Project 1: Prolate Spinning Body

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Angular Velocity Solution

The body has the following inertia matrix:

$$I = \begin{pmatrix} \mathrm{It} & 0 & 0 \\ 0 & \mathrm{It} & 0 \\ 0 & 0 & \mathrm{Ia} \end{pmatrix} \tag{1}$$

And has a torque applied u, where:

$$\frac{u}{\mathrm{It}} = 0.2 \tag{2}$$

Using Euler's Eqs. of Motion: Since it is an axissymmetric-prolate body, the angular acceleration in the body-3 direction is zero.

$$l_3 = \mathrm{Ia}\, \dot{\omega}_3 = 0 \tag{3}$$

So we can call ω_3=n which is a constant. Now simplifying the other expressions of Euler Eqs. of Motion:

$$l_2 = n\, \omega_1\, (\mathrm{It} - \mathrm{Ia}) + \mathrm{It}\, \dot{\omega}_2 = 0 \tag{4}$$

$$\dot{\omega}_2 = \frac{(\mathrm{Ia} - \mathrm{It})}{\mathrm{It}}\, n\, \omega_1 \tag{5}$$

$$u = n\, \omega_2\, (\mathrm{Ia} - \mathrm{It}) + \mathrm{It}\, \dot{\omega}_1 \tag{6}$$

$$\dot{\omega}_1 = \frac{u}{\mathrm{It}} + \frac{(\mathrm{It} - \mathrm{Ia})}{\mathrm{It}}\, n\, \omega_2 \tag{7}$$

Now, we define the following expressions:

$$\frac{u}{\mathrm{It}} \equiv l \tag{8}$$

$$\lambda \equiv \frac{\mathrm{It} - \mathrm{Ia}}{\mathrm{It}}\, n \tag{9}$$

Which will simplify the angular velocity equations:

$$\dot{\omega}_1 = \lambda\, \omega_2 + l \tag{10}$$

$$\dot{\omega}_2 = -\lambda\, \omega_1 \tag{11}$$

Using Euler's Eqs. of Motion: Since it is an axissymmetric-prolate body, the angular acceleration in the body-3 direction is zero.

$$l_3 = \mathrm{Ia}\, \dot{\omega}_3 = 0 \tag{12}$$

Differentiating the above and substituting:

$$\ddot{\omega}_2 = -\lambda\, \dot{\omega}_1 \tag{13}$$

$$\ddot{\omega}_2 + \lambda^2\, \omega_2 = \lambda \mathrm{l} \tag{14}$$

The differential equation can be solved by merging the homogeneous solution:

$$w_{2\,H} = a \sin(\lambda\, t) + b \cos(\lambda\, t) \tag{15}$$

With the particular solution $\omega_{2\,P} = c$:

$$c = -\frac{l}{\lambda} \tag{16}$$

So the general solution is:

$$\omega_2 = a \sin(\lambda\, t) + b \cos(\lambda\, t) - \frac{l}{\lambda} \tag{17}$$

Now we can differentiate when *t=0* and plug into the $\dot{\omega}_2$ equation:

$$\dot{\omega}_2 = a \cos(\lambda\, t(0)) - b \sin(\lambda\, t(0)) = -\lambda\, \omega_1(0) \tag{18}$$

$$a = -\lambda\, \omega_1(0) \tag{19}$$

$$\omega_2(0) = b \cos(\lambda\, t(0)) - \frac{l}{\lambda} + (-\lambda)\, \omega_1(0) \sin(\lambda\, t(0)) \tag{20}$$

$$b = \frac{l}{\lambda} + \omega_2(0) \tag{21}$$

Finally the complete solutions are:

$$\omega_1 = \omega_1(0) \cos(\lambda\, t) + \left(\frac{l}{\lambda} + \omega_2(0)\right) \sin(\lambda\, t) \tag{22}$$

$$\omega_2 = \omega_1(0) \sin(\lambda\, t) + \left(\frac{l}{\lambda} + \omega_2(0)\right) \cos(\lambda\, t) - \frac{l}{\lambda} \tag{23}$$

