Multivariable Calculus

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Introduction

Multivariable Calculus approaches the subject from a mathematical, but not overly technical, perspective. The key idea of calculus—chop things into little pieces and put them together again—is emphasized throughout.

Licensing

Chapter 1

Preliminaries

1.1 Mathematical Notation

Mathematics is a sophisticated and precise language, and we best not adventure into calculus without learning some basic words.

The most basic mathematical word is that of a *set*. A set is an unordered collection of objects. We won't try and pin it down more exactly than this—our intuition about collections of objects will suffice[1]. We write a set with curly-braces { and } and list the objects inside. For instance

$$\{1,2,3\}.$$

This would be read aloud as "the set containing the elements 1, 2, and 3." The symbol $\in$ is used to specify that some object is an element of a set, and $\notin$ is used to specify it is not. For example,

$$3 \in \{1,2,3\} \qquad 4 \notin \{1,2,3\}.$$

Sets can contain mixtures of objects, including other sets. For example,

$$\{1,2,a,\{-70,\infty\},x\}$$

is a perfectly valid set.

It is tradition to use capital letters to name sets. So we might say $A = \{6,7,12\}$ or $X = \{7\}$. There is, however, a special set which already has its own name—the empty set. The *empty set* is the set containing no elements and is written $\emptyset$ or $\{\}$. Note that $\{\emptyset\}$ is *not* the empty set. It is the set containing the empty set! It is also traditional to call elements of a set *points* regardless of whether you consider them "point-like" objects.

Operations on Sets

If the set A contains all the elements that the set B does, we call A a *superset* and B a *subset*. We'll give this a formal definition.

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[1] When you pursue more rigorous math, you rely on definitions to get yourself out of philosophical jams. For instance, with our definition of set, consider "the set of all sets that don't contain themselves." Such a set cannot exist! This is called *Russel's Paradox*, and shows that if we start talking about sets of sets, we may need more than intuition.

Definition 1.1.1 — Subset & Superset. The set B is a *subset* of the set A, written $B \subseteq A$ if for all $b \in B$ we also have $b \in A$. In this case, A is called a *superset* of B.[a]

[a] Some mathematicians use the symbol $\subset$ instead of $\subseteq$.

Some simple examples are $\{1,2,3\} \subseteq \{1,2,3,4\}$ and $\{1,2,3\} \subseteq \{1,2,3\}$. There's something funny about that last example though. Those two sets are not only subsets/supersets of each other, they're *equal*. As surprising as it seems, we actually need to define what it means for two sets to be equal.

Definition 1.1.2 — Set Equality. The sets A and B are *equal*, written $A = B$ if $A \subseteq B$ and $B \subseteq A$.

Having a definition of equality to lean on will help us when we need to prove things about sets.

■ **Example 1.1** Let A be the set of numbers that can be expressed as $2n$ for some whole number n and let B be the set of numbers that can be expressed as $m+1$ where m is an odd whole number. We will show $A = B$.

First, let us show $A \subseteq B$. If $x \in A$ then $x = 2n$ for some whole number n. Therefore $x = 2n = 2(n-1)+1+1 = m+1$ where $m = 2(n+1)+1$ is, by definition, an odd number. Therefore $x \in B$.

Now we will show $B \subseteq A$. Let $x \in B$. By definition, $x = m+1$ for some odd m and so by the definition of oddness, $m = 2k+1$ for some whole number k. Thus

$$\begin{aligned} x = m+1 &= (2k+1)+1 = 2k+2 \\ &= 2(k+1) = 2n, \end{aligned}$$

where $n = k+1$. Thus, $x \in A$. Since $A \subseteq B$ and $B \subseteq A$, by definition $A = B$. ■

Set-builder Notation

Specifying sets by listing all their elements can be a hassle, and if there are an infinite number of elements, it's impossible! Fortunately, *set-builder notation* solves these problems. If X is a set, we can define a subset

$$Y = \{a \in X : \text{some rule involving } a\},$$

which is read "Y is the set of a in X *such that* some rule involving a is true." If X is intuitive, we may omit it and simply write $Y = \{a : \text{some rule involving } a\}$[2]. You may equivalently use "|" instead of ":", writing $Y = \{a \,|\, \text{some rule involving } a\}$.

■ **Example 1.2** The set $\mathbb{Z}$ is the set of integers (positive, negative, and zero whole numbers). To define E as the even integers, we could write

$$E = \{n \in \mathbb{Z} : n = 2k \text{ for some } k \in \mathbb{Z}\}.$$

[2] If you want to get technical, to make this notation unambiguous, you define a *universe of discourse*. That is, a set $\mathcal{U}$ containing every object you might want to talk about. Then $\{a : \text{some rule involving } a\}$ is short for $\{a \in \mathcal{U} : \text{some rule involving } a\}$

To define P as the set of positive integers, we could write

$$P = \{n \in \mathbb{Z} \,:\, n > 0\}.$$

■

There are also some common operations we can do with two sets.

Definition 1.1.3 — Intersections & Unions. Let A and B be sets. Then the *intersection* of A and B, written $A \cap B$, is defined by

$$A \cap B = \{x \,:\, x \in A \text{ and } x \in B\}.$$

The *union* of A and B, written $A \cup B$, is defined by

$$A \cup B = \{x \,:\, x \in A \text{ or } x \in B\}.$$

For example, if $A = \{1,2,3\}$ and $B = \{-1,0,1,2\}$, then $A \cap B = \{1,2\}$ and $A \cup B = \{-1,0,1,2,3\}$. Set unions and intersections are *associative*, which means it doesn't matter how you apply parenthesis to an expression involving just unions or just intersections. For example $(A \cup B) \cup C = A \cup (B \cup C)$, which means we can give an unambiguous meaning to an expression like $A \cup B \cup C$ (just put the parenthesis wherever you like). But watch out, $(A \cup B) \cap C$ means something different than $A \cup (B \cap C)$!

Definition 1.1.4 — Set Subtraction. For sets A and B, the *set-wise difference* between A and B, written $A \backslash B$, is the set

$$A \backslash B = \{x \,:\, x \in A \text{ and } x \notin B\}.$$

Definition 1.1.5 — Cardinality. For a set A, the *cardinality* of A, written $|A|$ is the number of elements in A. If A contains infinitely many elements, we write $|A| = \infty$.

Let's define some notation for common sets.

$$\begin{aligned}
\emptyset &= \{\}, \text{ the empty set} \\
\mathbb{N} &= \{0,1,2,3,\ldots\} = \{\text{natural numbers}\} \\
\mathbb{Z} &= \{\ldots,-3,-2,-1,0,1,2,3,\ldots\} = \{\text{integers}\} \\
\mathbb{Q} &= \{\text{rational numbers}\} \\
\mathbb{R} &= \{\text{real numbers}\} \\
\mathbb{R}^n &= \{\text{vectors in } n\text{-dimensional Euclidean space}\}
\end{aligned}$$

Besides unions, there's another way to join sets together: *products*.

Definition 1.1.6 — Cartesian Product. Given two sets A and B, the *Cartesian product* (sometimes shortened to *product*) of the sets A and B is written $A \times B$ and defined to be

$$A \times B = \{(a,b) \,:\, a \in A \text{ and } b \in B\}.$$

The Cartesian product of two sets is the set of all ordered pairs of elements from those sets. For example,

$$\{1,2\} \times \{1,2,3\} = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\}.$$

You can repeat this operation more than once. $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ is the set of all triples of real numbers. Borrowing notation from numbers, if you take the Cartesian product of a set with itself some number of times, you can represent it with an exponent. Thus, $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ can be written as $\mathbb{R}^3$, which is a set we've seen before[3].

Functions

You're probably used to seeing functions like $f(x)=x^2$, but it's worth reviewing some of the concepts and terminology associated with functions.

Definition 1.1.7 — Function. A *function* with *domain* the set A and *co-domain* the set B is an object that associates every point in the set A with *exactly one* point in the set B.

If a function f has domain A and co-domain[4] B, we notate this by writing $f : A \to B$. If we want to further specify what the function f actually is, we need to express how f associates each point in A to a point in B. This can be done with an equation. For example, we could define the function $f : \mathbb{R}\to\mathbb{R}$ by

$$f(x)=2x,$$

which says that each real number gets associated to its double. We can notate the same thing using a special type of arrow: "$\mapsto$". Now we might write

$$f : \mathbb{R}\to\mathbb{R} \text{ where } x\mapsto 2x,$$

which is read "f is a function from $\mathbb{R}$ to $\mathbb{R}$ where $x\in\mathbb{R}$ gets mapped to $2x$."

Note that every point in the co-domain of a function doesn't need to get mapped to. For example $g:\mathbb{R}\to\mathbb{R}$ given by $g(x)=x^2$ outputs only non-negative numbers, but it is still valid to specify $\mathbb{R}$ as the co-domain. However, if we wanted to make a point of it, we are perfectly justified in writing $g:\mathbb{R}\to[0,\infty)$ when defining g.

Many common math operations give rise to functions. For example, $f(x)=\sqrt{x}$ is the familiar square root function. Sometimes, when we wish to talk about a function for which notation already exists, we will put a " $\cdot$ " where we would normally put a variable. Thus, we might say, "$\sqrt{\cdot}$ is the square root function."[5]

Definition 1.1.8 — Range. The *range* of a function $f : A\to B$ is the set of all outputs of f. That is

$$\operatorname{range} f = \{y\in B \,:\, y=f(x) \text{ for some } x\in A\}.$$

Definition 1.1.9 — Image. Let $f : A\to B$ be a function. The *image* of a set $X\subseteq A$, written $f(X)$ is defined by

$$f(X)=\{y\in B \,:\, y=f(x) \text{ for some } x\in X\}.$$

We see that if $f : A\to B$, $\operatorname{range} f = f(A)$. In words, the range of f is the image of its domain. This language will become useful when we think of functions as transformations that

[3] If you're scratching your head saying, "I thought $\mathbb{R}^3$ was vectors in 3-dimensional space. How do we know that's the same thing as triples of real numbers?" your mind is keen. This is a theorem of linear algebra.

[4] Some people use the word *range* interchangeably with co-domain.

[5] Since $\sqrt{x}$ is "the square root of the quantity x," it is technically a quantity and not a function. This is why we write $\sqrt{\cdot}$ instead of $\sqrt{x}$ when we want to refer to the square root *function*.

move or bend space. If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a function that warps the Cartesian plane, then the image of X under f could be visualized by painting X on the Cartesian plane, warping the whole plane, and then looking at the resulting, painted shape.

Closely related to images, we have the idea of *restriction*. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x, y) = xy$, but we were only really interested in f on the unit circle, $\mathcal{C}$. In this case, we might say f attains a maximum on $\mathcal{C}$, or f *restricted to* $\mathcal{C}$ attains a maximum, even though f itself is unbounded. This idea comes up often enough to deserve its own notation.

Definition 1.1.10 — Restriction. If $f : A \to B$ and $X \subseteq A$, the *restriction* of f to X is written $f\big|_X$ and is defined to be the function $g : X \to A$ where $x \mapsto f(x)$.

The last important function-related ideas for us are function composition and inverses. Given two function $f : A \to B$ and $g : B \to C$, we can *compose* g and f to get a new function.

Definition 1.1.11 — Composition. Given two functions $f : A \to B$ and $g : B \to C$, the *composition* of g and f, written $g \circ f$, is the function $h : A \to C$ where $x \mapsto g(f(x))$.

