Advanced Topics

T. J. Rogers

February, 2023

EXAMPLES TO COVER

1. Latent forces for load estimation
2. Nonlinear system identification, latent restoring forces
3. Alternative Views on SSI
4. Some adventures in SMC

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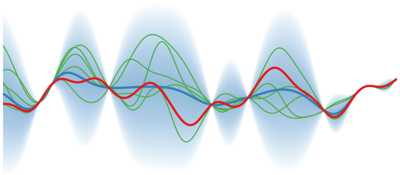
Gaussian Processes:

Flexible nonlinear Bayesian regression.

$$y = f(x) + \varepsilon \quad \varepsilon \sim \mathcal{N}(0, \sigma_n^2)$$

$$f(x) \sim \mathcal{GP}(m(x), k(x, x'))$$



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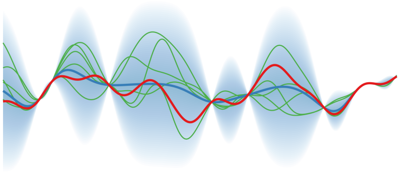

GP Latent Force Problem:

The GP-LFM (Alvarez 2009) solves problems in the form:

$$m\ddot{x} + c\dot{x} + kx = U \quad U \sim \mathcal{GP}\left(0, k(t, t')\right)$$

The GP can be rewritten as an LGSSM (Hartikainen 2010) and similarly for the GP-LFM (Hartikainen 2012).

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THE FUNDAMENTAL TRICK

Turn a Gaussian process ($\mathcal{O}(N^3)$) into a linear SSM ($\mathcal{O}(N)$)

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Turn a Gaussian process ($\mathcal{O}(N^3)$) into a linear SSM ($\mathcal{O}(N)$)

Temporal
Covariance
Function

$k\,(\mathrm{t}, \mathrm{t}')$

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THE FUNDAMENTAL TRICK

Turn a Gaussian process ($\mathcal{O}(N^3)$) into a linear SSM ($\mathcal{O}(N)$)

Temporal Covariance Function		Covariance Spectral Density
$k\left(t, t'\right)$	$\Rightarrow$	$S_k\left(\omega\right)$

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THE FUNDAMENTAL TRICK

Turn a Gaussian process ($\mathcal{O}(N^3)$) into a linear SSM ($\mathcal{O}(N)$)

Temporal Covariance Function		Covariance Spectral Density		Rational Transfer Function
$k\,(\mathrm{t}, \mathrm{t}')$	$\Rightarrow$	$S_k\,(\omega)$	$\Rightarrow$	$H(i\omega)$

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THE FUNDAMENTAL TRICK

Turn a Gaussian process ($\mathcal{O}(N^3)$) into a linear SSM ($\mathcal{O}(N)$)

Temporal Covariance Function		Covariance Spectral Density		Rational Transfer Function		Continuous Time SSM
$k\,(\mathrm{t},\mathrm{t}')$	→	$S_k\,(\omega)$	→	$H(i\omega)$	→	$F\ \mathrm{f}(t) + L\ w(t)$

Hartikainen, Jouni, and Simo Särkkä. "Kalman filtering and smoothing solutions to temporal Gaussian process regression models." 2010 *IEEE international workshop on machine learning for signal processing.* IEEE.

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LATENT FORCES ARE NATURAL SOLUTIONS IN DYNAMICS

$$M\ddot{x} + C\dot{x} + Kx = U$$

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LATENT FORCES ARE NATURAL SOLUTIONS IN DYNAMICS

$$M\ddot{\mathrm{x}} + C\dot{\mathrm{x}} + K\mathrm{x} = U$$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ -M^{-1}K & -M^{-1}C \end{pmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{pmatrix} 0 \\ M^{-1} \end{pmatrix} U$$

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$$U \sim \mathcal{GP}\left(0, k(\mathrm{t}, \mathrm{t}')\right)$$

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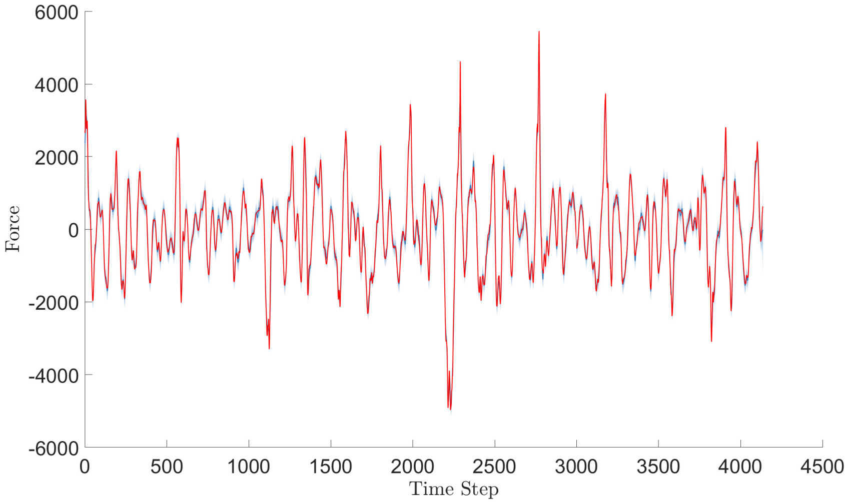
MDOF EXAMPLE

NMSE=1.05%



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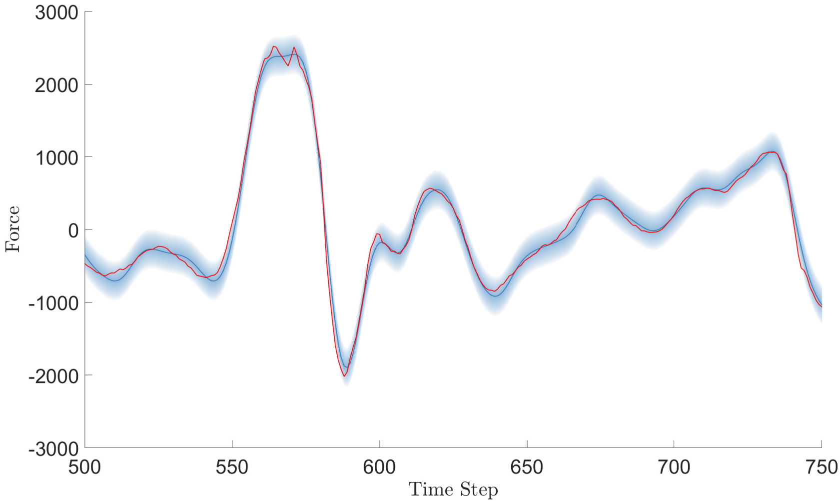

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Let's consider a nonlinear system:

$$m\ddot{x} + c\dot{x} + kx + f(x, \dot{x}) = U$$

Masri and Caughey introduced the Restoring Force Surface method (1979),

$$m\ddot{x} + c\dot{x} + kx = U - f(x, \dot{x})$$

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Pasadena, Calif. 91106. Mem. ASME

A Nonparametric Identification Technique for Nonlinear Dynamic Problems

A nonparametric identification technique is presented that uses information about the state variables of nonlinear systems to express the system characteristics in terms of orthogonal functions. The method can be used with deterministic or random excitation (stationary or otherwise) to identify dynamic systems with arbitrary nonlinearities, including those with hysteretic characteristics. The method is shown to be more efficient than the Weiner-kernel approach in identifying nonlinear dynamic systems of the type considered.

