called m-Clique based on a given meta-subgraph. As all the above models are developed for HINs, they cannot be used in cases when more than one type of relationship between homogeneous entities is considered.

3 PRELIMINARIES

In this section, we first introduce some preliminaries and notions related to the general multi-layer graph model, and then present our generalized core model together with its elegant properties. At last, the problems studied in this paper are proposed.

3.1 The General Multi-layer Graph Model

By joining multiple simple graphs $G = (V, E)$ with vertex set V and edge set E via the connections among them, we have a multi-layer graph, which is defined as follows.

Definition 3.1 ([24]). A *multi-layer graph* is a pair $\mathcal{M} = (\mathcal{G}, C)$, where $\mathcal{G} = \{G_1, G_2, \ldots, G_l\}$ is a set of simple graphs (also called *layers*) $G_i = (V_i, E_i)$, where $i \in \{1, 2, \cdots, l\}$, and $C = \{E_{i,j} | 1 \leq i < j \leq l\}$ is the collection of sets of edges $E_{i,j} \subseteq V_i \times V_j$ linking vertices of G_i to vertices of G_j.

We call the edges in $E_1, E_2, \ldots, E_l$ *intra-layer edges* and edges in $E_{i,j}$ *cross-layer edges*. In the following, let $E(\mathcal{G}) = \bigcup_{i=1}^{l} E_i$ be the set of all intra-layer edges and $E(C) = \bigcup_{E_{i,j} \in C} E_{i,j}$ the set of all cross-layer edges. Obviously, the pillar multi-layer graph model is a special case of the above general multi-layer graph model because all layers have identical vertex sets, and a cross-layer edge only exists between two copies of a vertex lying in different layers.

In a multi-layer graph, a vertex may have intra-layer and/or cross-layer neighbors. For a vertex $v \in V_i$, the set of neighbors of (adjacent to) v in G_j is denoted by $N_j(v)$. If $j = i$, the vertices in $N_j(v)$ are called *intra-layer neighbors*; Otherwise, they are called *cross-layer neighbors*. Let $deg_j(v) = |N_j(v)|$. We call $deg_j(v)$ the *intra-layer degree* of v if $j = i$, and otherwise the *cross-layer degree*.

A multi-layer graph $\mathcal{M}' = (\mathcal{G}', C')$ is a subgraph of $\mathcal{M}$ if $|\mathcal{G}| = |\mathcal{G}'|$, G'_i is a subgraph of G_i for $G'_i \in \mathcal{G}'$, and $E'_{i,j} \subseteq E_{i,j}$ for $E'_{i,j} \in C'$.

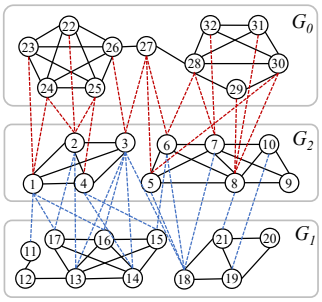


Figure 3: An example of a general multi-layer graph.

Given a simple graph $G = (V, E)$, the subgraph of G *induced* by $Q \subseteq V$ is $G[Q] = (Q, E[Q])$, where $E[Q] \subseteq E$ is the set of edges with both endpoints in Q. Similarly, let $\boldsymbol{Q} = \{Q_1, Q_2, \ldots, Q_l\}$, where $Q_i \subseteq V_i$ for $1 \le i \le l$, the subgraph of $\mathcal{M}$ induced by $\boldsymbol{Q}$ is $\mathcal{M}[\boldsymbol{Q}] = (\mathcal{G}[\boldsymbol{Q}], C[\boldsymbol{Q}])$, where $\mathcal{G}[\boldsymbol{Q}] = \{G_1[Q_1], G_2[Q_2], \ldots, G_l[Q_l]\}$ and $C[\boldsymbol{Q}] = \{E_{i,j}[Q_i, Q_j] | 1 \le i < j \le l\}$ with $E_{i,j}[Q_i, Q_j] \subseteq E_{i,j}$ being the set of edges having one endpoint in Q_i and the other in Q_j.

Besides, the cross-layer edges in a multi-layer graph enable us to define cross-layer induced subgraphs. Let $Q_i \subseteq V_i$. For $j \neq i$, let $Q_j = \bigcup_{v \in Q_i} N_j(v)$ be Q_i's cross-layer neighbors in G_j. The induced subgraph $G_j[Q_j]$ is called the *cross-layer subgraph* of G_j induced by Q_i, denoted by $G_j[Q_i]$.

3.2 gCores in General Multi-layer Graphs

As stated in Section 1, we expect that a generalized core Q on some layer G_i could show its cohesiveness in various aspects including the direct interactions within G_i and the indirect ones represented by $E_{i \to j}$ and G_j for $j = 1, 2, \cdots, l$ and $j \neq i$. The former can be easily characterized by the elegant k-core model [30]. Specifically, Q is a k-core in a graph G if each vertex $v \in Q$ is adjacent to at least k vertices in $G[Q]$, i.e., the degree of v in $G[Q]$ is at least k. To characterize the latter, we extend the concept of *fraction* proposed in [37] and rephrase it as *neighbor coverage fraction* in Definition 3.2.

Definition 3.2. For $v \in V_i$ and $Q_j \subseteq V_j$, the fraction of v's neighbors in G_j falling in Q_j, i.e., $\phi(v, Q_j) = \frac{|N_j(v) \cap Q_j|}{|N_j(v)|}$, is called the *neighbor coverage fraction* of v within Q_j.

A higher value of $\phi(v, Q_j)$ indicates that Q_j covers more neighbors of v in G_j, thereby capturing more information about v's cross-layer neighborhood. Based on Definition 3.2, we can define the concept of generalized core (gCore) on general multi-layer graphs.

Definition 3.3. Given an l-layer graph $\mathcal{M}$, a specific layer G_i, l thresholds $k_1, k_2, \ldots, k_l \in \mathbb{N}$ and $p_{i,j} \in [0, 1]$ for $j = 1, 2, \ldots, l$ and $j \neq i$, a vertex subset $Q_i \subseteq V_i$ is a *generalized core (gCore)* in $\mathcal{M}$ if

(1) Q_i is a k_i-core in G_i.
(2) For $j = 1, 2, \ldots, l$ and $j \neq i$, there exists a nonempty k_j-core Q_j in $G_j[Q_i]$ such that $\phi(v, Q_j) \ge p_{i,j}$ for all $v \in Q_i$.
(3) None of the proper supersets of Q_i satisfies (1) and (2).

Without loss of generality, let the selected layer in Definition 3.3 be G_l in the rest of the paper. For simplicity, let $\mathbf{k} = (k_1, k_2, \ldots, k_l)$ and $\mathbf{p} = (p_{l,1}, p_{l,2}, \ldots, p_{l,l-1})$. A gCore defined w.r.t. $\mathbf{k}$ and $\mathbf{p}$ is called a $(\mathbf{k}, \mathbf{p})$-core. Obviously, multi-layer $\mathbf{k}$-core studied on pillar multi-layer graphs [11] is a special $(\mathbf{k}, \mathbf{p})$-core where $\mathbf{p} = \mathbf{1}^{l-1}$.

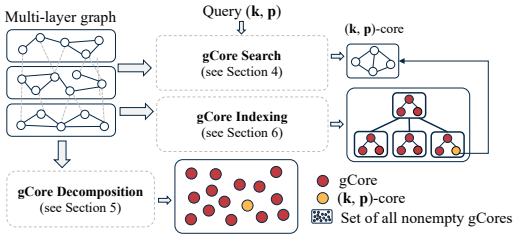


Figure 4: The relationships between the problems in this paper.

Example. Figure 3 shows a general multi-layer graph with 3 layers. Assume G_2 is of users' interest. Let $\mathbf{k} = (3, 3, 3)$, $\mathbf{p}_1 = (0, 0)$, $\mathbf{p}_2 = (1/2, 0)$, $\mathbf{p}_4 = (1/2, 2/3)$. The entire vertex set of G_2, V_2, forms the $(\mathbf{k}, \mathbf{p}_1)$-core as $\mathbf{p}_1$ does not impose any constraints on the cohesiveness of the vertices on G_0 and G_1, and V_2 itself is a 3-core. The $(\mathbf{k}, \mathbf{p}_2)$-core is $\{1, 2, 3, 4, 5, 6, 7, 8\}$. Vertices $\{9, 10\}$ are excluded from the $(\mathbf{k}, \mathbf{p}_2)$-core because they have no cross-layer neighbors in G_0. $Q = \{1, 2, 3, 4\}$ forms the $(\mathbf{k}, \mathbf{p}_3)$-core. Vertices $\{22, 23, 24, 25, 26\}$ on G_0 and Vertices $\{13, 14, 15, 16, 17\}$ on G_1 are both 3-cores and cover at least $1/2$ and $2/3$ neighbors of each vertex in Q, respectively.

Properties. The concept of gCore has several elegant properties.

Property 1. *Given* $\mathbf{k}$ *and* $\mathbf{p}$*, the* $(\mathbf{k}, \mathbf{p})$*-core is unique.*

As a notation, let $\mathbf{x}$ and $\mathbf{y}$ be two vectors of the same dimension, we say $\mathbf{x} \le \mathbf{y}$ if $\mathbf{x}[i] \le \mathbf{y}[i]$ for every dimension i, and we say $\mathbf{x} < \mathbf{y}$ if $\mathbf{x} \le \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$.

Property 2. *Given* $\mathbf{k}_1, \mathbf{k}_2$ *and* $\mathbf{p}$*, if* $\mathbf{k}_1 \le \mathbf{k}_2$*, the* $(\mathbf{k}_2, \mathbf{p})$*-core is a subset of the* $(\mathbf{k}_1, \mathbf{p})$*-core.*

Property 3. *Given* $\mathbf{k}$*,* $\mathbf{p}_1$ *and* $\mathbf{p}_2$*, if* $\mathbf{p}_1 \le \mathbf{p}_2$*, the* $(\mathbf{k}, \mathbf{p}_2)$*-core is a subset of the* $(\mathbf{k}, \mathbf{p}_1)$*-core.*

Due to limited space, we leave the proofs of all the properties, lemmas and theorems in this paper in Appendix H.

3.3 Problem Formulation

This paper studies the following three problems related to the proposed gCore model on general multi-layer graphs.

gCore Search (GCS). Given an l-layer graph $\mathcal{M}$, $\mathbf{k} \in \mathbb{N}^l$ and $\mathbf{p} \in [0, 1]^{l-1}$, find the $(\mathbf{k}, \mathbf{p})$-core in $\mathcal{M}$.

gCore Decomposition (GCD). Given an l-layer graph $\mathcal{M}$, enumerate all nonempty and distinct gCores in $\mathcal{M}$.

gCore Indexing (GCI). Given an l-layer graph $\mathcal{M}$, store all nonempty gCores in $\mathcal{M}$ in a compact data structure to support fast retrieval of the $(\mathbf{k}, \mathbf{p})$-core for given $\mathbf{k}$ and $\mathbf{p}$.

The relationships between these problems are illustrated in Figure 4. Given $\mathcal{M}$, $\mathbf{k}$ and $\mathbf{p}$, the GCS problem directly finds the $(\mathbf{k}, \mathbf{p})$-core in $\mathcal{M}$. A standalone GCS algorithm (without using any indexes) will be proposed in Section 4. The GCD problem enumerates all nonempty and distinct gCores in $\mathcal{M}$, which will be studied in Section 5. The GCI problem designs a data structure to compactly store all gCores found by the GCD problem (Section 6). This data structure enables fast retrieval of the $(\mathbf{k}, \mathbf{p})$-core for certain $\mathbf{k}$ and $\mathbf{p}$ and derives an index-based GCS algorithm (Section 6).

4 GCORE SEARCH (GCS)

Given an l-layer graph $\mathcal{M}$, $\mathbf{k} \in \mathbb{N}^l$ and $\mathbf{p} \in [0, 1]^{l-1}$, we propose a polynomial-time algorithm called GCS to find the $(\mathbf{k}, \mathbf{p})$-core

Algorithm 1 GCS (gCore Search)

Input: an l-layer graph $\mathcal{M}$, $\mathbf{k} \in \mathbb{N}^l$ and $\mathbf{p} \in [0,1]^{l-1}$
Output: the $(\mathbf{k},\mathbf{p})$-core in $\mathcal{M}$
1: $Q_l \leftarrow V_l$ ▷ V_l is the vertex set of G_l
2: **repeat**
3: $Q_{before} \leftarrow Q_l$
4: $Q_l \leftarrow \text{peel}(G_l[Q_l], \mathbf{k}[l])$
5: **for** $i \leftarrow 1, 2, \ldots, l-1$ **do**
6: $Q_i \leftarrow \text{peel}(G_i[Q_l], \mathbf{k}[i])$
7: **for** $v \in Q_l$ **do**
8: **if** $\phi(v, Q_i) < \mathbf{p}[i]$ **then**
9: $Q_l \leftarrow Q_l - \{v\}$
10: **until** $Q_l = Q_{before}$
11: **return** Q_l

in $\mathcal{M}$. The algorithm follows the *peeling paradigm* that has been successfully used by the existing core search/decomposition algorithms [1], that is, it iteratively removes from G_l the vertices that violate Constraint (1) or (2) specified in Definition 3.3.

Algorithm 1 presents the pseudocode of GCS. We use Q_l to keep the vertices remaining in G_l after vertex peeling. Initially, Q_l is set to the vertex set V_l of G_l in line 1. The **repeat** loop in lines 2–10 carries out the vertex peeling process on Q_l as follows. In line 3, a copy of Q_l is saved in Q_{before} to support checking the variation of Q_l in line 10. In line 4, we call the peel function to iteratively remove the vertices with degrees less than $\mathbf{k}[l]$ (i.e., violating Constraint (1) in Definition 3.3) from $G_l[Q_l]$. The output of peel, the $\mathbf{k}[l]$-core in $G_l[Q_l]$, is updated to Q_l. Then, for each other layer G_i, where $1 \le i < l$, we get the cross-layer subgraph $G_i[Q_l]$ of G_i induced by Q_l, and call peel to obtain the $\mathbf{k}[i]$-core in $G_i[Q_l]$, which is assigned to Q_i. Next, we check the neighbor coverage fraction $\phi(v, Q_i)$ for each vertex $v \in Q_l$ in lines 7–9. If $\phi(v, Q_i) < \mathbf{p}[i]$, v violates Constraint (2) in Definition 3.3, so we remove v from Q_l (line 9). If Q_l is changed in this iteration, that is, $Q_l \neq Q_{before}$, the **repeat** loop continues to peel vertices from Q_l; Otherwise, no more vertices can be peeled from Q_l, and thus, the algorithm returns Q_l and terminates.

Theorem 4.1. *The output of Algorithm 1 is the* $(\mathbf{k},\mathbf{p})$*-core in* $\mathcal{M}$.

We implement Algorithm 1 based on a set of well-designed arrays (see Appendix A), leading to a time complexity of $O(|\mathcal{M}| + l|V_l|)$ and a space complexity of $O(l \cdot V_{max})$, where $|\mathcal{M}| = \sum_{i=1}^{l} |V_i| + |E(\mathcal{G})| + |E(C)|$ and $V_{max} = \max_{1 \le i \le l} |V_i|$. Complexity Analysis for all algorithms is left in Appendix I for saving space.

5 GCORE DECOMPOSITION (GCD)

In this section, we propose an algorithm for the gCore decomposition (GCD) problem. As with the multi-layer cores [11], the number of distinct gCores can be exponential in the number of layers, and not all of them are nested into one another. Hence, we cannot expect to apply the vertex peeling paradigm used in core decomposition on simple graphs to find all gCores. To handle this problem, we first present a naïve solution in Section 5.1. By exploiting the containment relationships among gCores as described in Properties 2 and 3, a tree-based structure called *KP-tree* is introduced to model the search space of the GCD problem in Section 5.2. The KP-tree structure determines a good order for fast computing all gCores.

5.1 Naïve GCD Algorithm

A naïve solution to the GCD problem consists of two phases. In phase 1, we enumerate all $(\mathbf{k},\mathbf{p})$ pairs such that the $(\mathbf{k},\mathbf{p})$-core is nonempty. In phase 2, we use the GCS algorithm (Algorithm 1) to compute the $(\mathbf{k},\mathbf{p})$-core for each $(\mathbf{k},\mathbf{p})$ pair enumerated in phase 1. Obviously, phase 1 is the key to the algorithm design.

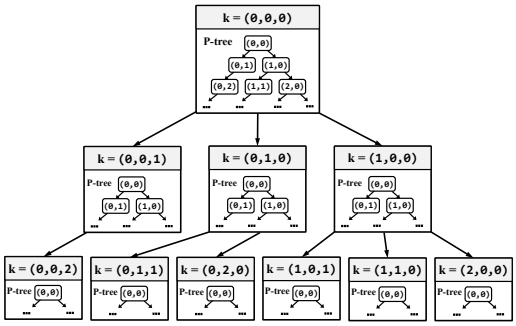


Figure 5: The structure of a KP-tree of a 3-layer graph.

All elements of $\mathbf{k}$ are integers. For $i = 1, 2, \ldots, l$, we have $0 \le \mathbf{k}[i] \le \kappa(G_i)$, where $\kappa(G_i)$ is the degeneracy of G_i, the largest integer k such that the k-core of G is nonempty. If $\mathbf{k}[i] > \kappa(G_i)$ for some $i \in \{1, 2, \ldots, l\}$, the $\mathbf{k}[i]$-core of G_i is certainly empty, thereby violating Constraint (1) or (2) specified in Definition 3.3. Therefore, we only need to consider possible values for $\mathbf{k}$ ranging from $(0, 0, \ldots, 0)$ to $(\kappa(G_1), \kappa(G_2), \ldots, \kappa(G_l))$.

Every element of $\mathbf{p}$ is a real number in $[0, 1]$. It is definitely infeasible to enumerate all values of $\mathbf{p}$. However, we found that it is sufficient to choose $\mathbf{p}[i]$ from the finite set F_i defined as below.

Lemma 5.1. *For any* $(\mathbf{k},\mathbf{p})$*-core* Q *in the multi-layer graph* $\mathcal{M}$*, there exists a vector* $\hat{\mathbf{p}}$ *with* $\hat{\mathbf{p}}[i] \in F_i$ *for* $i = 1, 2, \cdots, l-1$ *such that the* $(\mathbf{k},\hat{\mathbf{p}})$*-core in* $\mathcal{M}$ *is identical to* Q*, where*

$$F_i = \left\{ \frac{j}{deg_i(v)} \middle| v \in V_l, j = 0, 1, \ldots, deg_i(v) \right\}. \quad (1)$$

The generation of each F_i is obvious, and its pseudocode will be given in Procedure GenFrac in Algorithm 5 in Appendix B.

After introducing how to set possible values for $\mathbf{k}$ and $\mathbf{p}$, the naïve algorithm for the GCD problem is clear. Due to limited space, we leave the pseudocode in Algorithm 5 in Appendix B. The correctness of the algorithm is obvious. The time complexity of the naïve approach is $O(\prod_{i=1}^{l} \kappa(G_i) \cdot \prod_{i=1}^{l-1} |F_i| \cdot (|\mathcal{M}| + l|V_l|) + \sum_{i=1}^{l-1} d_i^2 \log d_i)$, and the space complexity is $O(l \cdot (\prod_{i=1}^{l} \kappa(G_i) + \prod_{i=1}^{l-1} |F_i| + V_{max}))$, where d_i is the maximum cross-layer degree of vertices in V_l on layer G_i.

The naïve approach has two main disadvantages. First, it computes the $(\mathbf{k},\mathbf{p})$-core for all enumerated $(\mathbf{k},\mathbf{p})$ pairs, although many of them are empty. Second, each $(\mathbf{k},\mathbf{p})$-core is independently computed by Algorithm 1 from scratch. A large number of repeated computations are carried out for different $(\mathbf{k},\mathbf{p})$ pairs.

5.2 GCD Algorithm based on KP-Trees

To overcome the disadvantages of the naïve approach, we design a "tree of trees" data structure called *KP-tree* to represent the search space of the GCD problem. We will show later that by traversing a KP-tree, it is possible to fast identify empty gCores and reduce repeated computations when computing different gCores.

5.2.1 **KP-Trees.** As illustrated in Figure 5, the overall structure of a KP-tree is a tree. Each node in the KP-tree is associated with

an l-dimensional vector $\mathbf{k} \in \mathbb{N}^l$. We call the node associated with a vector $\mathbf{k}$ the $\mathbf{k}$-*node*. The root of the KP-tree is the $\mathbf{0}^l$-node. In the KP-tree, the $\mathbf{k}$-node is the parent of the $\mathbf{k}'$-node if and only if $\mathbf{k}'$ is an immediate suffix successor of $\mathbf{k}$ as defined below.

Definition 5.2. Let $\mathbf{x}$ and $\mathbf{y}$ be two integer vectors of the same dimension. $\mathbf{y}$ is an *immediate suffix successor* of $\mathbf{x}$ if

(1) $\mathbf{x}$ and $\mathbf{y}$ are different in exactly one element, say $\mathbf{x}[i]$ and $\mathbf{y}[i]$;
(2) $\mathbf{x}[i] + 1 = \mathbf{y}[i]$;
(3) $\mathbf{x}[j] = \mathbf{y}[j] = 0$ for all $j > i$.

Obviously, if $\mathbf{y}$ is an immediate suffix successor of $\mathbf{x}$, we have $\mathbf{x} < \mathbf{y}$. Let $end0(\mathbf{k})$ be the number of consecutive zeros at the end of a vector $\mathbf{k}$. It's easy to see that the $\mathbf{k}$-node in the KP-tree has $end0(\mathbf{k}) + 1$ children if $\mathbf{k} \neq \mathbf{0}^l$ and l children if $\mathbf{k} = \mathbf{0}^l$.

Inside every node of the KP-tree is nested another tree called *P-tree*. Each node in the P-tree is associated with an $(l-1)$-dimensional real vector $\mathbf{p} \in [0, 1]^{l-1}$, and the node is called the $\mathbf{p}$-node. In the rest of the paper, we actually store $\mathbf{p}$ as an integer vector. As described in Section 5.1, the i-th element of $\mathbf{p}$ is chosen from the set F_i of fractional numbers formulated in Equation (1). Therefore, after sorting the elements in F_i in increasing order, we store in $\mathbf{p}[i]$ the index of the fractional value in F_i. It helps to define the P-tree and save storage. Specifically, in the P-tree, the root is the $\mathbf{0}^{l-1}$-node, the $\mathbf{p}$-node is the parent of the $\mathbf{p}'$-node if and only if $\mathbf{p}'$ is an immediate suffix successor of $\mathbf{p}$. Thus, the $\mathbf{p}$-node has $end0(\mathbf{p}) + 1$ children if $\mathbf{p} \neq \mathbf{0}^{l-1}$ and $l - 1$ children if $\mathbf{p} = \mathbf{0}^{l-1}$.

The following lemma ensures that the KP-tree is a systematic organization of gCores: a $\mathbf{p}$-node nested in a $\mathbf{k}$-node represents a distinct $(\mathbf{k}, \mathbf{p})$ pair that uniquely corresponds to the $(\mathbf{k}, \mathbf{p})$-core.

Lemma 5.3. *For each possible value of* $\mathbf{k}$*, there is exactly one* $\mathbf{k}$*-node in the KP-tree. And in every* $\mathbf{k}$*-node, for each possible value of* $\mathbf{p}$*, there is exactly one* $\mathbf{p}$*-node in the P-tree nested in the* $\mathbf{k}$*-node.*

5.2.2 **KP-tree-based GCD Algorithm.** The KP-tree has several properties that help overcome the disadvantages of the naïve method.

First, let N and N' be two nodes in a P-tree representing the $(\mathbf{k}, \mathbf{p})$-core Q and the $(\mathbf{k}, \mathbf{p}')$-core Q', respectively. If N is an ascendant of N', we have $\mathbf{p} < \mathbf{p}'$. According to Property 3, we have $Q' \subseteq Q$. This observation has two implications:

I1. If $Q = \emptyset$, we also have $Q' = \emptyset$, so it is unnecessary to compute Q' by Algorithm 1.
I2. Based on the following Lemma 5.4, Q' can be computed on the subgraph $\mathcal{M}_{\mathbf{k}}[Q]$ instead of on the entire $\mathcal{M}$, where $\mathcal{M}_{\mathbf{k}}[Q]$ is the subgraph of $\mathcal{M}$ induced by $\mathcal{Q} = \{Q_1, Q_2, \cdots, Q_l\}$, Q_i is the $\mathbf{k}[i]$-core of $G_i[Q]$ for $1 \leq i < l$, and $Q_l = Q$.

Lemma 5.4. *Given a multi-layer graph* $\mathcal{M}$*, for any pair of the* $(\mathbf{k}, \mathbf{p})$*-core* Q *and the* $(\mathbf{k}', \mathbf{p}')$*-core* Q' *in* $\mathcal{M}$*,* $\mathcal{M}_{\mathbf{k}'}[Q']$ *is a subgraph of* $\mathcal{M}_{\mathbf{k}}[Q]$ *if* $\mathbf{k} \leq \mathbf{k}'$ *and* $\mathbf{p} \leq \mathbf{p}'$*.*

Second, let N and N' be two nodes in the KP-tree with vectors $\mathbf{k}$ and $\mathbf{k}'$, respectively. If N is an ascendant of N', we have $\mathbf{k} < \mathbf{k}'$. By Property 2, the $(\mathbf{k}', \mathbf{p})$-core Q' is a subset of the $(\mathbf{k}, \mathbf{p})$-core Q for any vector $\mathbf{p}$. Thus, Implications I1 and I2 also hold for Q and Q'.

Implication I1 helps to fast identify empty $(\mathbf{k}, \mathbf{p})$-cores, and Implication I2 helps to reduce repeated computations when computing different gCores. Based on them, we propose an efficient GCD algorithm called GCD+ in Algorithm 2. The idea is to make a depth-first search (DFS) on the KP-tree, and for each visited $\mathbf{k}$-node, make a DFS on the P-tree nested in the $\mathbf{k}$-node, so that the implications above can be leveraged as much as possible. Notice that GCS used in line 5 of Algorithm 2 needs to be simply adapted to return Q_l as well as $Q_1, Q_2, \ldots, Q_{l-1}$, where Q_i is the set of vertices remaining on layer G_i when GCS terminates. We leave the description of Algorithm 2 and the discussion on alternatives of the KP-tree, for example, the 'lattice' structure used in [11], in Appendix C.

