# Multivariable Calculus

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# Contents

# Introduction

*Multivariable Calculus* approaches the subject from a mathematical, but not overly technical, perspective. The key idea of calculus—chop things into little pieces and put them together again—is emphasized throughout.

# Licensing

# Chapter 1

# Preliminaries

## 1.1 Mathematical Notation

Mathematics is a sophisticated and precise language, and we best not adventure into calculus without learning some basic words.

The most basic mathematical word is that of a *set*. A set is an unordered collection of objects. We won't try and pin it down more exactly than this—our intuition about collections of objects will suffice[1]. We write a set with curly-braces { and } and list the objects inside. For instance

$$\{1, 2, 3\}.$$

This would be read aloud as "the set containing the elements 1, 2, and 3." The symbol $\in$ is used to specify that some object is an element of a set, and $\notin$ is used to specify it is not. For example,

$$3 \in \{1, 2, 3\} \qquad 4 \notin \{1, 2, 3\}.$$

Sets can contain mixtures of objects, including other sets. For example,

$$\{1, 2, a, \{-70, \infty\}, x\}$$

is a perfectly valid set.

It is tradition to use capital letters to name sets. So we might say $A = \{6, 7, 12\}$ or $X = \{7\}$. There is, however, a special set which already has its own name—the

[1] When you pursue more rigorous math, you rely on definitions to get yourself out of philosophical jams. For instance, with our definition of set, consider "the set of all sets that don't contain themselves." Such a set cannot exist! This is called *Russel's Paradox*, and shows that if we start talking about sets of sets, we may need more than intuition.

empty set. The *empty set* is the set containing no elements and is written $\emptyset$ or $\{\}$. Note that $\{\emptyset\}$ is *not* the empty set. It is the set containing the empty set! It is also traditional to call elements of a set *points* regardless of whether you consider them "point-like" objects.

### Operations on Sets

If the set $A$ contains all the elements that the set $B$ does, we call $A$ a *superset* and $B$ a *subset*. We'll give this a formal definition.

**Definition 1.1.1 — Subset & Superset.** The set $B$ is a *subset* of the set $A$, written $B \subseteq A$ if for all $b \in B$ we also have $b \in A$. In this case, $A$ is called a *superset* of $B$.[a]

[a] Some mathematicians use the symbol $\subset$ instead of $\subseteq$.

Some simple examples are $\{1,2,3\} \subseteq \{1,2,3,4\}$ and $\{1,2,3\} \subseteq \{1,2,3\}$. There's something funny about that last example though. Those two sets are not only subsets/supersets of each other, they're *equal*. As surprising as it seems, we actually need to define what it means for two sets to be equal.

**Definition 1.1.2 — Set Equality.** The sets $A$ and $B$ are *equal*, written $A = B$ if $A \subseteq B$ and $B \subseteq A$.

Having a definition of equality to lean on will help us when we need to prove things about sets.

■ **Example 1.1** Let $A$ be the set of numbers that can be expressed as $2n$ for some whole number $n$ and let $B$ be the set of numbers that can be expressed as $m+1$ where $m$ is an odd whole number. We will show $A = B$.

First, let us show $A \subseteq B$. If $x \in A$ then $x = 2n$ for some whole number $n$. Therefore $x = 2n = 2(n-1)+1+1 = m+1$ where $m = 2(n+1)+1$ is, by definition, an odd number. Therefore $x \in B$.

Now we will show $B \subseteq A$. Let $x \in B$. By definition, $x = m+1$ for some odd $m$ and so by the definition of oddness, $m = 2k+1$ for some whole number $k$. Thus

$$\begin{aligned} x = m+1 &= (2k+1)+1 = 2k+2 \\ &= 2(k+1) = 2n, \end{aligned}$$

where $n = k+1$. Thus, $x \in A$. Since $A \subseteq B$ and $B \subseteq A$, by definition $A = B$. ■

### Set-builder Notation

Specifying sets by listing all their elements can be a hassle, and if there are an infinite number of elements, it's impossible! Fortunately, *set-builder notation* solves these problems. If $X$ is a set, we can define a subset

$$Y = \{a \in X \,:\, \text{some rule involving } a\},$$

which is read "$Y$ is the set of $a$ in $X$ *such that* some rule involving $a$ is true." If $X$ is intuitive, we may omit it and simply write $Y = \{a \,:\, \text{some rule involving } a\}$[2]. You may equivalently use "|" instead of ":", writing $Y = \{a \mid \text{some rule involving } a\}$.