Introduction

The identification of dynamic system models through the use of experimental data is a problem of considerable importance in the area of applied mechanics. The development of procedures to handle the problem has received wide attention in recent years because of the development of efficient computer-oriented systems identification techniques and the availability of sophisticated experimental apparatus for accurate, convenient gathering and analysis of test data.

The approaches used to handle different identification problems and the degree of difficulty in identification depend on the classification of the case:

1 Linear/nonlinear.
2 Stationary/nonstationary.
3 Discrete/continuous.
4 Single-input/multi-input.
5 Deterministic/stochastic.
6 The degree of *a priori* knowledge about the system [1–22].

System identification methods can also be classified on the basis of their search space: (*a*) parametric methods that search in parameter space and (*b*) nonparametric methods that search in function space. Basically, parametric methods seek to determine the value of parameters in an assumed model of the system to be identified, while nonparametric methods produce the best functional representation of the system without *a priori* assumptions about the system model.

So far, most of the identification work in applied mechanics has been parametric. A considerable amount of effort has been devoted to determining efficient algorithms and techniques for estimating the magnitude of various parameters in an assumed mathematical model of the dynamic system of interest. One of the limitations of this class of methods is that the type of model, once assumed at the onset of the investigation, cannot be changed. Thus, if the type of model does not closely resemble the characteristics of the physical system (which, in many practical dynamic problems, are not fully understood), the prediction of the future behavior of the identified system may be in substantial error.

The restriction of forcing the system characteristics to fit an assumed form can be eliminated by using nonparametric identification techniques such as the ones that use the Volterra-series, or Weiner-kernel, approach [23–25]. However, this approach has its own problems of

1 Greater mathematical complexity.
2 Serious difficulties with convergence rate.
3 Excessive computation time.
4 Very demanding (and usually unrealistic) storage requirements.
5 Restrictions on the nature of dynamic systems to be identified (nonhysteretic, stationary), and on the input signals that can be used (white noise).

In an effort to alleviate some of these problems, this paper presents a relatively simple and straightforward approach to identify a broad

Presented at the Eighth U.S. National Congress of Applied Mechanics, University of California at Los Angeles, Los Angeles, Calif., June 26–30, 1978.

Discussion on this paper should be addressed to the Editorial Department, ASME, United Engineering Center, 345 East 47th Street, New York, N.Y. 10017, and will be accepted until September 1, 1979. Readers who need more time to prepare a Discussion should request an extension of the deadline from the Editorial Department. Manuscript received by ASME Applied Mechanics Division, July, 1978.

RESTORING FORCES FOR SYSTEM ID

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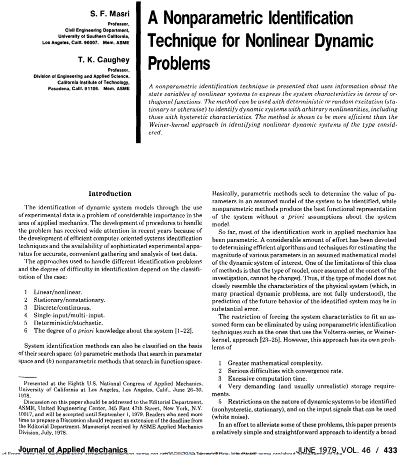
Unfortunately, if the displacement and velocity isn't known this can be hard.



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POSING RESTORING FORCES AS A LATENT FORCE MODEL

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The GP can be rewritten as an LGSSM (Hartikainen 2010) and similarly for the GP-LFM (Hartikainen 2012).

Latent Restoring Forces:

Model the restoring forcing of the nonlinear system:

$$m\ddot{x} + c\dot{x} + kx + f(x, \dot{x}) = U$$

As a GP in time,

$$m\ddot{x}+c\dot{x}+kx+R = U \quad R \sim \mathcal{GP}\left(0, k(t, t')\right)$$

Solve as a **linear** state-space model.

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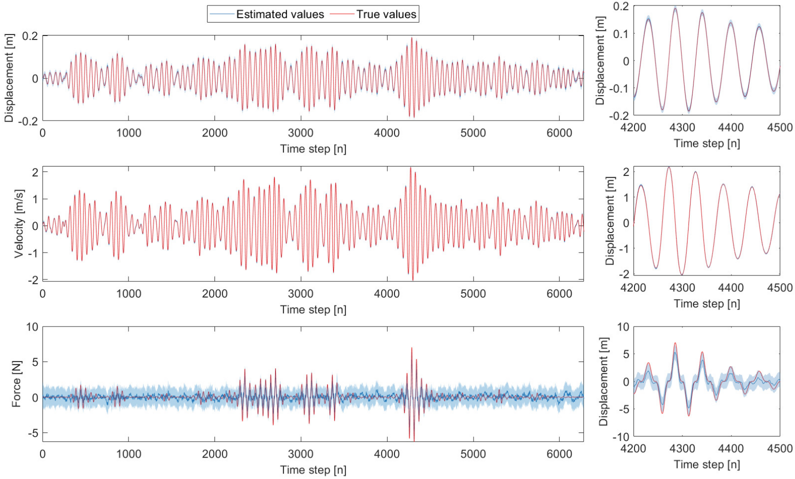
DOES IT WORK?



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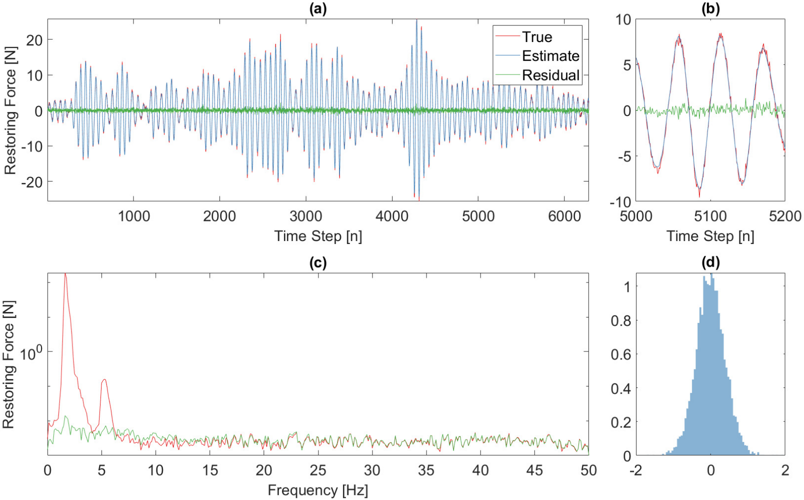