Algorithm 2 GCD+ (KP-tree-based gCore Decomposition)

Input: An l-layer graph $\mathcal{M}$
Output: The collection R of all nonempty gCores
1: $R \leftarrow \emptyset$
2: KPTreeDFS($\mathcal{M}, \mathbf{0}^l, R$)
3: **return** R
4: **procedure** PTreeDFS($\mathcal{M}, \mathbf{k}, \mathbf{p}, R$)
5: $\{Q_1, Q_2, \ldots, Q_l\} \leftarrow$ GCS($\mathcal{M}, \mathbf{k}$, ToFrac($\mathbf{p}$))
6: **if** $Q_l \neq \emptyset$ **then**
7: $R \leftarrow R \cup \{(Q_l, \mathbf{k}, \mathbf{p})\}$
8: **for** $i \leftarrow l-1, l-2, \ldots, l - end0(\mathbf{p}) - 1$ **do**
9: **if** $i > 0$ and $\mathbf{p}[i] < |F_i|$ **then**
10: $\mathbf{p}' \leftarrow \mathbf{p}$
11: $\mathbf{p}'[i] \leftarrow \mathbf{p}'[i] + 1$
12: PTreeDFS($\mathcal{M}[\{Q_1, Q_2, \ldots, Q_l\}], \mathbf{k}, \mathbf{p}', R$)
13: **return** $\{Q_1, Q_2, \ldots, Q_l\}$
14: **procedure** KPTreeDFS($\mathcal{M}, \mathbf{k}, R$)
15: $\{Q_1, Q_2, \ldots, Q_l\} \leftarrow$ PTreeDFS($\mathcal{M}, \mathbf{k}, \mathbf{0}^{l-1}, R$)
16: **if** $Q_l \neq \emptyset$ **then**
17: **for** $i \leftarrow l, l-1, \ldots, l - end0(\mathbf{k})$ **do**
18: **if** $i > 0$ and $\mathbf{k}[i] < \kappa(G_i)$ **then**
19: $\mathbf{k}' \leftarrow \mathbf{k}$
20: $\mathbf{k}'[i] \leftarrow \mathbf{k}'[i] + 1$
21: KPTreeDFS($\mathcal{M}[\{Q_1, Q_2, \ldots, Q_l\}], \mathbf{k}', R$)
22: **procedure** ToFrac($\mathbf{p}$)
23: **return** $(F_1[\mathbf{p}[1]], F_2[\mathbf{p}[2]], \ldots, F_{l-1}[\mathbf{p}[l-1]])$ ▷ Convert $\mathbf{p}$ to its fractional form

Theorem 5.5. *Algorithm 2 performs generalized core decomposition, returning all nonempty gCores in* $\mathcal{M}$*.*

The time complexity of Algorithm 2 is $O(\prod_{i=1}^{l} \kappa(G_i) \cdot \prod_{i=1}^{l-2} |F_i| \cdot (|\mathcal{M}| + l|V_l|) + l \cdot \prod_{i=1}^{l} \kappa(G_i) \cdot \prod_{i=1}^{l-1} |F_i| + \sum_{i=1}^{l-1} d_i^2 \log d_i$. The space complexity of Algorithm 2 is $O(l \cdot (V_{max} + \sum_{i=1}^{l-1} |F_i|))$.

6 GCORE INDEXING (GCI)

In this section, we study how to store and index the results of the GCD problem to enable fast search of any $(\mathbf{k}, \mathbf{p})$-core.

6.1 Storage and Index Structure

When GCD+ (Algorithm 2) terminates, it conceptually generates a subtree of the KP-tree where a $\mathbf{p}$-node nested in a $\mathbf{k}$-node uniquely represents a nonempty $(\mathbf{k}, \mathbf{p})$-core. By materializing this subtree of KP-tree, a storage and index structure can be designed.

Specifically, for each $\mathbf{k}$-node in it, we use a hash table to map $\mathbf{k}$ to the $\mathbf{k}$-node to support fast locating the node. To store all nonempty gCores in the P-tree nested in the $\mathbf{k}$-node, say T, we augment the structure of T in the following way:

(1) For any leaf node N in T, we add a dummy node L to represent an empty gCore and designate N as the parent of L.
(2) Let N and N' be two nodes in T representing gCores Q and Q', respectively, and N' is the *leftmost* child of N. We store the set $Q - Q'$ along with the edge between N and N'. The concept of the *leftmost* child of N is defined based on a total order $\prec$ on N's children as given below.

Definition 6.1. Let N' and N'' be two children of a node N. We have $N' \prec N''$ if $end0(\mathbf{p}') < end0(\mathbf{p}'')$, where $\mathbf{p}'$ and $\mathbf{p}''$ are the vectors associated with N' and N'', respectively.

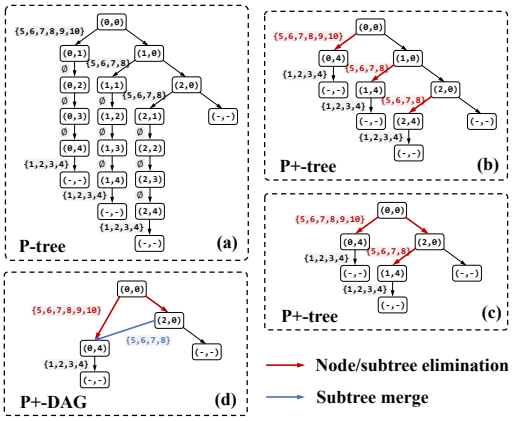


Figure 6: Augmented structures nested in the k-node of the KP-tree index of the example multi-layer graph in Figure 3, where $\mathbf{k} = (3, 3, 3)$. (a) The original augmented P-tree. (b-d) The compact structures obtained by applying different compaction techniques.

A node N' is the leftmost child of its parent N if $N' \prec N''$ for all siblings N'' of N'. Figure 6 (a) illustrates an augmented P-tree. Obviously, for any node N in the P-tree, all sets associated with the edges on N's leftmost path are retained, where the leftmost path of N is obtained by following the leftmost child of N, the leftmost child of the leftmost child of N ... until a leaf node is reached.

Theorem 6.2. *Let N be the node representing the gCore Q. We have that Q equals the union of the sets associated with the edges on the leftmost path from N to a leaf node.*

Theorem 6.2 ensures that the augmented P-tree is a compact storage of all gCores corresponding to the nodes in the P-tree. In the rest of the paper, we say KP-tree refers to the storage and index structure constructed in the above way.

6.2 Index-based GCS Algorithm

In this subsection, we propose a fast gCore search (GCS) algorithm (Algorithm 3) based on the KP-tree index proposed in Section 6.1. Given vectors $\mathbf{k} \in \mathbb{N}^l$ and $\mathbf{p} \in [0, 1]^{l-1}$, the algorithm retrieves the $(\mathbf{k}, \mathbf{p})$-core from the KP-tree. It works as follows:

(1) Find a node N in the P-tree nested in the $\mathbf{k}$-node of the KP-tree that represents the $(\mathbf{k}, \mathbf{p})$-core.
(2) Compute the union of all vertex sets associated with the edges on the leftmost path of N (Theorem 6.2).

The $\mathbf{k}$-node can be easily found from the hash table of the KP-tree with the key $\mathbf{k}$ (line 1 of Algorithm 3). Since $\mathbf{p}$ is a real vector, it is very likely that $\mathbf{p} \notin F_1 \times F_2 \times \cdots \times F_{l-1}$, and therefore the $\mathbf{p}$-node is not indexed in the KP-tree. To handle this, we find a $\hat{\mathbf{p}}$-node in the P-tree such that $(\mathbf{k}, \mathbf{p})$-core is identical to the $(\mathbf{k}, \hat{\mathbf{p}})$-core. The proof of Lemma 5.1 (Appendix H) provides us a method to efficiently identify $\hat{\mathbf{p}}$: For $i = 1, 2, \ldots, l-1$, $\hat{\mathbf{p}}[i]$ is the smallest element in F_i that is not less than $\mathbf{p}[i]$. If $\mathbf{p} \in F_1 \times F_2 \times \cdots \times F_{l-1}$, we have $\hat{\mathbf{p}} = \mathbf{p}$.

After knowing $\hat{\mathbf{p}}$, Algorithm 3 calls Procedure Search (line 5–10) in line 3 to search for the $\hat{\mathbf{p}}$-node (line 3). Then, Procedure Recover (lines 18–25) is called in line 4 to get the union of vertex sets on the leftmost path of the $\hat{\mathbf{p}}$-node, and the result is subsequently returned. The pseudocode for Procedure Search and Procedure Recover are clear, and we leave the detailed description in Appendix D.

Algorithm 3 GCS+ (Index-based gCore Search)

Input: The KP-tree for an l-layer graph $\mathcal{M}$, a vector $\mathbf{k} \in \mathbb{N}^l$ and a vector $\mathbf{p} \in [0, 1]^{l-1}$
Output: The $(\mathbf{k}, \mathbf{p})$-core in $\mathcal{M}$

```
1: Find the k-node from the hash table of the KP-tree with the key k
2: T ← the P-tree nested in the k-node
3: N ← Search(T, p)
4: return Recover(N)
5: procedure Search(T, p)
6:     N ← the root of the P-tree T
7:     i ← 1
8:     while i < l do
9:         (N, i) ← Forward(N, i, p)
10:    return N
11: procedure Forward(N, i, p)
12:    if p'[i] < p[i] then                ▷ p' is the vector (in the fractional form) associated with N
13:        for each child N' of N do
14:            if the vector associated with N' and p' are different in their i-th element then
15:                return (N', i)
16:    else
17:        return (N, i + 1)
18: procedure Recover(N)
19:    Q ← ∅
20:    while N is not a dummy leaf do
21:        N' ← the leftmost child of N
22:        S ← the set of vertices associated with the edge between N and N' in the P-tree
23:        Q ← Q ∪ S
24:        N ← N'
25:    return Q
```

Theorem 6.3. *Algorithm 3 returns the $(\mathbf{k}, \mathbf{p})$-core in $\mathcal{M}$.*

Algorithm 3 runs in $O(\sum_{i=1}^{l-1} |F_i| + |Q|)$ time and $O(|Q|)$ space, where $|Q|$ is the size of the target $(\mathbf{k}, \mathbf{p})$-core.

7 COMPACTION OF P-TREES

The gCores represented by the nodes in a P-tree may be not distinct. For example, in Figure 6 (a), the $\mathbf{p}$-nodes for $\mathbf{p} = (2, 1)$, $(2, 2)$, $(2, 3)$, and $(2, 4)$ all represent the gCore $\{1, 2, 3, 4\}$. Storing these nodes incurs unnecessary storage overheads and increases computational costs. In this section, we study how to eliminate unnecessary nodes from a P-tree with a guarantee of any gCore being correctly retrieved using Algorithm 3 on the obtained compact structure.

7.1 Foundations

Let us first introduce the foundations of our P-tree compaction methods. Two nodes N and N' in a P-tree are *redundant*, denoted by $N \cong N'$, if they represent the same gCore. The binary relation $\cong$ is obviously reflective, symmetric and transitive, so $\cong$ is an equivalence relation. The equivalence class of N under $\cong$ is $[N] = \{N' | N' \cong N\}$.

Definition 7.1. The vector $\mathbf{p}$ associated with a node $N \in [N]$ is *maximal* if $\mathbf{p}' \preceq \mathbf{p}$ for all vectors $\mathbf{p}'$ associated with nodes $N' \in [N]$.

Theorem 7.2. *The maximal vector for $[N]$ is unique.*

The following theorem helps to get the maximal vector for $[N]$.

Theorem 7.3. *Let Q be the $(\mathbf{k}, \mathbf{p})$-core represented by N in the P-tree and let $\hat{\mathbf{p}}$ be the maximal vector for $[N]$. We have $\hat{\mathbf{p}}[i] = \min_{v \in Q} \phi(v, Q_i)$ for $i = 1, 2, \ldots, l-1$, where Q_i is the $\mathbf{k}[i]$-core of $G_i[Q]$ and $\phi(v, Q_i)$ is defined in Definition 3.2.*

The maximal vector for $[N]$ can uniquely characterize $[N]$:

Lemma 7.4. *For two nodes N and N' in a P-tree, we have $N \cong N'$ if and only if the maximal vectors for $[N]$ and $[N']$ are identical.*

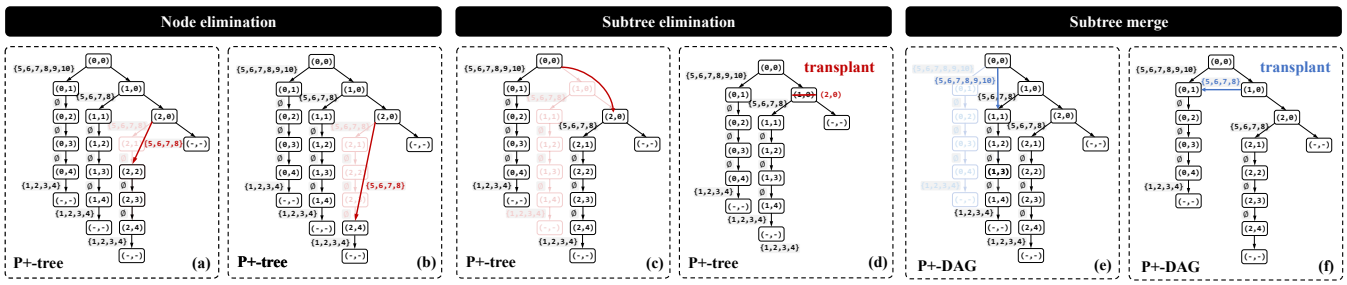


Figure 7: Illustration of P-tree compaction techniques: (a-b) node elimination, (c-d) subtree elimination and (e-f) subtree merge.

7.2 Node Elimination

Let N and N' be two nodes in a P-tree. If $N \cong N'$ and N' is N's only child, we can remove N because the gCore represented by N can be identically obtained by searching for the gCore represented by N'. If N has a parent, we link N' to N's parent as a new child. We call the process "node elimination".

Example. An example of node elimination is shown in Figure 7 (a). Let N_0, N_1 and N_2 be the nodes with $\mathbf{p} = (2,0)$, $(2,1)$ and $(2,2)$, respectively. We have $N_1 \cong N_2$ and N_2 is N_1's only child. Therefore, N_1 can be removed. After removing N_1, N_0 becomes N_2's parent.

After applying node elimination, the obtained tree is no longer a P-tree as the vector associated with N' is not an immediate suffix successor (Definition 5.2) of the new parent's vector. We name the new tree *P+-tree*, which satisfies the following property.

Property 4. *Let N and N' be two nodes in a P+-tree, and let $\mathbf{p}$ and $\mathbf{p}'$ be the vectors associated with N and N', respectively. If N is the parent of N', $\mathbf{p}'$ is a suffix successor of $\mathbf{p}$, that is,*

(1) $\mathbf{p}$ and $\mathbf{p}'$ are different in exactly one element, say $\mathbf{p}[i]$ and $\mathbf{p}'[i]$;
(2) $\mathbf{p}[i] < \mathbf{p}'[i]$;
(3) $\mathbf{p}[j] = \mathbf{p}'[j] = 0$ for all $j > i$.

The node elimination procedure presented above can be repeatedly applied to a P+-tree until no node can be removed anymore. The following theorem guarantees the correctness of applying Algorithm 3 on a P+-tree to solve the GCS problem.

Theorem 7.5. *Given a P-tree and the P+-tree obtained by repeatedly applying the node elimination procedure, when the $(\mathbf{k}, \mathbf{p})$-core represented by a node N in the P-tree is queried, Procedure Search in Algorithm 3 returns a node N^* in the P+-tree. We have (1) $N \cong N^*$; (2) The sets (except $\emptyset$) on the leftmost path from N' to a leaf node in the P-tree are all retained on the leftmost path from N^* to a leaf node in the P+-tree, where N' is the copy of N^* in the P-tree.*

Example. The P+-tree shown in Figure 7 (b) is obtained by eliminating the nodes with vectors $\mathbf{p} = (2,1)$, $(2,2)$ and $(2,3)$. When the $(\mathbf{k}, \mathbf{p})$-core for $\mathbf{p} = (2,2)$ is queried, Procedure Search in Algorithm 3 returns the $\mathbf{p}^*$-node N^*, where $\mathbf{p}^* = (2,4)$. The vertex set $\{1,2,3,4\}$ on the leftmost path of N^* is returned as the $(\mathbf{k}, \mathbf{p})$-core.

Redundant Node Detection. Lemma 7.4 implies an easier way to test the equivalence between two nodes N and N' in a P-tree if N' is N's only child as described in the following theorem.

Theorem 7.6. *Let N and N' be two nodes in a P-tree, and N' is N's only child. Let $\hat{\mathbf{p}}$ be the maximal vector for $[N]$ and $\mathbf{p}'$ be the vector associated with N'. We have $N \cong N'$ if and only if $\mathbf{p}' \leq \hat{\mathbf{p}}$.*

7.3 Subtree Elimination

If a node N in a P-tree has more than one child, node elimination can never be applied to N. To overcome this limitation, we extend node elimination to subtree elimination in this subsection.

First, let us define some concepts. For a node N in a P-tree, based on the order $\prec$ (Definition 6.1) on N's children, we have a node N' is the rightmost child of N if $N'' \prec N'$ for all siblings N'' of N'. $N_1, N_2, \cdots, N_m$ is the rightmost path of N if $N_1 = N$, N_{i+1} is the rightmost child of N_i for $1 \leq i < m$ and N_m is a leaf node.

Definition 7.7. For a non-leaf node N in a P-tree, let N' be the rightmost child of N. The subtree rooted at N except the subtree rooted at N' is called the preceding subtree rooted at N.

Definition 7.8. Let T and T' be two subtrees in a P-tree rooted at R and R', respectively. T and T' are redundant if

(1) $R \cong R'$, that is, R and R' are redundant;
(2) R and R' have the same number of children, and for the i-th child N_i of R and the i-th child N_i' of R', the subtrees rooted at N_i and N_i' are redundant.

In other words, T is isomorphic to T' in terms of the node redundancy relationship $\cong$, which we denote by $T \cong T'$.

If $T \cong T'$, T and T' have the same number of nodes, and there exists a bijection f from the nodes of T to the nodes of T':

$$f(N) = \begin{cases} R' & \text{if } N = R, \\ \text{the } i\text{-th child of } f(N') & \text{if } N \text{ is the } i\text{-th child of its parent } N' \end{cases} \tag{2}$$

Under the bijection f, we have

(1) $N \cong f(N)$ for all nodes N in T;
(2) N is the parent of N' in T if and only if $f(N)$ is the parent of $f(N')$ in T'.

The fact that every node N in T is redundant w.r.t. $f(N)$ in T' makes T a redundant subtree. The following *subtree elimination* procedure removes redundant preceding subtrees from a P-tree.

For two nodes N and N' in a P-tree, and N' is N's rightmost child, if the preceding subtree rooted at N, say T, is isomorphic to the preceding subtree rooted at N', say T', we remove T from the P-tree. If N has a parent, we link N' to N's parent as a new child. An example of subtree elimination is illustrated in Figure 7 (c).

Clearly, node elimination presented in Section 7.2 is a special case of subtree elimination when N' is N's only child. After subtree elimination, the obtained tree is also a P+-tree. Besides, subtree elimination can also be repeatedly applied to a P+-tree. Theorem 7.5 still holds if "node elimination" is replaced by "subtree elimination".

Redundant Subtree Detection. By extending the redundant node detection method described in Theorem 7.6, we introduce an efficient

way to identify redundant subtrees in a P-tree. The basic idea is to define a unique signature for a subtree and test if two subtrees are isomorphic by comparing their signatures.

Definition 7.9. Let T be a subtree in a P-tree. The signature of T is defined as the element-wise minimum of the maximal vectors of $[N]$ for all nodes N in T.

The following theorem provides a fast way to test whether two preceding subtrees are isomorphic.

Theorem 7.10. *Let N and N' be two nodes in a P-tree with vectors $\mathbf{p}$ and $\mathbf{p}'$, respectively, and N' is a descendant of N on N's rightmost path. Let T and T' be the preceding subtrees rooted at N and N', respectively. We have $T \cong T'$ if and only if $\mathbf{p}'[i] \leq \hat{\mathbf{p}}[i]$, where $\hat{\mathbf{p}}$ is the signature for T, and i is the dimension at which $\mathbf{p}$ differs from $\mathbf{p}'$.*

7.4 Subtree Transplant

Recall that nodes in a P-tree are generated in DFS order (Algorithm 2). For any two nodes N and N' in a P-tree, and N' is N's rightmost child, the preceding subtree rooted at N, say T, must be generated prior to N' and the preceding subtree T' rooted at N'. When N' is generated, Theorem 7.10 enables us to test the isomorphism between T and T' before T' has been generated. If $T \cong T'$, rather than removing T and taking additional time to generate T', we propose to reuse the structure of T to obtain T' directly.

Because $T \cong T'$, T and T' have the same structure but different vectors associated with their nodes. As shown in Equation 2, there is a bijection f from the nodes in T to the nodes in T'. Thus, in principle, we can update the vector associated with each node N in T to the vector associated with $f(N)$, and then transplant T to the position where T' is planted. We intuitively call this approach "*subtree transplant*". Interestingly, we found that only the vector on the root of T has to be substituted with the vector on the root of T', and the correctness of applying Algorithm 3 to solve the GCS problem on the obtained P+-tree won't be affected. (see Appendix H.4 for detailed analysis). We then have T' obtained in $O(1)$ time.

Example. An illustration of subtree transplant is given in Figure 7 (d). Let N and N' be the nodes with vectors $(1, 0)$ and $(2, 0)$, and let T and T' be the preceding subtrees rooted at N and N', respectively. We have $T \cong T'$. To realize subtree elimination, we transplant T to the location of T' by substituting the vector on N with the vector $(2, 0)$ on N'. When the $(\mathbf{k}, \mathbf{p})$-core for $\mathbf{p} = (2, 2)$ is queried, Algorithm 3 locates the node with vector $(1, 2)$ and returns $\{1, 2, 3, 4\}$.

An extension that allows multiple redundant subtrees to be removed and transplanted in one batch will be given in Appendix E.

7.5 Subtree Merge

The condition for subtree elimination is strict. For any preceding subtrees T and T' rooted at N and N', respectively, subtree elimination cannot be applied to T if N' is not N's rightmost child or T and T' are not isomorphic. The latter case is more likely to happen if N has more children. In this subsection, we propose *subtree merge*, an approach to further reduce the redundancy between T and T'.

The idea is to conceptually divide T and T' into smaller substructures, called "branches', and in which, seek redundant subtrees that can be eliminated. The concept of branch is defined as follows:

Definition 7.11. Given two nodes N and N' in a P-tree, where N' is a child of N, the subtree formed by N and the subtree rooted at N' is called a branch. Let $\mathbf{p}$ and $\mathbf{p}'$ be the vectors associated with N and N', respectively. We simply denote this branch as B_i^N, where i is the dimension at which $\mathbf{p}$ differs from $\mathbf{p}'$.

Obviously, all the branches B_i^N for $l - end0(\mathbf{p}) - \alpha_{\mathbf{p}} < i < l$ entirely cover the preceding subtree rooted at N, where $\alpha_{\mathbf{p}} = 1$ if $\mathbf{p} \neq \mathbf{0}^{l-1}$ and 0 if $\mathbf{p} = \mathbf{0}^{l-1}$.

Based on the branch structure of a P-tree, we propose the following *subtree merge* approach. Let N and R be two nodes in the P-tree, and N is a child of R. We consider each pair of branches B_i^R and B_i^N within preceding subtrees of R and N, respectively. Let $R_0, R_1, R_2, \ldots$ be the rightmost path in B_i^R where $R_0 = R$, and $N_0, N_1, N_2, \ldots$ be the rightmost path in B_i^N where $N_0 = N$. If the subtrees rooted at R_j and N_k, say T and T', are isomorphic, T is merged into T'. It can be realized by removing T and making the root of T', i.e., N_k, a child of the parent of T's root, i.e., R_{j-1}.

Example. Figure 7 (e) shows an example of subtree merge. Let R, R_1, N, N_1 be the nodes with vectors $\mathbf{p} = (0, 0), (0, 1), (1, 0), (1, 1)$, respectively. Let T and T' be the subtrees rooted at R_1 and N_1, respectively. R and T form the branch B_2^R, and N and T' form the branch B_2^N. In this example, these two branches are exactly the preceding subtrees rooted at R and N, respectively. It's obvious that they are not isomorphic because of $R \ncong N$. However, within them, T and T' are isomorphic. Therefore, we merge T into T' by removing T and making R to be the parent of N_1.

Subtree elimination and subtree merge complement each other. By applying subtree merge to a P+-tree obtained by applying subtree elimination in a similar way as applied to a P-tree, we can get a more succinct storage of all gCores.

As we can see from the above example, applying subtree merging leads to a node having more than one parent, so the obtained data structure is no longer a tree but a directed acyclic graph (DAG). We call it P+-DAG. Thus, to make the best of the compaction capabilities of node/subtree elimination, we only apply subtree merge to a P+-tree after doing a sequence of node/subtree elimination operations.

Subtree merge can be repeatedly applied to a P+-DAG until no subtrees can be merged. The following theorem ensures Algorithm 3 can be applied to solve the GCS problem on a P+-DAG.

Theorem 7.12. *Given a P+-tree and the P+-DAG obtained by repeatedly applying the subtree merge procedure, when the gCore represented by a node N in the P+-tree is queried, Procedure Search in Algorithm 3 returns a node N^* in the P+-DAG. We have (1) $N \cong N^*$; (2) The sets (except $\emptyset$) on the leftmost path from N' to a leaf node in the P+-tree are all retained on the leftmost path from N^* to a leaf node in the P+-DAG, where N' is the copy of N^* in the P+-tree.*

Isomorphic Subtrees Detection. Based on the signatures of subtrees (Definition 7.9), we derive a fast method to detect redundant subtrees that can be merged within two branches. Due to limited space, we leave the details in Appendix F.1. Furthermore, a pivot-based optimization of the detection method followed by a subtree-transplant-based implementation of subtree merge is also given in Appendix F, which leads to a $O(m + n)$ time complexity of applying subtree merge to any pair of branches, where m and n are the lengths of the rightmost paths in the two branches.

Putting it all together. Consider the P-tree shown in Figure 6 (a), which consists of 19 nodes and 15 edges associated with vertex

Table 1: Properties of graphs used in experiments.