Note that the composition $g \circ f$ has the domain of f and the co-domain of g. When a point is fed into $g \circ f$, it moves from $A \to B \to C$. The composition $g \circ f$ only makes sense because the outputs of f are allowed as inputs to g. If we wrote $f \circ g$, it wouldn't mean much, because g outputs points in C and f has no idea what to do with points in C.[6]

Inverses relate to composition and the *identity function*, the function that does nothing to its inputs.

Definition 1.1.12 — Identity Function. The *identity function* $\mathrm{id} : A \to A$ is defined by the relation

$$\mathrm{id}(x) = x$$

for all $x \in A$.

Notice that for every set, that set is the domain of an identity function. Since the domain and co-domain of a function are part of its definition, we don't want to confuse them. After all, $f : \{0, 1\} \to \{0, 1\}$ given by $f(x) = x^2$ is a different function from $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. For the special case of the identity function, we sometimes write the domain of the function as a subscript. That is, for $\mathrm{id} : A \to A$ we'd write id_A so it doesn't get confused with $\mathrm{id} : B \to B$, which we'd write id_B.

Definition 1.1.13 — Inverse Function. Let $f : A \to B$ be a function. If there exists a function $g : B \to A$ such that

$$f \circ g = \mathrm{id}_B \qquad \text{and} \qquad g \circ f = \mathrm{id}_A,$$

we say f is *invertible* and we call g the *inverse* of f. If f is invertible, we notate its inverse by f^{-1}.

Inverses can be tricky some times. For example, consider $f(x) = x^2$ and $g(x) = \sqrt{x}$. Here $g \circ f(x) = \sqrt{x^2} = |x|$ and $f \circ g(x) = \sqrt{x}^2 = x$. What's the deal? Well, it's all about domains.

[6] It seems a little backwards to write $f : A \to B$, $g : B \to C$ and then write $g \circ f$ instead of $f \circ g$. You can thank Euler for that. He decided to write functions with their input on the right instead of the left. If we wrote functions backwards, like $((x)f)g$ for "g of f of x," they we could just *follow the arrows* and life would be simpler.

$f : \mathbb{R} \to [0,\infty)$ and $g : [0,\infty) \to [0,\infty)$. So, the domain of $g \circ f$ is $\mathbb{R}$ and the domain of $f \circ g$ is $[0,\infty)$. The domains are different, and indeed f is not invertible. However, g *is* invertible, and $g^{-1} = f\big|_{[0,\infty)}$. If we only input non-negative numbers into f, then f exactly undoes what g did. This subtle domain trickery can cause us a lot of headaches if we're not used to thinking carefully, and many of our favorite functions that we're used to calling "inverse functions" are actually only inverses when paired with specific domains.

1.2 Proof

Mathematics has the highest standard of proof of any field. In the Platonic ideal of mathematics, we start from some basic assumptions, called *axioms*, that we have all agreed upon. Then from those axioms, using the rules of logic, we deduce *theorems*. Every single mathematical statement we make can be traced back from theorem to theorem and eventually to our initial axioms.

This is contrary to other disciplines, like physics. In physics, based on observation, we construct *laws*. Laws in physics are like axioms in mathematics, but they have an important difference—they can be disproven by observation. A mathematical axiom can never be disproven. One can certainly argue that an axiom is not *useful* or not *interesting*, but you cannot say its *wrong*[7]. Of course, as human practitioners, we may misuse logic and be wrong ourselves, but that is no fault of the axioms.

But now, let's deviate from philosophical perfection and visit reality. In reality, *mathematics is a human pursuit to understand relationships between ideas and their consequences*. The key there is that *humans* do mathematics to *understand* relationships. If a theorem in math can ultimately be reduced to logical statements about axioms, but the argument is 100000 steps long, it doesn't help a human understand why something is true. Instead, a shorter argument that skips over some steps is more useful to us. And, indeed, most of our mathematics to date skips over some steps[8].

We call a correct mathematical argument a *proof*. A proof starts from a set of assumptions, and following the rules of logic, arrives at a conclusion. Strictly speaking, a proof doesn't need to make sense or show motivation, applications, or examples. It just has to be a sequence of correct logical steps. However, for us, as humans studying mathematics, we prove things for two reasons: to understand why things are true and to avoid making mistakes.

Reconciling these two goals can be very hard for a novice mathematician. If you include *all* the steps, it won't help with understanding, but if you don't include enough steps, the argument may not be convincing and might contain mistakes. Even professionals struggle to balance these competing goals, and how you balance those goals depends on your audience—if

[7] There are multiple ways to axiomatize geometry. In Euclidean geometry every pair of lines either coincides, intersects in exactly one place, or does not intersect. In spherical geometry, every pair of lines either coincides or intersects in exactly two places. Euclidean geometry is useful when your space looks flat. Spherical geometry is useful when your space is the surface of a sphere (like the Earth). Is one of these more *right* than the other? They're certainly contradictory.

[8] There are some projects to prove all of mathematics directly from the axioms using computer assistance. They've made progress, but there are still theorems in calculus that have not been reduced to the axioms. We believe that they *could be* reduced to the axioms, but no one has taken the time to do so.

you're trying to convince your math professor of something your proof will need to have more detail than if you were trying to convince your friend (mathematicians are very skeptical!).

Enough talk, let's go through a 2000 year old example of a proof.

Theorem 1.2.1 There is no rational number p/q such that $(p/q)^2 = 2$.

Proof. If p/q is a rational number, it can be expressed in lowest terms. Suppose p/q is in lowest terms and $(p/q)^2 = 2$. Then $p^2 = 2q^2$ and so p^2 is even. Since p^2 is even, it must be that p is even, and so by definition, $p = 2m$ for some integer m. Now,

$$\frac{p^2}{q^2} = \frac{(2m)^2}{q^2} = \frac{4m^2}{q^2} = 2,$$

with the last equality following by assumption. Multiplying both sides by q^2 and dividing by 2 we arrive at the equation

$$2m^2 = q^2,$$

and so q^2 is even which means q is even. By definition, this means $q = 2n$ for some integer n. But now,

$$\frac{p}{q} = \frac{2m}{2n}$$

is not in lowest terms! This is a contradiction and so it cannot be that $(p/q)^2 = 2$. ■

This is nearly identical to the argument the ancient Greeks gave. It's elegant, beautiful, and convincing. But, if we look closer, it does skip some steps. For example, it relies on the fact that there is such a thing as *lowest terms*. This is something that would need to be proven—a priori, the conclusion of the proof could be that the assumption that p/q could be in lowest terms is false.

You will not, over night, become a master at understanding what steps you can leave out and what steps you must show. However, with feedback, you'll get better. For a detailed guide about writing good proofs, please see Appendix A.

Chapter 2

Vectors

A *vector* is a quantity which is characterized by a *magnitude* and a *direction*. Many quantities are best described by vectors rather than numbers. For example, when driving a car, it may be sufficient to know your speed, which can be described by a single number, but the motion of an airplane must be described by a vector quantity—velocity—which takes into account its direction as well as its speed.

Ordinary numerical quantities are called *scalars* when we want to emphasize that they are not vectors.

Whereas numbers allow us to specify relationships between single quantities (put in twice as much flour as sugar), vectors will allow us to specify relationships between goemetric objects in space[1]. If we have two points, $P = (1,1)$ and $Q = (3,2)$, we specify the *displacement* from P to Q as a vector.

XXX Figure

We notate the displacement vector form P to Q by $\overrightarrow{PQ}$. The magnitude of $\overrightarrow{PQ}$ is given by the Pythagorean theorem to be $\sqrt{5}$ and its direction is specified by the directed line segment from P to Q.

2.1 Vector Notation

There are many ways to represent vector quantities in writing. If we have two points, P and Q, $\overrightarrow{PQ}$ represents the vector from P to Q. Absent of points, bold-faced letters or a letter with an arrow over it are the most common typographical representations of vectors. For example, $\vec{a}$ or $\mathbf{a}$ may both be used to represent the vector quantity named "a." In this book we will use $\vec{a}$ to represent a vector. The notation $\|\vec{a}\|$ represents the magnitude of the vector $\vec{a}$, which is sometimes called the *norm* of $\vec{a}$.

Graphically, vectors are represented as directed line segments (a line segment with an arrow at one end). The endpoints of the segment are called the *initial point* (the base) and the *terminal point* (the tip) of the vector.

[1] Though in this book we will treat vectors as intertwined with Euclidean space, they are much more general. For instance, someone's internet browsing habits could be describe by a vector—the topics they find most interesting might be the "direction" and the amount of time they browse might be the "magnitude."

Let $A=(1,1)$, $B=(2,3)$, $X=(0,1)$, and $Y=(1,3)$ and consider the vectors $\vec{a}=\overrightarrow{AB}$ and $\vec{x}=\overrightarrow{XY}$. Are these the same or different vectors? If we drew them as directed line segments, the drawings would be distinct. However, both $\vec{a}$ and $\vec{x}$ have equivalent magnitudes and directions. Thus, $\vec{a}$ and $\vec{x}$ are *equivalent*, and we would be justified writing $\vec{a}=\vec{x}$.

Alternatively, we could consider the *rooted vector* $\vec{a}$ rooted at the point A. In this terminology, $\vec{a}$ rooted at A is *different* than $\vec{a}$ rooted at X. This idea of rooted vectors will occasionally be useful, but our primary study will be unrooted vectors.

Vectors and Points

The distinction between vectors and points is sometimes nebulous because they are so closely related to each other. A *point* in Euclidean space specifies an absolute position whereas a vector specifies a magnitude and direction. However, given a point P, one associates P with the vector $\vec{p}=\overrightarrow{OP}$, where O is the origin. Similarly, we associate the vector $\vec{v}$ with the point V so that $\overrightarrow{OV}=\vec{v}$. Thus, we have a way to unambiguously go back and forth between vectors and points[2]. As such, *we will treat vectors and points as interchangeable*.

2.2 Vector Arithmetic

Vectors provide a natural way to give directions. For example, suppose $\hat{\mathbf{x}}$ points one mile eastwards and $\hat{\mathbf{y}}$ points one mile northwards. Now, if you were standing at the origin and wanted to move to a location 3 miles east and 2 miles north, you might say: "Walk 3 times the length of $\hat{\mathbf{x}}$ in the $\hat{\mathbf{x}}$ direction and 2 times the length of $\hat{\mathbf{y}}$ in the $\hat{\mathbf{y}}$ direction." Mathematically, we express this as

$$3\hat{\mathbf{x}}+2\hat{\mathbf{y}}.$$

Of course, we've incidentally described a new vector. Namely, let P be the point at 3-east and 2-north. Then

$$\overrightarrow{OP}=3\hat{\mathbf{x}}+2\hat{\mathbf{y}}.$$

If the vector $\vec{r}$ points north but has a length of 10 miles, we have a similar formula:

$$\overrightarrow{OP}=3\hat{\mathbf{x}}+\tfrac{1}{5}\vec{r},$$

and we have the relationship $\vec{r}=10\hat{\mathbf{y}}$. Our notation here is very suggestive. Indeed, if we could make sense of what $\alpha\vec{v}$ is for any scalar α and vector $\vec{v}$, and we could make sense of what $\vec{v}+\vec{w}$ means for any vectors $\vec{v}$ and $\vec{w}$, we would be able to do algebra with vectors. We might even say we have *an algebra of vectors*.