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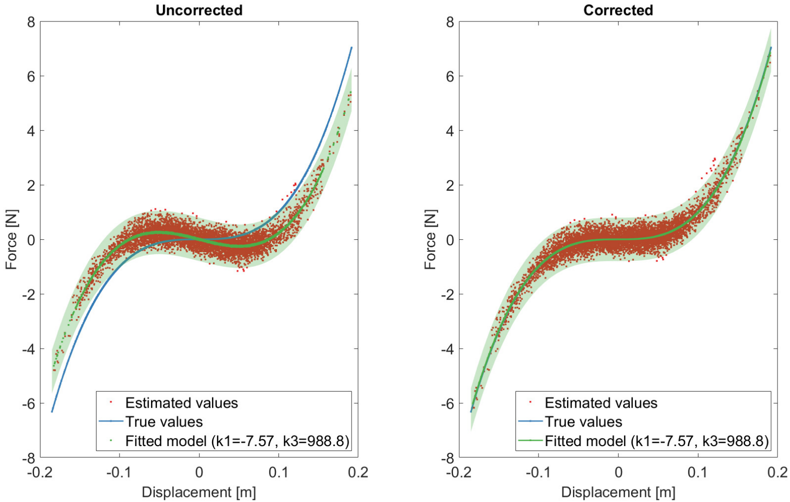

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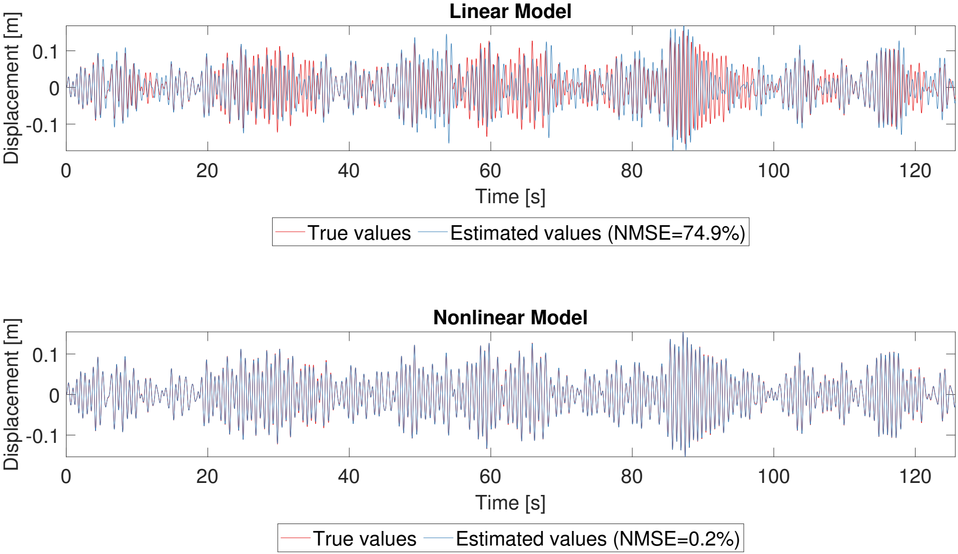
Linear Model
Displacement [m]
0.1
0
-0.1
0
20
40
60
80
100
120
Time [s]
True values
Estimated values (NMSE=74.9%)
Nonlinear Model
Displacement [m]
0.1
0
-0.1
0
20
40
60
80
100
120
Time [s]
True values
Estimated values (NMSE=0.2%)

REVISITING STOCHASTIC SUBSPACE

Remember that for SSI we had:

Time Series Data → Construct Hankel Matrices → CCA → Post Processing → Modal Properties

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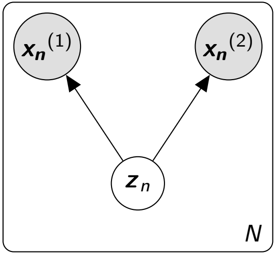

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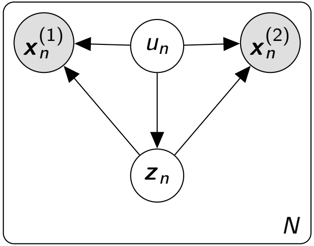
EXTENSIONS TO PROBABILISTIC SSI

Extend the model to be robust:

$$u_n \sim \mathcal{G}a\left(u_n \middle| \frac{\nu}{2}, \frac{\nu}{2}\right)$$
$$\boldsymbol{z}_n | u_n \sim \mathcal{N}\left(\boldsymbol{z}_n | 0, u_n^{-1}\mathbb{I}_d\right)$$
$$\boldsymbol{x}_n^{(m)} | \boldsymbol{z}_n, u_n \sim \mathcal{N}\left(\boldsymbol{x}_n^{(m)} \middle| W^{(m)}\boldsymbol{z}_n + \boldsymbol{\mu}^{(m)}, u_n^{-1}\Sigma^{(m)}\right)$$



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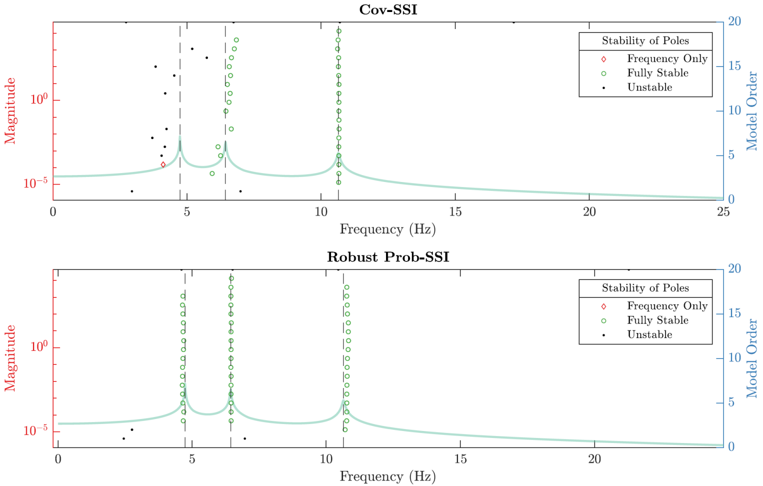
ROBUSTNESS IN SSI



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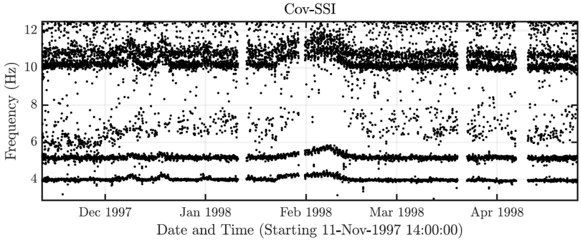