Graph	$\|V[\mathcal{G}]\|$	$\|E[\mathcal{G}]\|$	$\|E[C]\|$	#ET	$\|L\|$
SacchCere (SC) [11]	6750	247,152	39,420	1	7
ObamaInIsrael (Oii) [27]	2,279,535	3,827,964	4,559,070	1	3
Friendfeed (FF)[8]	505,104	18,673,521	1,010,208	1	3
6-NG [26]	4,500	15,787	24,001	5	5
9-NG [26]	6,750	24,264	36,015	5	5
DBLP [32]	41,892	280,707	381,176	2	2
Twitter[1]	47,280	445,287	89,775	3	3
Movie[2]	251,742	1,183,167	502,821	2	4
Aminer-5 [33]	2,890,443	14,536,094	7,730,034	3	5
Aminer-10 [33]	4,650,693	118,763,984	14,384,941	3	5

sets. Figure 6 (b), (c) and (d) show the P+-tree/P+DAG obtained by applying the compaction techniques proposed above. Repeatedly applying the node elimination results in a P+-tree with 10 nodes and 6 edges associated with vertex sets, as shown in Figure 6 (b). The P+-tree obtained by repeatedly applying subtree elimination is illustrated in Figure 6 (c), which attains a smaller size with 5 nodes and 4 edges with sets. By continuing to apply subtree merge to it, we finally have a P+-DAG illustrated in Figure 6 (d), in which only 5 nodes and 3 sets are retained.

8 EXPERIMENTS

8.1 Experiment Setup

Setup. All algorithms were implemented in C++. All experiments were performed on a server with an Intel Xeon Gold 5218R processor and 754GB of RAM, running 64-bit Ubuntu 22.04.

Datasets. We select 3 real-world pillar multi-layer graphs and construct 7 general multi-layer graphs using publicly available datasets. The statistics of these graphs are presented in Table 1. Column #ET denotes the number of entity types. Notice that $|E[C]|$ only counts cross-layer edges with one endpoint in the layer of users' interest. The details of these graphs are given in Appendix G.

Algorithms. To the best of our knowledge, no existing cohesive subgraph models can handle general multi-layer graphs. Therefore, we compare our gCore model with the k-core model [1], the multi-layer core (d-CC) model [39] and the relational community (r-com) model [17] designed for simple graphs, MPNs, and HINs, respectively. Specifically, we compare the following algorithms:

- KC: k-core computation algorithm in [1]. By default, we run KC only on the layer of users' interest.
- DCC: d-CC computation algorithm in [39]. In our implementation, we extend the algorithm to compute the multi-layer $\mathbf{k}$-core. Besides, we do not maintain vertices in decreasing order of their minimum degree among all layers.
- RCD: relational community detection algorithm in [17]. We extend the algorithm to handle the general multi-layer graphs in the following way. We assume each layer G_i corresponds to a type t_i, and use constraints $\langle t_i, t_i, k_i\rangle$ for $1 \le i \le l$ and $\langle t_l, t_i, 1\rangle$ for $1 \le i \le l-1$ as community schema. The output relational community is denoted by $\mathbf{k}$-rc, where $\mathbf{k} = (k_1, k_2, \cdots, k_l)$
- GCS: Our gCore search algorithm (Algorithm 1).
- GCS+: Our KP-tree-based gCore search algorithm (Algorithm 3).
- TN: GCD+ (Algorithm 2) with KP-tree constructed (Section 6.1).
- TE: TN with node and subtree elimination (Section 7.2–7.3).
- TM: TN with subtree merge (Section 7.5).

[1] https://www.twitter.com/
[2] https://www.themoviedb.org/

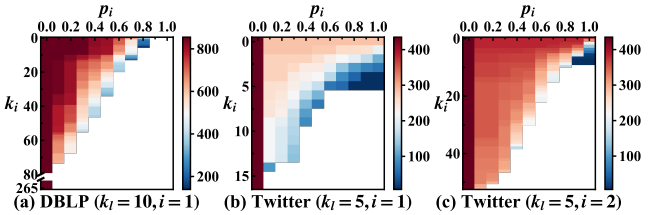


Figure 8: Size matrices of gCores on DBLP and Twitter.

- TEM: TN with both subtree elimination and subtree merge.

We do not evaluate the naïve GCD algorithm proposed in Section 5.1 because it is slow and cannot generate a KP-tree index for gCore search. Besides, we also do not provide a standalone evaluation for TN equipped with node elimination.

8.2 Effectiveness Evaluation

8.2.1 gCore Sizes. To analyze how the cohesiveness constraint imposed on different layers affects gCores, we examine the size of the $(\mathbf{k}, \mathbf{p})$-core w.r.t. k_i and p_i for $1 \le i < l$. Figure 8 visualizes the size matrices obtained on DBLP and Twitter with rows and columns representing k_i and p_i, respectively, and each cell (x, y) representing the size of the $(\mathbf{k}, \mathbf{p})$-core with $k_i = x$ and $p_i = y$. We fix $k_l = 10$ and 5 for DBLP and Twitter, respectively.

We can observe that the size of the $(\mathbf{k}, \mathbf{p})$-core is always monotonically decreasing when either k_i or p_i becomes larger. It's consistent with the containment relationships among gCores given in Property 2 and 3. Notice that all cells in the first column correspond to $p_i = 0$, meaning no constraint is imposed on the neighbor coverage fraction and thereby the cohesiveness of the subgraph w.r.t. G_i. In this case, the $(\mathbf{k}, \mathbf{p})$-core is exactly the k_l-core. A significant drop in size exhibits when increasing p_i from 0 to 0.1. This is because the k_l-core contains massive vertices with seldom, even no, cross-layer neighbors. These vertices show no interaction with other vertices on G_i. By properly setting p_i, the $(\mathbf{k}, \mathbf{p})$-core can well filter out these vertices. Besides, a trade-off between higher intra-layer cohesiveness and higher neighbor coverage fraction, i.e., a larger k_i and a larger p_i, can be seen. Each boundary cell corresponding to the last nonempty $(\mathbf{k}, \mathbf{p})$-core as k_i or p_i increases exhibits the highest cohesiveness w.r.t. G_i for different k_i/p_i trade-offs.

8.2.2 Closeness. We evaluate how closely vertices in a k-core, a relational community and a $(\mathbf{k}, \mathbf{p})$-core interact with each other w.r.t. G_i for $1 \le i < l$. To achieve this, two metrics are introduced:

(1) **k-number**: given a vertex subset $Q \subseteq V_l$ and a real number $p^* \in [0, 1]$, the k-number of a vertex $v \in Q$ w.r.t. G_i is defined as the P-th percentile of the core number of vertices in $N_i(v)$ within $G_i[Q]$, where $P = (1 - p^*) \times 100$. Here, the core number of a vertex v is the largest k such that there is a nonempty k-core containing v in the graph. If a fraction p^* of v's cross-layer neighbors in G_i can capture enough information of v's neighborhood in G_i, a large k-number of v indicates that v's neighbors strongly engage in the community formed by neighbors of other vertices in Q, implying the close interaction between v and other vertices in Q w.r.t. G_i.

(2) **p-number**: given a vertex subset $Q \subseteq V_l$ and an integer k^*, the p-number of a vertex $v \in Q$ w.r.t. G_i is defined as the neighbor coverage fraction of v within the k^*-core on $G_i[Q]$. Assume the k^*-core is enough to characterize a desired cohesiveness of a subgraph,

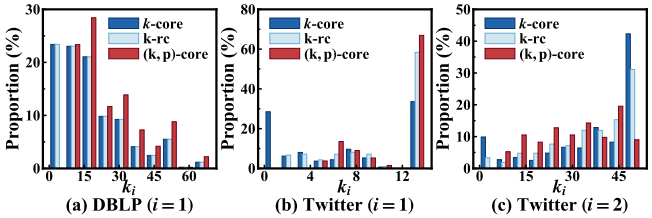


Figure 9: Distribution of the k-numbers of vertices in the k-core, the k-rc and the $(\mathbf{k}, \mathbf{p})$-core on DBLP and Twitter.

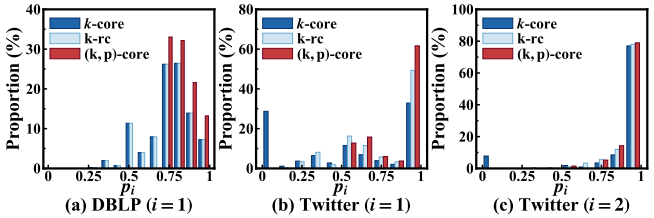
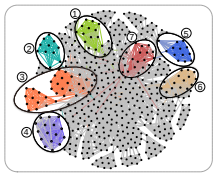


Figure 10: Distribution of the p-numbers of vertices in the k-core, the k-rc and the $(\mathbf{k}, \mathbf{p})$-core on DBLP and Twitter.



Model	No.	Size	Top-5 Keywords
k-core	-	566	data, system, database, query, databases
$(\mathbf{k}, \mathbf{p})$-core	①	11	demonstration, repository, xml, active, views
	②	11	application, design, technology, oracle, replication
	③	26	manager, performance, storage, database, memory
	④	11	system, incomplete, advanced, inconsistent, infomix
	⑤	11	hosting, sql, server, runtime, microsoft
	⑥	11	distributed, continuous, server, media, database
	⑦	12	distributed, stream, engine, operation, borealis

Figure 11: Visualization (left) and keywords (right) of the largest connected component of the k-core on DBLP and all connected components of the $(\mathbf{k}, \mathbf{p})$-core within it, where $k = 10, \mathbf{k} = (10, 10)$, and $\mathbf{p} = (0.757)$.

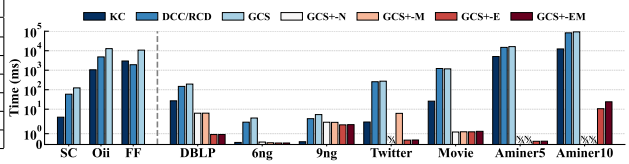


Figure 12: Runtime of cohesive subgraph search algorithms.

when the p-number of v is large, massive v's neighbors cohesively interact with neighbors of other vertices in Q within G_i, which also implies v's close interaction with other vertices in Q w.r.t. G_i.

We compute the k-number and p-number of vertices in the k-core, the $\mathbf{k}$-rc and the $(\mathbf{k}, \mathbf{p})$-core obtained on DBLP and Twitter. For DBLP, we set $k = 10$, $\mathbf{k} = (10, 10)$ and $\mathbf{p} = (0.7)$; and for Twitter, we set $k = 5$, $\mathbf{k} = (5, 5)$ and $\mathbf{p} = (0.5)$. When computing the k-number (resp. p-number) w.r.t G_i, we set $p^* = \mathbf{p}[i]$ (resp. $k^* = \mathbf{k}[i]$).

The distributions of the k-numbers are given in Figure 9. We can observe that the k-core contains massive vertices with small k-numbers, especially on DBLP and Twitter ($i = 1$), vertices with k-number no larger than 6 and 4 account for 23% and 29% of all vertices, respectively. The $\mathbf{k}$-rc also has vertices with small k-numbers. These vertices show weak interactions with other vertices w.r.t. G_i, which may be broken by removing several edges in G_i. However, all vertices in the $(\mathbf{k}, \mathbf{p})$-core have a relatively large k-number (no less than 10 for DBLP and 5 for Twitter). Notice that the k-core and the $\mathbf{k}$-rc may contain more proportion of vertices with large k-number than the $(\mathbf{k}, \mathbf{p})$-core, e.g. on Twitter ($i = 1$). This is because the massive vertices with small k-number enrich the subgraph induced by their cross-layer neighbors in G_i, and thereby many vertices have the core number increased.

Figure 10 shows the results on p-numbers, which exhibit similar distributions with the k-numbers. As DBLP is very dense in G_0 (the 'term' similarity layer), only a small proportion of vertices in the k-core and the $\mathbf{k}$-rc have small p-numbers (about 3% vertices with p-number less than 0.5). The k-core on Twitter has 28% and 7% vertices with p-number equal to 0 w.r.t. G_1 and G_2, respectively. These vertices have no interactions with other vertices w.r.t. G_i, making the k-core shown very sparse connections in terms of G_i. It is consistent with the significant $(\mathbf{k}, \mathbf{p})$-core size drop we observed in Section 8.2.1 when increasing p_i from 0 to 0.1. The $\mathbf{k}$-rc has a smaller proportion of vertices with small p-numbers than the k-core. However, all vertices in the $(\mathbf{k}, \mathbf{p})$-core have a large p-number (no less than 0.7 for DBLP and 0.5 for Twitter).

8.2.3 Case Study. We provide a more intuitive comparison between the k-core and the $(\mathbf{k}, \mathbf{p})$-core obtained on DBLP. Specifically, we set $k = 10$, $\mathbf{k} = (10, 10)$ and $\mathbf{p} = (p)$, where p is largest real number such that the $(\mathbf{k}, \mathbf{p})$-core is nonempty. In this case, $p = 0.757$. We extract the largest connected component of the k-core and all connected components of the $(\mathbf{k}, \mathbf{p})$-core within it. Each connected component corresponds to a group of cohesively cooperated authors. For each group, we compute the top-5 frequently connected terms (keywords) to represent the research interests of the authors. Figure 11 visualizes these connected components and reports their sizes and representative keywords. We have the following observations. 1) The connected component of the k-core is large and contains many connected components of the $(\mathbf{k}, \mathbf{p})$-core. 2) The connected component of the k-core characterizes a group of authors whose research interests are related to data management and database systems, which are both broad fields. However, each of the 7 small connected components of the $(\mathbf{k}, \mathbf{p})$-core captures a group of authors working on a specific fine-grained research field, e.g., the connected component of No. ③ corresponds to an author group working in the field of database storage management.

8.3 Efficiency Evaluation

8.3.1 Cohesive Subgraph Search. In this experiment, we compare the efficiency of our gCore search algorithm with other core-based cohesive subgraph search algorithms on both pillar and general multi-layer graphs. For pillar multi-layer graphs, we compare KC, DCC and GCS, and for general multi-layer graphs, we compare KC, RCD, GCS and GCS+. For each algorithm, 100 queries are performed and the total execution time is reported. Queries are generated as follows. We randomly sample 100 $(\mathbf{k}, \mathbf{p})$ pairs, and for each them, we feed $\mathbf{k}[l]$ to KC, feed $\mathbf{k}$ to DCC and RCD, and feed $(\mathbf{k}, \mathbf{p})$ to GCS and GCS+. To avoid obtaining massive empty results, we restrict $\mathbf{k}[i] \leq \kappa(G_i)/4$ for $1 \leq i \leq l$. Besides, we distinguish GCS+ using different KP-trees. GCS+-N, GCS+-M, GCS+-E and GCS+-EM represent GCS+ searching the KP-tree generated by TN, TM, TE and TEM, respectively. Figure 12 reports the results. Empty bars with 'N/A' mean the KP-tree used by GCS+ cannot be generated within 12 hours or due to memory exceeded.

For pillar multi-layer graphs, we can observe that KC uses the least time in most cases, and GCS takes the longest time. It's consistent with our expectations as KC only considers one layer. DCC

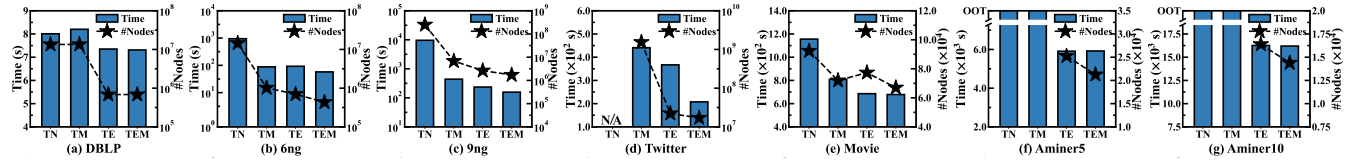


Figure 13: Construction time and scale of the KP-Tree.

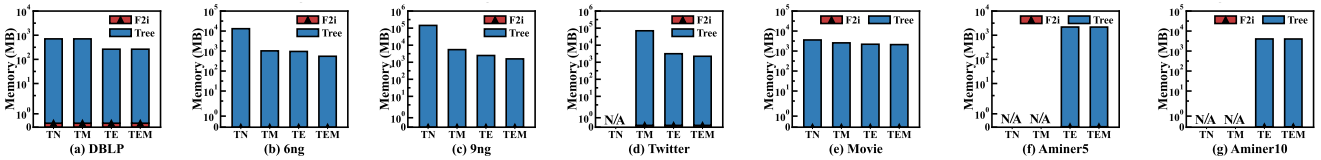


Figure 14: Memory consumption of the KP-Tree.

runs faster than GCS because we have optimized DCC to access the cross-layer neighbor of a vertex directly by id in a pillar multi-layer graph. However, GCS is designed for general multi-layer graphs, so it has to scan the adjacent list to obtain all cross-layer neighbors of any vertex.

For general multi-layer graphs, KC also executes the fastest among all algorithms without using the index as it only runs on one layer. GCS attains a slightly longer runtime than RCD because it has to check whether each vertex has its neighbor coverage fraction w.r.t. G_i no less than $\mathbf{p}[i]$ for $0 \leq i < l$. With a pre-computed KP-tree, GCS+ significantly outperforms GCS by $1-4$ orders of magnitude on the running time for all tested datasets except 9NG. On 9NG, GCS+ achieves a speedup of $2.4-2.9$ compared with GCS. We next inspect the efficiency of GCS+ with the use of different KP-trees. In most cases, GCS+-E uses the least time, while GCS+-N and GCS+-M show similarly the longest runtime. This is because applying subtree elimination results in P+-trees with a much smaller height than the original ones. The number of tree nodes traversed to retrieve the $(\mathbf{k},\mathbf{p})$-core is reduced. However, applying subtree merge do not reduce the length of any path from the root to a leaf node. Besides, we also notice that GCS+-EM is always slightly slower than GCS+-E. It is due to our P-tree storage strategy that tree nodes are stored in linked blocks in the order they are generated. GCS+-E can take advantage of the cache locality as each node in the P+-tree is stored followed by its leftmost child. It incurs more cache misses when GCS+-EM searches a P+-DAG because subtree merge destroys such sequential storage of the tree nodes.

8.3.2 KP-tree Construction. In this experiment, we evaluate our gCore decomposition (indexing) algorithm and P-tree compaction techniques. Specifically, we compare TN, TM, TE and TEM from 3 aspects, namely runtime, scale (number of nodes) and storage overhead of the output KP-tree index. Notice that datasets Movie, aminer-5 and aminer-10 have about 10^5, 10^9 and 10^{11} distinct values of $\mathbf{k}$, each of which corresponds to a P-tree. It's unlikely to construct a complete KP-tree at a reasonable time and memory cost. Besides, it makes no sense to build a P-tree for each distinct $\mathbf{k}$ as not all values of $\mathbf{k}$ are of interest to users. Thus, for each of these three datasets, we randomly sample 1000 values of $\mathbf{k}$ and compute corresponding P-trees. Each element $\mathbf{k}[i]$ of $\mathbf{k}$ is randomly sampled from $[0, \kappa(G_i)/4]$. Total runtime, scale and storage overhead are reported.

Runtime and Scale. Figure 13 reports the runtime and the scale of the generated KP-trees. We set a timeout (OOT) of 12h. Execution of TN on Twitter is aborted due to memory exceeded. We have the following observations. 1) TE outperforms TN in both runtime and scale of the output. It confirms the promising compaction ability of subtree elimination. 2) TM runs significantly faster than TN on 6ng, 9ng, Twitter and Movie, while falls behind on DBLP. This is because subtree merge only works for graphs with more than 2 layers. Meanwhile, the time for detecting isomorphic subtrees slows down the execution on DBLP. Besides, we also find that subtree merge shows more compaction ability on multi-layer graphs with more types of vertices. For example, on 9ng, compared with TN, TM reduces 95% runtime and 97% nodes. 3) By using both subtree merge and subtree elimination, TEM always yields the smallest KP-tree, and in most cases, uses the shortest time.

Storage Overhead. Figure 14 reports the storage overhead of the generated KP-tree index, including the space for storing the KP-tree structure and the space for storing F_i and the mapping that maps each element in F_i to its index. We denote the data structure to store the second part by 'f2i' (**f**raction to **i**ndex). We have the following observations. 1) F2i has neglected space cost, which is 3-9 orders of magnitude smaller than the KP-tree for all tested datasets. 2) Both subtree merge and subtree elimination contribute to reducing the storage overhead. By integrating both of them into TN, the KP-tree generated by TEM attains a $41\% - 98\%$ space reduction.

9 CONCLUSIONS

The generalized core (gCore) model is proposed on general multi-layer graphs by extending the k-core model designed for simple graphs. The gCore model has several elegant properties including uniqueness and containment hierarchy. Three related problems, gCore search (GCS), gCore decomposition (GCD) and gCore indexing (GCI), based on the gCore model are solved by designing efficient algorithms and data structures. The polynomial-time GCS algorithm is comparable to other algorithms for searching core-based cohesive subgraphs in multi-layer graphs in terms of execution time. The KP-tree structure represents the whole search space of the GCD problem and can serve as a compact storage and index (GCI) of all gCores. The KP-tree supports fast retrieval of any gCore Q in linear time in the size of Q and the height of the KP-tree. The compaction techniques including node/subtree elimination, subtree transplant and subtree merge significantly speed up the construction of GCI and reduce the storage overhead of the KP-tree. Extensive experiments show that the vertices in the gCores attain high closeness with each other, and the proposed algorithms are efficient in solving the three problems studied in the paper.

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A EFFICIENT IMPLEMENTATION OF ALGORITHM 1

In this section, we will present an efficient implementation of the GCS algorithm described in Algorithm 1. Let us first introduce the basic idea. For ease of notation, we use Q_l^t to denote the value of Q_l at the beginning of the t-th running of the **repeat** loop (lines 2–10) in Algorithm 1 and use Q_i^t to denote the value of Q_i obtained in line 6 of Algorithm 1 during the t-th running of the **repeat** loop.

Lemma A.1. *For $i = 1, 2, \cdots, l$, $Q_i^{t+1} \subseteq Q_i^t$ holds, where $t \in \mathbb{Z}$.*

Proof. The vertex peeling process ensures $Q_l^{t+1} \subseteq Q_l^t$. Suppose the value of Q_l^t and Q_l^{t+1} becomes C_l^t and C_l^{t+1} after line 4 in corresponding loop is executed, respectively. We have that C_l^t is the $\mathbf{k}[l]$-core of $G_l[Q_l^t]$ and C_l^{t+1} is the $\mathbf{k}[l]$-core of $G_l[Q_l^{t+1}]$. Apparently, $C_l^{t+1} \subseteq C_l^t$. According to the algorithm, Q_i^t is the $\mathbf{k}[i]$-core of $G_i[C_l^t]$, and Q_i^{t+1} is the $\mathbf{k}[i]$-core of $G_i[C_l^{t+1}]$. Obviously, we have $Q_i^{t+1} \subseteq Q_i^t$. Overall, the lemma holds. □

Based on Lemma A.1, we dynamically maintain each Q_i for $1 \leq i < l-1$ via vertex peeling besides Q_l, rather than calculating it from scratch. Arrays are utilized to realize this efficiently. Specifically, for each layer G_i, we set up two arrays $vert_i$ and pos_i to keep vertices in Q_i and the position of these vertices in $vert_i$, respectively. Besides, two offsets s_i and e_i are attached to $vert_i$, dividing it into three parts. We stipulate that: (1) $vert_i[0 : s_i)$ keeps all '*discarded*' vertices that have been removed from Q_i; (2) $vert_i[s_i : e_i)$ keeps the '*inactive*' vertices to be removed from Q_i, which will become *discarded* after a batch discard operation. (3) and the last part $vert_i[e_i : |V_i|)$ keeps all '*active*' vertices that currently remain in the Q_i. As stated in Section 4, once no vertices violating Definition 3.3 can be removed from $vert_l[e_i : |V_i|)$, all remaining vertices form the $(\mathbf{k}, \mathbf{p})$-core.

Next, we define two basic operations, namely Remove and Peel. Remove converts a vertex $v \in Q_i$ from active state to inactive state. It can be realized by exchanging the positions of v and $vert_i[e_i]$ within $vert_i$ and then increasing e_i by 1. Peel iteratively identifies vertices in Q_i with degrees smaller than $\mathbf{k}[i]$ and inactivates them. Peel can be realized by iteratively checking active neighbors of each inactive vertex, updating their degree, and inactivating those whose intra-layer degree drops below $\mathbf{k}[i]$ by applying Remove. Clearly, after applying Peel, vertices remained in Q_i form a $\mathbf{k}[i]$-core. The pseudocode for Remove and Peel is given in Procedure Remove (lines 29–34) and Peel (lines 35–44) in Algorithm 4, respectively.