Intuitively, for a vector $\vec{v}$ and a scalar $\alpha>0$, the vector $\vec{w}=\alpha\vec{v}$ should point in the same direction as $\vec{v}$ but have magnitude scaled up by α. That is, $\|\vec{w}\|=\alpha\|\vec{v}\|$. Similarly, $-\vec{v}$ should be the vector of the same length as $\vec{v}$ but pointing in the exact opposite direction.

For two vectors $\vec{u}$ and $\vec{v}$, the sum $\vec{w}=\vec{u}+\vec{v}$ should be the displacement vector created by first displacing along $\vec{u}$ and then displacing along $\vec{v}$.

XXX Figure

[2] Mathematically, we say there is an *isomorphism* between vectors and points.

Now, there is one snag. What should $\vec{v}+(-\vec{v})$ be? Well, first we displace along $\vec{v}$ and then we displace in the exact opposite direction by the same amount. So, we have gone nowhere. This corresponds to a displacement with zero magnitude. But, what direction did we displace? Here we make a philosophical stand.

Definition 2.2.1 — Zero Vector. The *zero vector*, notated as $\vec{0}$, is the vector with no magnitude.

We will be pragmatic about the direction of the zero vector and say, *the zero vector does not have a well-defined direction*[3]. That means sometimes we consider the zero vector to point in every direction and sometimes we consider it to point in no directions. It depends on our mood—but we must never talk about *the* direction of the zero vector, since it's not defined.

We need the zero vector if we are to make precise mathematical sense of vector arithmetic. Further along this line of thinking, we can define precisely how vector arithmetic should behave. Specifically, if $\vec{u}$, $\vec{v}$, $\vec{w}$ are vectors and α and β are scalars, the following conditions should be satisfied:

$$(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w}) \qquad \text{(Associativity)}$$
$$\vec{u}+\vec{v}=\vec{v}+\vec{u} \qquad \text{(Commutativity)}$$
$$\alpha(\vec{u}+\vec{v})=\alpha\vec{u}+\alpha\vec{v} \qquad \text{(Distributivity)}$$

and

$$(\alpha\beta)\vec{v}=\alpha(\beta\vec{v}) \qquad \text{(Associativity II)}$$
$$(\alpha+\beta)\vec{v}=\alpha\vec{v}+\beta\vec{v} \qquad \text{(Distributivity II)}$$

Indeed, if we intuitively think about vectors in flat (Euclidean) space, all of these properties are satisfied[4]. From now on, these properties of vector operations will be considered the *laws (or axioms) of vector arithmetic*.

We'll be talking about these vector operations (scalar multiplication and vector addition) a lot. So much so that the concept is worth naming.

Definition 2.2.2 — Linear Combination. A *linear combination* of the vectors $\vec{v}_1,\dots\vec{v}_n$ is any vector expressible as

$$\alpha_1\vec{v}_1+\cdots+\alpha_n\vec{v}_n,$$

where $\alpha_1,\alpha_2,\dots,\alpha_n$ are scalars.

We've given laws for linear combinations of vectors, but what about for magnitudes of vectors? We'd like the magnitude (or norm) of a vector to obey the following laws.

$$\|\vec{v}\|\geq 0 \qquad \text{(Non-negativity)}$$
$$\|\vec{v}\|=0 \text{ only when} \vec{v}=\vec{0} \qquad \text{(Definiteness)}$$
$$\|\alpha\vec{v}\|=|\alpha|\|\vec{v}\| \qquad \text{(Homogeneity)}$$
$$\|\vec{v}+\vec{w}\|\leq\|\vec{v}\|+\|\vec{w}\| \qquad \text{(Triangle Inequality)}$$

[3] In the mathematically precise definition of vector, the idea of "magnitude" and "direction" are dropped. Instead, a set of vectors is defined to be a set over which you can reasonably define addition and scalar multiplication.

[4] If we deviate from flat space, some of these rules are no longer respected. Consider moving 100 miles north then 100 miles east on a sphere. Is this the same as moving 100 miles east and then 100 miles north?

for all $\vec{v}$, $\vec{w}$, and scalars α. Any function on vectors satisfying those four properties is called a *norm*, and our usual notion of length in three-dimensional space indeed obeys those properties[5].

Homogeneity is a particularly special property of a norm. It allows us to easily create *unit vectors*.

Definition 2.2.3 — Unit Vector. A *unit vector* is a vector $\vec{u}$ satisfying $||\vec{u}|| = 1$.

Unit vectors are handy because if $\vec{u}$ is a unit vector, then $k\vec{u}$ has length $|k|$. Further, we can always turn a vector into a unit vector.

■ **Example 2.1** The vector $\vec{v}/||\vec{v}||$ is always a unit vector in the direction of $\vec{v}$. Computing,

$$\left|\left|\frac{\vec{v}}{||\vec{v}||}\right|\right| = \left|\frac{1}{||\vec{v}||}\right| ||\vec{v}|| = \frac{1}{||\vec{v}||}||\vec{v}|| = 1.$$

■

2.3 Coordinates

Recall that a coordinate system in the plane is specified by choosing an origin O and then choosing two perpendicular axes meeting at the origin. These axes are chosen in some order so that we know which axis (usually the x-axis) comes first and which (usually the y-axis) second. Note that there are many different coordinate systems which could be used although we often draw pictures as if there were only one.

In physics, one often has to think carefully about the coordinate system because choosing one appropriately may greatly simplify the analysis. Note that axes for coordinate systems are usually drawn with *right-hand orientation*, where the right angle from the positive x-axis to the positive y-axis is in the counter-clockwise direction. However, it would be equally valid to use the *left-hand orientation* in which that angle is in the clockwise direction. One can easily switch the orientation of a coordinate system by reversing one of the axes[6].

XXX Figure

For any coordinate system, there are special vectors associated with it. For the plane, the vector pointing one unit along the positive x-axis is called $\hat{\mathbf{x}}$ and the vector pointing one unit along the positive y-axis is called $\hat{\mathbf{y}}$. The vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are called the *standard basis* vectors for $\mathbb{R}^2$.

Notice that every point (or vector) in the plane can be represented as a linear combination of $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, and the vector $\alpha\hat{\mathbf{x}}+\beta\hat{\mathbf{y}}$ is the vector $\overrightarrow{OP}$ where $P = (\alpha, \beta)$. Now, to state an intuitive fact: if $\vec{w}$ is a vector in the plane, *there is only one way to write a vector as a linear combination of* $\hat{\mathbf{x}}$ *and* $\hat{\mathbf{y}}$. This means, if $\vec{w} = \alpha\hat{\mathbf{x}} + \beta\hat{\mathbf{y}}$, the pair (α, β) captures all information[7] about $\vec{w}$.

[5] The Euclidean norm comes from the Pythagorean theorem $a^2 + b^2 = c^2$. However, by changing the exponent, we have a whole family of norms coming from the equations $|a|^p + |b|^p = |c|^p$.

[6] The concept of orientation is quite fascinating and it arises in mathematics, physics, chemistry, and even biology in many interesting ways. Note that almost all of us base our intuitive concept of orientation on our inborn notion of "right" versus "left".

[7] Maybe you already knew this because the point (α, β) is described by the pair of numbers (α, β), duh! But consider, what would we do if we didn't know about coordinates at all? One approach is to *define* coordinates in terms of vectors, which is really what we're doing.

For a vector $\vec{w} = \alpha\hat{\mathbf{x}} + \beta\hat{\mathbf{y}}$, we call the pair (α, β) the *components* of the vector $\vec{w}$. There are many equivalent notations used to represent components.

(α, β)	parenthesis
$\langle \alpha, \beta \rangle$	angle brackets
$\begin{bmatrix} \alpha & \beta \end{bmatrix}$	square brackets in a row (a row matrix)
$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$	square brackets in a column (a column matrix)

Given what we now know about representing vectors and there equivalency with points, we can now dissect the notation $\mathbb{R}^2$. On the one hand, $\mathbb{R}^2$ is the set of vectors in two-dimensional Euclidean space. On the other hand $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is the set of all pairs of real numbers. Via the use of coordinates, we know these concepts represent the same thing! Further, since vectors in $\mathbb{R}^2$ are equivalent to their representation in coordinates, we will often write

$$\vec{v} = (\alpha, \beta)$$

as a shorthand for $\vec{v} = \alpha\hat{\mathbf{x}} + \beta\hat{\mathbf{y}}$.

Breaking vectors into components, and in particular, viewing vectors as linear combinations of the standard basis vectors, allows us to solve problems that were difficult before. For instance, suppose we have vectors $\vec{v}$ and $\vec{w}$. How can we compute $\|\vec{v} + \vec{w}\|$? With components, it's easy.

■ **Example 2.2** Suppose $\vec{v} = \alpha_1\hat{\mathbf{x}} + \beta_1\hat{\mathbf{y}}$ and $\vec{w} = \alpha_2\hat{\mathbf{x}} + \beta_2\hat{\mathbf{y}}$. By the laws of vector arithmetic we have

$$\vec{v} + \vec{w} = (\alpha_1\hat{\mathbf{x}} + \beta_1\hat{\mathbf{y}}) + (\alpha_2\hat{\mathbf{x}} + \beta_2\hat{\mathbf{y}}) = (\alpha_1 + \alpha_2)\hat{\mathbf{x}} + (\beta_1 + \beta_2)\hat{\mathbf{y}}.$$

Now, since $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are orthogonal to each other, the Pythagorean theorem gives

$$\|\vec{v} + \vec{w}\| = \sqrt{(\alpha_1 + \alpha_2)^2 + (\beta_1 + \beta_2)^2}.$$

■

Writing things in terms of the standard basis allowed us to make easy work of computing $\|\vec{v} + \vec{w}\|$ in Example 2.2. We can use the laws of vector arithmetic to produce rules for working with components.

The rules are are likely familiar:

$$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} a + \alpha \\ b + \beta \end{bmatrix} \qquad \text{and} \qquad \alpha \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \alpha a \\ \alpha b \end{bmatrix}.$$

Exercise 2.1 Prove the rules for adding the component representation of vectors and multiplying the component representation of vectors directly from the laws of vector arithmetic. ■

Armed with these rules, we will be able to tackle sophisticated vector problems.

Three-dimensional Coordinates

In three-dimensional space, the story is very similar. Again, we imagine three perpendicular axes, the x, y, and z axes. To draw consistent pictures, we have an notion of a right-handed three-dimensional coordinate system given by the *right-hand rule*.

XXX Figure

We now have three standard basis vectors, $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$, each pointing one unit in the positive direction of their respective axes. Any vector in three-dimensional space can be represented in exactly one way as a linear combination $\alpha\hat{\mathbf{x}} + \beta\hat{\mathbf{y}} + \gamma\hat{\mathbf{z}}$. Thus, vectors in three-dimensional space, notated $\mathbb{R}^3$, are synonymous with triplets (α, β, γ) of real numbers. With some clever geometry, we deduce

$$||\alpha\hat{\mathbf{x}} + \beta\hat{\mathbf{y}} + \gamma\hat{\mathbf{z}}|| = \sqrt{\alpha^2 + \beta^2 + \gamma^2}.$$

Historically, three-dimensional space has been studied a lot and there are several notations for the standard basis vectors still in use.