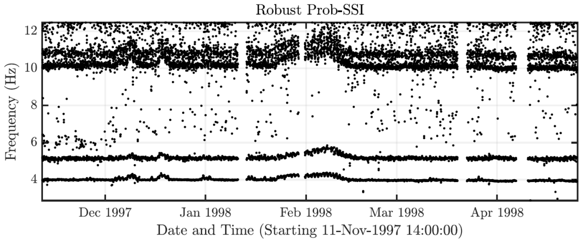

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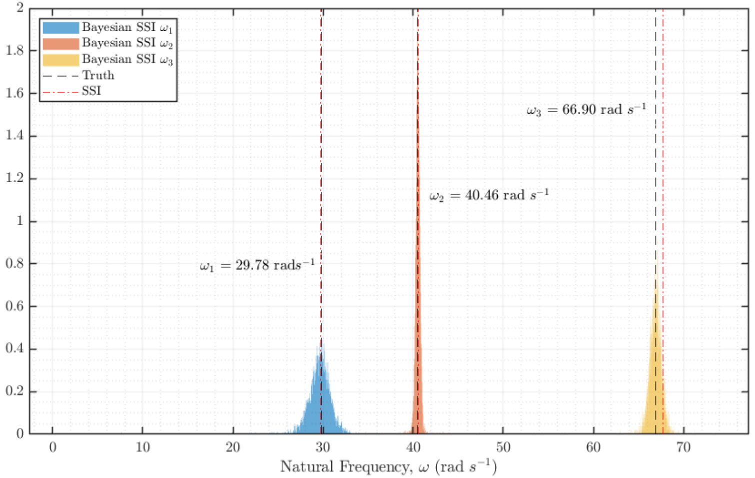
OF COURSE BAYESIAN SSI



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IDENTIFYING A NONLINEAR SYSTEM

The Bouc-Wen system is defined by a second-order continuous time ODE:

$$
\begin{aligned}
m\ddot{q}(t) + R(q,\dot{q}) + Z(q,\dot{q}) &= F(t) \\
R(q,\dot{q}) &= k_L q(t) + c_L \dot{q}(t) \\
\dot{Z}(q,\dot{q}) &= \alpha \dot{q}(t) - \beta\left(\gamma|\dot{q}(t)||Z(t)|^{\nu-1}Z(t) + \delta\dot{q}(t)|Z(t)|^{\nu}\right)
\end{aligned}
$$

Which we can write in the state space by choosing some states:

$$
\boldsymbol{y} = \begin{bmatrix} q \\ \dot{q} \\ z \end{bmatrix}
$$

And rearranging to get $\dot{\boldsymbol{y}}$.

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THE PROBLEM WITH CONTINUOUS TIME

We want to infer systems where we know $\boldsymbol{y}_{1:T}$ and $\boldsymbol{u}_{1:T}$ which are governed by some set of ODES:

$$\dot{\boldsymbol{y}} = f(\boldsymbol{y}, \boldsymbol{u}, t)$$

If we know some initial conditions $\boldsymbol{y}_0$ we are solving an IVP and could use e.g. Euler:

$$\boldsymbol{y}_{t+1} = \boldsymbol{t}_t + h\, f(\boldsymbol{y}_{t-1}, \boldsymbol{u}, t-1)$$

Or extend to Runge-Kutta by raising the order... Then you can assess the quality of the simulated $\boldsymbol{y}_{1:T}$

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NUMERICALLY INTEGRATING PROBABILISTICALLY

Philipp Hennig *et al.* have been recasting numerical integration methods as probabilistic algorithms.

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Enter the probabilistic ODE solver, we know:

$$\dot{\boldsymbol{y}} = f(\boldsymbol{y}) \implies \dot{\boldsymbol{y}} - f(\boldsymbol{y}) = 0$$

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If I set $\boldsymbol{x} = [\boldsymbol{y}, \dot{\boldsymbol{y}}]$ and choose some dynamic model which I can integrate exactly then:

$$\dot{\boldsymbol{x}} = A_c\boldsymbol{x} + Lw(t) \to p\left(\boldsymbol{x}_{t+1} \,|\, \boldsymbol{x}_t\right) = \mathcal{N}\left(A\boldsymbol{x}, Q\right)$$

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and at every time point of interest:

$$p\left(\boldsymbol{z}_t \,|\, \boldsymbol{x}_t\right) = \mathcal{N}\left(0 \,|\, \boldsymbol{x}_t^{(1)} - f\left(\boldsymbol{x}_t^{(0)}\right), R\right)$$

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SOLVING THE PROBABILISTIC ODE

You may notice that $p(\boldsymbol{x}_{t+1} \mid \boldsymbol{x}_t)$ and $p(\boldsymbol{z}_t \mid \boldsymbol{x}_t)$ describe a (nonlinear) probabilistic SSM.

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SOLVING THE PROBABILISTIC ODE

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So the solution to the IVP for the ODE is given by the filtering/smoothing solution of the SSM...

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One outstanding question is: how to choose A and Q? We rewrite a Gaussian process as an SSM. Importantly, the transition model remains linear and available in closed form because we define a linear SDE.

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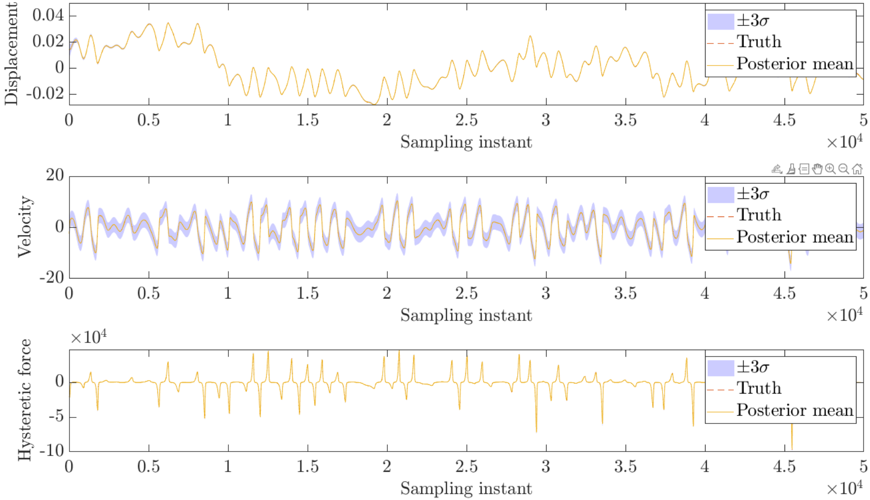
DOES IT WORK?



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EXTENDING TO SYSTEM ID

Since this is now just a (nonlinear) SSM we can include additional dimensions in the observations, e.g. measured data $\boldsymbol{d}_{1:T}$, so $\hat{\boldsymbol{z}}_t = [\boldsymbol{z}_t, \boldsymbol{d}_t]^\mathsf{T} = [0, \boldsymbol{d}_t]^\mathsf{T}$ and update $p(\hat{\boldsymbol{z}}_t \mid \boldsymbol{x}_t)$

Importantly as a side effect of solving the SSM we have an estimate of $p(\hat{\boldsymbol{z}}_{1:T} \mid \boldsymbol{\theta}_i) = \int p(\hat{\boldsymbol{z}}_{1:T} \mid \boldsymbol{x}_{1:T}) p(\boldsymbol{x}_{1:T} \mid \boldsymbol{\theta}_i) \, \mathrm{d}\boldsymbol{x}_{1:T}$ from the filter.

This means we can use this as our cost function in an optimisation…or even better a Bayesian identification.

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A BAYESIAN SYSTEM ID APPROACH

In fact we can use the estimate of $p\left(\hat{\boldsymbol{z}}_{1:t} \,\middle|\, \boldsymbol{\theta}\right)$ inside another Sequential Monte Carlo scheme as the weighting on a set of particles if each particle is a set of parameters.