Based on the array structures and operations introduced above, we present our efficient implementation of Algorithm 1 in Algorithm 4. For efficiency consideration, we also dynamically maintain intra-layer degrees and cross-layer degrees of each vertex. Specifically, deg_i keeps the intra-layer degree of vertices in Q_i, and $deg_{i,j}$ keeps the cross-layer degree of vertices in Q_i within Q_j. Lines 1–9 perform essential initialization. For each layer G_i, $vert_i$, pos_i, s_i and e_i are set in lines 2–6. Vertices with degrees less than $\mathbf{k}[i]$ are identified as the first batch of inactive vertices, so Remove is applied to them (lines 7–9). The **while** loop in lines 10–27 performs vertex peeling in each layer. In each iteration of the loop, vertices violating Constraint (1) in Definition 3.3 are iteratively removed from Q_l by applying Peel in line 11. Lines 12–18 identify and remove vertices not in $N_i[Q_l]$ from Q_i. Specifically, for each inactive vertex in layer

Algorithm 4 EffGCS (Efficient Generalized Core Search)

Input: an l layer graph $\mathcal{M}$, $\mathbf{k} \in \mathbb{N}^l$ and $\mathbf{p} \in [0, 1]^{l-1}$.
Output: the $(\mathbf{k}, \mathbf{p})$-core in $\mathcal{M}$.

```
        /* Lines 1–9 perform initialization. */
 1: for i ← 1, 2, ··· , l do
 2:     j ← 0
 3:     for v ∈ V_i do
 4:         vert_i[j] ← v, pos_i[v] = j
 5:         j ← j + 1
 6:     s_i ← 0, e_i ← 0
 7:     for j ← 0, 1, ··· , |V_i| − 1 do
 8:         if deg_i[vert_i[j]] < k[i] then
 9:             Remove(vert_i[j], i)
        /* Lines 10–27 perform vertex peeling. */
10: while s_l ≠ e_l do
11:     Peel(l)
12:     for i ← 1, 2, ··· , l − 1 do
13:         for j ← s_l, ··· , e_l − 1 do
14:             for v ∈ N_i(vert_l[j]) do
15:                 if pos_i[v] ≥ e_i then
16:                     deg_{i,l}[v] ← deg_{i,l}[v] − 1
17:                     if deg_{i,l}[v] = 0 then
18:                         Remove(v, i)
19:     s_l ← e_l
20:     for i ← 1, 2, ··· , l − 1 do
21:         Peel(i)
22:         for v ∈ N_l(vert_i[j]) do
23:             if pos_l[v] ≥ e_l then
24:                 deg_{l,i}[v] ← deg_{l,i}[v] − 1
25:                 if deg_{l,i}[v]/|N_i(v)| < p[i] then
26:                     Remove(v, l)
27:         s_i ← e_i
28: return {vert_l[i] | e_l ≤ i < |V_l|}
29: procedure Remove(v, i)
30:     vert_i[pos_i[v]] = vert_i[e_i]
31:     pos_i[vert_i[e_i]] = pos_i[v]
32:     vert_i[e_i] = v
33:     pos_i[v] = e_i
34:     e_i ← e_i + 1
35: procedure Peel(i)
36:     j ← s_i
37:     while j ≠ e_i do
38:         v ← vert_i[j]
39:         for u ∈ N_i(v) do
40:             if pos_i[u] ≥ e_i then
41:                 deg_i[u] ← deg_i[u] − 1
42:                 if deg_i[u] < k[i] then
43:                     Remove(u, i)
44:         j ← j + 1
```

G_l, i.e., in $vert_l[s_l, e_l)$, lines 14–16 update the cross-layer degree of its neighbors in every other layer. If a neighbor v in some Q_i has its cross-layer degree dropping to 0, v will no longer belong to Q_i, and hence Remove is applied to it (lines 17-18). After that, all inactive vertices in G_l are batch discarded (converted to discarded state) by increasing s_l to e_l (line 19). Then, for each layer G_i with $1 \leq i < l$, line 21 computes the $\mathbf{k}[i]$-core of $G_i[Q_l]$ by applying Peel. For each inactive vertex in G_i, i.e., in $vert_i[s_i, e_i]$, lines 22–24 update the cross-layer degree of its neighbors in Q_l. If it leads to the neighbor coverage fraction of a vertex $v \in Q_l$ becoming smaller

Algorithm 5 NaïveGCD (Naïve Generalized Core Decomposition)

Input: An l-layer graph $\mathcal{M}$
Output: The collection R of all $(\mathbf{k}, \mathbf{p})$-cores (and $\mathbf{k}$ and $\mathbf{p}$)
1: $R \leftarrow \emptyset$
2: $K \leftarrow \{\mathbf{k} | 0 \leq \mathbf{k}[i] \leq \kappa(G_i)$ for $i = 1, 2, \ldots, l\}$
3: **for** $i \leftarrow 1, 2, \ldots, l-1$ **do**
4: $\quad F_i \leftarrow$ GenFrac$(\mathcal{M}, i)$
5: $P \leftarrow \{\mathbf{p} | \mathbf{p}[i] \in F_i$ for $i = 1, 2, \ldots, l-1\}$
6: **for** $\mathbf{k} \in K$ **do**
7: $\quad$ **for** $\mathbf{p} \in P$ **do**
8: $\quad\quad Q \leftarrow \text{GCS}(\mathcal{M}, \mathbf{k}, \mathbf{p})$
9: $\quad\quad R \leftarrow R \cup \{(Q, \mathbf{k}, \mathbf{p})\}$
10: **return** R
11: **procedure** GenFrac$(\mathcal{M}, i)$
12: $\quad F_i \leftarrow \emptyset$
13: $\quad$ **for** $d \in \{deg_i(v) | v \in V_l\}$ **do**
14: $\quad\quad$ **for** $i \leftarrow 0, 1, \ldots, d$ **do**
15: $\quad\quad\quad F_i \leftarrow F_i \cup \{i/d\}$
16: $\quad$ **return** F_i

than $\mathbf{p}[i]$ (violating Constraint (2) in Definition 3.3), Remove is applied to v (lines 25–26). Thereafter, all inactive vertices in G_i are batched discarded by increasing s_i to e_i (line 27). The **while** loop terminates as soon as $s_l = e_l$ holds at the end of some iteration. Finally, vertices in $vert_l[e_l : |V_l|)$ are returned as the $(\mathbf{k}, \mathbf{p})$-core in line 28.

B PSEUDOCODE FOR THE NAÏVE GCD ALGORITHM

The pseudocode for the naïve GCD algorithm introduced in Section 5.1 is given in Algorithm 5, which is self-explanatory. For each $(\mathbf{k}, \mathbf{p})$-core Q, the algorithm outputs Q as well as $\mathbf{k}$ and $\mathbf{p}$.

C SUPPLEMENTARY EXPLANATION TO GCD+ (ALGORITHM 2)

In this section, we give some supplementary explanations to the KP-tree-based GCD algorithm GCD+ (Algorithm 2) introduced in Section 5.2.2.

C.1 Description of Algorithm 2

Algorithm 2 includes two key procedures: PTreeDFS (lines 4–13) and KPTreeDFS (lines 14–21).

PTreeDFS is a recursive procedure to realize a DFS on a P-tree. Given vectors $\mathbf{k}$ and $\mathbf{p}$ as input, PTreeDFS first calls GCS (Algorithm 1) to compute the $(\mathbf{k}, \mathbf{p})$-core (line 5). Here, GCS needs to be simply adapted to return Q_l as well as $Q_1, Q_2, \ldots, Q_{l-1}$, where Q_i is the set of vertices remaining on layer G_i when GCS terminates. If $Q_l = \emptyset$, the $(\mathbf{k}, \mathbf{p}')$-cores corresponding to the descendant $\mathbf{p}'$-nodes of the $\mathbf{p}$-node are all empty according to Implication I1, so lines 7–12 can be skipped. If $Q_l \neq \emptyset$, we add $(Q_l, \mathbf{k}, \mathbf{p})$ to the result set R (line 7). Now, we completed the traversal to the $\mathbf{p}$-node in this P-tree. Next, we proceed with a DFS traversal on the descendants of the $\mathbf{p}$-node. To this end, we enumerate immediate suffix successors of $\mathbf{p}$ (lines 8–11). For each valid immediate suffix successor $\mathbf{p}'$ of $\mathbf{p}$ ($\mathbf{p}'$ is valid if $\mathbf{p}'[i] \leq |F_i|$ for all i), we recursively call PTreeDFS given $\mathcal{M}[\{Q_1, Q_2, \ldots, Q_l\}]$, $\mathbf{k}$ and $\mathbf{p}'$ as input to make a DFS starting from the $\mathbf{p}'$-node (line 12). Note that this recursive call to PTreeDFS is performed on the subgraph $\mathcal{M}[\{Q_1, Q_2, \ldots, Q_l\}]$ of the input multi-layer graph $\mathcal{M}$ induced by $Q_1, Q_2, \ldots, Q_l$ (defined in Section 3.1), i.e., $\mathcal{M}_{\mathbf{k}}[Q_l]$, instead of on $\mathcal{M}$ because the $(\mathbf{k}, \mathbf{p}')$-core can be correctly computed on $\mathcal{M}_{\mathbf{k}}[Q_l]$ according to Implication I2, thereby reducing computational overhead. Finally, PTreeDFS returns $\{Q_1, Q_2, \ldots, Q_l\}$ (line 13).

KPTreeDFS is a recursive procedure to realize a DFS on the KP-tree, which is very similar to PTreeDFS. When the $\mathbf{k}$-node in the KP-tree is traversed, we start a DFS on the P-tree nested in the $\mathbf{k}$-node by calling PTreeDFS$(\mathcal{M}, \mathbf{k}, \mathbf{0}^{l-1}, R)$ (line 15), where $\mathbf{0}^{l-1}$ represents the $(l-1)$-dimensional vector of zeros. The output $\{Q_1, Q_2, \ldots, Q_l\}$ of PTreeDFS$(\mathcal{M}, \mathbf{k}, \mathbf{0}^{l-1}, R)$ corresponds to the $(\mathbf{k}, \mathbf{0}^{l-1})$-core. If $Q_l = \emptyset$, all descendants of the $\mathbf{k}$-node need not be traversed according to Implication I1, so lines 17–21 can be skipped. If $Q_l \neq \emptyset$, any $(\mathbf{k}', \mathbf{p}')$-core in the descendants of the $\mathbf{k}$-node is completely contained in $\mathcal{M}[\{Q_1, Q_2, \ldots, Q_l\}]$ according to Implication I2. Thus, line 21 recursively calls KPTreeDFS with $\mathcal{M}[\{Q_1, Q_2, \ldots, Q_l\}]$ given as input.

The GCD+ algorithm simply calls KPTreeDFS$(\mathcal{M}, \mathbf{0}^l, R)$ to start the DFS traversal starting from the root of the KP-tree. All nonempty gCores discovered during the DFS are kept in the result set R.

C.2 Discussion on Alternatives of KP-tree

Except for KP-trees, there are alternative ways to represent the search space of the GCD problem. Similar to Galimberti et al. [11], the search space can be represented by a lattice, where a $(\mathbf{k}, \mathbf{p})$-core can have multiple parents. There are two main disadvantages to representing the search space by a lattice. First, a $(\mathbf{k}, \mathbf{p})$-core can be generated multiple times based on its parents [11]. Therefore, a complicated strategy needs to be designed to avoid duplicated generation. Second, people usually store some useful information about the traversed part of the search space to speed up the subsequent search process. Since the structure of the lattice is much more complex than the KP-tree, more information needs to be stored, thereby the storage costs are higher. Both of these disadvantages are avoided by KP-trees.

D SUPPLEMENTARY EXPLANATION TO GCS+ (ALGORITHM 3)

In this section, we give some supplementary explanations to the KP-tree-based GCS algorithm GCS+ (Algorithm 3) introduced in Section 6.2.

As stated in Section 6.2, to retrieve the $(\mathbf{k}, \mathbf{p})$-core from the KP-tree, GCS+ first search the $\hat{\mathbf{p}}$-node in the P-tree nested in the $\mathbf{k}$-node of the KP-tree, where $\hat{\mathbf{p}}[i]$ is the smallest element in F_i (Equation 1) that is not less than $\mathbf{p}[i]$ for $i = 1, 2, \cdots, l-1$, and then compute the core represented by the $\hat{\mathbf{p}}$-node as the $(\mathbf{k}, \mathbf{p})$-core.

The $\hat{\mathbf{p}}$-node search process is described in Procedure Search in Algorithm 3. It repeatedly calls Procedure Forward to find the path from the root of the P-tree, i.e., the $\mathbf{0}^{l-1}$-node, to the $\hat{\mathbf{p}}$-node. Let N be the node under investigation, and let i be a counter. Initially, N is set to the root of the P-tree, and i is set to 1. Procedure Forward works as follows: Let $\mathbf{p}'$ be the vector (in the fractional form) associated with N. If $\mathbf{p}'[i] < \mathbf{p}[i]$, Procedure Forward finds the child N' of N such that the vectors associated with N and N' are different in their i-th elements (line 14), and then returns N' and i (line 15). Otherwise, it returns N and $i+1$ (line 17). Procedure Search repeatedly calls Forward to update N and i until $i = l$ (lines 8–9). When $i = l$, N is eventually the $\hat{\mathbf{p}}$-node. The correctness

of repeatedly applying Forward to find the path from the root of the P-tree to the $\hat{\mathbf{p}}$-node is guaranteed by the following invariant.

Lemma D.1. *Given N and i as input, let N' and i' be the output of Procedure Forward. N' must be on the path from the root to the $\hat{\mathbf{p}}$-node in the P-tree. Let $\mathbf{p}'$ be the vector (in fractional form) associated with N'. If $i' = i+1$, we have that $\mathbf{p}'[j]$ is the smallest element in F'_j that is not less than $\mathbf{p}[j]$ for $j = 1, 2, \ldots, i$.*

After that, Procedure Recover (lines 18–25) is called in line 4 to collect the vertex sets on the leftmost path of the $\hat{\mathbf{p}}$-node down to a dummy leaf node and the union of these vertex sets is returned as the result. Theorem 6.2 ensures the correctness of Procedure Recover.

E BATCH REDUNDANT SUBTREE TRANSPLANT.

The subtree transplant technique introduced in Section 7.4 transplants one redundant subtree at a time. In some situations, there may exist more than one redundant subtree. It is possible to remove multiple redundant subtrees in one batch, thus reducing unnecessary computations. In this section, we propose a "batch subtree transplant" approach to realize this. It works as follows.

Let $R_1, R_2, \ldots, R_n$ be the rightmost path starting from the node R_1 in a P-tree. For $i = 1, 2, \ldots, n$, let T_i be the preceding subtree rooted at R_i, and $\mathbf{p}_i$ be the vector associated with R_i. Of course, $\mathbf{p}_i$ is an immediate suffix successor of $\mathbf{p}_{i-1}$ for $i \geq 2$. Let k be the dimension at which $\mathbf{p}_1$ differs from $\mathbf{p}_2$. By Definition 5.2, we have $\mathbf{p}_2[k] = \mathbf{p}_1[k] + 1$, as well as $\mathbf{p}_{i+1}[k] = \mathbf{p}_i[k] + 1$ for $2 \leq i < n$. Let $\hat{\mathbf{p}}$ be the signature of T_1 (Definition 7.9). There must exist $m \in \{1, 2, \ldots, n\}$ such that $\mathbf{p}_m[k] = \hat{\mathbf{p}}[k]$. According to Theorem 7.10, we have $T_1 \cong T_2 \cong \cdots \cong T_m$, so $T_1, T_2, \ldots, T_{m-1}$ are redundant and therefore can be eliminated. By extending the subtree transplant method proposed in Section 7.4, the elimination of $T_1, T_2, \ldots, T_{m-1}$ as well as the construction of T_m can be simply realized in one subtree transplant process in $O(1)$ time, i.e., changing $\mathbf{p}_1$ to $\mathbf{p}_m$ and directly transplanting T_1 to the position where T_m is planted. In this way, we can avoid cascaded execution of subtree transplant, that is, T_1 is transplanted to the position of T_2; T_2 is transplanted to the position of T_3; ... Finally, T_{m-1} is transplanted to the position of T_m.

F SUPPLEMENTS FOR SUBTREE MERGE.

F.1 Redundant subtree detection

In Section 7.5, we have proposed a subtree merge technique targeting at reducing the redundancy between preceding subtrees rooted at a node and any of its children. Specifically, we conceptually divide a P-tree into branches (Definition 7.11) and identify redundant subtrees within corresponding branches. After that, redundant subtrees are merged into one another.

A supplementary illustration of the branch is given in Figure 15. By extending Definition 7.11 into P+-tree, we can certainly use a similar subtree merge procedure to compact a P+-tree as to compact a P-tree. However, it is worth noticing that whether a subtree T_1 rooted at a node N_1 can be merged into the subtree T_2 rooted at a node N_2 in the P+-tree depends on the isomorphism between subtrees rooted at N_1 and N_2 in the original P-tree, say T'_1 and T'_2.

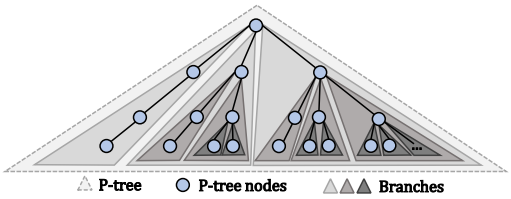

Figure 15: Illustration of branches.

In fact, $T_1 \cong T_2$ not necessarily guarantees $T'_1 \cong T'_2$. For ease of distinguishing, we say T_1 and T_2 are *strong isomorphic* if T'_1 and T'_2 are isomorphic.

Let us briefly describe the process of applying subtree merge to a P+-tree. Let N and N' be two nodes in a P+-tree, and N' be a child of N. We consider each pair of branches B_i^N and $B_i^{N'}$, where i ranges from $l - end0(\mathbf{p}')$ to $l-1$, and $\mathbf{p}'$ is the vector associated with N'. Let $R_0, R_1, R_2, \ldots$ be the rightmost path in B_i^N where $R_0 = N$, and $R'_0, R'_1, R'_2, \ldots$ be the rightmost path in $B_i^{N'}$ where $R'_0 = N'$. We use T_j and T'_k to denote the subtree rooted at R_j and R'_k, respectively. For all j and k, we test whether the subtrees rooted at R_j and R'_k are strong isomorphic. And if it is true, we merge T_j into T'_k by removing T and making R'_k a new child of R_{j-1} instead of R_j. Based on the signature (Definition 7.9) of T_j, a fast test method is given in the theorem below.

Theorem F.1. *Let $\mathbf{p}_j$ and $\mathbf{p}'_k$ be the vectors associated with R_j and R'_k, respectively. Let $\hat{\mathbf{p}}_j$ be the signature of T_j. T_j is strong isomorphic to T'_k if and only if*

(1) $\mathbf{p}_j[i] = \mathbf{p}'_k[i]$,
(2) $\mathbf{p}'_k[i'] \leq \hat{\mathbf{p}}_j[i']$ *for* $i' = 1, 2, \cdots, i-1$.

What is the order for testing whether the subtrees rooted at R_j and R'_k can be merged, i.e., T_j and T'_k are strong isomorphic? We have a very important observation: when T_j is merged into T'_k, all subtrees rooted at $R_{j'}$ for $j' > j$ are consequently merged into T'_k as they are contained in T_j. Therefore, we examine the subtree rooted at R_j in increasing order of j, and test if T_j and T'_k are strong isomorphic for each T'_k. Whenever subtree merge is applicable, we get the most succinct branch.

F.2 Pivot-based Optimization of Redundant Subtree Detection

Using Theorem F.1 to check the strong isomorphism between subtrees rooted at each pair of R_j and R'_k in branches B_i^N and $B_i^{N'}$ is still not efficient because of the following twofold limitations. (1) Assume the lengths of the rightmost paths in B_i^N and $B_i^{N'}$ are m and n, respectively. In the worst case (when no nodes on the rightmost path of B_i^N have its subtree strong isomorphic to the subtree rooted at nodes on the rightmost path of in $B_i^{N'}$), it takes $O(mn)$ time to run the testing procedure presented in Section F.1. In fact, many isomorphic subtree detection operations are unnecessary and therefore can be skipped. (2) To check the Condition (1) in Theorem F.1, we must know the vectors associated with both R_j and R'_k in advance, which requires the preceding subtrees rooted at R_j and R'_k are entirely generated. However, once we found the subtrees rooted at R_j and R'_k are strong isomorphic, the subtree

of one of them in the P+-tree will be removed, and thereby the computation is wasted.

To overcome the above limitations, we devise a sufficient condition to determine the strong isomorphism between subtrees rooted at R_j and R'_k, which is shown in the following Theorem F.2. Notice that the theorem is applicable even before R'_k and the subtree rooted at R'_k has been generated.

Theorem F.2. *The subtree rooted at R_j is strong isomorphic to the subtree rooted at R'_k if*

(1) $\mathbf{p}_{j-1}[i] \leq \mathbf{p}'_{k-1}[i] < \mathbf{p}_j[i]$,
(2) $\mathbf{p}'_{k-1}[i'] \leq \hat{\mathbf{p}}_j[i']$ *for* $i' = 1, 2, \cdots, i-1$.

Based on the theorem, we devise a pivot-based optimization of the strong isomorphic subtree detection approach. Recall Property 4 in Section 7.2: for a vector $\mathbf{p}$ and any of its suffix successors $\mathbf{p}'$, we have $\mathbf{p} < \mathbf{p}'$. Therefore, for the vectors $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_m$ associated with the nodes on the rightmost path in B_i^N, we have $\mathbf{p}_0 < \mathbf{p}_1 < \cdots < \mathbf{p}_m$. For the vectors $\mathbf{p}'_0, \mathbf{p}'_1, \ldots, \mathbf{p}'_n$ associated with the nodes on the rightmost path in $B_i^{N'}$, we have $\mathbf{p}'_0 < \mathbf{p}'_1 < \cdots < \mathbf{p}'_n$. Then, for each node R'_k on the rightmost path of $B_i^{N'}$, we can define a node R_j on the rightmost path of B_i^N as the pivot node if j is the smallest integer such that $\mathbf{p}_j[i] > \mathbf{p}'_{k-1}[i]$. It is easy to see that the subtree T_j rooted at R_j is the only subtree rooted at a node on the rightmost path of B_i^N that satisfies Condition (1) in Theorem F.2. If T_j also satisfies Condition (2) in Theorem F.2, T_j is strong isomorphic to the subtree T'_k rooted at R'_k; Otherwise, no subtree rooted at a node on the rightmost path of B_i^N is strong isomorphic to T'_k.

A useful property of pivot nodes is given as follows: Let R'_{k_1} and R'_{k_2} be two nodes on the rightmost path of $B_i^{N'}$. Suppose the pivot nodes for R'_{k_1} and R'_{k_2} are R_{j_1} and R_{j_2}, respectively. If $k_1 < k_2$, we certainly have $j_1 \leq j_2$. Therefore, for all nodes on the rightmost path of $B_i^{N'}$, their pivot nodes can be found by scanning the nodes on the rightmost path of B_i^N only once. Subsequently, the time complexity of strong isomorphic subtree detection is reduced to $O(m+n)$ in the worst case.

F.3 Implementation.

A straightforward implementation of subtree merge is to first generate a P+-tree and then repeatedly apply subtree merge to the P+-tree to obtain a P+-DAG. However, such implementation usually performs lots of wasted computations as many generated subtrees are discarded later if they are merged into other subtrees. Therefore, like pushing subtree elimination into the P-tree generation process using subtree transplant (Section 7.4), we propose to push subtree merge into the P+-tree generation process to avoid the unnecessary generation of merged subtrees.

Remember that the nodes of a P+-tree are generated in a depth-first order. Such order leads to the following facts. (1) For each node N with vector $\mathbf{p}$, the branch B_i^N must be generated prior to its children N' with vector $\mathbf{p}'$ if $end0(\mathbf{p}') > i$ and therefore prior to the generation of $B_i^{N'}$. (2) For the nodes $R'_0, R'_1, \cdots$ on the rightmost path of $B_i^{N'}$, the preceding subtree rooted at R'_{k-1} must be generated prior to R'_k for $k > 1$. When R'_{k-1} and its preceding subtree have been generated, we are able to test whether the subtree T' rooted at R'_k is strong isomorphic to some subtree T in B_i^N using the pivot-based approach proposed in Section F.2. If $T \cong T'$, we merge T into T' by removing T and making R'_{k-1} to be the parent of T.

However, T' has not been generated yet at this time. To avoid generating T' from scratch, we make use of the isomorphism between T and T' and apply subtree transplant (see Section 7.4) to obtain T'. Specifically, we transplant T to the position where T' is planted. Although the vectors associated with the nodes in T and T' are different, we found that the correctness of applying Algorithm 4 to solve the GCS problem on the P+-DAG obtained after subtree transplant will not be affected (see Appendix H for detailed analysis).

Example. Figure 7 (f) illustrates the process of using subtree transplant to realize subtree merge. Let N_1, N_2, N'_1, N'_2 be the nodes with $\mathbf{p} = (0,0), (0,1), (1,0), (1,1)$, respectively. As we have already known that the subtree T rooted at N_2 is strong isomorphic to the subtree T' rooted at N'_2, we transplant T to the position where T' is planted by simply making N_2 to be a child of N'_1. When the $(\mathbf{k}, \mathbf{p})$-core for $\mathbf{p} = (1,2)$ is queried, Procedure Search in Algorithm 3 returns the node associated with vector $(0,2)$. Finally, Algorithm 3 returns $\{1,2,3,4\}$ as the result.

Using the pivot-based approach to detect strong isomorphic subtrees (Section F.2) and the above subtree-transplant-based subtree merge approach bring extra $O(\sum_{i=2}^{l-1} \prod_{j=1}^{i} |F_j|)$ time overhead to the P+-tree generation process (see Appendix I for the complexity analysis). However, it prevents a lot of redundant subtrees from being generated and obtains a P+-DAG directly, which always takes much less time in practice.

G DATASETS CONSTRUCTION

We select 3 real-world pillar multi-layer graphs and construct 7 general multi-layer graphs using publicly available datasets as presented in Table 1. The details are given as follows:

SacchCere [11], **ObamaInIsrael** [27] and **Friendfeed** [8] are multiplex networks. **SacchCere** has 7-layers describing different types of genetic interactions between genes in Saccharomyces cerevisiae [11]. **ObamaInIsrael**[3] has 3 layers representing re-tweeting, mentioning, and replying relationships among Twitter users during the visit to Israel by Barack Obama in 2013 [27]. **Friendfeed**[4] has 3 layers defined by commenting, liking and following interactions among users of Friendfeed collected over two months[8].

DBLP is built from the DBLP (four-area) dataset [32], consisting of an author co-authoring layer and a term similarity layer. We use frequently appeared venues for each term as a feature to compute the Tanimoto similarities between pair-wise terms. Two terms are connected in the term similarity layer if their similarity is no less than 0.5. An author is connected to a term through a cross-layer edge if he (or she) has published a paper containing the term. By default, the author layer is of users' interest.

6-NG and **9-NG** are extracted from the 20-Newsgroup dataset[5] basically following [26]. Both graphs have 5 news similarity layers. Specifically, 6-NG consists of news from 6 newsgroups. From each

[3] https://manliodedomenico.com/data.php
[4] http://multilayer.it.uu.se/datasets.html
[5] http://qwone.com/~jason/20Newsgroups/

newsgroup, {100, 125, 150, 175, 200} distinct news are randomly selected and fed into 5 layers, respectively. After computing the cosine similarity between *tf-idf* vectors of pair-wise news, we regard two news are closely related if their similarity is at least 0.5 or one falls into the other's top-5 similar news. Each news is connected to its closely related news through either intra-layer edges or cross-layer edges. 9-NG is built in the same way but using news from 9 newsgroups.

Twitter is a 3-layer graph describing the co-occurring relationships between components of tweets. Following the idea in [31] with slight modifications, we collect about 20000 tweets concerning 4 hot events happened in 2016[6]: Rio2016, Election2016, PokemonGo and Brexit from Twitter[7]. Then, three types of key components, namely *hashtags*, *keywords* and *mentions*, are extracted from each tweet and fed into 3 layers, respectively. Every pair of components are connected by an edge (either an intra-layer edge or a cross-layer edge) if they have co-occurred in a tweet. By default, the mentions layer is of users' interest.