The following is a non-exhaustive list.

$\hat{\mathbf{x}}$	$\hat{\mathbf{y}}$	$\hat{\mathbf{z}}$
$\hat{\imath}$	$\hat{\jmath}$	$\hat{k}$
$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\vec{e}_1$	$\vec{e}_2$	$\vec{e}_3$

Keep these notations in the back of your mind. You might see them in other classes.

Higher dimensions

One can't progress very far in the study of science and mathematics without encountering a need for higher dimensional "vectors". For example, physicists have known since Einstein that the physical universe is best thought of as a four-dimensional entity called spacetime in which time plays a role close to that of the three spatial coordinates. Since, we don't have any way to deal with $\mathbb{R}^n$ intuitively, we must proceed by analogy with two and three dimensions. The easiest way to proceed is to generalize the idea of a standard basis. From there, we can represent vectors in $\mathbb{R}^n$ as n-tuples of real numbers. We then define

$$||(x_1, x_2, \ldots, x_n)|| = \sqrt{x_1 2 + x_2^2 + \cdots + x_n^2}.$$

We've now unified our theory of vectors across all integer dimensions $n > 0$. The case $n = 1$ yields "geometry" on a line, the cases $n = 2$ and $n = 3$ geometry in the plane and in space, and the case $n = 4$ yields the geometry of "4-vectors" which are used in the special theory of relativity. Larger values of n are used in a variety of contexts, some of which we shall encounter later.

Exercises for 2.3

1. Find $||a||$, $5\vec{a} - 2\vec{b}$, and $-3\vec{b}$ for each of the following vector pairs.

   a) $\vec{a} = 2\hat{\mathbf{x}} + 3\hat{\mathbf{y}}$, $\vec{b} = 4\hat{\mathbf{x}} - 9\hat{\mathbf{y}}$

b) $\vec{a} = (1, 2, -1)$, $\vec{b} = (2, -1, 0)$

2. Let $P = (7, 2, 9)$ and $Q = (-2, 1, 4)$. Find $\overrightarrow{PQ}$ as a linear combination of $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$.

2.4 Dot Products & Projections

Dot Product

Let $\vec{a}$ and $\vec{b}$ be vectors. We assume they are placed so their tails coincide. Let θ denote the *smaller* of the two angles between them, so $0 \leq \theta \leq \pi$. The *dot product* of $\vec{a}$ and $\vec{b}$ is defined to be

$$\vec{a} \cdot \vec{b} = ||\vec{a}|| ||\vec{b}|| \cos\theta.$$

We will call this the *geometric definition of the dot product*. The dot product is also sometimes called the *scalar product* because the result is a scalar. Note that $\vec{a} \cdot \vec{b} = 0$ when either $\vec{a}$ or $\vec{b}$ is zero or, more interestingly, if their directions are perpendicular.

XXX Figure showing angle theta

Algebraically, we can define the dot product in terms of components:

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

We will call this the *algebraic definition of the dot product*[8].

By switching between algebraic and geometric definitions, we can use the dot product to find quantities that are otherwise difficult.

■ **Example 2.3** Find the angle between the vectors $\vec{v} = (1, 2, 3)$ and $\vec{w} = (1, 1, -2)$.

From the algebraic definition of the dot product, we know

$$\vec{v} \cdot \vec{w} = 1(1) + 2(1) + 3(-2) = -3.$$

From the geometric definition, we know

$$\vec{v} \cdot \vec{w} = ||\vec{v}|| ||\vec{w}|| \cos\theta = \sqrt{14}\sqrt{6}\cos\theta = \sqrt{21}\cos\theta.$$

Equating the two definitions of $\vec{v} \cdot \vec{w}$, we see

$$\cos\theta = \frac{-3}{\sqrt{21}}$$

and so $\theta = \arccos(-3/\sqrt{21})$. ■

[8] Philosophically, every object should have only one definition from which equivalent characterizations can be deduced as theorems. If you're bothered, pick your favorite definition to be the "true" definition and consider the other definition a theorem.

Recall that for vectors $\vec{a}$, $\vec{b}$ the relationship $\vec{a} \cdot \vec{b} = 0$ can hold for two reasons: (i) either $\vec{a} = \vec{0}$, $\vec{b} = \vec{0}$, or both or (ii) $\vec{a}$ and $\vec{b}$ meet at 90°. Thus, the dot product can be used to tell if two vectors are perpendicular. There is some strangeness with the zero vector here, but it turns out this strangeness simplifies our lives mathematically.

Definition 2.4.1 — Orthogonal. The vectors $\vec{u}$ and $\vec{v}$ are *orthogonal* if $\vec{u} \cdot \vec{v} = 0$.

The definition of orthogonal encapsulates both the idea of two vectors being perpendicular and the idea of one of them being $\vec{0}$.

Before we continue, let's pin down the idea of one vector pointing in the *direction* of another. There are many ways we could define this idea, but we'll go with this one.

Definition 2.4.2 The vector $\vec{u}$ points in the *direction* of the vector $\vec{v}$ if $k\vec{u} = \vec{v}$ for some scalar k.

A simple example is that $2\hat{\mathbf{x}}$ points in the direction of $\hat{\mathbf{x}}$ since $\frac{1}{2}(2\hat{\mathbf{x}}) = \hat{\mathbf{x}}$. However, nothing in the definition says the scalar needs to be positive, so $-\hat{\mathbf{x}}$ also points in the direction $\hat{\mathbf{x}}$.

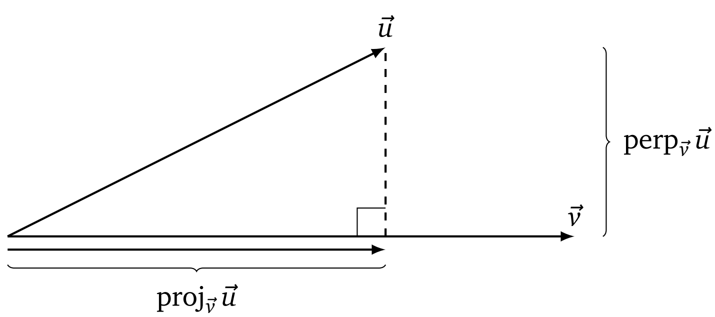
Projection

Another common vector operation is *projection*. Projection measures how much a vector points in the direction of another. This quantity is encoded as a vector. We make this definition mathematically precise as follows.

Definition 2.4.3 — Projection. For a vector $\vec{u}$ and a non-zero vector $\vec{v}$, the *projection* of $\vec{u}$ onto $\vec{v}$ is written as $\operatorname{proj}_{\vec{v}} \vec{u}$ and is a vector in the direction of $\vec{v}$ with the property that $\vec{u} - \operatorname{proj}_{\vec{v}} \vec{u}$ is orthogonal to $\vec{v}$.

The vector $\vec{u} - \operatorname{proj}_{\vec{v}} \vec{u}$ is called the *perpendicular component* of the projection of $\vec{u}$ onto $\vec{v}$ and is notated $\operatorname{perp}_{\vec{v}} \vec{u}$.

We can visualize projections with the following diagram.



From the picture it appears that $\vec{u}$, $\operatorname{proj}_{\vec{v}} \vec{u}$, and $\operatorname{perp}_{\vec{v}} \vec{u}$ form a right triangle. Of course, we shouldn't trust the picture. We should verify this mathematically.

Theorem 2.4.1 If $\vec{u}$ and $\vec{v}$ are non-zero vectors, then $\vec{v}$, $\operatorname{proj}_{\vec{v}} \vec{u}$, and $\operatorname{perp}_{\vec{v}} \vec{u}$ form a (possibly degenerate) right triangle.

Proof. We need to verify that the sides $\operatorname{proj}_{\vec{v}} \vec{u}$ and $\operatorname{perp}_{\vec{v}} \vec{u}$ meet at a right angle and that the hypotenuse $\vec{u}$ meets the sides. That is, $\operatorname{perp}_{\vec{v}} \vec{u} + \operatorname{proj}_{\vec{v}} \vec{u} = \vec{u}$.

By the definition of projection, $\text{perp}_{\vec{v}}\,\vec{u} = \vec{u} - \text{proj}_{\vec{v}}\,\vec{u}$ is orthogonal to $\vec{v}$. Since $\text{proj}_{\vec{v}}\,\vec{u}$ points in the direction of $\vec{v}$, we have $\text{proj}_{\vec{v}}\,\vec{u} = k\vec{v}$ and so $\text{perp}_{\vec{v}}\,\vec{u}$ is orthogonal to $\text{proj}_{\vec{v}}\,\vec{u}$.

Finally, consider

$$\text{perp}_{\vec{v}}\,\vec{u} + \text{proj}_{\vec{v}}\,\vec{u} = (\vec{u} - \text{proj}_{\vec{v}}\,\vec{u}) + \text{proj}_{\vec{v}}\,\vec{u} = \vec{u},$$

so indeed the vectors form a right triangle. ■

Now that we've proved $\vec{u}$, $\text{proj}_{\vec{v}}\,\vec{u}$, and $\text{perp}_{\vec{v}}\,\vec{u}$ form a right triangle, we are free to use trigonometry to compute projections. If θ is the angle between $\vec{u}$ and $\vec{v}$ and $0 \leq \theta \leq \pi/2$, we know $\|\text{proj}_{\vec{v}}\,\vec{u}\| = \|\vec{u}\|\cos\theta$. This means

$$\text{proj}_{\vec{v}}\,\vec{u} = k\vec{v} = \|\vec{u}\|\cos\theta\,\frac{\vec{v}}{\|\vec{v}\|}$$

(Recall that $\vec{v}/\|\vec{v}\|$ is a unit vector in the direction of $\vec{v}$). But $\cos\theta$ appears in the formula for the dot product. Solving for $\cos\theta$ in the dot product formula, we see $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$. Thus,

$$\text{proj}_{\vec{v}}\,\vec{u} = \|\vec{u}\|\cos\theta\,\vec{v} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\left(\frac{\vec{v}}{\|\vec{v}\|}\right) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}.$$

Upon close inspection, we see $\|\vec{v}\|^2 = \vec{v}\cdot\vec{v}$ (since $\cos 0 = 1$) and so we finally arrive at the formula

$$\text{proj}_{\vec{v}}\,\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}$$

Incredibly, if we use the algebraic definition of the dot product, we can compute a projection without computing cosine of anything!

Exercises for 2.4

2.5 The Cross Product

For vectors $\vec{a}$ and $\vec{b}$, the dot $\vec{a}\cdot\vec{b}$ measures how close $\vec{a}$ and $\vec{b}$ are to being orthogonal. In contrast, the *cross product* of $\vec{a}$ and $\vec{b}$, written $\vec{a}\times\vec{b}$ will measure the *area* of the parallelogram whose sides are given by $\vec{a}$ and $\vec{b}$.

Let's explore this idea. Since the cross product is a *product*, we will demand it follow reasonable distribution laws[9]:

$$\begin{aligned}
\vec{a}\times(\vec{b}+\vec{c}) &= \vec{a}\times\vec{b}+\vec{a}\times\vec{c}\\
(\vec{a}+\vec{b})\times\vec{c} &= \vec{a}\times\vec{c}+\vec{b}\times\vec{c}\\
(\alpha\vec{a})\times\vec{b} &= \alpha(\vec{a}\times\vec{b})\\
\vec{a}\times(\alpha\vec{b}) &= \alpha(\vec{a}\times\vec{b})
\end{aligned}$$

for vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and scalars α.