■ **Example 1.2** The set $\mathbb{Z}$ is the set of integers (positive, negative, and zero whole numbers). To define $E$ as the even integers, we could write

$$E = \{n \in \mathbb{Z} \,:\, n = 2k \text{ for some } k \in \mathbb{Z}\}.$$

To define $P$ as the set of positive integers, we could write

$$P = \{n \in \mathbb{Z} \,:\, n > 0\}.$$

■

There are also some common operations we can do with two sets.

**Definition 1.1.3 — Intersections & Unions.** Let $A$ and $B$ be sets. Then the *intersection* of $A$ and $B$, written $A \cap B$, is defined by

$$A \cap B = \{x \,:\, x \in A \text{ and } x \in B\}.$$

The *union* of $A$ and $B$, written $A \cup B$, is defined by

$$A \cup B = \{x \,:\, x \in A \text{ or } x \in B\}.$$

For example, if $A = \{1,2,3\}$ and $B = \{-1,0,1,2\}$, then $A \cap B = \{1,2\}$ and $A \cup B = \{-1,0,1,2,3\}$. Set unions and intersections are *associative*, which means it doesn't matter how you apply parenthesis to an expression involving just unions or just intersections. For example $(A \cup B) \cup C = A \cup (B \cup C)$, which means we can give an unambiguous meaning to an expression like $A \cup B \cup C$ (just put the parenthesis wherever you like). But watch out, $(A \cup B) \cap C$ means something different than $A \cup (B \cap C)$!

**Definition 1.1.4 — Set Subtraction.** For sets $A$ and $B$, the *set-wise difference*

[2] If you want to get technical, to make this notation unambiguous, you define a *universe of discourse*. That is, a set $\mathcal{U}$ containing every object you might want to talk about. Then $\{a \,:\, \text{some rule involving } a\}$ is short for $\{a \in \mathcal{U} \,:\, \text{some rule involving } a\}$

between $A$ and $B$, written $A \backslash B$, is the set

$$A \backslash B = \{x \,:\, x \in A \text{ and } x \notin B\}.$$

**Definition 1.1.5 — Cardinality.** For a set $A$, the *cardinality* of $A$, written $|A|$ is the number of elements in $A$. If $A$ contains infinitely many elements, we write $|A| = \infty$.

Let's define some notation for common sets.

$$\begin{aligned}
\emptyset &= \{\}, \text{ the empty set} \\
\mathbb{N} &= \{0, 1, 2, 3, \ldots\} = \{\text{natural numbers}\} \\
\mathbb{Z} &= \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} = \{\text{integers}\} \\
\mathbb{Q} &= \{\text{rational numbers}\} \\
\mathbb{R} &= \{\text{real numbers}\} \\
\mathbb{R}^n &= \{\text{vectors in } n\text{-dimensional Euclidean space}\}
\end{aligned}$$

Besides unions, there's another way to join sets together: *products*.

**Definition 1.1.6 — Cartesian Product.** Given two sets $A$ and $B$, the *Cartesian product* (sometimes shortened to *product*) of the sets $A$ and $B$ is written $A \times B$ and defined to be

$$A \times B = \{(a, b) \,:\, a \in A \text{ and } b \in B\}.$$

The Cartesian product of two sets is the set of all ordered pairs of elements from those sets. For example,

$$\{1, 2\} \times \{1, 2, 3\} = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}.$$

You can repeat this operation more than once. $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is the set of all triples of real numbers. Borrowing notation from numbers, if you take the Cartesian product of a set with itself some number of times, you can represent it with an exponent. Thus, $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ can be written as $\mathbb{R}^3$, which is a set we've seen before[3].

## Functions

You're probably used to seeing functions like $f(x) = x^2$, but it's worth reviewing some of the concepts and terminology associated with functions.

[3] If you're scratching your head saying, "I thought $\mathbb{R}^3$ was vectors in 3-dimensional space. How do we know that's the same thing as triples of real numbers?" your mind is keen. This is a theorem of linear algebra.

**Definition 1.1.7 — Function.** A *function* with *domain* the set $A$ and *co-domain* the set $B$ is an object that associates every point in the set $A$ with *exactly one* point in the set $B$.