Movie is a 4-layer graph describing relationships among movies and casts. Specifically, we collect information about 68000 movies from TMDB[8], including the overview, similar movies list, recommended movies list and casts. Three movie similarity layers and a cast cooperation layer are thereby constructed. We compute the cosine similarity of the overview of pair-wise movies and connect two movies in the first movie similarity layer if one falls into the other's top-5 similar movies or their similarity is at least 0.5. In the second and third movie similarity layers, each movie is connected to the ones in its top-10 similar movies list and top-10 recommended movies list, respectively. Two casts are connected in the cast cooperation layer if they have cooperated in at least 2 movies. Each cast is connected to his/hers top-5 most popular movies through cross-layer edges. In the experiment, we randomly selected the movie similarity layer constructed based on the similar movies list as the layer of users' interest.

Aminer-5 and **Aminer-10** [33] are both 5-layer graphs extracted from the DBLP-Citation-network V13 dataset from Aminer[9]. We select papers published in 2011–2015 and 2011–2020 to construct the two graphs, respectively. Each graph contains three paper layers, an author layer and a venue layer. After computing the cosine similarity of the abstract of pair-wise papers, we build a paper similarity layer where two papers are connected if one falls into the other's top-5 similar papers or their similarity is at least 0.5. Besides, a paper citation layer and a paper co-citation layer are also built. Note that two papers are connected in the co-citation layer only if they have co-cited at least 3 papers. The author layer is defined by the co-authoring relationships between authors. We measure the closeness between venues by the number of their paper citations. Two venues are connected in the venue layer if one has cited the other at least 5 (resp. 10) times in Aminer-5 (resp. Aminer-10). With cross-layer edges, each paper is connected to its authors and the venue it has been published in. During the experimental evaluation, we randomly selected the paper similarity layer constructed based on the similarity of the abstract as the layer of users' interest.

[6]https://blog.twitter.com/en_us/a/2016/thishappened-in-2016
[7]https://www.twitter.com/
[8]https://www.themoviedb.org/
[9]https://www.aminer.cn/citation

H MISSING PROOFS

This section provides the proofs for properties, lemmas and theorems proposed in this paper. For ease of notation, we let $d_G(v)$ denote the degree of vertex v in G.

H.1 Properties

Proof. **(Property 1)** We prove the property by contradiction. Suppose Q and Q' are two distinct $(\mathbf{k}, \mathbf{p})$-cores of $\mathcal{M}$. By Definition 3.3, we have (1) both Q and Q' are $\mathbf{k}[l]$-core; (2) there exist $\mathbf{k}[i]$-cores Q_i and Q'_i on $G_i[Q]$ such that $\phi(v, Q_i) \geq \mathbf{p}[i]$ for $v \in Q$ and $\phi(v', Q'_i) \geq \mathbf{p}[i]$ for $v' \in Q'$, respectively; (3) both Q and Q' are maximal. Let $Q'' = Q \cup Q'$. We have $d_{G_l[Q'']}(v) \geq \min\{d_{G_l[Q]}(v), d_{G_l[Q']}(v)\} \geq \mathbf{k}[l]$ for $v \in Q''$. Therefore, Q'' is a $\mathbf{k}[l]$-core. Similarly, we have $Q''_i = Q_i \cup Q'_i$ is a $\mathbf{k}[i]$-core on $G_i[Q'']$ for $1 \leq i < l$. Besides, it holds that for $v \in Q''$, $\phi(v, Q''_i) \geq \min\{\phi(v, Q_i), \phi(v, Q'_i)\} \geq \mathbf{p}[i]$. By Definition 3.3, Q'' is also a $(\mathbf{k}, \mathbf{p})$-core. It contradicts with the hypothesis of the maximality of Q and Q'. Thus, the property holds. □

Proof. **(Property 2)** Let Q' and Q be the $(\mathbf{k}_1, \mathbf{p})$-core and the $(\mathbf{k}_2, \mathbf{p})$-core, respectively. By Definition 3.3, we have (1) $d_{G_l[Q]}(v) \geq \mathbf{k}_2[l] \geq \mathbf{k}_1[l]$ for $v \in Q$; (2) there exists a $\mathbf{k}_2[i]$-core Q_i on $G_i[Q]$ such that $\phi(v, Q_i) \geq \mathbf{p}[i]$ for $v \in Q$. Certainly, Q_i is also a $\mathbf{k}_1[i]$-core. According to the definition of the $(\mathbf{k}_1, \mathbf{p})$-core and the maximality of Q', we have $Q \subseteq Q'$. Thus, the property holds. □

Proof. **(Property 3)** Let Q' and Q be the $(\mathbf{k}, \mathbf{p}_1)$-core and the $(\mathbf{k}, \mathbf{p}_2)$-core, respectively. By Definition 3.3, we have (1) $d_{G_l[Q]}(v) \geq \mathbf{k}[l]$ for $v \in Q$; (2) there exists a $\mathbf{k}[i]$-core Q_i on $G_i[Q]$ such that $\phi(v, Q_i) \geq \mathbf{p}_2[i] \geq \mathbf{p}_1[i]$ for $v \in Q$. According to the definition of the $(\mathbf{k}, \mathbf{p}_1)$-core and the maximality of Q', we have $Q \subseteq Q'$. Thus, the property holds. □

Proof. **(Property 4)** Let T be the P+-tree and T^c be the original P-tree of T. Let N'' be the child of N in T^c whose vector $\mathbf{p}''$ is an immediate suffix successor of $\mathbf{p}$ and differs from $\mathbf{p}$ only in the i-th element. Suppose $P = N_0, N_1, \cdots$ are nodes on the rightmost path of N'' in T^c, where $N_0 = N''$, and $\mathbf{p}_j$ is the vector associated with N_j for $j \in \mathbb{N}$. The structure of T^c ensures $\mathbf{p}_j$ is an immediate suffix successor of $\mathbf{p}_{j-1}$ for $j > 1$, and they differ in only the i-th element. As N' must be one of the nodes in P, we certainly have $\mathbf{p}'$ and $\mathbf{p}$ are different only the i-th element, $\mathbf{p}[i] < \mathbf{p}'[i]$ and $\mathbf{p}[i'] = \mathbf{p}'[i'] = 0$ for $i' > i$. Thus, the property holds. □

H.2 Lemmas

Proof. **(Lemma 5.1)** We construct each element $\hat{\mathbf{p}}[i]$ of $\hat{\mathbf{p}}$ as follows. Let $f_{i,1}, f_{i,2}, \cdots, f_{i,n_i}$ be the fractions in F_i sorted in increasing order. It's obvious that $f_{i,1} = 0$ and $f_{i,n} = 1$. Therefore, there exists an f_{i,j_i} where $1 \leq j_i \leq n_i$ such that $\mathbf{p}[i] \leq f_{i,j_i}$ and $\mathbf{p}[i] > f_{i,j_i-1}$ if $j_i \neq 1$. We set $\hat{\mathbf{p}}[i]$ to f_{i,j_i}, i.e., the smallest fraction in F_i no less than $\mathbf{p}[i]$. Let $\hat{Q}$ be the $(\mathbf{k}, \hat{\mathbf{p}})$-core. We next prove $\hat{Q} = Q$.

Apparently, we have $\mathbf{p} \leq \hat{\mathbf{p}}$. Based on Property 2, it holds that $\hat{Q} \subseteq Q$. By Definition 3.3, Q is a $\mathbf{k}[l]$-core, and for $i = 1, 2, \cdots, l-1$, there is a $\mathbf{k}[i]$-core Q_i in $G_i[Q]$ such that $\phi(v, Q_i) \geq \mathbf{p}[i]$ for all $v \in Q$. For each vertex $v \in Q$, the number of its cross-layer neighbors falling in $Q_i \subseteq V_i$ ranges from 0 to $deg_i(v)$, so $\phi(v, Q_i)$ must be in the set $\left\{0, \frac{1}{deg_i(v)}, \frac{2}{deg_i(v)}, \ldots, 1\right\} \subseteq F_i$. As $\phi(v, Q_i) \geq \mathbf{p}[i]$, we

must have $\phi(v, Q_i) \geq f_{i,j_i}$. Thus, we have $\phi(v, Q_i) \geq \hat{\mathbf{p}}[i]$ for $v \in Q$. According to the definition of the $(\mathbf{k}, \hat{\mathbf{p}})$-core and the maximality of $\hat{Q}$, $Q \subseteq \hat{Q}$ holds. Overall, we have $Q = \hat{Q}$. The lemma holds. □

Proof. **(Lemma 5.3)** We first prove for each possible value of $\mathbf{k}$, there is exactly one $\mathbf{k}$-node in the KP-tree. Let $\text{sum}(\mathbf{k})$ denote the sum of all elements of vector $\mathbf{k}$. The possible values for $\mathbf{k}$ range from $\mathbf{0}^l$ to $(\kappa(G_1), \kappa(G_2), \cdots, \kappa(G_l))$. Let K be the set of all possible values of $\mathbf{k}$. We have $0 \leq \text{sum}(\mathbf{k}) \leq \sum_{i=1}^{l} \kappa(G_i)$ for any $\mathbf{k} \in K$. Thus, we only need to prove by induction that there is exactly one $\mathbf{k}$-node in the KP-tree for each possible $\mathbf{k} \in K$ sum to s for $s = 0, 1, \cdots, \sum_{i=1}^{l} \kappa(G_i)$.

Base case: When $s = 0$, $\mathbf{k} = \mathbf{0}^l$ is the only vector in K sum to s. As we have introduced in Section 5.2.1, the $\mathbf{0}^l$-node is the root of the KP-tree, which is certainly unique. Therefore, the inductive case holds.

Induction Step: Let $s^* \in \{1, 2, \cdots, \sum_{i=1}^{l} \kappa(G_i) - 1\}$ be given and suppose when $s = s^*$, there is exactly one $\mathbf{k}$-node for each possible $\mathbf{k} \in K$ sum to s. Let $\mathbf{k}'$ be any vector in K sum to $s + 1$. We construct a l-dimensional vector $\mathbf{k}''$ by decreasing the i-th element in $\mathbf{k}'$ by 1, where i is the last non-zero element in $\mathbf{k}'$. It's obvious that $\text{sum}(\mathbf{k}'') = s$. By the induction hypothesis, there is exactly one $\mathbf{k}''$-node in the KP-tree. According to Definition 5.2, $\mathbf{k}'$ is a immediate suffix successor of $\mathbf{k}''$. Therefore, the $\mathbf{k}'$-node is also in the KP-tree, specifically, is a child of the $\mathbf{k}''$-node. It's obvious that $\mathbf{k}''$ is the only vector in K of which $\mathbf{k}'$ is an immediate suffix successor. Therefore, there is exactly one $\mathbf{k}'$-node in the KP-tree, and the proof for the induction step is complete.

Conclusion: By the principle of induction, there is exactly one $\mathbf{k}$-node in the KP-tree for each possible $\mathbf{k} \in K$ sum to s for $s = 0, 1, \cdots, \sum_{i=1}^{l} \kappa(G_i)$, which certainly covers all possible values of $\mathbf{k}$.

We next prove that in every $\mathbf{k}$-node of the KP-tree, for each possible value of $\mathbf{p}$, there is exactly one $\mathbf{p}$-node in the P-tree nested in the $\mathbf{k}$-node.

As stated in Section 5.1, each element $\mathbf{p}[i]$ of $\mathbf{p}$ is chosen from the set F_i fractional numbers formulated in Equation 1. Besides, we actually store the integral form of each $\mathbf{p}$, i.e., each i-th element of $\mathbf{p}$ is replaced by its index in F_i. Therefore, the possible values of $\mathbf{p}$ range from $\mathbf{0}^{l-1}$ to $(|F_1| - 1, |F_2| - 1, \cdots, |F_{l-1}| - 1)$. Let P be the set of all possible values of $\mathbf{p}$. We have $0 \leq \text{sum}(\mathbf{p}) \leq \sum_{i=1}^{l-1}(|F_i| - 1)$ for any $\mathbf{p} \in P$, where $\text{sum}(\mathbf{p})$ is the sum of all elements of $\mathbf{p}$. Thus, we only need to prove by induction that there is exactly one $\mathbf{p}$-node in the P-tree nested in each $\mathbf{k}$-node of the KP-tree for every possible $\mathbf{p} \in P$ sum to s for $s = 0, 1, \cdots, \sum_{i=1}^{l-1}(|F_i| - 1)$.

Base case: When $s = 0$, $\mathbf{p} = \mathbf{0}^{l-1}$ is the only vector in P_{int} sum to s. As we have introduced in Section 5.2.1, the $\mathbf{0}^{l-1}$-node is the root of P-tree nested in each $\mathbf{k}$-node of the KP-tree, which is definitely unique. Therefore, the inductive case holds.

Induction Step: Let $s^* \in \{1, 2, \cdots, \sum_{i=1}^{l-1}(|F_i| - 1) - 1\}$ be given and suppose when $s = s^*$, there is exactly one $\mathbf{p}$-node in P-tree nested in each $\mathbf{k}$-node of the KP-tree for every possible $\mathbf{p} \in P$ sum to s. Let $\mathbf{p}'$ be any vector in P sum to $s + 1$. We construct a $(l-1)$-dimensional vector $\mathbf{p}''$ by decreasing the i-th element in $\mathbf{p}'$ by 1, where i is the last non-zero element in $\mathbf{p}'$. As $\text{sum}(\mathbf{p}'') = s$, we have there is exactly one $\mathbf{p}''$-node in each P-tree by the induction hypothesis. Definition 5.2 ensures that $\mathbf{p}'$ is and only is a immediate suffix successor of $\mathbf{p}''$. Thus, the $\mathbf{p}'$-node is unique in the P-tree as a child of the $\mathbf{p}''$-node. The proof for the induction step is complete.

Conclusion: By the principle of induction, there is exactly one $\mathbf{p}$-node in the P-tree nested in each $\mathbf{k}$-node of the KP-tree for every possible $\mathbf{p} \in P$ sum to s for $s = 0, 1, \cdots, \sum_{i=1}^{l-1}(|F_i| - 1)$, which covers all possible values of $\mathbf{p}$.

Overall, lemma 5.3 holds. □

Proof. **(Lemma 5.4)** Suppose that for $1 \leq i < l - 1$, Q_i is the $\mathbf{k}[i]$-core of $G_i[Q]$ and Q_i' is the $\mathbf{k}'[i]$-core of $G_i[Q']$, respectively. We have $\mathcal{M}_{\mathbf{k}}[Q] = (\mathcal{G}, C)$ is the subgraph of $\mathcal{M}$ induced by $\mathcal{Q} = \{Q_1, Q_2, \cdots, Q_{l-1}, Q\}$, and $\mathcal{M}_{\mathbf{k}'}[Q] = (\mathcal{G}', C')$ is the subgraph of $\mathcal{M}$ induced by $\mathcal{Q}' = \{Q_1', Q_2', \cdots, Q_{l-1}', Q'\}$. As $\mathbf{k} \leq \mathbf{k}'$ and $\mathbf{p} \leq \mathbf{p}'$, we have $Q' \subseteq Q$ based on Property 2 and Property 3. Therefore, $G_l[Q']$ is a subgraph of $G_l[Q]$. For each $i = 1, 2, \cdots, l-1$, let Q_i'' be the $\mathbf{k}[i]$-core of $G_i[Q']$. We have $Q_i' \subseteq Q_i'' \subseteq Q_i$ by the hierarchy of the k-core model. Thus, $G_i[Q_i']$ is a subgraph of $G_i[Q_i]$. For $E_{i,j}' \in C'$, as each edge $e \in E_{i,j}'$ has its endpoints in $Q_i' \subseteq Q_i$ and $Q_j' \subseteq Q_j$, respectively, e is certainly contains in $E_{i,j}$. Therefore, $E_{i,j}' \subseteq E_{i,j} \in C$. By definition of the multi-layer subgraph, $\mathcal{M}_{\mathbf{k}'}[Q']$ is a subgraph of $\mathcal{M}_{\mathbf{k}}[Q]$. The lemma holds. □

Proof. **(Lemma 7.4)** We first prove the **sufficiency**. Suppose that the P-tree is nested in the $\mathbf{k}$-node of the KP-tree. All nodes in $[N]$ correspond to the $(\mathbf{k}, \mathbf{p})$-core, say Q, and all nodes in $[N']$ correspond to the $(\mathbf{k}, \mathbf{p}')$-core, say Q'. If $\hat{\mathbf{p}} = \hat{\mathbf{p}}'$, we have $Q = Q'$ by Property 1, and therefore have $N \cong N'$. The sufficiency holds.

We next prove the **necessity**. If $N \cong N'$, N and N' correspond to the same generalized core. For any node in $N'' \in [N]$, we have $N'' \cong N \cong N'$, and hence have $N'' \in [N']$. Thus, $[N] \subseteq [N']$. Similarly, we have $[N'] \subseteq [N]$, which finally leads to $[N'] = [N]$. The maximal vector $\hat{\mathbf{p}}$ for $[N]$ is certainly equal to the maximal vector $\hat{\mathbf{p}}'$ for $[N']$. The necessity holds. The proof for Lemma 7.4 is complete. □

Proof. **(Lemma D.1)** We prove the lemma by induction on N and i. For ease of notation, we use $\mathbf{p}_N$ to denote the vector associated with N for any node N in the P-tree T.

Base case: In the first call to Procedure Forward, N is the root of the P-tree T and $i = 1$. The integral form of $\mathbf{p}_N$ corresponds to $\mathbf{0}^{l-1}$. If $\mathbf{p}_N[i] < \mathbf{p}[i]$ (the condition in line 12) is true, the **for** loop in lines 13–15 locates the child of N, say N', whose vector $\mathbf{p}_{N'}$ differs from $\mathbf{p}_N$ in the first element, and then returns N' and i. We can easily have that N' is N's rightmost child, and thereby the integral form of $\mathbf{p}_{N'}$ corresponds to $(1, 0, \cdots, 0)$. By definition of $\hat{\mathbf{p}}$, $\hat{\mathbf{p}}[i]$ is the smallest element in F_i no less than $\mathbf{p}[i]$. Thus, $\mathbf{p}_{N'}[i] \leq \hat{\mathbf{p}}[i]$ holds because $\mathbf{p}_{N'}[i]$ and $\mathbf{p}_N[i]$ are adjacent in F_i. Furthermore, we have $\mathbf{p}_{N'} \leq \hat{\mathbf{p}}$. According to Property 3, the $(\mathbf{k}, \hat{\mathbf{p}})$-core is a subset of the $(\mathbf{k}, \mathbf{p}_{N'})$-core, which implies that N' is on the path from the root to the $\hat{\mathbf{p}}$-node. Otherwise, we have $\mathbf{p}_N[i] \geq \mathbf{p}[i]$, and as the result, N and $i + 1$ are returned in line 17. N is on the path from the root to the $\hat{\mathbf{p}}$-node as itself is the root. Besides, $\mathbf{p}_N[i]$ must be the smallest element in F_i no less than $\mathbf{p}[i]$ because the index of $\mathbf{p}_N[i]$ is 0. Overall, the base case holds.

Induction step: Suppose that when Procedure Forward is called with input N and i, N' and i' are returned and the following is ensured: (1) N' is on the path from the root to the $\hat{\mathbf{p}}$-node; (2)

if $i' = i+1$, $\mathbf{p}_N[j]$ is the smallest element in F_j that is not less than $\mathbf{p}[j]$ for $j < i$. As the $\hat{\mathbf{p}}$-node is a descendant of N', we have $\mathbf{p}_{N'} \leq \hat{\mathbf{p}}$. When Procedure Forward is next called with input N' and i', if $\mathbf{p}_{N'}[i'] < \mathbf{p}[i']$ (the condition in line 12) holds, the **for** loop in lines 13–15 locates the child of N', say N'', whose vector $\mathbf{p}_{N''}$ differs from $\mathbf{p}_{N'}$ in the i'-th element, and then returns N'' and i'. Similar to the base case, N'' is the rightmost child of N'. Due to the structure of the P-tree, the integral form of $\mathbf{p}_{N''}$ is an immediate suffix successor of the integral form of $\mathbf{p}_{N'}$, which implies (1) $\mathbf{p}_{N''}[j] = \mathbf{p}_{N'}[j]$ for $j \neq i'$ and (2) $\mathbf{p}_{N'}[i']$ and $\mathbf{p}_{N''}[i']$ are adjacent in $F_{i'}$. As we have already known $\mathbf{p}_{N'}[i'] < \mathbf{p}[i']$, we certainly have $\mathbf{p}_{N''}[i'] \leq \hat{\mathbf{p}}[i']$, and thereby have $\mathbf{p}_{N''} < \hat{\mathbf{p}}$. Thus, the $(\mathbf{k}, \hat{\mathbf{p}})$-core is a subset of the $(\mathbf{k}, \mathbf{p}_{N''})$-core, which ensuring that N'' is on the path from the root to the $\hat{\mathbf{p}}$-node. Otherwise, we have $\mathbf{p}_{N'}[i'] \geq \mathbf{p}[i']$. N' and $i'+1$ will be returned in line 17. According to the induction hypothesis, N' is on the path from the root to the $\hat{\mathbf{p}}$-node. If $i' = i+1$, it holds that $\mathbf{p}_{N'}[i'] = 0$. Thus, $\mathbf{p}_{N'}[i']$ must be the smallest in element in $F_{i'}$ no less than $\mathbf{p}[i']$. Otherwise, we have $i' = i$ and $\mathbf{p}_N[i] < \mathbf{p}[i]$. In this case, $\mathbf{p}_{N'}[i']$ is also the smallest in element in $F_{i'}$ no less than $\mathbf{p}[i']$. Overall, the proof for the induction step is complete.

Conclusion: By the principle of induction, Lemma D.1 holds. □

H.3 Theorems

H.3.1 Proof for Theorem 4.1. Let us first introduce the foundation of our proof. Recall that in Appendix A, we have introduced two notations, namely Q_l^t and Q_i^t. Q_l^t denotes the value of Q_l at the beginning of the t-th running of the **repeat** loop (line 2–9) in Algorithm 1, and Q_i^t denotes the value of Q_i obtained in line 6 of Algorithm 1 during the t-th running of the **repeat** loop.

During the execution of GCS (Algorithm 1), we regard a vertex v as *qualified* w.r.t. Q_l^t if it satisfies both the Constraint (1) and the Constraint (2) in Definition 3.3, i.e., $d_{G_l[Q_l^t]}(v) \geq \mathbf{k}[l]$ and $\phi(v, Q_i^t) \geq \mathbf{p}[i]$ for $1 \leq i < l-1$. Otherwise, v is *unqualified* and will be removed later. We have the following lemma.

Lemma H.1. *Each vertex v unqualified w.r.t. Q_l^x is also unqualified w.r.t. Q_l^y for $x, y \in \mathbb{Z}$ and $x \leq y \leq T$, where T is the number of times the **repeat** loop (line 2–9 in Algorithm 1) runs.*

Proof. As $x \leq y$, we have $Q_i^y \subseteq Q_i^x$ for $1 \leq i \leq l$ by Lemma A.1. If a vertex v is unqualified w.r.t. Q_l^x, it holds either (1) $d_{G_l[Q_l^x]}(v) < \mathbf{k}[l]$ or (2) $\phi(v, Q_i^x) < \mathbf{p}[i]$ for some $i \in \{1, 2, \cdots, l-2\}$. When the first case is true, we have $d_{G_l[Q_l^y]}(v) \leq d_{G_l[Q_l^x]}(v) < \mathbf{k}[l]$, and thereby v is unqualified w.r.t. Q_l^y. When the second case is true, we have $\phi(v, Q_i^y) \leq \phi(v, Q_i^x) < \mathbf{p}[i]$, and v is also unqualified w.r.t. Q_l^y. Overall, the lemma holds. □

With Lemma H.1, the proof for Theorem 4.1 is given below.

Proof. **(Theorem 4.1)** Algorithm 1 iteratively removes each currently unqualified vertex, until all remaining ones in Q_l become qualified. Let Q be the value of Q_l when Algorithm 1 terminates. By Lemma H.1, each removed vertex is unqualified w.r.t. Q, which ensures the maximality of Q. According to Definition 3.3, Q is the $(\mathbf{k}, \mathbf{p})$-core. Therefore, the theorem holds. □

H.3.2 Proof for Theorem 5.5.

Proof. **(Theorem 5.5)** Lemma 5.3 ensures that for each nonempty and distinct $(\mathbf{k}, \mathbf{p})$-core in $\mathcal{M}$, there is a $\mathbf{p}$-node in the P-tree nested in the $\mathbf{k}$-node of the KP-tree corresponding to it. Lemma 5.4 and Implications I1 and I2 guarantee the correctness of the computation of each $(\mathbf{k}, \mathbf{p})$-core. Overall, the theorem holds. □

H.3.3 Proof for Theorem 6.2.

Proof. **(Theorem 6.2)** Let $N_1, N_2, \cdots, N_m$ be the leftmost path starting from N to a leaf node, where $N_1 = N$ and N_m is a leaf node. Suppose that the generalized core represented by N_i is Q_i for $1 \leq i \leq m$. It's obvious that $Q_m = \emptyset$. We use V_i to denote the set of vertices associated with the edge connecting N_i and N_{i+1}. By the augmentations to the P-tree that we have introduced in Section 6.1, all sets V_i are stored, and $V_i = Q_i - Q_{i+1}$. The union of the sets associated with all edges on the path, therefore, is: $\sum_{i=1}^{m-1} |V_i| = \sum_{i=1}^{m-1} (Q_i - Q_{i+1}) = Q_1$. Thus, the theorem holds. □

H.3.4 Proof for Theorem 6.3.

Proof. **(Theorem 6.3)** Line 5 finds the $\mathbf{k}$-node in the KP-tree through a hash table lookup. Lemma D.1 ensures Procedure Search (line 3) returns the $\hat{\mathbf{p}}$-node, say N, in the P-tree nested in the $\mathbf{k}$-node. Theorem 6.2 guarantees Procedure Recover (line 4) correctly retrieves the generalized core corresponding to N, which is identical to $(\mathbf{k}, \mathbf{p})$-core. Overall, the theorem holds. □

H.3.5 Proof for Theorem 7.2.