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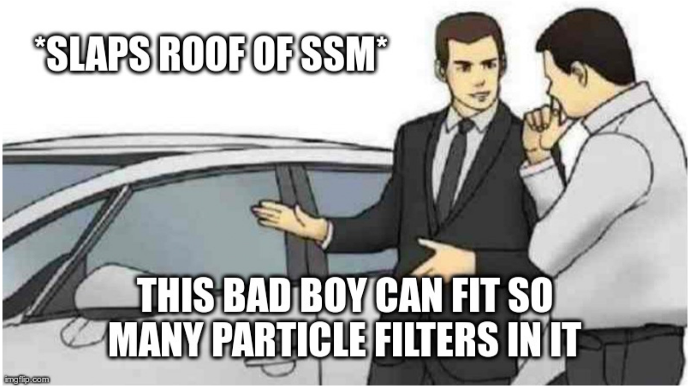

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Components:

1. An inner SSM which is the probabilistic ODE solver augmented with observation data
2. An outer SMC where the particle system is the set of parameters
3. A transition kernel which targets $\pi\left(\boldsymbol{\theta} \mid \boldsymbol{d}_{1:T}\right)$

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At each time point $t = 1, \ldots, T$ introduce $\boldsymbol{d}_t$, compute incremental weight $\propto p\left(\boldsymbol{d}_t \mid \boldsymbol{d}_{1:t-1}, \boldsymbol{\theta}_i\right)$ for $i = 1, \ldots, N$, if ESS too low resample and rerun filter from $t = 0$

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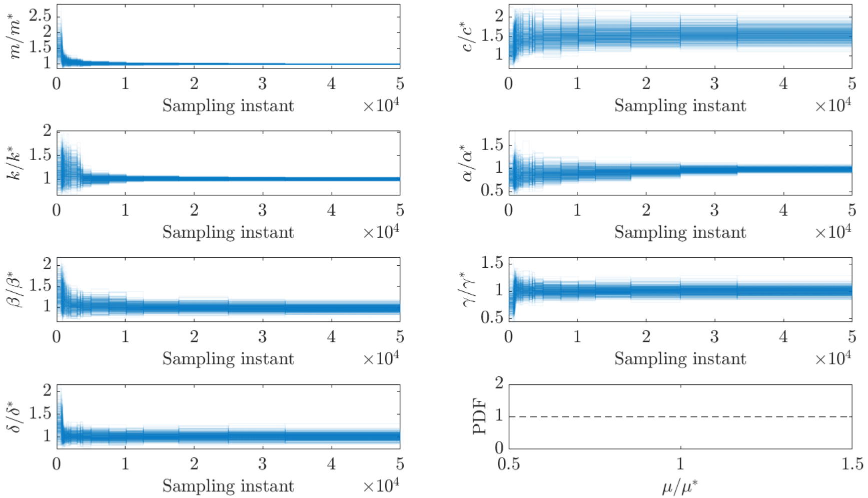
WHAT DOES IT LOOK LIKE?



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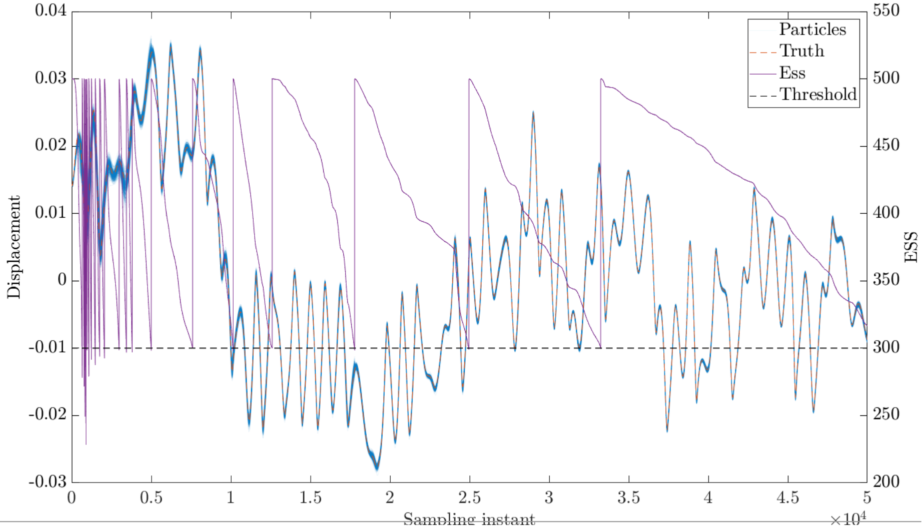

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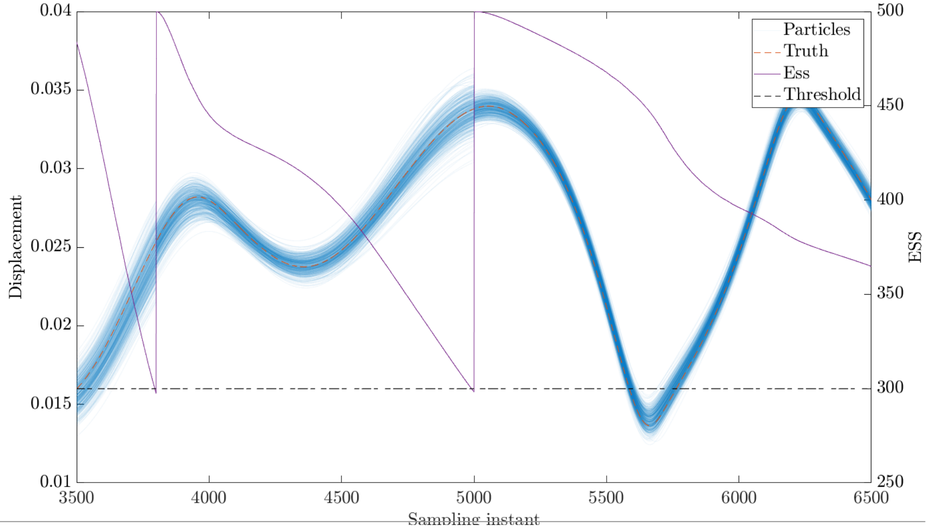

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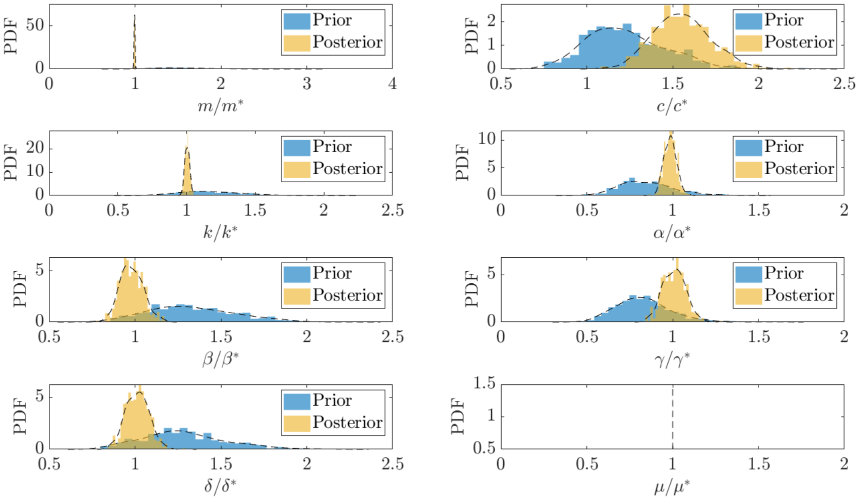
THE FINAL POSTERIORS