If a function $f$ has domain $A$ and co-domain[4] $B$, we notate this by writing $f : A \to B$. If we want to further specify what the function $f$ actually is, we need to express how $f$ associates each point in $A$ to a point in $B$. This can be done with an equation. For example, we could define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = 2x,$$

which says that each real number gets associated to its double. We can notate the same thing using a special type of arrow: "$\mapsto$". Now we might write

$$f : \mathbb{R} \to \mathbb{R} \text{ where } x \mapsto 2x,$$

which is read "$f$ is a function from $\mathbb{R}$ to $\mathbb{R}$ where $x \in \mathbb{R}$ gets mapped to $2x$."

Note that every point in the co-domain of a function doesn't need to get mapped to. For example $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^2$ outputs only non-negative numbers, but it is still valid to specify $\mathbb{R}$ as the co-domain. However, if we wanted to make a point of it, we are perfectly justified in writing $g : \mathbb{R} \to [0, \infty)$ when defining $g$.

**Definition 1.1.8 — Range.** The *range* of a function $f : A \to B$ is the set of all outputs of $f$. That is

$$\operatorname{range} f = \{y \in B \;:\; y = f(x) \text{ for some } x \in A\}.$$

**Definition 1.1.9 — Image.** Let $f : A \to B$ be a function. The *image* of a set $X \subseteq A$, written $f(X)$ is defined by

$$f(X) = \{y \in B \;:\; y = f(x) \text{ for some } x \in X\}.$$

We see that if $f : A \to B$, $\operatorname{range} f = f(A)$. In words, the range of $f$ is the image of its domain. This language will become useful when we think of functions as transformations that move or bend space. If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a function that warps the Cartesian plane, then the image of $X$ under $f$ could be visualized by painting $X$ on the Cartesian plane, warping the whole plane, and then looking at the resulting, painted shape.

Closely related to images, we have the idea of *restriction*. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x, y) = xy$, but we were only really interested in $f$ on the unit

[4] Some people use the word *range* interchangeably with co-domain.

circle, $\mathcal{C}$. In this case, we might say $f$ attains a maximum on $\mathcal{C}$, or $f$ *restricted to* $\mathcal{C}$ attains a maximum, even though $f$ itself is unbounded. This idea comes up often enough to deserve its own notation.

**Definition 1.1.10 — Restriction.** If $f : A \to B$ and $X \subseteq A$, the *restriction* of $f$ to $X$ is written $f\big|_X$ and is defined to be the function $g : X \to A$ where $x \mapsto f(x)$.

The last important function-related ideas for us are function composition and inverses. Given two function $f : A \to B$ and $g : B \to C$, we can *compose* $g$ and $f$ to get a new function.

**Definition 1.1.11 — Composition.** Given two functions $f : A \to B$ and $g : B \to C$, the *composition* of $g$ and $f$, written $g \circ f$, is the function $h : A \to C$ where $x \mapsto g(f(x))$.

Note that the composition $g \circ f$ has the domain of $f$ and the co-domain of $g$. When a point is fed into $g \circ f$, it moves from $A \to B \to C$. The composition $g \circ f$ only makes sense because the outputs of $f$ are allowed as inputs to $g$. If we wrote $f \circ g$, it wouldn't mean much, because $g$ outputs points in $C$ and $f$ has no idea what to do with points in $C$.[5]

Inverses relate to composition and the *identity function*, the function that does nothing to its inputs.

**Definition 1.1.12 — Identity Function.** The *identity function* $\mathrm{id} : A \to A$ is defined by the relation

$$\mathrm{id}(x) = x$$

for all $x \in A$.

Notice that for every set, that set is the domain of an identity function. Since the domain and co-domain of a function are part of its definition, we don't want to confuse them. After all, $f : \{0,1\} \to \{0,1\}$ given by $f(x) = x^2$ is a different function from $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. For the special case of the identity function, we sometimes write the domain of the function as a subscript. That is, for $\mathrm{id} : A \to A$ we'd write $\mathrm{id}_A$ so it doesn't get confused with $\mathrm{id} : B \to B$, which we'd write $\mathrm{id}_B$.

**Definition 1.1.13 — Inverse Function.** Let $f : A \to B$ be a function. If there

[5] It seems a little backwards to write $f : A \to B$, $g : B \to C$ and then write $g \circ f$ instead of $f \circ g$. You can thank Euler for that. He decided to write functions with their input on the right instead of the left. If we wrote functions backwards, like $((x)f)g$ for "$g$ of $f$ of $x$," they we could just *follow the arrows* and life would be simpler.

exists a function $g : B \to A$ such that

$$f \circ g = \mathrm{id}_B \qquad \text{and} \qquad g \circ f = \mathrm{id}_A,$$

we say $f$ is *invertible* and we call $g$ the *inverse* of $f$. If $f$ is invertible, we notate its inverse by $f^{-1}$.