Proof. **(Theorem 7.2)** We prove the theorem by contradiction. Given P-tree T nested in the $\mathbf{k}$-node of the KP-tree, suppose that $[N]$, where N is any node in T, has two distinct maximal vectors $\mathbf{p}_1$ and $\mathbf{p}_2$. Due to the maximality of $\mathbf{p}_1$ and $\mathbf{p}_2$, there must exist two distinct integers $j_1, j_2 \in \{1, 2, \cdots, l-1\}$ such that $\mathbf{p}_1[j_1] > \mathbf{p}_2[j_1]$ and $\mathbf{p}_2[j_2] > \mathbf{p}_1[j_2]$. Let Q be the generalized core corresponding to nodes in $[N]$. By Definition 3.3, for $i = 1, 2, \cdots, l-1$, there exists a nonempty $\mathbf{k}[i]$-core Q_i in $G_i[Q]$ such that $\phi(v, Q_i) \geq \mathbf{p}_1[i]$ and $\phi(v, Q_i) \geq \mathbf{p}_2[i]$ for $v \in Q$. Let $\mathbf{p}$ be the element-wise maximum of $\mathbf{p}_1$ and $\mathbf{p}_2$, i.e., $\mathbf{p}[i] = \max\{\mathbf{p}_1[j], \mathbf{p}_2[j]\}$ for $1 \leq j \leq l-1$, and N^* be the node in T with which $\mathbf{p}$ is associated. It's obvious that $\phi(v, Q_i) \geq \mathbf{p}[i]$ for $v \in Q$, making Q also a $(\mathbf{k}, \mathbf{p})$-core. Hence, we have $N^* \in [N]$, and the fact that $\mathbf{p}'$ dominates both $\mathbf{p}_1$ and $\mathbf{p}_2$ contradicts with the maximality of $\mathbf{p}_1$ and $\mathbf{p}_2$. The theorem thus holds. □

H.3.6 Proof for Theorem 7.3.

Proof. **(Theorem 7.3)** We prove the theorem by contradiction. For ease of notation, let $\phi_i = \min_{v \in Q} \phi(v, Q_i)$, where Q_i is the $\mathbf{k}[i]$-core of $G_i[Q]$. Suppose that $\hat{\mathbf{p}}[i] \neq \phi_i$ for some $i \in \{1, 2, \cdots, l-1\}$. If $\hat{\mathbf{p}}[i] < \phi_i$, we can construct a $(l-1)$-dimensional vector $\mathbf{p}'$ by replacing the i-th element of $\mathbf{p}$ with ϕ_j. As Q is a $(\mathbf{k}, \mathbf{p})$-core, there must exists a nonempty $\mathbf{k}[i]$-core Q_i of $G_i[Q]$ such that $\phi(v, Q_i) \geq \mathbf{p}[i]$ according to Definition 3.3. It hence also holds that $\phi(v, Q_i) \geq \mathbf{p}'[i]$ for $1 \leq i \leq l-1$ since $\mathbf{p}'[i] = \max\{\mathbf{p}[i], \phi_i\}$. Q therefore is also a $(\mathbf{k}, \mathbf{p}')$-core. Let N' be the node in T with which $\mathbf{p}'$ is associated with. We have $N' \in [N]$, and the fact that $\mathbf{p}'$ dominates $\mathbf{p}$ contradicts with the maximality of $\mathbf{p}$. Otherwise, we have $\hat{\mathbf{p}}[i] > \phi_i$.

According to Definition 3.3, it holds that $\min_{v \in Q} \phi(v, Q_i) \geq \mathbf{p}[i] > \phi_i$, which is also a contradiction. Overall, the theorem holds. □

H.3.7 Proof for Theorem 7.5. Let us first introduce the foundation of our proof. As node elimination is a special case of subtree elimination, we prove Theorem 7.5 holds after replacing the "node elimination" in the theorem with "subtree elimination".

Let T and T' denote any P-tree and the P+-tree obtained by repeatedly applying the subtree elimination procedure. When a $(\mathbf{k}, \mathbf{p})$-core represented by a node N in T is queried using Algorithm 3, Procedure Search locates N by traversing a series of nodes in T and comparing their vectors against $\mathbf{p}$. Let $S = N_1, N_2, \cdots, N_l$ be the sequence of nodes returned at line 17 during the execution of Search. Clearly, N_l is N. However, when using Algorithm 3 to query the $(\mathbf{k}, \mathbf{p})$-core in T', as some nodes in T are removed by the subtree elimination procedure, and thereby no longer exist in T', we suppose $S' = N'_1, N'_2, \cdots, N'_l$ are the sequence of nodes returned at line 17 instead, where $N'_l = N^*$. The following Lemma H.2 builds a bridge between S and S', which serves as a key to prove $N \cong N^*$.

Lemma H.2. *The preceding subtree rooted at N_i is isomorphic to the preceding subtree rooted at N'_i for $i = 1, 2, \cdots, l$.*

Proof. We prove the lemma by induction on i.

Base case: The inductive base is for $i = 1$. When Algorithm 3 runs on T and T', before N_1 and N'_1 are returned, it repeatedly calls Procedure Forward in line 9 to check the node under investigation, say R and R', and visit the rightmost child of R and R', respectively. N_1 is the first node visited in T whose vector has the i-th element no smaller than $\mathbf{p}[i]$, and N'_1 is the first node visited in T' whose vector has the i-th element no smaller than $\mathbf{p}[i]$. Let $P_0 = N_{0,0}, N_{0,1}, \cdots$ be the rightmost path of the root node of T, where $N_{0,0}$ is T's root, and let $P'_0 = N'_{0,0}, N'_{0,1}, \cdots$ be the rightmost path of the root node of T', where $N'_{0,0}$ is the root of T'. Obviously, there exists a node $N_{0,j}$ such that $N_{0,j} = N_1$. If $N_{0,j}$ is not eliminated and remains in P'_0, it must also be returned as N'_1 in line 17. The preceding subtrees rooted at N_1 and N'_1 are thereby isomorphic. Otherwise, suppose $N_{0,j}, N_{0,j+1}, \cdots, N_{0,k-1}$ is the continuous subsequence of P_0 starting from $N_{0,j}$ being eliminated. Certainly, $N_{0,k}$ will be returned as N'_1. As the premises of subtree elimination, the preceding subtrees rooted at each node in $\{N_{0,j}, N_{0,j+1}, \cdots, N_{0,k-1}\}$ are isomorphic and are isomorphic to the subtree rooted at $N_{0,k}$. Therefore, the preceding subtrees rooted at N_1 and N'_1 are also isomorphic in this case. Overall, the base case holds.

Induction step: Let $i^* \in \{1, 2, \cdots, l-1\}$ be given and suppose when $i = i^*$, the preceding subtree rooted at N_i is isomorphic to the preceding subtree rooted at N'_i. When N_i and N'_i are returned at line 17, Algorithm 3 continues calling Procedure Forward to visit and check the nodes on the rightmost path of N_i and N'_i, respectively. The process repeats until the first nodes in P_i and P'_i with the $(i+1)$-th element of their vectors no smaller than $\mathbf{p}[i+1]$ are visited, where P_i is the rightmost path of N_i in T and P'_i is the rightmost path of N'_i in T. Then, such nodes will be returned as N_{i+1} and N'_{i+1} in line 17, respectively. Suppose $P_i = N_{i,0}, N_{i,1}, \cdots$, where $N_{i,0} = N_i$, and $P'_i = N'_{i,0}, N'_{i,1}, \cdots$, where $N'_{i,0} = N'_i$. The induction hypothesis ensures the preceding subtrees rooted at N_i and N'_i are isomorphic. By Definition 7.8, we can derive that the preceding subtrees rooted at each pair of $N_{i,j}$ and $N'_{i,j}$ are isomorphic for $j \in \mathbb{N}$. Besides, the structure of the P-tree ensures the $(i+1)$-th elements in each pair of $N_{i,j}$ and $N'_{i,j}$ are identical. Similar to the base case, there must be a node $N_{i,j}$ in P_i that is N_{i+1}. If $N'_{i,j}$ is not removed by the subtree elimination procedure, it will definitely be returned as N'_{i+1} because $\mathbf{p}'_{i,j}[i] = \mathbf{p}_{i,j}[i]$, where $\mathbf{p}_{i,j}$ and $\mathbf{p}'_{i,j}$ are the vectors associated with $N_{i,j}$ and $N'_{i,j}$, respectively. In this case, the isomorphism between the preceding subtrees rooted at N_{i+1} and N'_{i+1} is clear. Otherwise, let $N'_{1,j}, N'_{1,j+1}, \cdots, N'_{1,k-1}$ be the continuous subsequence of P'_i starting from $N'_{1,j}$ being eliminated. Obviously, $N'_{1,k} \in P'_i$ will be returned as N'_{i+1}. As the premise of the subtree elimination, the preceding subtrees rooted at all $N'_{1,j'}$ for $j \leq j' \leq k$ are isomorphic. Thus, by the transitivity of $\cong$, we have the preceding subtrees rooted at N_{i+1} and N'_{i+1} are isomorphic. The proof for the induction step is complete.

Conclusion: By the principle of induction, the preceding subtrees rooted at each pair of N_i and N'_i are isomorphic, where $1 \leq i \leq l$. □

With Lemma H.2, the proof for Theorem 7.5 is presented below.

Proof. **(Theorem 7.5)** By Lemma H.2, the preceding subtrees rooted at N_l and N'_l are isomorphic, which certainly implies $N_l \cong N'_l$, i.e., $N \cong N^*$. Let $P = R_0, R_1, \cdots$ be nodes on the leftmost path starting from N' in T, where $R_0 = N'$. It's clear that R_j has only one child R_{j+1} for $j \in \mathbb{Z}$. If some $R_j \in P$ is removed, where $j \in \mathbb{Z}$, $R_j \cong R_{j+1}$ must hold, and the vertex set associated with the edge connecting R and R' is empty. In other words, applying subtree elimination to nodes in P involves only removing edges with an empty vertex set on the leftmost path of N'. Thus, all nonempty vertex sets on the leftmost path of N' in T are retained on the leftmost path of N^* in T'. Overall, the theorem holds. □

H.3.8 Proof for Theorem 7.6. Let us first introduce the foundation of our proof. The following Lemma H.3 gives a sufficient condition to determine the redundancy between any two nodes in a P-tree.

Lemma H.3. *Given any two nodes N and N' in a P-tree, let $\mathbf{p}$ and $\mathbf{p}'$ be the vectors associated with N and N', respectively, and let $\hat{\mathbf{p}}$ and $\hat{\mathbf{p}}'$ be the maximal vectors for $[N]$ and $[N']$, respectively. We have $N \cong N'$ if either $\mathbf{p} \leq \mathbf{p}' \leq \hat{\mathbf{p}}$ or $\mathbf{p}' \leq \mathbf{p} \leq \hat{\mathbf{p}}'$ holds.*

Proof. **(Lemma H.3)** Without loss of generality, we suppose $\mathbf{p} \leq \mathbf{p}' \leq \hat{\mathbf{p}}$ holds. Let Q be the generalized core corresponding to nodes in $[N]$, and Q' be the generalized core corresponding to nodes in $[N']$. By Property 3, the left and right inequalities imply $Q' \subseteq Q$ and $Q \subseteq Q'$, respectively. Thus, we have $Q = Q'$. The lemma holds. □

With Lemma H.3, the proof for Theorem 7.6 is given below.

Proof. **(Theorem 7.6)** Let $\mathbf{p}$ be the vector associated with N. We first prove the sufficiency. As N' is a child of N, we have $\mathbf{p} < \mathbf{p}'$. Together with the condition that $\mathbf{p}' \leq \hat{\mathbf{p}}$, we have $N \cong N'$ by Lemma H.3. The sufficiency holds.

We next prove the **necessity**. If $N \cong N'$, we have $N' \in [N]$. By Definition 7.1, $\mathbf{p}'' \leq \hat{\mathbf{p}}$ for all vectors $\mathbf{p}''$ associated with nodes in $[N]$. Thus, it certainly holds that $\mathbf{p}' \leq \hat{\mathbf{p}}$. The necessity thereby holds. Overall, the proof for Theorem 7.6 is complete. □

H.3.9 Proof for Theorem 7.10. Let us first introduce the foundation of our proof. For ease of notation, let $sd(\mathbf{p})$ denote the **s**mallest **d**imension of a vector $\mathbf{p}$ at which $\mathbf{p}$ differs from its immediate suffix successors. According to Definition 5.2, it can be easily derived that $\mathbf{p}[i] = 0$ for all $i > sd(\mathbf{p})$. We then present several observations on the structures of P-trees.

Observation 1. *Given any node N in a P-tree, let $\mathbf{p}$ be the vector associated with N. We have that N has $l - sd(\mathbf{p})$ children, and for $i = sd(\mathbf{p}), sd(\mathbf{p}) + 1, \cdots, l - 1$, there is a child N' of N such that the vector associated with N' differs from $\mathbf{p}$ in the i-th element.*

Observation 2. *Given any node N in a P-tree, let $\mathbf{p}$ be the vector associated with N, and let T be the subtree rooted at N. N' is a node in T if and only if $\mathbf{p}[i] = \mathbf{p}'[i]$, where $\mathbf{p}'$ for $i = 1, 2, \cdots, sd(\mathbf{p}) - 1$ where $\mathbf{p}'$ is the vector associated with N', i.e., $\mathbf{p}$ and $\mathbf{p}'$ share a prefix of length $sd(\mathbf{p}) - 1$.*

Based on the above observations, we introduce the following two lemmas. Lemma H.4 gives a sufficient condition to determine the isomorphism between any two subtrees in a P-tree, and Lemma H.5 gives a sufficient condition to determine the isomorphism between any two preceding subtrees in a P-tree.

Lemma H.4. *Given any two nodes N_1 and N_2 in a P-tree, let $\mathbf{p}_1$ and $\mathbf{p}_2$ be the vectors associated with N_1 and N_2, respectively, and let $\hat{\mathbf{p}}_1$ and $\hat{\mathbf{p}}_2$ be the signature of the subtree rooted at N_1 and N_2, say T_1 and T_2, respectively. We have $T_1 \cong T_2$ if (1) $\#sd(\mathbf{p}_1) = \#sd(\mathbf{p}_2)$, say i; (2) $\mathbf{p}_1[i] = \mathbf{p}_2[i]$; and (3) either $\mathbf{p}_1[j] \le \mathbf{p}_2[j] \le \hat{\mathbf{p}}_1[j]$ for $1 \le j \le i - 1$ or $\mathbf{p}_2[j] \le \mathbf{p}_1[j] \le \hat{\mathbf{p}}_2[j]$ for $1 \le j \le i - 1$ hold.*

Proof. **(Lemma H.4)** Let h be the height of T_1, we next prove the lemma by induction on h. Obviously, h ranges from 1 to h_T, where h_T is the height of the P-tree T.

Base case: When $h = 1$, T_1 consists of a single root node N_1. Obviously, N_1 is a leaf node of T and thereby represents an empty gCore. Without loss of generality, suppose $\mathbf{p}_1[j] \le \mathbf{p}_2[j] \le \hat{\mathbf{p}}_1[j]$ for $1 \le j \le i - 1$ in Condition (3) holds. Based on Condition (1), we have $\mathbf{p}_1[j] = \mathbf{p}_2[j] = 0 \le \hat{\mathbf{p}}_1[j]$ for $j > i$. By Condition (2), we have $\mathbf{p}_1[i] = \mathbf{p}_2[i] \le \hat{\mathbf{p}}_1[i]$. Thus, $\mathbf{p}_1 \le \mathbf{p}_2 \le \hat{\mathbf{p}}_1$, and $N_1 \cong N_2$ by Lemma H.3. Thus, N_2 also represents an empty gCore, and serves as a leaf node of T. T_2 thereby consists of N_2 only. By Definition 7.8, T_1 is isomorphism to T_2 because $N_1 \cong N_2$. The base case holds.

Induction step: Let $h^* \in \{2, 3, \cdots, h_T - 1\}$ be given and suppose Lemma H.4 holds if the height of T is h, where $1 \le h \le h^*$. Assume T_1 rooted at N_1 is a subtree of height $h + 1$ in T. As we have proved in the base case, $N_1 \cong N_2$. According to Condition (1) and Observation 1, N_1 and N_2 have the same number of children. Let N_1' be the child of N_1 with vector $\mathbf{p}_1'$ different from $\mathbf{p}_1$ in the j-th element, and N_2' be the child of N_2 with vector $\mathbf{p}_2'$ different from $\mathbf{p}_2$ in the j-th element, where $i \le j \le l - 1$. We use T_1' and T_2' to denote the subtrees rooted at N_1' and N_2', respectively. It's obvious that $sd(\mathbf{p}_1') = sd(\mathbf{p}_2') = j$. If $j = i$, $\mathbf{p}_1'[j] = \mathbf{p}_2'[j] = \mathbf{p}_1[j] + 1$. Otherwise, we have $j > i$, and hence $\mathbf{p}_1'[j] = \mathbf{p}_2'[j] = 1$. Therefore, $\mathbf{p}_1'[j] = \mathbf{p}_2'[j]$ always holds. Without loss of generality, we suppose that $\mathbf{p}_1[j] \le \mathbf{p}_2[j] \le \hat{\mathbf{p}}_1[j]$ for $1 \le j \le i - 1$ in Condition (3) holds. The left inequality implies $\mathbf{p}_1'[k] \le \mathbf{p}_2'[k]$ for $1 \le k \le i - 1$. For $i \le k < j$, we have $\mathbf{p}_1'[k] = \mathbf{p}_2'[k] = 0$. Thus, $\mathbf{p}_1'[k] \le \mathbf{p}_2'[k]$ for all $1 \le k \le j - 1$. Let $\hat{\mathbf{p}}_1'$ be the signature of T_1'. As all nodes in T_1' must be in T_1, by Definition 7.9, we have $\hat{\mathbf{p}}_1 \le \hat{\mathbf{p}}_1'$. With the right inequality in the above Condition (3), it holds for $1 \le k < i$ that $\mathbf{p}_2'[k] = \mathbf{p}_2[k] \le \hat{\mathbf{p}}_1'[k]$. For $i \le k < j$, we have $\mathbf{p}_2'[k] = 0 \le \hat{\mathbf{p}}_1'[k]$. Thus, $\mathbf{p}_2'[k] \le \hat{\mathbf{p}}_1'[k]$ for all $1 \le k \le j - 1$. Overall, we have $\mathbf{p}_1'[k] \le \mathbf{p}_2'[k] \le \hat{\mathbf{p}}_1'[k]$ for each $k < j$. As T_1' and T_2' must have heights no larger than h, the induction hypothesis ensures that $T_1' \cong T_2'$. Notice N_1' and N_2' are the $(j - i + 1)$-th child of N_1 and N_2 in sorted order of $\prec$ (Definition 6.1), respectively. Due to the arbitrariness of j and Definition 7.8, T_1 and T_2, both with height $h + 1$, are isomorphic. The proof for the induction step is complete.

Conclusion: By the principle of induction, Lemma H.4 holds for T_1 with height ranging from 1 to h_T, which certainly covers all subtrees in T. Overall, the lemma holds. □

Lemma H.5. *Given any two nodes N_1 and N_2 in a P-tree, let $\mathbf{p}_1$ and $\mathbf{p}_2$ be the vectors associated with N_1 and N_2, respectively, let $\hat{\mathbf{p}}_1$ and $\hat{\mathbf{p}}_2$ be the maximal vectors for $[N_1]$ and $[N_2]$, respectively, and let $\hat{\mathbf{p}}_1'$ and $\hat{\mathbf{p}}_2'$ be the signature of the preceding subtree rooted at N_1 and N_2, say T_1 and T_2, respectively. We have $T_1 \cong T_2$ if (1) $sd(\mathbf{p}_1) = sd(\mathbf{p}_2)$, say i; and (2) either $\mathbf{p}_1[i] \le \mathbf{p}_2[i] \le \hat{\mathbf{p}}_1[i]$ and $\mathbf{p}_1[j] \le \mathbf{p}_2[j] \le \hat{\mathbf{p}}_1'[j]$ for $1 \le j < i$ or $\mathbf{p}_2[i] \le \mathbf{p}_1[i] \le \hat{\mathbf{p}}_2[i]$ and $\mathbf{p}_2[j] \le \mathbf{p}_1[j] \le \hat{\mathbf{p}}_2'[j]$ for $1 \le j < i$ hold.*

Proof. **(Lemma H.5)** According to Condition (1) and Observation 1, $\mathbf{p}_1[j] = \mathbf{p}_2[j] = 0$ for $j > i$. Thus, we have $\mathbf{p}_1[j] \le \mathbf{p}_2[j] \le \hat{\mathbf{p}}_1[j]$ for $i < j \le l - 1$. Without loss of generality, suppose that $\mathbf{p}_1[i] \le \mathbf{p}_2[i] \le \hat{\mathbf{p}}_1[i]$ and $\mathbf{p}_1[j] \le \mathbf{p}_2[j] \le \hat{\mathbf{p}}_1'[j]$ for $1 \le j < i$ in Condition (2) are true. As N_1 is contained in T_1, we have $\hat{\mathbf{p}}_1' \le \hat{\mathbf{p}}_1$ by Definition 7.1. Thus, $\mathbf{p}_1[j] \le \mathbf{p}_2[j] \le \hat{\mathbf{p}}_1[j]$ for $1 \le j \le i$, and moreover, $\mathbf{p}_1 \le \mathbf{p}_2 \le \hat{\mathbf{p}}_1$. By Lemma H.3, $N_1 \cong N_2$.

Obviously, N_1 and N_2 have the same number of children in T_1 and T_2, respectively. Let N_1' be the child of N_1 with vector $\mathbf{p}_1'$ different from $\mathbf{p}_1$ in the j-th element, and N_2' be the child of N_2 with vector $\mathbf{p}_2'$ different from $\mathbf{p}_2$ in the j-th element, where $i < j \le l - 1$. And let T_1' and T_2' be the subtrees rooted at N_1' and N_2', respectively. We have $sd(\mathbf{p}_1') = sd(\mathbf{p}_2') = j$, and $\mathbf{p}_1'[j] = \mathbf{p}_2'[j] = 1$. As proved before, $\mathbf{p}_1 \le \mathbf{p}_2$, we have $\mathbf{p}_1'[k] \le \mathbf{p}_2'[k]$ for $k < j$. Let $\hat{\mathbf{p}}_1'$ be the signature of T_1'. All nodes in T_1' must be in T_1, $\hat{\mathbf{p}}_1 \le \hat{\mathbf{p}}_1'$ is thus ensured by Definition 7.1. Then based on Condition (2), we have $\mathbf{p}_2'[k] = \mathbf{p}_2[k] \le \hat{\mathbf{p}}_1'[k]$ for $1 \le k < i$, and $\mathbf{p}_2'[k] = 0 \le \hat{\mathbf{p}}_1'[k]$ for $1 \le k < j$. That is, $\mathbf{p}_2'[k] \le \hat{\mathbf{p}}_1'[k]$ for all $k < j$. Overall, $\mathbf{p}_1[k] \le \mathbf{p}_2'[k] \le \hat{\mathbf{p}}_1'[k]$ is true for $1 \le k < j$. According to Lemma H.4, T_1' is isomorphic to T_2', i.e., $T_1' \cong T_2'$. Notice N_1' and N_2' are the $(j - i + 1)$-th child of N_1 and N_2 in sorted order of $\prec$ (Definition 6.1), respectively. Due to the arbitrariness of j and Definition 7.8, we have $T_1 \cong T_2$. The proof for Lemma H.5 is thereby complete. □

With the above observations and lemmas, the proof for Theorem 7.10 is given below.

Proof. **(Theorem 7.10)** We first prove the **sufficiency**. As N' is a descendant of N on N's rightmost path, we have $\mathbf{p} < \mathbf{p}'$ and $sd(\mathbf{p}) = sd(\mathbf{p}') = i$. If the condition $\mathbf{p}'[i] < \hat{\mathbf{p}}[i]$ is true, it holds that $\mathbf{p}[i] \le \mathbf{p}'[i] < \hat{\mathbf{p}}[i]$. According to Observation 2, $\mathbf{p}$ and $\mathbf{p}'$ share a prefix of length $i - 1$. We then have $\mathbf{p}'[j] = \mathbf{p}[j] \le \hat{\mathbf{p}}[j]$ for $1 \le j \le i - 1$. Thus, $\mathbf{p}[j] \le \mathbf{p}'[j] \le \hat{\mathbf{p}}[j]$ holds for each $j < i$. By Lemma H.5, T is isomorphic to T'. The sufficiency thereby holds.

We next prove the **necessity**. If $T \cong T'$, for each node R' in T', there is a node R in T such that $R' \cong R$. Therefore, according to Definition 7.9, we have $\hat{\mathbf{p}}' = \hat{\mathbf{p}}$, where $\hat{\mathbf{p}}'$ is the signature of T'. For each node R' in T' with vector $\mathbf{p}''$, R' is a either N' or a descendant of N'. It thus holds that $\mathbf{p}' \leq \mathbf{p}'' \leq \hat{\mathbf{p}}''$, where $\hat{\mathbf{p}}''$ is the maximal vector for $[R']$. Also by Definition 7.9, we have $\mathbf{p}' \leq \hat{\mathbf{p}}'$. Overall, $\mathbf{p}' \leq \hat{\mathbf{p}}$, which certainly implies that $\mathbf{p}'[i] \leq \hat{\mathbf{p}}[i]$. The necessity holds, and the proof for Theorem 7.10 is complete. □

H.3.10 Proof for Theorem 7.12. Let us first introduce the foundation of our proof. We use T to denote any P+-tree and T' to denote the P+-DAG obtained by repeatedly applying the subtree merge procedure to T. Let T^c be the original P-tree of T. We suppose T is obtained by repeatedly applying the subtree elimination procedure to T^c until no more nodes can be removed. The assumption is rational because (1) as we have stated in Section 7.5, to make T' more compact, we apply subtree merge after doing a sequence of subtree elimination operation; (2) such assumption simplifies the proof for Theorem 7.12. In fact, Theorem 7.12 holds for any P+-tree, which can be proved by a more complicated induction.

When the $(\mathbf{k}, \mathbf{p})$-core represented by a node N in T is queried using Algorithm 3, Procedure Search repeatedly calls Procedure Forward to traverse a series of nodes in T and compare their vectors against $\mathbf{p}$. The process repeats until N has been found. Let $O = O_1, O_2, \cdots, O_n$ be the sequence of outputs of Forward at either line 15 or line 17 during the execution of Search, where each O_j with $1 \leq j \leq n$ is the pair (N_j, i_j) returned at the j-th call to Forward. We have $O_n = (N, l)$. After applying the subtree merge procedure to T, some subtrees in T may be removed and replaced by other subtrees. Thus, using Algorithm 3 to search the $(\mathbf{k}, \mathbf{p})$-core on T' may lead to another sequence of outputs of Procedure Forward, say $O' = O'_1, O'_2, \cdots, O'_{n'}$, where $O'_j = (N'_j, i'_j)$ for $1 \leq j \leq n'$ and $O'_{n'} = (N^*, l)$. The following lemma builds a bridge between O and O', which is key to the proof for $N \cong N^*$.