Distributions at $t = T$



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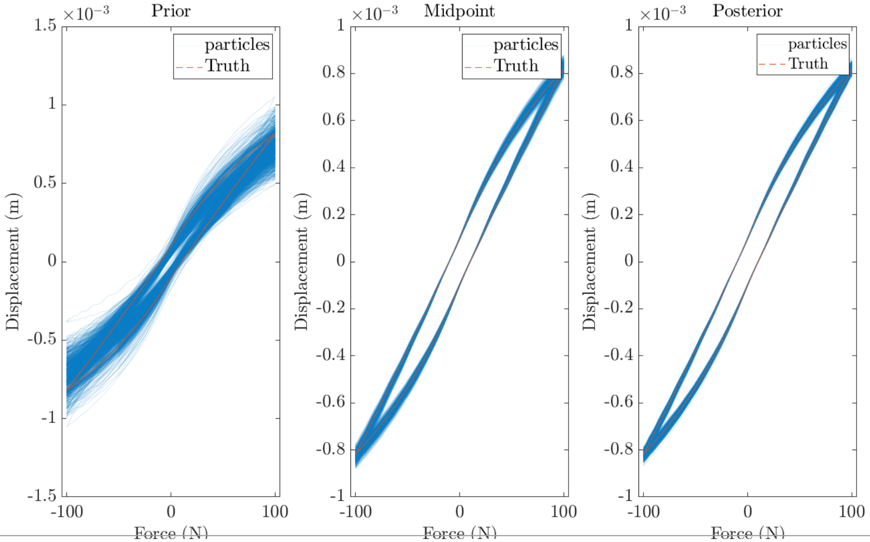

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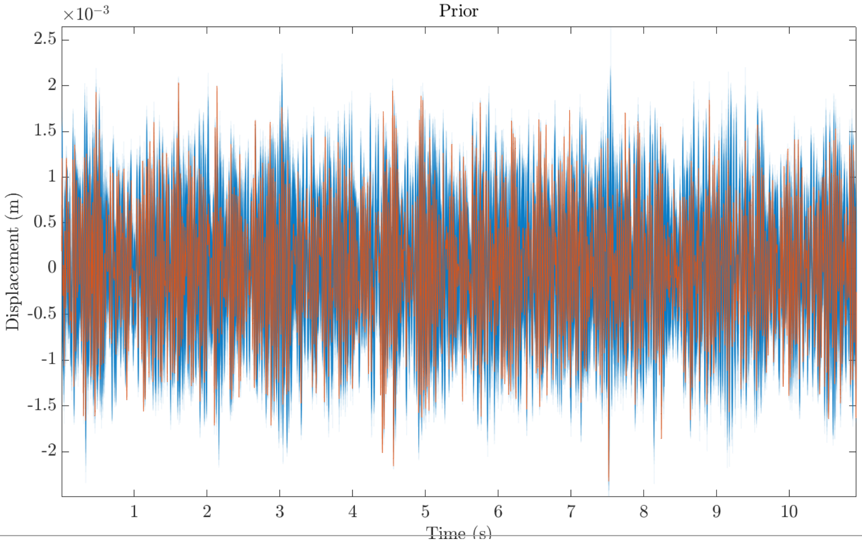
MULTISINE TESTING



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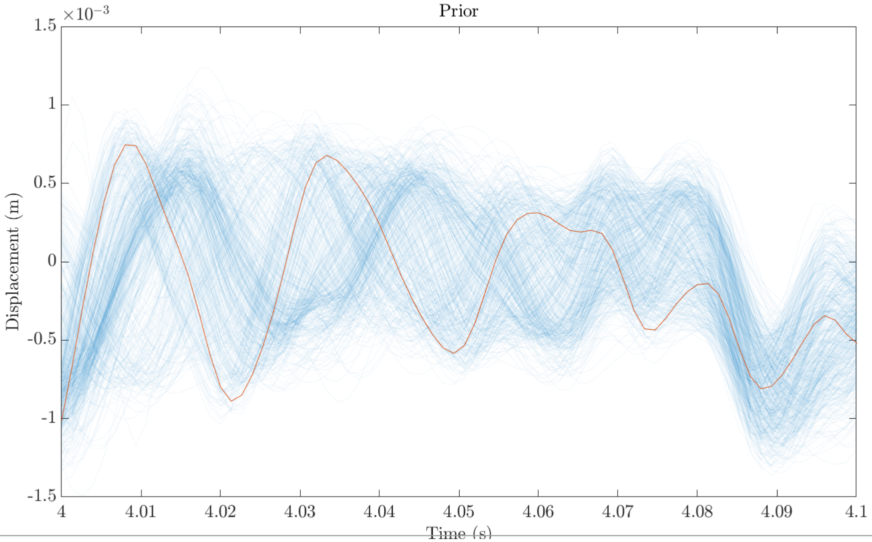

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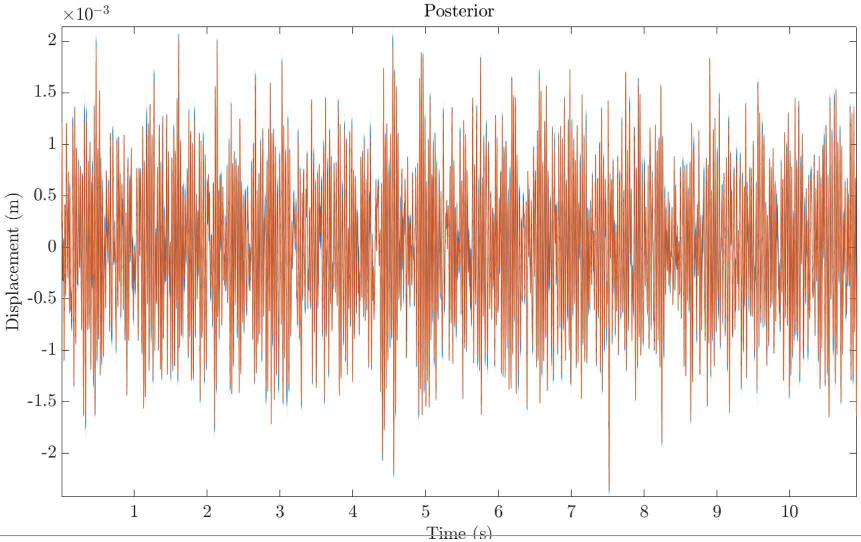