Orientation Angles Solution

Using a 1-2-3 (α , β, γ) rotation sequence, we get the following attitude kinematics matrix:

$$\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} = \frac{1}{\mathrm{Cos}[\beta]} \begin{pmatrix} \mathrm{Cos}[\gamma] & -\mathrm{Sin}[\gamma] & 0 \\ \mathrm{Cos}[\beta]\,\mathrm{Sin}[\gamma] & \mathrm{Cos}[\gamma]\,\mathrm{Cos}[\beta] & 0 \\ -\mathrm{Sin}[\beta]\,\mathrm{Cos}[\gamma] & \mathrm{Sin}[\gamma]\,\mathrm{Sin}[\beta] & \mathrm{Cos}[\beta] \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \tag{24}$$

Notice that $\dot{\gamma}$ can be simplified in the following way:

$$\dot{\gamma} = n - \tan(\beta)\,(\omega_1 \cos(\gamma) - \omega_2 \sin(\gamma)) \tag{25}$$

Substituting for $\dot{\alpha}$ in the equation above:.

$$\dot{\gamma} = n - \dot{\alpha} \sin(\beta) \tag{26}$$

Since we have a prolate body, we can approximate the orientation angles solution with small angle approximations, where $\mathrm{Cos}[\beta] \approx 1$ and $\mathrm{Sin}[\beta] \approx \beta$:

$$\dot{\alpha} = \omega_1 \cos(\gamma) - \omega_2 \sin(\gamma) \tag{27}$$

$$\dot{\beta} = \omega_1 \sin(\gamma) + \omega_2 \cos(\gamma) \tag{28}$$

Now we integrate with respect to time to find closed form solutions of the orientation angles:

$$\int \dot{\gamma}\, dt = \gamma = n\, t \tag{29}$$

$$\int [\omega_1 \cos(\gamma) - \omega_2 \sin(\gamma)]\, dt = \alpha = \frac{\mathrm{It}\, l \left(\mathrm{Ia}\,\mathrm{Cos}[n\, t] - \mathrm{It}\,\mathrm{Cos}\left[\frac{\mathrm{Ia}\, n\, t}{\mathrm{It}}\right]\right)}{\mathrm{Ia}\,(\mathrm{Ia} - \mathrm{It})\, n^2} \tag{30}$$

$$\int [\omega_1 \sin(\gamma) + \omega_2 \cos(\gamma)]\, dt = \beta = \frac{\mathrm{It}\, l \left(\mathrm{Ia}\,\mathrm{Sin}[n\, t] - \mathrm{It}\,\mathrm{Sin}\left[\frac{\mathrm{Ia}\, n\, t}{\mathrm{It}}\right]\right)}{\mathrm{Ia}\,(\mathrm{Ia} - \mathrm{It})\, n^2} \tag{31}$$

Numerical Integration

The full nonlinear system is numerically integrated with Matlab ODE45() to provide a high accuracy solution of the prolate body’s rotation.

Results

The α and β angles as a function or time are shown in Figure 1. The parameters used are: γ=5 rad/s, It=1, Ia=0.05, initial conditions: all zero except γ.

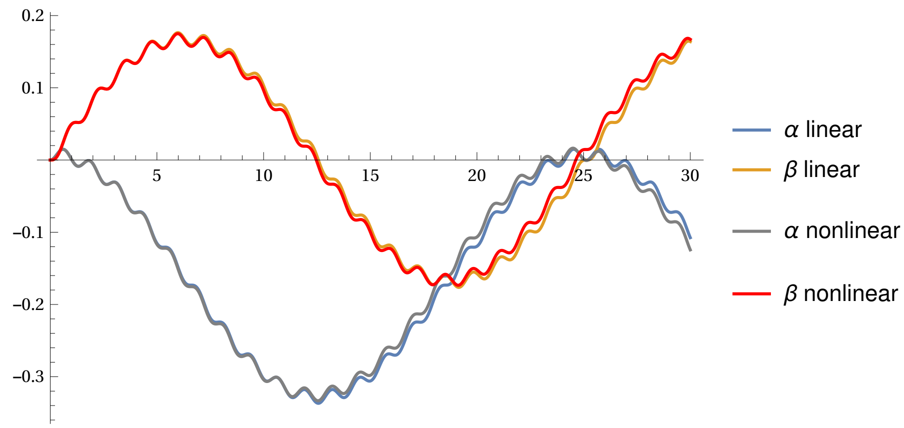

Figure 1. Orientation angles over frequency

As it can be seen. The linear solution tracks the nonlinear solution very well. `

Fourier Analysis

The linear nutational frequency (F_n) equals $\omega_3(0)$, F_n=0.796 Hz.