[9] The technical term for satisfying these laws is *bilinearity*.

Now, suppose $\vec{a} \times \vec{b}$ indeed encapsulates the area of the parallelogram with sides $\vec{a}$ and $\vec{b}$. If we slide the tip of $\vec{b}$ parallel to the vector $\vec{a}$, we should not change the area. Thus the cross product of $\vec{a}$ and $\vec{b}$ should be the same as that of $\vec{a}$ and $\vec{b} + \alpha\vec{a}$.

XXX Figure

Using this invariance along with our distributive rules, we now see

$$\vec{a} \times \vec{b} = \vec{a} \times (\vec{b} + \alpha\vec{a}) = \vec{a} \times \vec{b} + \alpha(\vec{a} \times \vec{a}),$$

and so $\vec{a} \times \vec{a} = 0$. We can apply this newly-found fact to the vector $\vec{a} + \vec{b}$ to deduce

$$\begin{aligned} 0 = (\vec{a} + \vec{b}) \times (\vec{a} + \vec{b}) &= \vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{b} \times \vec{a} + \vec{b} \times \vec{b} \\ &= 0 + \vec{a} \times \vec{b} + \vec{b} \times \vec{a} + 0 \\ &= \vec{a} \times \vec{b} + \vec{b} \times \vec{a}, \end{aligned}$$

and so

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}.$$

Products with this property are called *anti-commutative*. Now for an incredible fact of the universe: the result of the cross product of two vectors in $\mathbb{R}^3$ can be represented by another vector in $\mathbb{R}^3$ whose magnitude corresponds to the area of the parallelogram with sides $\vec{a}$ and $\vec{b}$.[10] Using trigonometry, we deduce

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\|\|\vec{b}\| \sin\theta$$

where $0 \leq \theta \leq \pi$ is smaller of the two angles between $\vec{a}$ and $\vec{b}$. What remains to be seen is what direction $\vec{a} \times \vec{b}$ points in. For this, we use the standard basis for $\mathbb{R}^3$ as a launching point. Recall $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are all unit vectors and all orthogonal to each other. Thus, any of them cross each other must result in a unit vector. By convention,

$$\begin{aligned} \hat{\mathbf{x}} \times \hat{\mathbf{y}} &= \hat{\mathbf{z}}, \\ \hat{\mathbf{y}} \times \hat{\mathbf{z}} &= \hat{\mathbf{x}} \\ \hat{\mathbf{z}} \times \hat{\mathbf{x}} &= \hat{\mathbf{y}}. \end{aligned}$$

Let $\vec{a} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}}$ and $\vec{b} = b_x\hat{\mathbf{x}} + b_y\hat{\mathbf{y}} + b_z\hat{\mathbf{z}}$. Using the distributive laws of the cross product we see,

$$\begin{aligned} \vec{a} \times \vec{b} = (a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}}) &\times (b_x\hat{\mathbf{x}} + b_y\hat{\mathbf{y}} + b_z\hat{\mathbf{z}}) \\ = \quad & a_x b_x \hat{\mathbf{x}} \times \hat{\mathbf{x}} + a_x b_y \hat{\mathbf{x}} \times \hat{\mathbf{y}} + a_x b_z \hat{\mathbf{x}} \times \hat{\mathbf{z}} \\ & + a_y b_x \hat{\mathbf{y}} \times \hat{\mathbf{x}} + a_y b_y \hat{\mathbf{y}} \times \hat{\mathbf{y}} + a_y b_z \hat{\mathbf{y}} \times \hat{\mathbf{z}} \\ & + a_z b_x \hat{\mathbf{z}} \times \hat{\mathbf{x}} + a_z b_y \hat{\mathbf{z}} \times \hat{\mathbf{y}} + a_z b_z \hat{\mathbf{z}} \times \hat{\mathbf{z}} \\ = \quad & \vec{0} + a_x b_y \hat{\mathbf{z}} - a_x b_z \hat{\mathbf{y}} \\ & - a_y b_x \hat{\mathbf{z}} + \vec{0} + a_y b_z \hat{\mathbf{x}} \\ & + a_z b_x \hat{\mathbf{y}} - a_z b_y \hat{\mathbf{x}} + \vec{0} \end{aligned}$$

[10] This is *only* true in $\mathbb{R}^3$. In $\mathbb{R}^4$ a product that produces area-like quantities does exist, but the output cannot be described by a vector. In higher dimensions, the cross product is called the *wedge product*.

so

$$\vec{a} \times \vec{b} = (a_y b_z - a_z b_y)\hat{\mathbf{x}} - (a_x b_z - a_z b_x)\hat{\mathbf{y}} + (a_x b_y - a_y b_x)\hat{\mathbf{z}}.$$

Exercise 2.2 Verify that $\|\vec{a} \times \vec{b}\| = \|\vec{a}\|\|\vec{b}\| \sin\theta$. (Hint: you can use $\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos\theta$ to solve for θ and the proceed using components.) ■

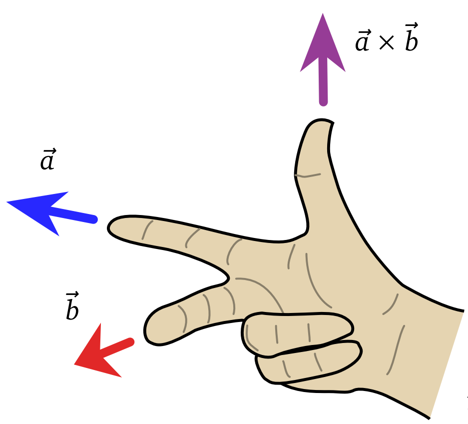
Now that we know what the cross product is and how to compute it, let's explore some of its incredible properties. First,

$$(\vec{a} \times \vec{b}) \cdot \vec{a} = (a_y b_z - a_z b_y)a_x - (a_x b_z - a_z b_x)a_y + (a_x b_y - a_y b_x)a_z = 0$$

and

$$(\vec{a} \times \vec{b}) \cdot \vec{b} = (a_y b_z - a_z b_y)b_x - (a_x b_z - a_z b_x)b_y + (a_x b_y - a_y b_x)b_z = 0.$$

Thus, $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$. Just based on this property, since the length of $\vec{a} \times \vec{b}$ is fixed, $\vec{a} \times \vec{b}$ can be one of two vectors in space. If we investigate further, we'll see that $\vec{a} \times \vec{b}$ is the vector that satisfies the *right-hand rule*.



[11]

A vector that encodes area, points orthogonally to others, and obeys the right-hand rule is handy indeed, and the cross product will be a useful tool for solving many problems.

Exercises for 2.5

2.6 Lines and Planes

With a handle on vectors, we can now use them to describe some common geometric objects: lines and planes.

[11] Image credit: Acdx, from Wikipedia https://en.wikipedia.org/wiki/Cross_product

Lines

Consider for a moment the line ℓ through the points P and Q. If $P,Q \in \mathbb{R}^2$, we could describe this line in $y = mx + b$ form (provided it isn't a vertical line), but if $P,Q \in \mathbb{R}^3$ it's much harder to describe ℓ with an equation. Using vectors provides an easier way.

Let $\vec{d} = \overrightarrow{PQ}$ and consider the set of points (or vectors)

$$\vec{x} = t\vec{d} + P$$

for $t \in \mathbb{R}$. Geometrically, this is the set of all points we get by starting at P and displacing by some multiple of the direction $\vec{d}$. This is a line! Call this line ℓ. In set-builder notation, we would write

$$\ell = \{\vec{x} \,:\, \vec{x} = t\vec{d} + P \text{ for some } t \in \mathbb{R}\}.$$

Notice that in set-builder notation we write "for some $t \in \mathbb{R}$." Make sure you understand why replacing "for some $t \in \mathbb{R}$" with "for all $t \in \mathbb{R}$" would be incorrect.

Writing lines with set-builder notation all the time can be overkill, so we will allow ourselves to describe lines in a shorthand called *vector form*[12].

Definition 2.6.1 — Vector form of a Line. A line ℓ is described in *vector form* if there are two vectors $\vec{d} \neq \vec{0}$ and $\vec{p}$ so that

$$\vec{x} = t\vec{d} + \vec{p}$$

satisfies $\vec{x} \in \ell$ for all $t \in \mathbb{R}$. In this case we call $\vec{d}$ the *direction* of ℓ and the equation $\vec{x} = t\vec{d} + \vec{p}$ the *vector equation* of ℓ.

Note that if $\vec{x} = t\vec{d} + \vec{p}$ is the vector equation of a line ℓ, by setting $t = 0$ we necessarily have $\vec{p} \in \ell$.

The direction of a line is easily obtained by finding the displacement vector between two points on the line. Thus, given a line in another form, computing its vector form is straightforward.

■ **Example 2.4** Find vector form of the line ℓ in $\mathbb{R}^2$ with equation $y = 2x + 3$. First, we find two points on the line. By guess-and-check we see $P = (0,3)$ and $Q = (1,5)$ are on ℓ. Thus, a direction vector for ℓ is given by

$$\vec{d} = (1,5) - (0,3) = (1,2).$$

We may now write the vector equation of ℓ as

$$\vec{x} = t\vec{d} + P$$

or, in components,

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

■

[12] $y = mx + b$ form of a line is also shorthand. The line ℓ described by the equation $y = mx + b$ is actually the set $\{(x,y) \in \mathbb{R}^2 \,:\, y = mx + b\}$.

The downside of writing lines in vector form is that there are multiple direction vectors and multiple points for every line. Thus, merely by looking at the vector equation for two lines, it can be hard to tell if they're equal.

For example,

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \qquad \begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

all represent the same line. In the second equation, the direction is parallel but scaled, and in the third equation, a different point on the line was chosen.

In vector form, the variable t is called the *parameter variable*. It is an instance of a *dummy variable*; that is, it is mostly there as a placeholder. Remember, vector form is shorthand for set-builder notation.

Let $\vec{d}_1, \vec{d}_2 \neq \vec{0}$ and $\vec{p}_1, \vec{p}_2$ be vectors and define the lines

$$\ell_1 = \{\vec{x} : \vec{x} = t\vec{d}_1 + \vec{p}_1 \text{ for some } t \in \mathbb{R}\}$$

$$\ell_2 = \{\vec{x} : \vec{x} = t\vec{d}_2 + \vec{p}_2 \text{ for some } t \in \mathbb{R}\}.$$

These lines have vector equations $\vec{x} = t\vec{d}_1 + \vec{p}_1$ and $\vec{x} = t\vec{d}_2 + \vec{p}_2$. However, declaring that $\ell_1 = \ell_2$ if and only if $t\vec{d}_1 + \vec{p}_1 = t\vec{d}_2 + \vec{p}_2$ does *not* make sense. Instead $\ell_1 = \ell_2$ if $\ell_1 \subseteq \ell_2$ and $\ell_2 \subseteq \ell_1$. If $\vec{x} \in \ell_1$ then $\vec{x} = t\vec{d}_1 + \vec{p}_1$ for some $t \in \mathbb{R}$. If $\vec{x} \in \ell_2$ then $\vec{x} = t\vec{d}_2 + \vec{p}_2$ for some *possibly different* $t \in \mathbb{R}$. This can get confusing really quickly. The easiest solution is to use different parameter variables if we want to compare lines in vector form.