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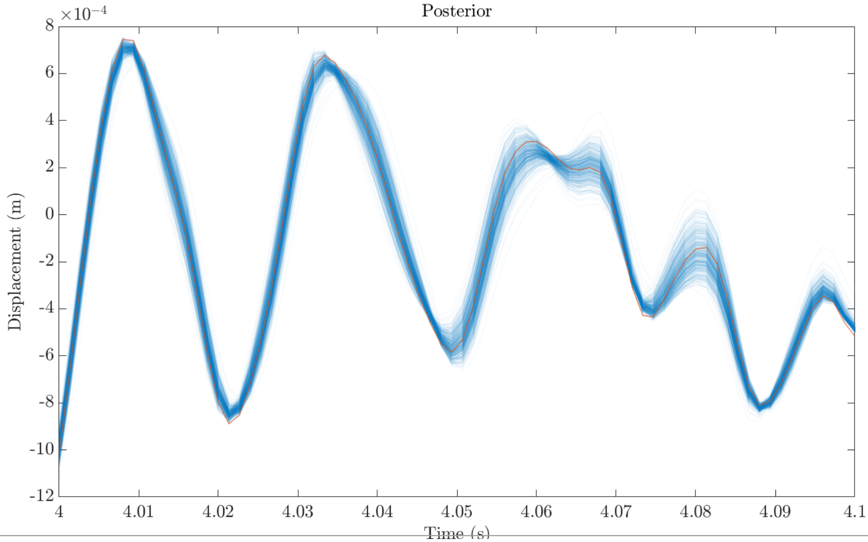

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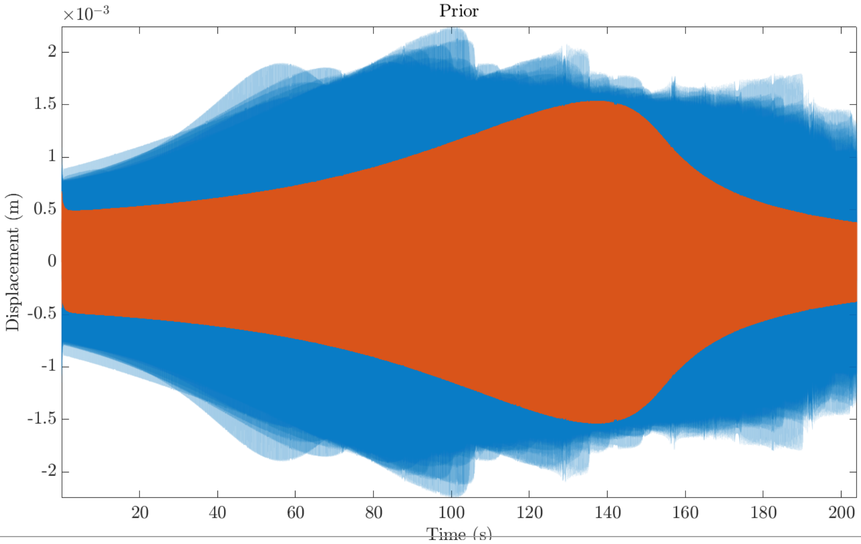
SINE SWEEP TESTING



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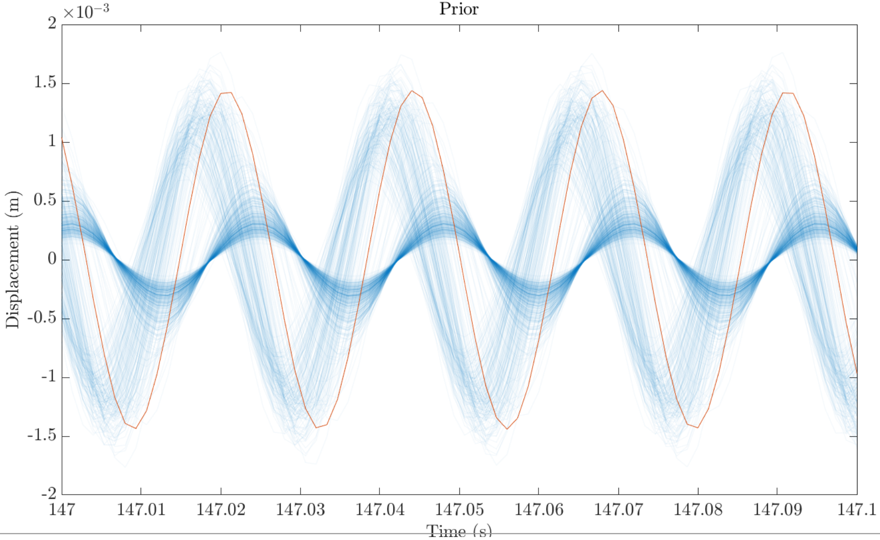

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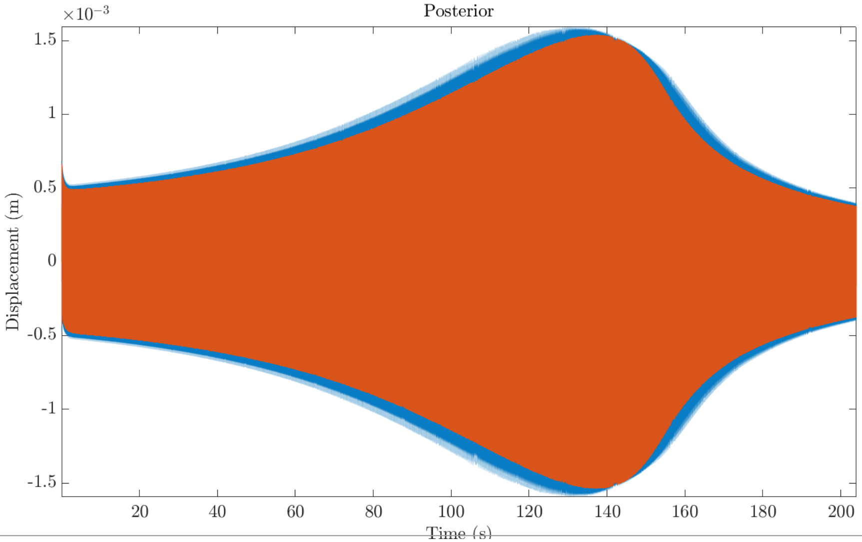

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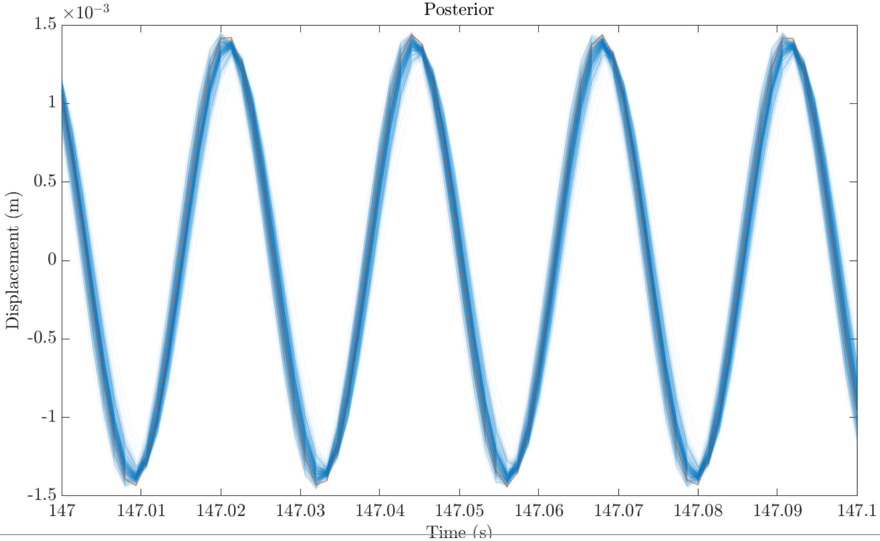

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The University Of Sheffield.