Lemma H.6. *It holds for $j = 1, 2, \cdots$ that (1) either $\min\{n, n'\} > j$ or $n = n' = j$; (2) $i_j = i'_j$, say i; (3) $\mathbf{p}_j[i] = \mathbf{p}'_j[i]$, where $\mathbf{p}_j$ and $\mathbf{p}'_j$ are vectors associated with N_j and N'_j, respectively, and (4) the subtree rooted at N_j in T^c is isomorphic to the subtree rooted at N'_j in T^c.*

To facilitate the proof for Lemma H.6, Lemma H.7 is given next.

Lemma H.7. *Given a P-tree T and a P+-tree T' obtained by repeatedly applying the subtree elimination procedure on T until no more nodes can be removed. For any two nodes N_1 and N_2 in T, let $\mathbf{p}_1$ and $\mathbf{p}_2$ be the vectors associated with N_1 and N_2, respectively. If the subtree rooted at N_1 is isomorphic to the subtree rooted at N_2 in T and $\mathbf{p}_1[i] = \mathbf{p}_2[i]$, where $i = sd(\mathbf{p}_1)$, we have the subtree rooted at N'_1 is isomorphic to the subtree rooted at N'_2 in T and $\mathbf{p}'_1[j] = \mathbf{p}'_2[j]$, where N'_1 is a child of N_1 whose vector $\mathbf{p}'_1$ differs from $\mathbf{p}_1$ in the j-th element in T', and N'_2 is a child of N_2 whose vector $\mathbf{p}'_2$ differs from $\mathbf{p}_2$ in the j-th element in T'.*

Proof. **(Lemma H.7)** Let us first introduce some notations. We use R_1 and R_2 to denote the children of N_1 and N_2 in T with vectors differ from $\mathbf{p}_1$ and $\mathbf{p}_2$ in the j-th element, respectively. Let $P_1 = R_{1,0}, R_{1,1}, \cdots, R_{1,n_2}$ be the rightmost path starting from R_1 in T, where $R_{1,0} = R_1$, and let $P_2 = R_{2,0}, R_{2,1}, \cdots, R_{2,n_2}$ be the rightmost path starting from R_2 in T, where $R_{2,0} = R_2$. We use $\mathbf{p}_{1,k_1}$ and $\mathbf{p}_{2,k_2}$ denote the vectors associated with R_{1,k_1} and R_{2,k_2}, respectively, where $0 \leq k_1 \leq n_1$ and $0 \leq k_2 \leq n_2$.

By Definition 7.8, the isomorphism between subtrees rooted at N_1 and N_2 in T ensures $n_1 = n_2$, say n, and the isomorphism between subtrees rooted at each pair of $R_{1,k}$ and $R_{2,k}$ in T for $0 \leq k \leq n$. Besides, according to the structure of the P-tree, the j-th elements of vectors of nodes in P_1 and P_2 are monotonically increasing, holding that $\mathbf{p}'[j] = \mathbf{p}[j] + 1$ for any pair of adjacent nodes R and R' with vectors $\mathbf{p}$ and $\mathbf{p}'$ in P_1 or P_2, respectively. As $\mathbf{p}_1[i] = \mathbf{p}_2[i]$ and $j \geq i$, we have $\mathbf{p}_{1,j}[j] = \mathbf{p}_{2,j}[j] = p$, where $p = \mathbf{p}_1[i] + j + 1$ when $j = i$, and $p = j + 1$ when $j > i$. Thus, the j-th elements of each pair of $R_{1,k}$ and $R_{2,k}$ are always identical. Obviously, there exist two integers k_1 and k_2, where $0 \leq k_1, k_2 \leq n$, such that $R_{1,k_1} = N'_1$ and $R_{2,k_2} = N'_2$. To prove the isomorphism between subtrees rooted at N'_1 and N'_2 and the identity between the j-th elements in their vectors, we only need to prove $k_1 = k_2$.

Next, we prove $k_1 = k_2$ by a simple contradiction. If $k_1 \neq k_2$, without loss of generality, we suppose $k_1 < k_2$. Let R be the node R_{2,k_1}. Clearly, R is in the sequence $P'_2 = R_{2,0}, R_{2,1}, \cdots, R_{2,k_2-1}$. Let us consider two nodes R_{1,k_1+1} and R_{2,k_1+1}, the child node of R_{1,k_1} and R, respectively. We have (1) $\mathbf{p}_{1,k_1+1}[j] = \mathbf{p}_{2,k_1+1}[j]$ (2) the subtrees rooted at R_{1,k_1+1} and R_{1,k_2+1} are isomorphic. Certainly, the isomorphism between the subtrees rooted at R_{1,k_1} and R ensures the isomorphism between the preceding subtrees rooted at R_{1,k_1} and R, and the isomorphism between the subtrees rooted at R_{1,k_1+1} and R_{1,k_2+1} ensures the isomorphism between the preceding subtrees rooted at them. As all nodes in P'_2 are removed by the subtree elimination procedure, as the premises of which, the preceding subtrees rooted at all nodes in P'_2 are isomorphic, and are isomorphic to the preceding subtree rooted at R_{2,k_2}. It's obvious that R_{2,k_1+1} is either in P'_2 or exactly is R_{2,k_2}. We hence have the preceding subtrees rooted at R and R_{2,k_1+1} are isomorphic. By transitivity of $\cong$, the preceding subtrees rooted at R_{1,k_1} and R_{1,k_1+1} are isomorphic. R_{1,k_1}, i.e., N'_1 will therefore be removed by the subtree elimination procedure. It contradicts with the hypothesis that no nodes can be removed from T'. Overall, we have $k_1 = k_2$, and hence the proof for the lemma is complete. □

With Lemma H.7, the proof for Lemma H.6 is given as below.

Proof. **(Lemma H.6)** We prove the lemma by induction on j. As the terminal condition at line 8 in Search is reached when i is increased to l from 1, and each increase of i corresponds to a call to Forward and return at line 17, both n and n' are no less than $l - 1$. Without loss of generality, we suppose $l \geq 2$.

Base case: The inductive base is for $j = 1$. Let R and R' be the root of T and T', respectively, and $\mathbf{p}_0$ and $\mathbf{p}'_0$ be the vector associated with R and R', respectively. The first call to Procedure Forward on T is given R and 1 as input, and the call on T' is given R' and 1 as input. Apparently, the subtrees rooted at R and all nodes on the rightmost path starting from R cannot be removed by the subtree merge procedure. We, therefore, have $R = R'$, and the rightmost child of R is identical to the rightmost child of R', say R''. If $\mathbf{p}_0[i] < \mathbf{p}[i]$, the rightmost child of R and R' will be returned at line 15. We then have $N_j = N'_j = R''$ and $i_j = i'_j = 1$. The isomorphism between subtrees rooted at N_j and N'_j in T^c is

obvious, and $\mathbf{p}_j[i] = \mathbf{p}'_j[i]$. Besides, both n and n' are larger than j since $i < l$.

Otherwise, we have $\mathbf{p}_0[i] \geq \mathbf{p}[i]$. In this case, R and R' will be returned at line 17. Thus, it holds $N_j = N'_j = R$ and $i_j = i'_j = 2$. The isomorphism between subtrees rooted at N_j and N'_j in T^c is also obvious. And the i-th elements of $\mathbf{p}_j$ and $\mathbf{p}'_j$ are identical. If $l > 2$, Procedure Forward will be continually called to test new nodes, and both n and n' are larger than j. Otherwise, we have $l = 2$. The **while** loop in lines 8–9 terminates, and N_j and N'_j are returned as N and N^* at line 10. It holds that $n = n' = j$. Overall, the base case holds.

Inductive step: Let $j^* \in \mathbb{Z}$ be given and suppose when $j = j^*$, it holds $j < \min\{n, n'\}$, $i_j = i'_j$, $\mathbf{p}_j[i] = \mathbf{p}'_j[i]$ with $i = i_j$, and the subtree rooted at N_j in T^c is isomorphic to the subtree rooted at N'_j in T^c. In the $(j+1)$-th call to Procedure Forward on T and T', the input are (N_j, i) and (N'_j, i), respectively. Let R be the child of N_j with vector different from $\mathbf{p}_j$ only in the i-th element in T, and R' be the child of N'_j with vector different from $\mathbf{p}'_j$ in the i-th element in T. Notice that the vector associated with R' may differ from $\mathbf{p}'_j$ in multiple elements because of subtree merge, while it ensures that they are different only in the first i-th elements. There are two cases:

(1) If $\mathbf{p}_j[i] \leq \mathbf{p}[i]$, (R, i) and (R', i) will be returned at line 15. It holds $i_{j+1} = i'_{j+1} = i$. Besides, $i < l$ definitely holds, otherwise Procedure Search terminates before $(j+1)$-th call to Procedure Forward. Thus, we have $\min\{n, n'\} > j$. Let R'' be the child of N'_j in T with vector only different from $\mathbf{p}'_j$ in the j-th element. By induction hypothesis, the subtrees rooted at N_j and N'_j in T^c are isomorphic and $\mathbf{p}_j[i] = \mathbf{p}'_j[i]$ holds. According to Lemma H.7, we have the subtrees rooted at R and R'' in T^c are also isomorphic, and the i-th elements of vectors associated with R and R'' are identical. If R'' is not removed by the subtree merge procedure, it exactly is R'. We then certainly have $\mathbf{p}_{j+1}[i] = \mathbf{p}'_{j+1}[i]$, and the subtrees rooted at N_{j+1} and N'_{j+1} in T^c are certainly isomorphic. Otherwise, the subtree rooted at R'' is removed and finally replaced by the subtree rooted at R'. Thus, there must exists a sequence of nodes $R_1, R_2, \cdots, R_m$ in T, where $R_1 = R''$ and $R_m = R'$, such that for $1 \leq k < m$, R_k and R_{k+1} are on the rightmost paths of branches B_i^N and $B_i^{N'}$, respectively, with N' being a child of N, and the subtrees rooted at R_k and R_{k+1} in T^c are isomorphic. By transitivity of $\cong$, the subtrees rooted at R' and R'' in T^c are isomorphic and are thereby isomorphic to the subtree rooted at R in T^c. Moreover, according to Theorem F.1, the i-th elements of vectors associated with each pair of R'_k and R'_{k+1} are identical. By transitivity, we have the i-th elements of vectors associated with R' and R'' are the same, and are further equal to the i-th element of the vector associated with the R. Overall, we have that the subtrees rooted at N_{j+1} and N'_{j+1} in T^c are isomorphic and $\mathbf{p}_{j+1}[i] = \mathbf{p}'_{j+1}[i]$.

(2) Otherwise, we have $\mathbf{p}_j[i] \geq \mathbf{p}[i]$, and consequently, $(N_j, i+1)$ and $(N'_j, i+1)$ will be returned at line 17. Thus, $N_{j+1} = N_j$, $N'_{j+1} = N'_j$ and $i_{j+1} = i'_{j+1} = i+1$. The isomorphism between subtrees rooted at N_{j+1} and N'_{j+1} in T^c is ensured by the induction hypothesis. Besides, we can easily derive that $\mathbf{p}_{j+1}[i+1] = \mathbf{p}'_{j+1}[i+1] = 0$. If $i_{j+1} < l$, the **while** loop (lines 8–9) in Procedure Search continues executing and call Forward. We have $\min\{n, n'\} > j+1$. Otherwise, the terminal condition of the **while** loop is reached. Search returns with $n = n' = j+1$. Overall, we have the induction step holds.

Conclusion: By the principle of induction, Lemma H.6 holds for each j. The proof is complete. □

By leveraging the above Lemma H.6, we present the proof for Theorem 7.12 as follows.

Proof. **(Theorem 7.12)** By Lemma H.6, we have the subtree rooted at N_n in T^c and the subtree rooted at $N'_{n'}$ in T^c are isomorphic, which certainly implies $N_n \cong N_{n'}$, i.e., $N \cong N^*$. Let $P = R_1, R_2, \cdots, R_n$ be the nodes on the leftmost path starting from N^* in T', where $R_1 = N^*$, and let $P' = R'_1, R'_2, \cdots, R_n$ be the nodes on the leftmost path starting from N' in T, where $R'_1 = N'$. For ease of notation, we use $\mathbf{p}_j$ and T_j be the vector associated with R_j and the subtree rooted at R_j in T^c, respectively, where $1 \leq j \leq n$, and $\mathbf{p}'_k$ and T'_k be the vector associated with R'_k and the subtree rooted at R'_k in T^c, respectively, where $1 \leq k \leq n'$. We next prove by induction that for $j = 1, 2, \cdots$, it holds (1) either $\min\{n, n'\} > j$ or $n = n' = j$; (2) $\mathbf{p}_j[l-1] = \mathbf{p}'_j[l-1]$; and (3) $T_j \cong T'_j$.

Base case: When $j = 1$, $R_j = N^*$ and $R'_j = N'$. As N' is the copy of N^*, it certainly holds that $\mathbf{p}_j[l-1] = \mathbf{p}'_j[l-1]$ and $T_j \cong T'_j$. Suppose R_j and R'_j represent the gCore Q. If $Q = \emptyset$, R_j and R'_j are leaf nodes of T and T', respectively. Therefore, we have $n = n' = j$. Otherwise, there is at least one node (leaf) in the P and P' after R_j and R'_j, respectively. We then have $\min\{n, n'\} > j$. Overall, the base case holds.

Induction step: Let $j^* \in \mathbb{Z}$ be given and suppose when $j = j^*$, it holds that $\min\{n, n'\} > j$, $\mathbf{p}_j[l-1] = \mathbf{p}'_j[l-1]$ and $T_j \cong T'_j$. R_{j+1} is the child of R_j whose vector $\mathbf{p}_{j+1}$ differs from $\mathbf{p}_j$ in the $(l-1)$-th element. R'_{j+1} is the child of R'_j whose vector $\mathbf{p}'_{j+1}$ differs from $\mathbf{p}'_j$ in the $(l-1)$-th element. According to Lemma H.7, we have $\mathbf{p}_{j+1}[l-1] = \mathbf{p}'_{j+1}[l-1]$ and $T_{j+1} \cong T'_{j+1}$. Certainly, $R_{j+1} \cong R'_{j+1}$. Let Q be the gCore represented by them. Similar to the base case, if $Q = \emptyset$, R_{j+1} and R'_{j+1} are leaf nodes of T and T', respectively, and we have $n = n' = j+1$. Otherwise, there is at least one node (leaf) in the P and P' after R_{j+1} and R'_{j+1}, respectively. Thus, $\min\{n, n'\} > j+1$. Overall, the inductive step holds.

Conclusion: By the principle of induction, we have $n = n'$ and for $j = 1, 2, \cdots, n$; (2) $\mathbf{p}_j[l-1] = \mathbf{p}'_j[l-1]$; and (3) $T_j \cong T'_j$.

$T_j \cong T'_j$ for $1 \leq j \leq n$ implies $R_j \cong R'_j$ for $1 \leq j \leq n$. Thus, for any pair of adjacent nodes R'_j and R'_{j+1} in P' representing gCore Q and Q', respectively, the vertex set $Q - Q'$ associated with the edge between R'_j and R'_{j+1} are retained on the edge between R_j and R_{j+1}. That is, the sets on the leftmost path from N' to a leaf node in T are all retained on the leftmost path from N^* to a leaf in T'. Overall, the proof is completed. □

H.3.11 Proof for Theorem F.1.

Proof. **(Theorem F.1)** Let T^c be the original P-tree. Let T_j and T'_k denote the subtrees rooted R_j and R'_k in T^c, respectively.

We first prove the **sufficiency**. By definition of branch (Definition 7.11), we can easily have that $sd(\mathbf{p}_j) = sd(\mathbf{p}'_k) = i$ (defined in

Section H.3.9). As N' is a child of N, $\mathbf{p}_0 \le \mathbf{p}'_0$. According to Observation 2, $\mathbf{p}_j$ and $\mathbf{p}_0$ share a prefix of length $i-1$, and $\mathbf{p}'_k$ and $\mathbf{p}'_0$ share a prefix of length $i-1$. We thus have $\mathbf{p}_j[i'] \le \mathbf{p}'_k[i'] \le \hat{\mathbf{p}}_j[i']$ for $1 \le i' < i$. Together with Condition (2), we have $\mathbf{p}_j[i'] \le \mathbf{p}'_k[i'] \le \hat{\mathbf{p}}_j[i']$ for $i' = 1, 2, \cdots, i-1$. Besides, $\mathbf{p}_j[i] = \mathbf{p}'_k[i]$ (Condition (1)). According to Lemma H.4, $T_j \cong T'_k$. The sufficiency thereby holds.

We next prove the **necessity**. For short, we denote B_i^N and $B_i^{N'}$ by B_N and $B_{N'}$, respectively, and denote their corresponding branch in T^c by B_N^c and $B_{N'}^c$, respectively. Let P_N^c and $P_{N'}^c$ be the rightmost path in B_N^c and $B_{N'}^c$, respectively.

If $T_j \cong T'_k$, we prove Condition (2) is true, which serves as the foundation of the following proof for Condition (1). By Definition 7.8, for each node R' in T'_k, there is a node R in T_j such that $R' \cong R$. According to Definition 7.9, we can derive that $\hat{\mathbf{p}}'_k = \hat{\mathbf{p}}_j$, where $\hat{\mathbf{p}}'_k$ is the signature of T'_k. As each node R' in T'_k is a descendants of R'_k, it holds that $\mathbf{p}'_k \le \mathbf{p}' \le \hat{\mathbf{p}}'$, where $\hat{\mathbf{p}}'$ is the maximal vector for $[R']$. Thus, we have $\mathbf{p}'_k \le \hat{\mathbf{p}}'_k$, and further have $\mathbf{p}'_k \le \hat{\mathbf{p}}_j$. It certainly implies that $\mathbf{p}'_k[i'] \le \hat{\mathbf{p}}_j[i']$ for $i' = 1, 2, \cdots, i-1$.

We then prove Condition (1) by contradiction. Suppose $\mathbf{p}_j[i] \ne \mathbf{p}'_k[i]$, it holds either $\mathbf{p}_j[i] < \mathbf{p}'_k[i]$ or $\mathbf{p}_j[i] > \mathbf{p}'_k[i]$. When the first case is true, there must exist a node R' in the subsequence of $P_{N'}^c$ before R'_k such that $\mathbf{p}_j[i] = \mathbf{p}'[i]$, where $\mathbf{p}'$ is the vector associated with R'. Obviously, we have $\mathbf{p}'[i'] = \mathbf{p}'_k[i'] \le \hat{\mathbf{p}}_j[i']$ for $1 \le i' < i$. According to the sufficiency we have proved above, the subtree rooted at R_j is isomorphic to the subtree rooted at R'. By transitivity of $\cong$, the subtrees rooted at R' and R'_k are isomorphic, which contradicts with the fact that R'_k is a descendant of R'. When the second case is true, there must exist a node R' with vector $\mathbf{p}'$ in the subsequence of $P_{N'}^c$, after R'_k such that $\mathbf{p}_j[i] = \mathbf{p}'[i]$. Similar to the first case, the subtree rooted at R' is isomorphic to the subtree rooted at R_j, and is further isomorphic to the subtree rooted at R'_k. It is clearly also a contradiction. Overall, we have $\mathbf{p}_j[i] = \mathbf{p}'_k[i]$, and thereby the necessity holds. The proof for Theorem F.1 is complete. □

H.3.12 Proof for Theorem F.2.

Proof. **(Theorem F.2)** For ease of explanation, let us first introduce some notations and facts:

(1) Let T be the P+-tree and T^c be its original P-tree. For short, we denote B_i^N and $B_i^{N'}$ by B_N and $B_{N'}$, respectively, and denote their corresponding branch in T^c by B_N^c and $B_{N'}^c$, respectively. Let P_N and $P_{N'}$ be the rightmost path in B_N and $B_{N'}$, respectively, and let P_N^c and $P_{N'}^c$ be the rightmost path in B_N^c and $B_{N'}^c$, respectively. As described in Section 7.3, subtree elimination procedure may remove some vertices from P_N^c and $P_{N'}^c$, making P_N and $P_{N'}$ being a subsequence of P_N^c and $P_{N'}^c$, respectively. We use S_j to denote the subsequence in B_N^c between R_j and R_{j+1}, and use S'_k to denote the subsequence in $B_{N'}^c$ between R_k and R_{k+1}. All nodes in S_j and S'_k are eliminated, as the premises of which, the subtrees rooted at all nodes in S_j (in the original P-tree T) are isomorphic, which are isomorphic to the subtree rooted at R_{j+1} in T, and the subtrees rooted at all nodes in S'_k (in the original P-tree T) are isomorphic, which are isomorphic to the subtree rooted at R'_{k+1} in T.
(2) In the following, we will use notations R, R' and R'' be denote nodes in P_N^c or $P_{N'}^c$ in the rest of the proof. Unless otherwise stated, their vectors are denoted by $\mathbf{p}$, $\mathbf{p}'$ and $\mathbf{p}''$, respectively.
(3) Vectors of all nodes in both P_N^c and $P_{N'}^c$ are monotonic with regard to the relationship $<$ between vectors. Specifically, by Observation 2, the vectors of all nodes in P_N^c share a prefix of length $i-1$, and the vectors of all nodes in $P_{N'}^c$ share a prefix of length $i-1$. However, the i-th elements of vectors of nodes in P_N^c and $P_{N'}^c$ are monotonically increasing, holding that $\mathbf{p}'[i] = \mathbf{p}[i] + 1$ for any pair of adjacent nodes R and R' in P_N^c or $P_{N'}^c$.

Let T_j and T'_k denote the subtrees rooted R_j and R'_k in T^c, respectively. Based on Condition (2), we have $\mathbf{p}'_k[i'] = \mathbf{p}'_{k-1}[i'] \le \hat{\mathbf{p}}_j[i']$ for $1 \le i' < i$. Thus, to prove $T_j \cong T'_k$, according to Theorem F.1, we only need to prove $\mathbf{p}_j[i] = \mathbf{p}'_k[i]$.

We next prove $\mathbf{p}_j[i] = \mathbf{p}'_k[i]$ by contradiction. Suppose $\mathbf{p}_j[i] \ne \mathbf{p}'_k[i]$, it holds either $\mathbf{p}_j[i] < \mathbf{p}'_k[i]$ or $\mathbf{p}_j[i] > \mathbf{p}'_k[i]$. When the first case is true, by Condition (1), there is a node R' in S'_{k-1} such that $\mathbf{p}'[i] = \mathbf{p}_j[i]$. Based on the Condition (2), we have $\mathbf{p}'[i'] = \mathbf{p}'_{k-1}[i'] \le \hat{\mathbf{p}}_j[i']$ for $i' < i$. Thus, Theorem F.1 ensures that T_j is isomorphic to the subtree rooted at R' in T^c. Let R and R'' be the child of R_j and R' in P_N^c and $P_{N'}^c$, respectively. It clearly holds that $\mathbf{p}[i] = \mathbf{p}''[i]$. As R is in the subtree rooted at R_j, $\hat{\mathbf{p}}_j \le \hat{\mathbf{p}}$ holds, where $\hat{\mathbf{p}}$ is the signature of the subtree rooted at R in T^c. We can then derive that $\mathbf{p}''[i'] = \mathbf{p}'_{k-1}[i'] \le \hat{\mathbf{p}}[i']$ based on Condition (2). By Theorem F.1, the subtrees rooted at R and R' in T^c are isomorphic. Since $R \in S'_{k-1}$, R'' must be also in S'_{k-1} or exactly is R'_k. The preceding subtree rooted at R'' in T^c is thereby isomorphic to the preceding subtree rooted at R' in T^c. Notice that the isomorphism between subtrees rooted at R and R'', and R_j and R'' in T^c also imply the isomorphism between preceding subtrees rooted at them in T^c. By transitivity of $\cong$, we have the preceding subtrees rooted at R_j and R in T^c are isomorphic. That is, R_j will be removed by the subtree elimination procedure and no longer exists in P_N, which clearly is a contradiction.

When the second case is true, we have $\mathbf{p}_j[i] > \mathbf{p}'_k[i]$. According to Condition (1), there is a node R in S_{j-1} such that $\mathbf{p}[i] = \mathbf{p}'_k[i]$. Let S be the subsequence of S_{j-1} staring from R. Obviously, the preceding subtrees of all nodes in S (in the original P-tree T^c) are isomorphic and are isomorphic to the preceding subtree rooted at R_j in T^c. We use T_R to denote the subtree rooted at R in T^c. T_j and all preceding subtrees of nodes in S in T constitute T_R. For any node R' in T_R but not in T_j, it must be in the preceding subtree of some node in S, and there exists a node R'' in T_j such that $R' \cong R''$ by Definition 7.8. Based on Definition 7.9, we can derive that $\hat{\mathbf{p}} = \hat{\mathbf{p}}_j$, where $\hat{\mathbf{p}}$ is the signature of T_R in T^c. With Condition (2), we have $\mathbf{p}'_k[i'] = \mathbf{p}'_{k-1}[i'] \le \hat{\mathbf{p}}[i']$ for $i' < i$. Hence, the subtree rooted at R in T^c and T'_k are isomorphic by Theorem F.1. Let R'' and R' be the child of R and R'_k in P_N^c and $P_{N'}^c$, respectively. Clearly, the i-th elements of $\mathbf{p}''$ and $\mathbf{p}'$ are identical. Since R'' is a child of R, it holds that $\hat{\mathbf{p}}'' \le \hat{\mathbf{p}}$, where $\hat{\mathbf{p}}''$ is the signature of the subtree rooted at R'' in T^c. Theorem F.1 then ensures that the subtrees rooted at R and R'' in T^c are isomorphic. Since $R \in S_{j-1}$, R'' must be either also in S_{j-1} or exactly R_j. Thus, the preceding subtrees rooted at R and R'' in T^c are isomorphic. Notice that the isomorphism between subtrees rooted at R and R_k, and R'' and R' in T^c also imply the

isomorphism between preceding subtrees rooted at them in T^c. By transitivity of $\cong$, the preceding subtrees rooted at R'_k and R' in T^c are isomorphic. That is, R'_k will be eliminated, which is certainly another contradiction.