■ **Example 2.5** Determine if the lines ℓ_1 and ℓ_2, represented in vector form by the equations

$$\vec{x} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \vec{x} = t \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

are the same line. To determine this, we need to figure out if $\vec{x} \in \ell_1$ implies $\vec{x} \in \ell_2$ and if $\vec{x} \in \ell_2$ implies $\vec{x} \in \ell_1$.

If $\vec{x} \in \ell_1$, then $\vec{x} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ for some $t \in \mathbb{R}$. If $\vec{x} \in \ell_2$, then $\vec{x} = s \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ for some $s \in \mathbb{R}$. Thus if

$$t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \vec{x} = s \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

always has a solution, $\ell_1 = \ell_2$. Moving everything to one side we see

$$\begin{aligned} \vec{0} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 2 \end{bmatrix} - t \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} 2 \\ 2 \end{bmatrix} + s \begin{bmatrix} 2 \\ 2 \end{bmatrix} - t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= (s+1) \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \tfrac{t}{2} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\ &= (s + 1 - \tfrac{t}{2}) \begin{bmatrix} 2 \\ 2 \end{bmatrix}. \end{aligned}$$

This has a solution whenever $0 = s + 1 - t/2$. Since for every $t \in \mathbb{R}$ we can find an $s \in \mathbb{R}$ and for every $s \in \mathbb{R}$ we can find a $t \in \mathbb{R}$ satisfying this equation, we know $\ell_1 = \ell_2$. ■

The geometry of lines in space is a bit more complicated than that of lines in the plane. Lines in the plane either intersect or are parallel. In space, we have to be a bit more careful about what we mean by "parallel lines," since lines with entirely different directions can still fail to intersect.

XXX Figure

■ **Example 2.6** Consider the lines described by

$$\begin{aligned}\vec{x} &= t(1,3,-2)+(1,2,1)\\ \vec{x} &= t(-2,-6,4)+(3,1,0).\end{aligned}$$

They have parallel directions since $(-2,-6,4) = -2(1,3,-2)$. Hence, in this case we say the lines are *parallel*. (How can we be sure the lines are not the same?) ■

■ **Example 2.7** Consider the lines described by

$$\begin{aligned}\vec{x} &= t(1,3,-2)+(1,2,1)\\ \vec{x} &= t(0,2,3)+(0,3,9).\end{aligned}$$

They are not parallel because neither of the direction vectors is a multiple of the other. They may or may not intersect. (If they don't, we say the lines are *skew*.) How can we find out? Mirroring our earlier approach, we can set their equations equal and see if we can solve for the point of intersection *after ensuring we give their parametric variables have different names*. We'll keep one parametric variable named t and name the other one s. Thus, we want

$$\vec{x} = t(1,3,-2)+(1,2,1) = s(0,2,3)+(0,3,9),$$

which after collecting terms yields

$$(t+1, 3t+2, -2t+1) = (0, 2s+3, 3s+9).$$

Picking out the components yields three equations

$$\begin{aligned}t+1 &= 0\\ 3t+2 &= 2s+3\\ -2t+1 &= 3s+9\end{aligned}$$

in 2 unknowns s and t. This is an *overdetermined* system, and it may or may not have a consistent solution. The first two equations yield $t = -1$ and $s = -2$. Putting these values in the last equation yields $(-2)(-1)+1 = 3(-2)+9$, which is indeed true. Hence, the equations are consistent, and the lines intersect. To find the point of intersection, put $t = -1$ in the equation for the first line (or $s = -2$ in that for the second) to obtain $(0,-1,3)$. ■

Planes

Any two distinct points define a line. To define a plane, we need three points. But there's a caveat: the three points cannot be on the same line, otherwise they'd define a line and not a plane. Let $A, B, C \in \mathbb{R}^3$ be three points that are not collinear and let $\mathcal{P}$ be the plane that passes through A, B, and C.

Just like lines, planes have direction vectors. For $\mathcal{P}$, both $\vec{d}_1 = \overrightarrow{AB}$ and $\vec{d}_2 = \overrightarrow{AC}$ are direction vectors for $\mathcal{P}$. Of course, $\vec{d}_1$, $\vec{d}_2$ and their multiples are not the only direction vectors for $\mathcal{P}$. There are infinitely many more, including $\vec{d}_1 + \vec{d}_2$, and $\vec{d}_1 - 7\vec{d}_2$, and so on. However, since a plane is a *two* dimensional object, we only need two different direction vectors to describe it.

Again like lines, planes have a vector form. $\mathcal{P}$ can be written in vector form as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\vec{d}_1 + s\vec{d}_2 + A.$$

Vector form of $\mathcal{P}$ is not unique. Any two different directions in $\mathcal{P}$ suffice for defining $\mathcal{P}$ in vector form.

Definition 2.6.2 — Vector form of a plane. The plane $\mathcal{P}$ is describe in *vector form* if there are three vectors $\vec{d}_1$, $\vec{d}_2$, and $\vec{p}$ where $\vec{d}_1, \vec{d}_2 \neq \vec{0}$ point in different directions and

$$\vec{x} = t\vec{d}_1 + s\vec{d}_2 + \vec{p}$$

satisfies $\vec{x} \in \mathcal{P}$ for all scalars $t, s \in \mathbb{R}$. The vectors $\vec{d}_1$ and $\vec{d}_2$ are called *direction vectors* for the plane $\mathcal{P}$.

Since we will commonly be working in $\mathbb{R}^3$ there is another way to define a plane. Given any vector $\vec{n} \in \mathbb{R}^3$, we can consider the set $\mathcal{Q} \subseteq \mathbb{R}^3$ of vectors orthogonal to $\vec{n}$. If $\vec{n} = \vec{0}$, then $\mathcal{Q} = \mathbb{R}^3$. Otherwise, $\mathcal{Q}$ is a plane through the origin. In this case, $\vec{n}$ is called the *normal vector* of the plane $\mathcal{Q}$.

Definition 2.6.3 — Normal form of a plane. The plane $\mathcal{P}$ is described in *normal form* if for some $\vec{n}$ and $\vec{p}$, the equation

$$\vec{n} \cdot (\vec{x} - \vec{p}) = 0$$

if and only if $\vec{x} \in \mathcal{P}$. Equivalently, $\mathcal{P}$ is described in normal form if for some $\vec{n}$ and scalar $\alpha \in \mathbb{R}$ the equation

$$\vec{n} \cdot \vec{x} = \alpha$$

is satisfied if and only if $\vec{x} \in \mathcal{P}$. In either case, the vector $\vec{n}$ is call a *normal vector* for $\mathcal{P}$.

Normal form of a plane only exists in $\mathbb{R}^3$, but it is often useful[13]. The equivalence of the two ways to write a normal form of a plane is straight forward.

$$\vec{n} \cdot (\vec{x} - \vec{p}) = 0$$

if and only if

$$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p} = \alpha.$$

Since $\vec{n}$ and $\vec{p}$ are fixed, α is a constant. Expanding normal form in terms of components we see

$$\vec{n} \cdot (\vec{x} - \vec{p}) = \vec{n} \cdot \vec{x} - \alpha = n_x x + n_y y + n_z z - \alpha = 0$$

[13] Just like $y = mx + b$ form of a line only exists in $\mathbb{R}^2$.

and so

$$n_x x + n_y y + n_z z = \alpha \tag{2.1}$$

is another way to write a plane. Equation (2.1) is sometimes called *scalar form* of a plane. For us, it will not be important to distinguish between scalar and normal form.

It should be noted that like vector form of a plane, normal form of a plane is not unique. For example, the plane described by $\vec{n}\cdot(\vec{x}-\vec{p})=0$ is the same as the plane $(2\vec{n})\cdot(\vec{x}-\vec{p})=0$.

■ **Example 2.8** Find vector form and normal form of the plane $\mathcal{P}$ passing through the point $A=(1,0,0)$, $B=(0,1,0)$ and $C=(0,0,1)$.

To find vector form of $\mathcal{P}$, we need a point on the plane and two direction vectors. We have three points on the plain, so we can obtain two direction vectors by subtracting these points in different ways. Let

$$\vec{d}_1=\overrightarrow{AB}=\begin{bmatrix}-1\\1\\0\end{bmatrix} \qquad \vec{d}_2=\overrightarrow{AC}=\begin{bmatrix}-1\\0\\1\end{bmatrix}.$$

Using the point A, we may now write vector form of $\mathcal{P}$ as

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=t\begin{bmatrix}-1\\1\\0\end{bmatrix}+s\begin{bmatrix}-1\\0\\1\end{bmatrix}+\begin{bmatrix}1\\0\\0\end{bmatrix}.$$

To write normal form we need to find a normal vector to $\mathcal{P}$. By symmetry, we can see that $\vec{n}=(1,1,1)$ is a normal vector to $\mathcal{P}$. If we weren't so insightful, we could also compute $\vec{d}_1\times\vec{d}_2=(1,1,1)$ to find a normal vector. Now, we may express $\mathcal{P}$ in normal form as

$$\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\left(\begin{bmatrix}x\\y\\z\end{bmatrix}-\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=0$$

or equivalently,

$$x+y+z=1.$$

■

■ **Example 2.9** Find the line $\mathcal{P}_1\cap\mathcal{P}_2$ where $\mathcal{P}_1$ is the plane given by the equation

$$x+y+z=2$$

and $\mathcal{P}_2$ is the plane given by the equation

$$2x-y+z=0.$$

Let $\ell=\mathcal{P}_1\cap\mathcal{P}_2$. Since $\ell\subseteq\mathcal{P}_1$ and $\ell\subseteq\mathcal{P}_2$, every direction vector for ℓ is also a direction vector for $\mathcal{P}_1$ and $\mathcal{P}_2$.

Let $\vec{n}_1=(1,1,1)$ be a normal vector for $\mathcal{P}_2$ and $\vec{n}_2=(2,-1,1)$ be a normal vector for $\mathcal{P}_2$. If $\vec{d}$ is a direction vector for ℓ, then $\vec{n}_1\cdot\vec{d}=0$ and $\vec{n}_2\cdot\vec{d}=0$. Thus,

$$\vec{d}=\vec{n}_1\times\vec{n}_2=\begin{bmatrix}2\\1\\-3\end{bmatrix}$$

is a direction vector for ℓ. By guess and check we find that $\vec{p} = (0,1,1)$ satisfies $\vec{p} \in \mathcal{P}_1$ and $\vec{p} \in \mathcal{P}_2$ and so $\vec{p} \in \ell$. Thus, we may write ℓ in vector form as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

■

Exercises for 2.6

Chapter 3

Parameterization

Parameterization is a mouthful, but the fundamental idea of a parameterization is to describe one object in terms of another. For example, consider the line ℓ described by the equation $y = 2x$. By its nature, ℓ is a set. Using set-builder notation, we could write

$$\ell = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \,:\, y = 2x \right\}.$$

But, we could also write ℓ in vector form as

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Writing ℓ in vector form shows a pairing between scalars $t \in \mathbb{R}$ and points on ℓ. In many ways, ℓ is the same as $\mathbb{R}$, it's just sitting in two-dimensional space instead of being on its own.