Advanced Topics

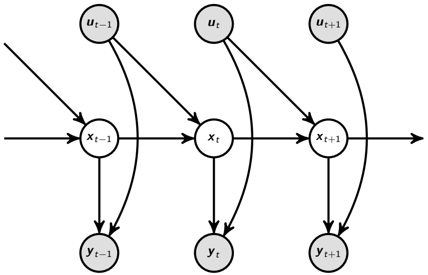
T. J. Rogers

February, 2023

NONLINEAR SSMS

$$m\ddot{x} + c\dot{x} + kx + k_3 x^3 = u$$

$$x_t \sim f_\theta\left(x_t \,\middle|\, x_{t-1}, u_{t-1}\right)$$
$$y_t \sim g_\theta\left(y_t \,\middle|\, x_t, u_t\right)$$



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NONLINEAR SSMS

$$m\ddot{x} + c\dot{x} + kx + k_3 x^3 = u \qquad u \sim \mathcal{GP}\left(0, k(t, t')\right)$$

$$x_t \sim f_\theta\left(x_t \mid x_{t-1}, u_{t-1}\right)$$
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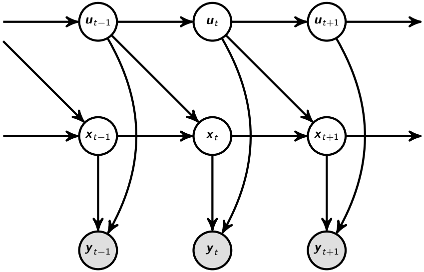

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PARTICLE SMOOTHING

A slight complication. . . particle filters can give us the filtering distribution but really want the smoothing distribution $p\left(x_{1:T} \mid y_{1:T}\right)$.

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PARTICLE SMOOTHING

A slight complication... particle filters can give us the filtering distribution but really want the smoothing distribution $p\left(x_{1:T} \mid y_{1:T}\right)$.

We will use an MCMC scheme to sample from this using the particle filter.

In particular we use a Particle Gibbs with Ancestor Sampling approach (Lindsten 2014) to sample from $p\left(x_{1:T} \mid y_{1:T}\right)$.

Details:

Rogers, Timothy J., Worden, Keith and Cross Elizabeth J.. "Bayesian Joint Input-State Estimation for Nonlinear Systems." *Vibration 3.3 (2020): 281-303.*

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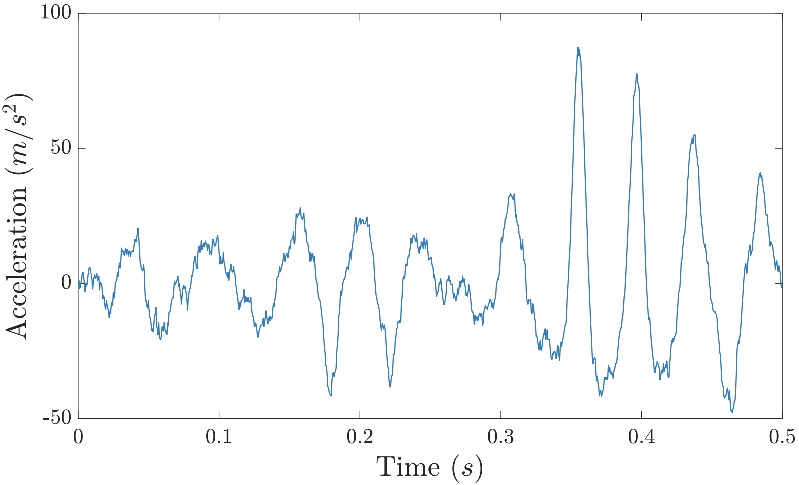
A DUFFING EXAMPLE



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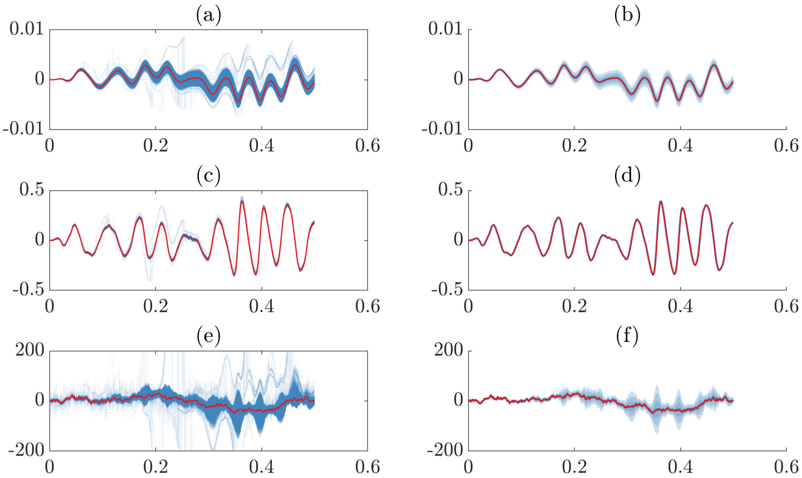

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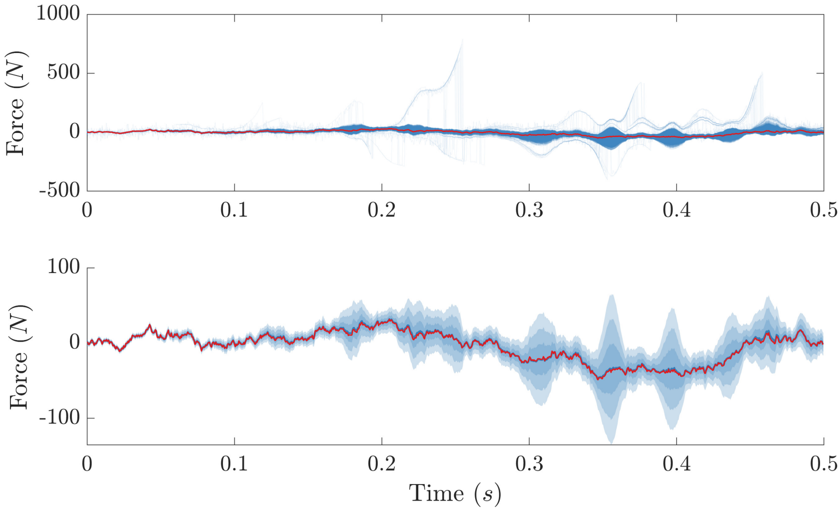

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