The linear precessional frequency (F_p) = $\frac{\text{Ia}}{\text{It}} F_n$ =0.0398 Hz

The nonlinear solution can be transformed with a Fast Fourier Transform and then the frequency spectrum plotted. Below, the frequency spectrum

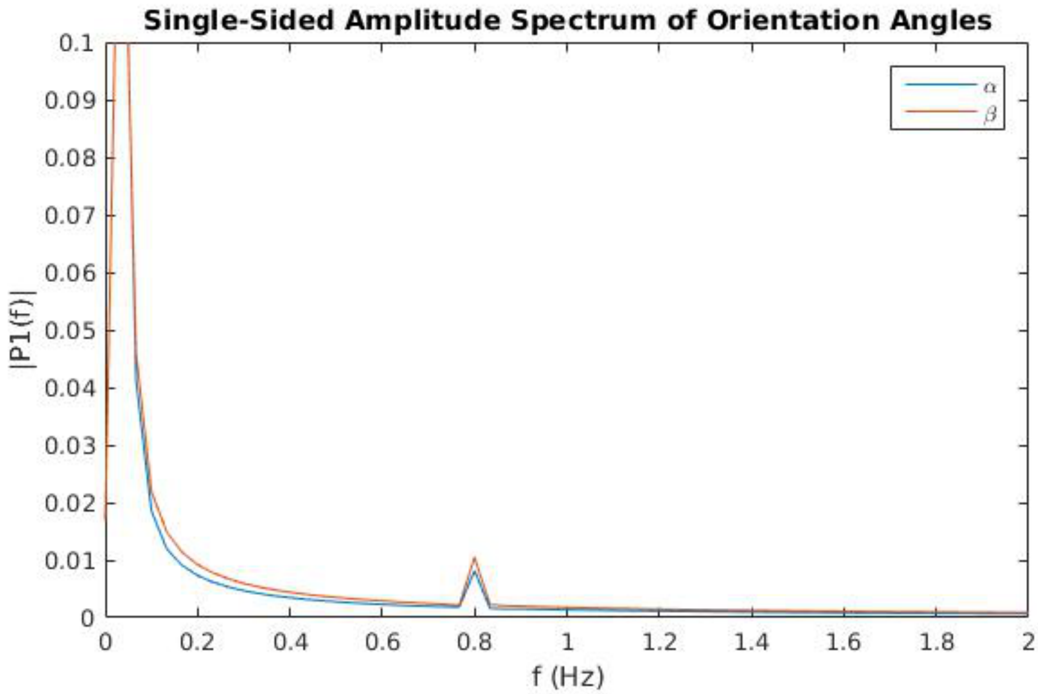

Figure 2. Orientation angles over frequency

As it can be seen. The spikes in the frequency domain graph align neatly with the calculated frequencies in the closed form solutions. This can mean that for the $\omega_3(0) = 5$, the linearized analysis correlates with high accuracy to the nonlinear solution.

Varying the Spin Rate

We can investigate the linear solution's fidelity by varying the initial ω_3. The slider below can be used to see the change in the time histories of the orientation angles as ω_3 changes.