Taking a more technical viewpoint, we may consider ℓ to be the range of a vector-valued function. Define $\vec{p}(t) = t \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then,

$$\ell = \text{range}(\vec{p}) = \{\vec{x} \,:\, \vec{x} = \vec{p}(t) \text{ for some } t \in \mathbb{R}\}.$$

Now we have something special. The function $\vec{p} : \mathbb{R} \to \mathbb{R}^2$ has domain $\mathbb{R}$ and outputs every point on the line ℓ exactly once. In other words, we've described ℓ in terms of $\mathbb{R}$ and $\vec{p}$. We could make a further assertion that *anything that you could learn by studying ℓ, you could learn by studying $\mathbb{R}$ and $\vec{p}$.*

However, there are other ways to create functions that describe ℓ. For example, consider $\vec{q} : \mathbb{R} \to \mathbb{R}^2$ where $\vec{q}(t) = 2t\vec{d}$. Again, $\ell = \text{range}(\vec{q})$ and so everything we could possibly learn about ℓ, we could learn by studying $\mathbb{R}$ and $\vec{q}$. We call both $\vec{p}$ and $\vec{q}$ *parameterizations of ℓ by $\mathbb{R}$.*

Definition 3.0.1 — Parameterization. A *parameterization* of an object X by an object Y is a continuous function $p : Y \to X$ with the added conditions that p is one-to-one[a] and $\text{range}(p) = Y$. In this case p is called a *parameterization* and Y is called the *parameter*.

[a] Sometimes we will drop the requirement that a parameterization be one-to-one, but for now we'll be strict about it.

This definition is fairly abstract, which will come in handy later, but for now, we will think of X as being some curve in $\mathbb{R}^n$ and Y as being an interval of real numbers.

■ **Example 3.1 — A Circle.** Let $\mathcal{C} \subseteq \mathbb{R}^2$ be the unit circle centered at the origin. We can parameterize $\mathcal{C}$ by angles in $[0, 2\pi)$. Consider the function $\vec{p} : [0, 2\pi) \to \mathcal{C}$ defined by

$$\vec{p}(\theta) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}.$$

Here, $\vec{p}$ traces out $\mathcal{C}$ starting at the point $(1, 0)$ and moving counter clockwise as the parameter θ increases. ■

■ **Example 3.2 — A Circle Again.** Let $\mathcal{C} \subseteq \mathbb{R}^2$ be the unit circle centered at the origin. We will parameterize $\mathcal{C}$ by the interval $[0, 1)$. Here we might imagine that our parameter $t \in [0, 1)$ represents a point that is t-percentage around the circle.

Recall $\vec{p}(\theta) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$, which parameterizes $\mathcal{C}$ based on angles. Now, consider the function $w(t) = 2\pi t$. w inputs numbers in $[0, 1)$ and outputs angles in $[0, 2\pi)$. We should now be able to use w to parameterize $\mathcal{C}$ in the desired way. After all, if we convert $[0, 1)$ to $[0, 2\pi)$ to $\mathcal{C}$, we win!

Let the parameterization $\vec{q} : [0, 1) \to \mathcal{C}$ be defined as $\vec{q} = \vec{p} \circ w$. Explicitly,

$$\vec{q}(t) = \vec{p} \circ w(t) = \vec{p}(2\pi t) = \begin{bmatrix} \sin 2\pi t \\ \cos 2\pi t \end{bmatrix}.$$

■

Exercise 3.1 Parameterize the unit circle $\mathcal{C} \subseteq \mathbb{R}^2$ by the interval $[1/2, 1)$. ■

Exercise 3.2 Let ℓ be the line segment connecting $(0, 0)$ and $(1, 1)$. Explain why $\vec{p}$: $[-1, 1] \to \ell$ given by $p(t) = (t^2, t^2)$ is *not* a parameterization. ■

3.1 Speed and Velocity of a Parameterization

In our day-to-day life, almost without thinking, we make a comparison between real numbers and time. Time has a forwards and backwards, which we equate to the real number's increasing and decreasing. We might even say we parameterize *time* by the real numbers. Thus, if $\vec{p} : [a, b] \to \mathcal{S}$ is a parameterization of the curve $\mathcal{S}$ by the interval $[a, b]$, we could think of $\vec{p}$ as describing the motion of a particle—at time $t \in [a, b]$ the particle is at $\vec{p}(t)$.

Interpreting parameterizations in this way, the *speed* of a parameterization should be the rate of change of distance with respect to time and the *velocity* of a parameterization should be the rate of change of displacement with respect to time.

Suppose $\vec{p} : [a, b] \to \mathcal{S}$ is a parameterization of $\mathcal{S}$ and $t \in [a, b]$ represents time. The *displacement* of $\vec{p}$ from time t to time $t + \Delta t$ is $\vec{p}(t + \Delta t) - \vec{p}(t)$ and the change in *distance* is $\|\vec{p}(t + \Delta t) - \vec{p}(t)\|$. Thus, if Δt is small, the velocity at time t can be approximated by

$$\text{velocity}\,\vec{p}(t) \approx \frac{\vec{p}(t + \Delta t) - \vec{p}(t)}{\Delta t}$$

and the speed[1] by

$$\operatorname{speed}\vec{p}(t) \approx \frac{\|\vec{p}(t+\Delta t)-\vec{p}(t)\|}{|\Delta t|}.$$

Taking limits, we arrive at exact rates of change, which leads us to the following definitions.

Definition 3.1.1 — Speed. Let $\vec{p} : [a,b] \to \mathcal{S}$ be a parameterization of $\mathcal{S}$. The *speed* of $\vec{p}$ at the time $t \in [a,b]$ is

$$\operatorname{speed}\vec{p}(t) = \lim_{\Delta t\to 0} \frac{\|\vec{p}(t+\Delta t)-\vec{p}(t)\|}{|\Delta t|}.$$

Definition 3.1.2 — Velocity. Let $\vec{p} : [a,b] \to \mathcal{S}$ be a parameterization of $\mathcal{S}$. The *velocity* of $\vec{p}$ at the time $t \in [a,b]$ is

$$\operatorname{velocity}\vec{p}(t) = \lim_{\Delta t\to 0} \frac{\vec{p}(t+\Delta t)-\vec{p}(t)}{\Delta t}.$$

Both the definition of speed and the definition of velocity look a lot like the definition of the derivative. In fact, if $\vec{p}$ were a scalar valued function, the velocity of $\vec{p}$ would be exactly the derivative of $\vec{p}$. For this reason, we will define a notation similar to that of the derivative you're familiar with. From now on, the following notations mean the same thing:

$$\operatorname{velocity}\vec{p}(t) = \vec{p}\,'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\vec{p}(t) = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}(t).$$

Let's try to use our new definition. Let $\vec{r}(t) = \begin{bmatrix}\cos t\\ \sin t\end{bmatrix}$. Now,

$$\begin{aligned}\operatorname{velocity}\vec{r}(t) &= \lim_{\Delta t\to 0} \frac{\begin{bmatrix}\cos(t+\Delta t)\\ \sin(t+\Delta t)\end{bmatrix} - \begin{bmatrix}\cos t\\ \sin t\end{bmatrix}}{\Delta t}\\ &= \lim_{\Delta t\to 0} \begin{bmatrix}\dfrac{\cos(t+\Delta t)-\cos t}{\Delta t}\\ \dfrac{\sin(t+\Delta t)-\sin t}{\Delta t}\end{bmatrix}.\end{aligned}$$

At this point we should pause. We don't know how to take limits of vectors. Fortunately the rule is simple enough—to take a limit of a vector, take the limit of each of its components[2].

[1] Recall that speed is always positive; if a particle is moving with speed 2 and we then ran the particle back in time, it would still move at speed 2, so speed is not distance/Δt, it is distance/$|\Delta t|$.

[2] As intuitive as it sounds, this rule actually has a proof which relys on the definition of limit and the continuity of $\|\cdot\|$.

Thus we see

$$\begin{aligned}
\text{velocity}\,\vec{r}(t) &= \lim_{\Delta t\to 0}\begin{bmatrix}\frac{\cos(t+\Delta t)-\cos t}{\Delta t}\\ \frac{\sin(t+\Delta t)-\sin t}{\Delta t}\end{bmatrix}\\
&= \begin{bmatrix}\lim_{\Delta t\to 0}\frac{\cos(t+\Delta t)-\cos t}{\Delta t}\\ \lim_{\Delta t\to 0}\frac{\sin(t+\Delta t)-\sin t}{\Delta t}\end{bmatrix}\\
&= \begin{bmatrix}\cos'(t)\\ \sin'(t)\end{bmatrix} = \begin{bmatrix}-\sin t\\ \cos t\end{bmatrix}.
\end{aligned}$$

Our use of the notation $\vec{r}\,'(t)$ for velocity $\vec{r}(t)$ seems further justified.

Speed also appears to be a derivative. From physics we know speed is the magnitude of velocity. Mathematically, we can prove it.

Theorem 3.1.1 For a parameterization $\vec{p}:\mathbb{R}\to\mathbb{R}^n$ where velocity $\vec{p}(t)$ exists, we have

$$\text{speed}\,\vec{p}(t) = \|\text{velocity}\,\vec{p}(t)\| = \|\vec{p}\,'(t)\|.$$

Proof. The proof relies on the continuity of $\|\cdot\|$. Since $\|\cdot\|$ is continuous, we may freely move limits in and out. Thus

$$\begin{aligned}
\text{speed}\,\vec{p}(t) &= \lim_{\Delta t\to 0}\frac{\|\vec{p}(t+\Delta t)-\vec{p}(t)\|}{|\Delta t|}\\
&= \lim_{\Delta t\to 0}\left\|\frac{\vec{p}(t+\Delta t)-\vec{p}(t)}{\Delta t}\right\|\\
&= \left\|\lim_{\Delta t\to 0}\frac{\vec{p}(t+\Delta t)-\vec{p}(t)}{\Delta t}\right\| = \|\text{velocity}\,\vec{p}(t)\|.
\end{aligned}$$

■

Arc-length

Let $\mathcal{S}\subseteq\mathbb{R}^n$ be a curve. The *arc-length* of $\mathcal{S}$ should be the length of $\mathcal{S}$ if you somehow untwisted $\mathcal{S}$ into a straight line without stretching anything. One of the big ideas of calculus is that we can handle curvy things by chopping them up into little pieces, computing for each piece, and then adding them back together. We use the same principle to define arc-length.

In essence, we will divide our curve $\mathcal{S}$ into many tiny line segments, add up the lengths of those line segments and take a limit as our line segments get tinier. This means we need some mathematical definition of "approximating $\mathcal{S}$ with tiny line segments." The word we will use for this is *mesh*.

Definition 3.1.3 — Mesh. Let $\mathcal{S}\subseteq\mathbb{R}^n$ is a curve with endpoints $\vec{s}$ and $\vec{e}$. A *mesh for $\mathcal{S}$ of diameter ε* is a sequence of points $\vec{s}_0,\vec{s}_1,\ldots,\vec{s}_k\in\mathbb{R}^n$ such that

(i) $\vec{s}_i\in\mathcal{S}$ for $i=0,\ldots,k$,

(ii) $\vec{s}_0 = \vec{s}$ and $\vec{s}_k = \vec{e}$, and

(iii) $\|\vec{s}_i - \vec{s}_{i-1}\| \leq \varepsilon$ for $i = 1, \ldots, k$.

Using a mesh, we can precisely define arc-length.