Overall, we have $\mathbf{p}_j[i] = \mathbf{p}'_k[i]$. The proof for Theorem F.2 is complete. □

H.4 Correctness of Subtree Transplant

It's simple but tedious to give the full proof of the correctness of using Algorithm 3 to solve the GCS problem on a P+-tree or P+-DAG obtained after subtree transplant. Therefore, in the following, we present the proof sketch only.

H.4.1 Subtree transplant for subtree elimination. Let T^c and T be any P-tree and the P+-tree obtained by applying the subtree elimination procedure introduced in Section 7.3. Theorem 7.5 ensures the correctness of applying Algorithm 3 to solve the GCS problem on the T.

Let T' be the P+-tree obtained by applying the same subtree elimination procedure as obtaining T but realized by subtree transplant (introduced in Section 7.4). We can prove by induction on the height of T that $T \cong T'$, which ensures that the vertex sets on the leftmost path of any node R in T are all retained on the leftmost path of its corresponding node $f(R)$ in T', where f is a bijection from nodes of T to nodes of T' defined in Equation 2. Thus, to prove the correctness of using subtree transplant to realize subtree elimination, we only need to show for the query to the gCore corresponding to any node N in T, Procedure Search in Algorithm 3 returns $f(N)$ in T'.

Suppose Procedure Search in Algorithm 3 returns N^* when running on T'. Let $S = N_1, N_2, \cdots, N_l$ be the sequence of nodes returned at line 17 during the execution of Search to locating N in T, and $S' = N'_1, N'_2, \cdots, N'_l$ be the sequence of nodes returned at line 17 during the execution of Search to locating N^* in T'. Obviously, $N_l = N$ and $N'_l = N^*$. We can prove $f(N_i) = N'_i$ for $i = 1, 2, \cdots, l$ by using a similar induction to the one used in the proof for Lemma H.2. Thus, we obtain that $f(N_l) = N'_l$, i.e., $f(N) = N^*$. Overall, Algorithm 3 can be applied to solve the GCS problem correctly on the P+-tree obtained after subtree transplant.

H.4.2 Subtree transplant for subtree merge. Let T^c be any P-tree and T^+ be the P+-tree obtained by repeatedly applying the subtree elimination procedure to T^c until no more nodes can be removed. After applying the subtree merge procedure to T^+, we obtain a P+-DAG, called T. Theorem 7.12 ensures the correctness of applying Algorithm 3 to solve the GCS problem on the T.

Let T' be the P+-DAG obtained by applying the same subtree emerge procedure to T^+ as obtaining T but realized by subtree transplant. $T \cong T'$ can be proved by induction on the height of T. Thus, the vertex sets on the leftmost path of any node R in T are all retained on the leftmost path of $f(R)$ in T', where f is a bijection from nodes of T to nodes of T' defined in Equation 2. We next show that for the query to the gCore corresponding to any node N in T, Procedure Search in Algorithm 3 returns $f(N)$ in T'.

Assume that Procedure Search in Algorithm 3 returns N^* when running on T'. Let $O = (N_1, i_1), (N_2, i_2), \cdots, (N_n, i_n)$ be the sequence of outputs of Procedure Forward at either line 15 or line 17 during the execution of Search to locating N in T, and $O' = (N'_1, i'_1), (N'_2, i'_2), \cdots, (N'_{n'}, i'_{n'})$ be the sequence of outputs of Procedure Forward at either line 15 or line 17 during the execution of Search to locating N^* in T'. We have $N_{n'} = N$ and $N'_{n'} = N^*$. It's easy to prove $n = n'$, $i_j = i'_j$ and $f(N_j) = N'_j$ for $1 \le j \le n$ by induction on j. Therefore, we have $f(N) = N^*$. By collecting all nodes in the vertex sets on the leftmost path of N^*, we certainly obtain the target core corresponding to N. Overall, Algorithm 3 can be applied to solve the GCS problem correctly on the P+-DAG obtained after subtree transplant.

I COMPLEXITY ANALYSIS

I.1 Time and Space Complexity of Algorithm 1

I.1.1 Time Complexity. We will show that Algorithm 1 (GCS) runs in time $O(|\mathcal{M}| + l|V_l|)$, where $|\mathcal{M}| = \sum_{i=1}^{l} |V_i| + |E(\mathcal{G})| + |E(C)|$. In the following, we consider its efficient implementation presented in Algorithm 4 in Appendix A.

Let us first analyze the time overhead of the two subroutines Remove (lines 29-34) and Peel (lines 35-44). It's obvious that Remove runs in constant $O(1)$ time. In Peel, line 36 contributes $O(1)$ to the algorithm. We add superscripts *old* and *new* to s_i and e_i to distinguish their values at the beginning and end of the execution of Peel, respectively. Clearly, the **while** loop in lines 37–44 repeats $e_i^{new} - s_i^{old}$ times. Therefore, line 38 and line 44 cost $O(e_i^{new} - s_i^{old})$ time. Within the **while** loop, there is a **for** loop in lines 39–43 repeating $O(|N_i(v)|)$ times for $v \in vert_i[s_i^{old}, e_i^{new})$, with each using $O(1)$ time to run the conditional statements in lines 40–43. Overall, the **while** loop (lines 37–44), also Peel, costs $O(\sum_{s_i^{old} \le j < e_i^{new}} |N_i(vert_i[j])|)$ time.

We next consider the main body of Algorithm 4. In the initialization phase (lines 1–9), line 2 and line 6 run in $O(1)$ time, and hence contributes $O(l)$ to the algorithm. The **for** loops in lines 3–5 and lines 7–9 both have size $|V_i|$ with constant time $O(1)$ bodies. Therefore, the two loops cost $O(\sum_{i=1}^{l} |V_i|)$ time. The initialization phase runs in $O(\sum_{i=1}^{l} |V_i|)$ time.

In the vertex peeling phase (lines 10–27), suppose that the **while** loop in lines 10–27 repeats T times. As each run removes at least one vertex from V_l, we have $T \le |V_l|$. Therefore, line 19 and line 27 contribute $O(|V_l|)$ and $O(l|V_l|)$ time to the algorithm, respectively. Assume that after calling Peel in line 11 in the $(t-1)$-th and the t-th runs of the **while** loop, the value of e_l becomes e_l^{t-1} and e_l^t, respectively. We certainly have that the value of s_l equals e_l^{t-1} before calling Peel in the line 11 in the t-th run because of line 19. Therefore, line 11 takes $O(\sum_{e_l^{t-1} \le j < e_l^t} |N_l(vert_l[j])|)$ time in the t-th run, and totally uses time:

$$O\left(\sum_{t=1}^{T} \sum_{j=e_l^{t-1}}^{e_l^t} |N_l(vert_l[j])|\right) = O\left(\sum_{j=e_l^0}^{e_l^T} |N_l(vert_l[j])|\right) = O\left(\sum_{v \in V_l} |N_l(v)|\right) = O(|E_l|).$$

The conditional statements in lines 15–18 run in constant $O(1)$ time. It is executed for each cross layer edge from layer G_l to each G_i at most once, and hence the **for** loop in lines 12–18 contributes time $O(\sum_{i=1}^{l-1} |E_{l,i}|)$ in all repetitions of the **while** loop (lines 10–27). Similar to line 11, Peel in line 21 costs $O(\sum_{i=1}^{l-1} |E_i|)$ time across all runs of the **for** loop in lines 20–26. The conditional statements in lines 23–26 run in constant $O(1)$ time, and are executed for each

cross layer edge from each layer G_i to G_l at most once. Therefore, the **for** loop in lines 22–26 contributes time $O(\sum_{i=1}^{l-1}|E_{i,l}|)$ in total. Overall, the vertex peeling phase takes $O((l+1)|V_l| + \sum_{i=1}^{l}|E_i| + \sum_{i=1}^{l-1}|E_{l,i}| + \sum_{i=1}^{l-1}|E_{i,l}|)$ time.

To sum up, we have the time complexity of Algorithm 4 (Algorithm 1) is:

$$O\left(\sum_{i=1}^{l}|V_i| + (l+1)|V_l| + \sum_{i=1}^{l}|E_i| + \sum_{i=1}^{l-1}|E_{l,i}| + \sum_{i=1}^{l-1}|E_{i,l}|\right)$$
$$= O\left(\sum_{G_i\in\mathcal{G}}|V_i| + |E(\mathcal{G})| + |E(C| + l|V_l|\right) = O\left(|\mathcal{M}| + l|V_l|\right)$$

I.1.2 Space Complexity. We will show that Algorithm 1 runs in space $O(l \cdot V_{max})$, where $V_{max} = \max_{1\le i\le l}|V_i|$. In the following, we also consider its implementation in Algorithm 4.

Algorithm 4 sets up two arrays $vert_i$ and pos_i and two offsets s_i and e_i for each layer $G_i \in \mathcal{G}$, which takes $O(\sum_{i=1}^{l}|V_i|)$ space. Maintaining the intra-layer degree for each vertex in $\mathcal{M}$ requires $Q(\sum_{i=1}^{l}|V_i|)$ space. Maintaining each cross-layer degree $d_i(v)$ for $v \in V_l$ takes $Q(\sum_{i=1}^{l-1}|V_l|)$ space. Maintaining the cross-layer degree $d_l(v)$ for $v \in V_i, 1 \le i < l$ takes $Q(\sum_{i=1}^{l-1}|V_i|)$ space. Overall, we have the space complexity of Algorithm 4 (Algorithm 1) is:

$$O\left(\sum_{i=1}^{l}|V_i| + \sum_{i=1}^{l-1}|V_l| + \sum_{i=1}^{l-1}|V_i|\right) = O\left(l \cdot V_{max}\right)$$

I.2 Time and Space Complexity of Algorithm 2

Let us first introduce the foundation of our analysis. Given P-tree T nested in the **k**-node in the KP-tree and let R be the root of T, we conceptually divide T into a series of leftmost paths by the following rules:

- R1 If $l = 2$, T itself is R's leftmost path and is therefore no longer divided.
- R2 Otherwise, suppose that R has n children, namely $R_1, R_2, \cdots, R_n$, and among which R_1 is the leftmost child of R. T is then divided into the leftmost path of R and a series of subtrees rooted at $R_2, \cdots, R_n$, respectively, say $T_2, \cdots, T_n$. Thereafter, each T_i with $2 \le i \le n$ is divided recursively using the same way.

Recall that Algorithm 2 generates the nodes of a P-tree in depth-first order, which ensures that each node is generated followed by the generation of its leftmost child. That is, for any node N in a P-tree T, all cores corresponding to nodes on the leftmost path of N are generated consecutively. We next consider the computation on each leftmost path as a unit. Lemma I.1 presents the computational overhead for Algorithm 2 to process a leftmost path. And Lemma I.2 gives the total number of leftmost paths that constitute T.

Lemma I.1. *Given P-tree T nested in the* **k***-node in the KP-tree, let N be any node in T, Algorithm 2 computes all cores on the leftmost path of N in $O(|\mathcal{M}| + l|V_l| + l|F_{l-1}|)$ time and $O(|l \cdot V_{max}|)$ space.*

Proof. Let $N_1, N_2, \cdots, N_m$ be the leftmost path starting from N in T, in which $N_1 = N$ and N_m is a leaf node. It's obvious that m is bounded by $O(|F_{l-1}|)$. For $1 \le j \le m$, we use $\mathbf{p}_j$ and Q_j to denote the vector associated with N_i and the $(\mathbf{k}, \mathbf{p}_j)$-core corresponding to N_j, respectively. It's easy to see that $Q_1, Q_2, \cdots, Q_m$ are computed by a series of consecutively recursive calls of procedure PTreeDFS (lines 4–13) in Algorithm 2.

Time complexity. We first analyze the time overhead. Recall that GCS in line 5 is implemented by Algorithm 4 introduced in Appendix A based on a series of arrays. Specifically, for each layer G_i, we have set up arrays $vert_i$ and pos_i to facilitate vertex peeling. When the GCS terminates, $vert_i[e_i, |V_i|)$ consists all vertex remaining in layer G_i. It enables Algorithm 2 to share these arrays among all calls to GCS to compute $Q_1, Q_2, \cdots, Q_m$. For ease of notation, we use e_i^{old} to denote the value of e_i when the execution of GCS terminates. When Q_j, where $1 < j \le m$, is about to be computed in line 5, we simply set both s_i and e_i to the value of e_i^{old} obtained in the last call of GCS to compute Q_{j-1} for $1 \le i \le l$. After that, vertex peeling (lines 10–27) in Algorithm 4 can be continually performed by using the existing arrays $vert_i$ and pos_i. When the call of GCS to compute Q_m is finished, e_i becomes $|V_i|$. It's easy to see that during the execution of line 5 across all recursive calls to PTreeDFS to compute $Q_1, Q_2, \cdots Q_m$, lines 40–43 in Algorithm 4 are executed twice for each intra-layer edge in $\mathcal{M}$, lines 15–18 are executed once for each cross-layer edge from G_l to each G_i, and lines 23–26 are executed once for each cross-layer edge from each G_i to G_l. Therefore, these three parts contribute $|E(\mathcal{G})| + |E(C)|$ time to the algorithm in total. As the **while** loop in lines 10–27 repeats at most $|V_l|$ times, line 19 and line 27 cost $|V_l|$ and $l|V_l|$ time, respectively. Overall, the vertex peeling phase (lines 10–27) in the execution of GCS (line 5) needs $O((l+1)\cdot|V_l| + |E(\mathcal{G})| + |E(C|)$ time. Lines 1–6 of the initialization phase (lines 1–9) of GCS is only performed at most once (for computing Q_1), which requires $O(\sum_{i=1}^{l}|V_i|)$ time as stated in Appendix A. The conditional statement in lines 8–9 of GCS are executed at most once for each vertex in $\mathcal{M}$, and therefore cost $O(\sum_{i=1}^{l}|V_i|)$ time. Together with $O(l \cdot m)$ time to assign s_i and e_i each time before performing vertex peeling, the initialization phase needs $O(\sum_{i=1}^{l}|V_i| + l \cdot m)$ time in total. In addition, each execution of line 5 also corresponds to a call to Procedure ToFrac in lines 22-23, which can be implemented in $O(l)$ time by keeping a reverse index of each F_i. To sum up, we have that the overall time overhead of line 5 is:

$$O\left((l+1)\cdot|V_l| + |E(\mathcal{G})| + |E(C| + \sum_{i=1}^{l}|V_i| + l\cdot m + l\cdot m\right) = O(|\mathcal{M}| + l|V_l| + l|F_{l-1}|)$$

Besides line 5, lines 10–11 are executed for each computed Q_i, and therefore contribute $O(l\cdot m)$ time. Finally, we have that Algorithm 2 computes all cores on the leftmost path of N in $O(|\mathcal{M}| + l|V_l| + l|F_{l-1}| + l\cdot m) = O(|\mathcal{M}| + l|V_l| + l|F_{l-1}|)$ time.

Space complexity. We next analyze the space overhead. Line 5 takes $O(l \cdot V_{max})$ space as all arrays set up for facilitating vertex peeling are shared among calls to GCS to compute $Q_1, Q_2, \cdots, Q_m$. Lines 10–11 use $O(l)$ space to keep the newly generated vector $\mathbf{p}'$. Therefore, we have the space complexity for Algorithm 2 to compute all cores on the leftmost path of N is $O(l \cdot V_{max})$.

Overall, the lemma holds. □

Lemma I.2. *Any P-tree T will be divided into $O(\prod_{i=1}^{l-2}|F_i|)$ leftmost paths by applying rule R1 or R2.*

Proof. When $l = 2$, by rule R1, T consists of only one leftmost path, i.e., itself. Therefore, the lemma holds.

We next consider the case when $l > 2$. For any node N in T, let $\mathbf{p}$ be the vector associated with N. It's obvious that the leftmost path starting from N contributes to T if and only if $end0(\mathbf{p}) \ge 1$. Therefore, the number of leftmost paths constituting T is equal to the number of **p**-nodes in T such that $end0(\mathbf{p}) \ge 1$, and is further

equal to the number of vectors $\mathbf{v} \in F_1 \times F_2 \times \cdots \times F_{l-2}$, which is $O(\prod_{j=1}^{l-2} |F_j|)$. The lemma thus holds. □

I.2.1 Time Complexity of Algorithm 2. By Lemma I.1 and Lemma I.2, it takes $O(\prod_{i=1}^{l-2} |F_i| \cdot (|\mathcal{M}| + l|V_l| + l|F_{l-1}|))$ time to process the P-tree nested in each $\mathbf{k}$-node in the KP-tree. Thus, line 15 runs in $O(\prod_{i=1}^{l} \kappa(G_i) \cdot \prod_{i=1}^{l-2} |F_i| \cdot (|\mathcal{M}| + l|V_l| + l|F_{l-1}|))$ time. Lines 19–20 are executed once for each $\mathbf{k}$-node in the KP-tree, and therefore consume $O(l \cdot \prod_{i=1}^{l} \kappa(G_i))$ time. Notice that the pseudocode for generating each F_i is omitted in Algorithm 2, however it takes $O(l \cdot |V_l| + \sum_{i=1}^{l-1} d_i^2 \log d_i)$ time which will be analyzed in Section I.4.1. Overall, we have the time complexity of Algorithm 2 is:

$$O\left(\prod_{i=1}^{l} \kappa(G_i) \cdot \prod_{i=1}^{l-2} |F_i| \cdot (|\mathcal{M}| + l|V_l| + l|F_{l-1}|) + l \cdot \prod_{i=1}^{l} \kappa(G_i) + l \cdot |V_l| + \sum_{i=1}^{l-1} d_i^2 \log d_i\right)$$
$$= O\left(\prod_{i=1}^{l} \kappa(G_i) \cdot \prod_{i=1}^{l-2} |F_i| \cdot (|\mathcal{M}| + l|V_l|) + l \cdot \prod_{i=1}^{l} \kappa(G_i) \cdot \prod_{i=1}^{l-1} |F_i| + \sum_{i=1}^{l-1} d_i^2 \log d_i\right)$$

I.2.2 Space Complexity of Algorithm 2. As the space for GCS (line 5) to process each leftmost path, also each P-tree, can be reused, line 15 takes $O(l \cdot V_{max})$ space according to Lemma I.1. Lines 19–20 use $O(l)$ space to keep the newly generated vector $\mathbf{k}'$. In addition, we also need $O(l \cdot \sum_{i=1}^{l-1} |F_i|)$ space to store each F_i and its reverse index. Overall, we have that the space complexity of Algorithm 2 is:

$$O\left(l \cdot V_{max} + 1 + l \cdot \sum_{i=1}^{l-1} |F_i|\right) = O\left(l \cdot \left(V_{max} + \sum_{i=1}^{l-1} |F_i|\right)\right)$$

I.3 Time and Space complexity of Algorithm 3

Time Complexity. We will show that Algorithm 3 runs in $O(\sum_{i=1}^{l-1} |F_i| + |Q|)$ time, where $|Q|$ is the size of the output $(\mathbf{k}, \mathbf{p})$-core.

Line 1 costs $O(1)$ time to perform a hash table lookup. Procedure Search (line 3) visits each node on the path from the root node of the P-tree to the $\hat{\mathbf{p}}$-node in $O(h_1)$ time, where h_1 is the length of the path. Procedure Recover (line 4) visits each edge on the leftmost path starting from the $\hat{\mathbf{p}}$-node to a dummy leaf node and collects vertex sets on each edge using $O(h_2 + |Q|)$, where h_2 is the length of the leftmost path of the $\hat{\mathbf{p}}$-node. Obviously, $h_1 + h_2$ is bounded by the height of the P-tree $O(\sum_{i=1}^{l-1} |F_i|)$. We then have the time complexity of Algorithm 3 is $O(\sum_{i=1}^{l-1} |F_i| + |Q|)$.

Space Complexity. Obviously, except for the space for storing the KP-tree, Procedure Search runs in $O(1)$ space, and Procedure Recover runs in $O(|Q|)$ space. We then have the space complexity of Algorithm 3 is $O(|Q|)$.

I.4 Time and Space Complexity of Algorithm 5

I.4.1 Time Complexity. We will show the time complexity of Algorithm 5 is $O(\prod_{i=1}^{l} \kappa(G_i) \cdot \prod_{i=1}^{l-1} |F_i| \cdot (|\mathcal{M}| + l|V_l|) + \sum_{i=1}^{l-1} d_i^2 \log d_i)$.

Let us first analyze the time overhead of the subroutine GenFrac (lines 11–16) in Algorithm 5. Line 12 runs in $O(1)$ time to initialize F_i. In our realization, we first identify distinct values of possible cross-layer degrees of vertices in V_l to avoid generating duplicate fractions. It can be finished in $O(|V_l|)$ time. After that, the **for** loop in lines 13–15 repeats at most d_i times, where d_i is the maximum cross-layer degree. For each distinct cross-layer degree d, the **for** loop in lines 14–15 insert $d + 1$ fractions into set F_i. Therefore, $O(d_i^2)$ insertions are performed in all repetitions of **for** loop in lines 13–15, which requires $O(d_i^2 \log d_i)$ time. Overall, GenFrac runs in $O(|V_l| + d_i^2 \log d_i)$ time.

We next consider the main body of Algorithm 5. Line 1 runs in time $O(1)$ to initialize R. Line 2 generates all possible values of $\mathbf{k}$ with each costing $O(l)$ time, and therefore contributes $O(l \cdot \prod_{i=1}^{l} \kappa(G_i))$ time. The **for** loop in lines 3–4 repeats for each layer G_i, where $1 \le i < l$, and hence takes $O(\sum_{i=1}^{l-1}(|V_l| + d_i^2 \log d_i))$. Line 5 generates all possible values of $\mathbf{p}$ with each costing $O(l-1)$ time, and altogether contributes $O(l \cdot \prod_{i=1}^{l-1} |F_i|)$ time. For each combination of $\mathbf{k} \in K$ and $\mathbf{p} \in P$, line 8 is executed once in $O(|\mathcal{M}| + l|V_l|)$ time. To sum up, line 8 contributes $O(\prod_{i=1}^{l} \kappa(G_i) \cdot \prod_{i=1}^{l-1} |F_i| \cdot (|\mathcal{M}| + l|V_l|)$ time. Overall, we have the time complexity of Algorithm 5 is:

$$O\left(1 + l \cdot \prod_{i=1}^{l} \kappa(G_i) + \sum_{i=1}^{l-1} \left(|V_l| + d_i^2 \log d_i\right) + l \cdot \prod_{i=1}^{l-1} |F_i| + \prod_{i=1}^{l} \kappa(G_i) \cdot \prod_{i=1}^{l-1} |F_i| \cdot (|\mathcal{M}| + l|V_l|)\right)$$
$$= O\left(\prod_{i=1}^{l} \kappa(G_i) \cdot \prod_{i=1}^{l-1} |F_i| \cdot (|\mathcal{M}| + l|V_l|) + \sum_{i=1}^{l-1} d_i^2 \log d_i\right)$$

I.4.2 Space Complexity. We will show that the space complexity of Algorithm 5 is $O(l \cdot (\prod_{i=1}^{l} \kappa(G_i) + \prod_{i=1}^{l-1} |F_i| + V_{max}))$, where $V_{max} = \max_{1 \le i \le l} |V_i|$.

Let us first analyze the space overhead of the subroutine GenFrac. As stated in Section I.4.1, we detect distinct values of cross-layer degrees of vertices in V_l, which can be realized in $O(|V_l|)$ space. Maintaining all generated fractions in F_i costs $O((l-1) \cdot |F_i|)$ space. Overall, GenFrac takes $O(|V_l| + l|F_i|)$ space.

We next consider the main body of Algorithm 5. As it has to keep all generated vectors $\mathbf{k} \in K$ and $\mathbf{p} \in P$, $O(l \cdot (\prod_{i=1}^{l} \kappa(G_i) + \prod_{i=1}^{l-1} |F_i|))$ space is needed. Besides, it consumes $O(|V_l| + \max_{1 \le i < l} l|F_i|)$ space to run GenFrac (line 4) as the space for each run can be reused. Similarly, all runs of GCS in line 8 cost $O(l \cdot V_{max})$ space. To sum up, we have the space complexity of Algorithm 5 is:

$$O\left(l \cdot \left(\prod_{i=1}^{l} \kappa(G_i) + \prod_{i=1}^{l-1} |F_i|\right) + |V_l| + \max_{1 \le i < l} l \cdot |F_i| + l \cdot V_{max}\right) =$$
$$= O\left(l \cdot \left(\prod_{i=1}^{l} \kappa(G_i) + \prod_{i=1}^{l-1} |F_i| + V_{max}\right)\right)$$

I.5 Time Complexity of Subtree Merge

We will show that the time complexity to apply subtree merge to compact a P+-tree is $O\left(\sum_{i=2}^{l-1} \prod_{j=1}^{i} |F_j|\right)$.

Given P+-tree T nested in the $\mathbf{k}$-node in the KP-tree, for any branch B_i^N in T, the subtree merge procedure introduced in Section 7.5 detects strong isomorphic subtrees within B_i^N and $B_i^{N'}$, where N' is the parent of N. As we have analyzed in Section F.2, it can be realized by the pivot-based optimization in $O(m + n)$ time, where m and n are lengths of the rightmost path of N and N' in B_i^N and $B_i^{N'}$, respectively. It's obvious that both m and n are bounded by $O(|F_i|)$. If strong isomorphic subtrees are identified, the merge operation, which is implemented by the subtree transplant approach introduced in Section 7.4, can be applied at most once to B_i^N, and therefore costs $O(1)$ time. Thus, it takes $O(|F_i|)$ time for subtree merge to process branch B_i^N.

We conceptually divide branches in T to be examined by subtree merge into the following sets: $R_2, R_3, \cdots, R_{l-1}$, in which R_i consists

of all branches B_i^N of $\mathbf{p}$-nodes N such that $end0(\mathbf{p}) \geq l - i$. As analyzed in the proof of Lemma I.2, the number of $\mathbf{p}$-nodes in the P-tree satisfying $end0(\mathbf{p}) \geq l - i$ is equal to the number of vectors $\mathbf{v} \in F_1 \times F_2 \times \cdots \times F_{i-1}$, which is $\prod_{j=1}^{i-1} |F_j|$. Therefore, we have $|R_i| = \prod_{j=1}^{i-1} |F_j|$. Since each branch in R_i contributes $|F_i|$ time to the whole subtree merge process, we have the time complexity of applying subtree merge is:

$$O\left(\sum_{i=2}^{l-1}\left(\prod_{j=1}^{i-1}|F_j|\cdot|F_i|\right)\right) = O\left(\sum_{i=2}^{l-1}\prod_{j=1}^{i}|F_j|\right)$$

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