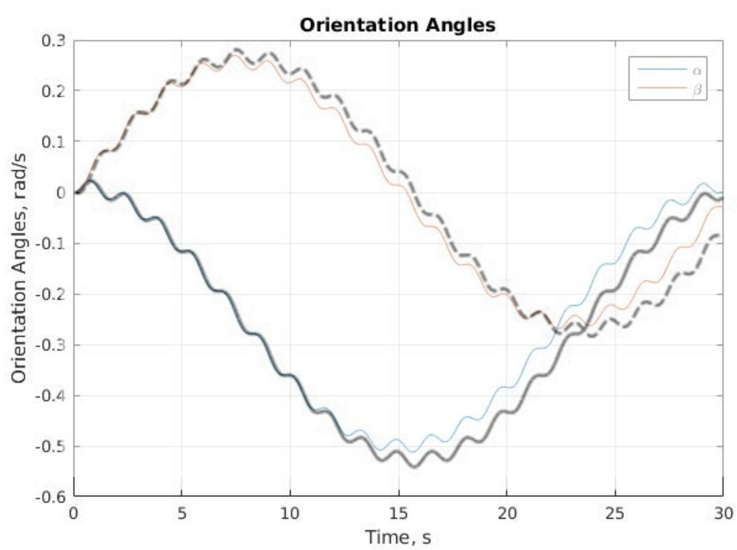

Figure 3. Overlay of both nonlinear and linear solutions. $\omega_3 = 4$

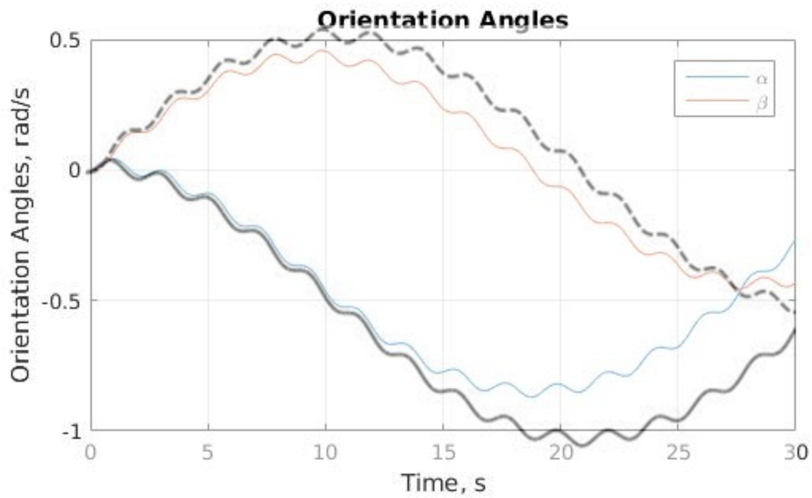

Figure 4. Overlay of both nonlinear and linear solutions. ω_3 = 3

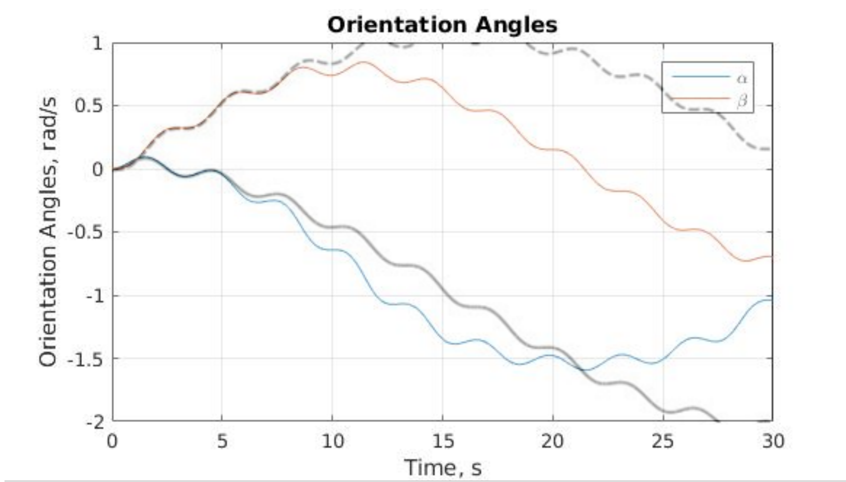

Figure 5. Overlay of both nonlinear and linear solutions. ω_3 = 2

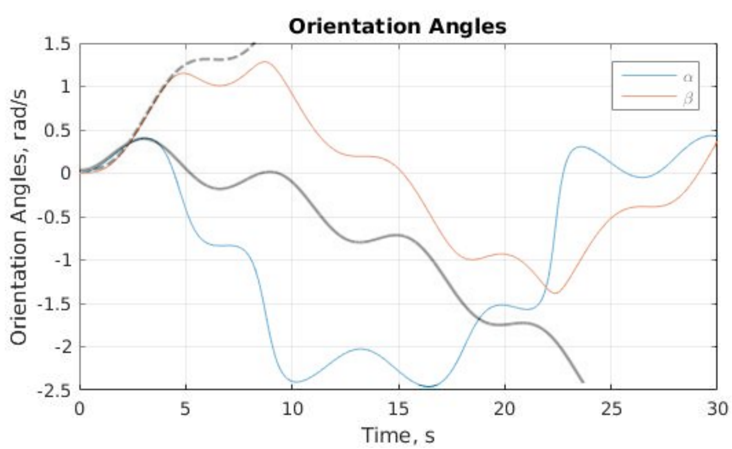

Figure 6. Overlay of both nonlinear and linear solutions. ω_3 = 1

As it can be seen, the linear solution starts losing the periodic oscillation at ω_3 = 3, when it reaches 2 rad/s. the linear solution's slow period is twice that of the nonlinear solution. When it reaches 1 rad/s, the linear solution is completely erroneous and loses track of all information within 5 seconds.