Definition 3.1.4 — Arc-length. Let $\mathcal{S} \subseteq \mathbb{R}^n$ be a curve with endpoints and let $\vec{s}_{m,0}, \vec{s}_{m,1}, \ldots, \vec{s}_{m,k_m}$ be a mesh for $\mathcal{S}$ with diameter $1/m$. The *arc-length* of $\mathcal{S}$

$$\operatorname{arclen} \mathcal{S} = \lim_{m\to\infty} \sum_{i=1}^{k_m} \|\vec{s}_{m,i} - \vec{s}_{m,i-1}\|.$$

This definition, though conceptually straightforward, is quite technical and very hard to compute with. Fortunately, using parameterizations, there's an easier way.

The main difficulty with using the definition of arc-length directly is getting a hold of a mesh. However, suppose $\vec{p} : [a,b] \to \mathcal{S}$ is a parameterization of $\mathcal{S}$. Now, letting $\Delta t = \frac{b-a}{k}$,

$$\vec{s}_i = \vec{p}(a + i\Delta t)$$

is a mesh for $\mathcal{S}$ with some diameter! Now, if we let $k \to \infty$ and consequently $\Delta t \to 0$, it is reasonable to expect that the diameter of our mesh tends towards zero as well. The following theorem ensures this property.

Theorem 3.1.2 Let $\mathcal{S} \subseteq \mathbb{R}^n$ is a curve with endpoints and let $\vec{p} : [a,b] \to \mathcal{S}$ be a parameterization of $\mathcal{S}$. Define $\Delta t = (b-a)/k$ and $\vec{s}_i = \vec{p}(a + i\Delta t)$. Then, if $\operatorname{arclen} \mathcal{S} < 0$, we have that the diameter of the mesh $\vec{s}_0, \vec{s}_1, \ldots \vec{s}_k$ tends to zero as $k \to \infty$ (equivalently, $\Delta t \to 0$).

A proof of Theorem 3.1.2 is beyond the scope of this book, but we can leverage it to generate meshes of small diameter from parameterizations.

Exercises for 3.1

Appendix A

Proofs

Below are some guidelines to help you write proofs. The following rules apply whenever you write a proof[1].

1. **The burden of communication lies on you, not on your reader.** It is your job to explain your thoughts; it is not your reader's job to guess them from a few hints. You are trying to convince a skeptical reader who doesn't believe you, so you need to argue with airtight logic in crystal clear language; otherwise the reader will continue to doubt. If you didn't write something on the paper, then (a) you didn't communicate it, (b) the reader didn't learn it, and (c) the grader has to assume you didn't know it in the first place.

2. **Tell the reader what you're proving.** The reader doesn't necessarily know or remember what "Theorem 2.13" is. Even a professor grading a stack of papers might lose track from time to time. Therefore, the statement you are proving should be on the same page as the beginning of your proof. For an exam this won't be a problem, of course, but on your homework, recopy the claim you are proving. This has the additional advantage that when you study for exams by reviewing your homework, you won't have to flip back in the notes/textbook to know what you were proving.

3. **Use English words.** Although there will usually be equations or mathematical statements in your proofs, use English sentences to connect them and display their logical relationships. If you look in your notes/textbook, you'll see that each proof consists mostly of English words.

4. **Use complete sentences.** If you wrote a history essay in sentence fragments, the reader would not understand what you meant; likewise in mathematics you must use complete sentences, with verbs, to convey your logical train of thought.

   Some complete sentences can be written purely in mathematical symbols, such as equations (e.g., $a^3 = b^{-1}$), inequalities (e.g., $x < 5$), and other relations (like $5 \mid 10$ or $7 \in \mathbb{Z}$). These statements usually express a relationship between two mathematical *objects*, like

[1] This list is an adaptation of *The Elements of Style for Proofs* written by Anders Hendrickson of St. Norbert College and modified by Dana Ernst of Northern Arizona University.

numbers or sets. However, it is considered bad style to begin a sentence with symbols. A common phrase to use to avoid starting a sentence with mathematical symbols is "We see that..."

5. **Show the logical connections among your sentences.** Use phrases like "Therefore" or "because" or "if. . . , then. . . " or "if and only if" to connect your sentences.

6. **Know the difference between statements and objects.** A mathematical object is a *thing*, a noun, such as a group, an element, a vector space, a number, an ordered pair, etc. Objects either exist or don't exist. Statements, on the other hand, are mathematical *sentences*: they can be true or false.

   When you see or write a cluster of math symbols, be sure you know whether it's an object (e.g., "$x^2 + 3$") or a statement (e.g., "$x^2 + 3 < 7$"). One way to tell is that every mathematical statement includes a verb, such as $=$, $\leq$, "divides", etc.

7. **"$=$" means equals.** Don't write $A = B$ unless you mean that A actually equals B. This rule seems obvious, but there is a great temptation to be sloppy. In calculus, for example, some people might write $f(x) = x^2 = 2x$ (which is false), when they really mean that "if $f(x) = x^2$, then $f'(x) = 2x$."

8. **Don't interchange $=$ and $\implies$.** The equals sign connects two *objects*, as in "$x^2 = b$"; the symbol "$\implies$" is an abbreviation for "implies" and connects two *statements*, as in "$ab = a \implies b = 1$." You should avoid using $\implies$ in your formal write-ups.

9. **Say exactly what you mean.** Just as the $=$ is sometimes abused, so too people sometimes write $A \in B$ when they mean $A \subseteq B$, or write $a_{ij} \in A$ when they mean that a_{ij} is an entry in matrix A. Mathematics is a very precise language, and there is a way to say exactly what you mean; find it and use it.

10. **Don't write anything unproven.** Every statement on your paper should be something you *know* to be true. The reader expects your proof to be a series of statements, each proven by the statements that came before it. If you ever need to write something you don't yet know is true, you *must* preface it with words like "assume," "suppose," or "if" (if you are temporarily assuming it), or with words like "we need to show that" or "we claim that" (if it is your goal). Otherwise the reader will think they have missed part of your proof.

11. **Write strings of equalities (or inequalities) in the proper order.** When your reader sees something like
$$A = B \leq C = D,$$
    he/she expects to understand easily why $A = B$, why $B \leq C$, and why $C = D$, and he/she expects the *point* of the entire line to be the more complicated fact that $A \leq D$. For example, if you were computing the distance d of the point $(12, 5)$ from the origin, you could write
$$d = \sqrt{12^2 + 5^2} = 13.$$

In this string of equalities, the first equals sign is true by the Pythagorean theorem, the second is just arithmetic, and the *point* is that the first item equals the last item: $d = 13$.

A common error is to write strings of equations in the wrong order. For example, if you were to write "$\sqrt{12^2+5^2} = 13 = d$", your reader would understand the first equals sign, would be baffled as to how we know $d = 13$, and would be utterly perplexed as to why you wanted or needed to go through 13 to prove that $\sqrt{12^2+5^2} = d$.

12. **Avoid circularity.** Be sure that no step in your proof makes use of the conclusion!

13. **Don't write the proof backwards.** Beginning students often attempt to write "proofs" like the following, which attempts to prove that $\tan^2(x) = \sec^2(x) - 1$:

$$\begin{aligned}
\tan^2(x) &= \sec^2(x) - 1 \\
\left(\frac{\sin(x)}{\cos(x)}\right)^2 &= \frac{1}{\cos^2(x)} - 1 \\
\frac{\sin^2(x)}{\cos^2(x)} &= \frac{1-\cos^2(x)}{\cos^2(x)} \\
\sin^2(x) &= 1 - \cos^2(x) \\
\sin^2(x) + \cos^2(x) &= 1 \\
1 &= 1
\end{aligned}$$

Notice what has happened here: the student *started* with the conclusion, and deduced the true statement "$1 = 1$." In other words, he/she has proved "If $\tan^2(x) = \sec^2(x) - 1$, then $1 = 1$," which is true but highly uninteresting.

Now this isn't a bad way of *finding* a proof. Working backwards from your goal often is a good strategy *on your scratch paper*, but when it's time to *write* your proof, you have to start with the hypotheses and work to the conclusion.

14. **Be concise.** Most students err by writing their proofs too short, so that the reader can't understand their logic. It is nevertheless quite possible to be too wordy, and if you find yourself writing a full-page essay, it's probably because you don't really have a proof, but just an intuition. When you find a way to turn that intuition into a formal proof, it will be much shorter.

15. **Introduce every symbol you use.** If you use the letter "k," the reader should know exactly what k is. Good phrases for introducing symbols include "Let $n \in \mathbb{N}$," "Let k be the least integer such that...," "For every real number a...," and "Suppose that X is a counterexample."

16. **Use appropriate quantifiers (once).** When you introduce a variable $x \in S$, it must be clear to your reader whether you mean "for all $x \in S$" or just "for some $x \in S$." If you just say something like "$y = x^2$ where $x \in S$," the word "where" doesn't indicate whether you mean "for all" or "some".

Phrases indicating the quantifier "for all" include "Let $x \in S$"; "for all $x \in S$"; "for every $x \in S$"; "for each $x \in S$"; etc. Phrases indicating the quantifier "some" (or "there exists") include "for some $x \in S$"; "there exists an $x \in S$"; "for a suitable choice of $x \in S$"; etc.

On the other hand, don't introduce a variable more than once! Once you have said "Let $x \in S$," the letter x has its meaning defined. You don't *need* to say "for all $x \in S$" again, and you definitely should *not* say "let $x \in S$" again.

17. **Use a symbol to mean only one thing.** Once you use the letter x once, its meaning is fixed for the duration of your proof. You cannot use x to mean anything else.

18. **Don't "prove by example."** Most problems ask you to prove that something is true "for all"—You *cannot* prove this by giving a single example, or even a hundred. Your answer will need to be a logical argument that holds for *every example there possibly could be*.

19. **Write "Let $x = \ldots$," not "Let $\cdots = x$."** When you have an existing expression, say a^2, and you want to give it a new, simpler name like b, you should write "Let $b = a^2$," which means, "Let the new symbol b mean a^2." This convention makes it clear to the reader that b is the brand-new symbol and a^2 is the old expression he/she already understands.

    If you were to write it backwards, saying "Let $a^2 = b$," then your startled reader would ask, "What if $a^2 \neq b$?"

20. **Make your counterexamples concrete and specific.** Proofs need to be entirely general, but counterexamples should be absolutely concrete. When you provide an example or counterexample, make it as specific as possible. For a set, for example, you must name its elements, and for a function you must give its rule. Do not say things like "θ could be one-to-one but not onto"; instead, provide an actual function θ that *is* one-to-one but not onto.

21. **Don't include examples in proofs.** Including an example very rarely adds anything to your proof. If your logic is sound, then it doesn't need an example to back it up. If your logic is bad, a dozen examples won't help it (see rule 18). There are only two valid reasons to include an example in a proof: if it is a *counterexample* disproving something, or if you are performing complicated manipulations in a general setting and the example is just to help the reader understand what you are saying.

22. **Use scratch paper.** Finding your proof will be a long, potentially messy process, full of false starts and dead ends. Do all that on scratch paper until you find a real proof, and only then break out your clean paper to write your final proof carefully. *Do not hand in your scratch work!*

    Only sentences that actually contribute to your proof should be part of the proof. Do not just perform a "brain dump," throwing everything you know onto the paper before showing the logical steps that prove the conclusion. *That is what scratch paper is for.*

Appendix B

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