AERO 628: Project 1

Prolate Rigid Body Rotation

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Angular Velocity Solution

The body has the following inertia matrix:

$$\bar{I} = \begin{pmatrix} \text{It} & 0 & 0 \\ 0 & \text{It} & 0 \\ 0 & 0 & \text{Ia} \end{pmatrix}$$

And has a torque applied u:

$$\frac{u}{\text{It}} = 0.2$$

Using Euler's Eqs. of Motion: Since it is an axissymmetric-prolate body, the angular acceleration in the body-3 direction is zero.

$$l_3 = \text{Ia}\,\dot{\omega}_3 = 0$$

So we can call ω_3=n which is a constant. Now simplifying the other expressions of Euler Eqs. of Motion:

$$l_2 = n\,\omega_1\,(\text{It} - \text{Ia}) + \text{It}\,\dot{\omega}_2 = 0$$

$$\dot{\omega}_2 = \frac{(\text{Ita} - \text{It})}{\text{It}}\,n\,\omega_1$$

$$u = n\,\omega_2\,(\text{Ia} - \text{It}) + \text{It}\,\dot{\omega}_1$$

$$\dot{\omega}_1 = \frac{u}{\text{It}} + \frac{(\text{It} - \text{Ia})}{\text{It}}\,n\,\omega_2$$

Now, we define the following expressions:

$$\frac{u}{\text{It}} \equiv l$$

$$\lambda \equiv \frac{\text{It} - \text{Ia}}{\text{It}}\,n$$

$$\dot{\omega}_1 = \lambda\,\omega_2 + l$$

$$\dot{\omega}_2 = -\lambda\,\omega_1$$

Differentiating the above and substituting:

$$\ddot{\omega}_2 = -\lambda\,\dot{\omega}_1$$

$$\ddot{\omega}_2 + \lambda^2\,\omega_2 = \lambda l$$

The differential equation can be solved by merging the homogeneous solution:

$$w_{2\,H} = a\sin(\lambda\,t) + b\cos(\lambda\,t)$$

With the particular solution:

$\omega_{2\,P} = c,\ \therefore$

$c\,\lambda^2 = \lambda\, l$

$c = -\dfrac{l}{\lambda}$

So the general solution is:

$$\omega_2 = a \sin(\lambda\, t) + b \cos(\lambda\, t) - \frac{l}{\lambda}$$

Now we can differentiate when *t=0* and plug into the $\dot{\omega}_2$ equation:

$$\dot{\omega}_2 = a \cos(\lambda\, t(0)) - b \sin(\lambda\, t(0)) = -\lambda\, \omega_1(0)$$

$$a = -\lambda\, \omega_1(0)$$

$$\omega_2(0) = b \cos(\lambda\, t(0)) - \frac{l}{\lambda} + (-\lambda)\, \omega_1(0) \sin(\lambda\, t(0))$$

$$b = \frac{l}{\lambda} + \omega_2(0)$$

Orientation Angles Solution

Using a 1-2-3 (α , β, γ) rotation sequence, we get the following attitude kinematics matrix:

$$\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} = \frac{1}{\text{Cos}[\beta]} \begin{pmatrix} \text{Cos}[\gamma] & -\text{Sin}[\gamma] & 0 \\ \text{Cos}[\beta]\,\text{Sin}[\gamma] & \text{Cos}[\gamma]\,\text{Cos}[\beta] & 0 \\ -\text{Sin}[\beta]\ \text{Cos}[\gamma] & \text{Sin}[\gamma]\,\text{Sin}[\beta] & \text{Cos}[\beta] \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

$$\text{R1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{pmatrix};$$

$$\text{R2} = \begin{pmatrix} \cos(\beta) & 0 & -\sin(\beta) \\ 0 & 1 & 0 \\ \sin(\beta) & 0 & \cos(\beta) \end{pmatrix};$$

$$\text{R3} = \begin{pmatrix} \cos(\gamma) & \sin(\gamma) & 0 \\ -\sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

$\omega = \{\omega_1,\ \omega_2,\ \omega_3\};$

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A = {0, 0, ω3} + R3.{0, ω2, 0} + R3.R2.{ω1, 0, 0} // MatrixForm
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$$\begin{pmatrix} \text{Cos}[\beta]\ \text{Cos}[\gamma]\ \omega_1 + \text{Sin}[\gamma]\ \omega_2 \\ -\text{Cos}[\beta]\ \text{Sin}[\gamma]\ \omega_1 + \text{Cos}[\gamma]\ \omega_2 \\ \text{Sin}[\beta]\ \omega_1 + \omega_3 \end{pmatrix}$$

Notice that $\dot{\gamma}$ can be simplified in the following way:

$\dot{\gamma} = -\text{Tan}[\beta]\ (\text{Cos}[\gamma]\ \omega_1 - \text{Sin}[\gamma]\ \omega_2)\ + \text{n}$

Substituting for $\dot{\alpha}$ in the equation above:

$\dot{\gamma} =$ -Sin[β] $\dot{\alpha}$ + n

Since we have a prolate body, we can approximate the orientation angles solution with small angle approximations, where Cos[β] ≈ 1 and Sin[β] ≈ β:

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$$\dot{\alpha} = \omega_1 \operatorname{Cos}[\gamma] - \omega_2 \operatorname{Sin}[\gamma]$$
$$\dot{\beta} = \omega_1 \operatorname{Sin}[\gamma] + \omega_2 \operatorname{Cos}[\gamma]$$