Inverses can be tricky some times. For example, consider $f(x) = x^2$ and $g(x) = \sqrt{x}$. Here $g \circ f(x) = \sqrt{x^2} = |x|$ and $f \circ g(x) = \sqrt{x}^2 = x$. What's the deal? Well, it's all about domains. $f : \mathbb{R} \to [0, \infty)$ and $g : [0, \infty) \to [0, \infty)$. So, the domain of $g \circ f$ is $\mathbb{R}$ and the domain of $f \circ g$ is $[0, \infty)$. The domains are different, and indeed $f$ is not invertible. However, $g$ *is* invertible, and $g^{-1} = f\big|_{[0,\infty)}$. If we only input non-negative numbers into $f$, then $f$ exactly undoes what $g$ did. This subtle domain trickery can cause us a lot of headaches if we're not used to thinking carefully, and many of our favorite functions that we're used to calling "inverse functions" are actually only inverses when paired with specific domains.

## 1.2 Proof

Mathematics has the highest standard of proof of any field. In the Platonic ideal of mathematics, we start from some basic assumptions, called *axioms*, that we have all agreed upon. Then from those axioms, using the rules of logic, we deduce *theorems*. Every single mathematical statement we make can be traced back from theorem to theorem and eventually to our initial axioms.

This is contrary to other disciplines, like physics. In physics, based on observation, we construct *laws*. Laws in physics are like axioms in mathematics, but they have an important difference—they can be disproven by observation. A mathematical axiom can never be disproven. One can certainly argue that an axiom is not *useful* or not *interesting*, but you cannot say its *wrong*[6]. Of course, as human practitioners, we may misuse logic and be wrong ourselves, but that is no fault of the axioms.

But now, let's deviate from philosophical perfection and visit reality. In reality, *mathematics is a human pursuit to understand relationships between ideas and their consequences*. The key there is that *humans* do mathematics to *understand* relationships. If a theorem in math can ultimately be reduced to logical state-

[6] There are multiple ways to axiomatize geometry. In Euclidean geometry every pair of lines either coincides, intersects in exactly one place, or does not intersect. In spherical geometry, every pair of lines either coincides or intersects in exactly two places. Euclidean geometry is useful when your space looks flat. Spherical geometry is useful when your space is the surface of a sphere (like the Earth). Is one of these more *right* than the other? They're certainly contradictory.

ments about axioms, but the argument is 100000 steps long, it doesn't help a human understand why something is true. Instead, a shorter argument that skips over some steps is more useful to us. And, indeed, most of our mathematics to date skips over some steps[7].

We call a correct mathematical argument a *proof*. A proof starts from a set of assumptions, and following the rules of logic, arrives at a conclusion. Strictly speaking, a proof doesn't need to make sense or show motivation, applications, or examples. It just has to be a sequence of correct logical steps. However, for us, as humans studying mathematics, we prove things for two reasons: to understand why things are true and to avoid making mistakes.

Reconciling these two goals can be very hard for a novice mathematician. If you include *all* the steps, it won't help with understanding, but if you don't include enough steps, the argument may not be convincing and might contain mistakes. Even professionals struggle to balance these competing goals, and how you balance those goals depends on your audience—if you're trying to convince your math professor of something your proof will need to have more detail than if you were trying to convince your friend (mathematicians are very skeptical!).

Enough talk, let's go through a 2000 year old example of a proof.

**Theorem 1.2.1** There is no rational number $p/q$ such that $(p/q)^2 = 2$.

*Proof.* If $p/q$ is a rational number, it can be expressed in lowest terms. Suppose $p/q$ is in lowest terms and $(p/q)^2 = 2$. Then $p^2 = 2q^2$ and so $p^2$ is even. Since $p^2$ is even, it must be that $p$ is even, and so by definition, $p = 2m$ for some integer $m$. Now,

$$\frac{p^2}{q^2} = \frac{(2m)^2}{q^2} = \frac{4m^2}{q^2} = 2,$$

with the last equality following by assumption. Multiplying both sides by $q^2$ and dividing by 2 we arrive at the equation

$$2m^2 = q^2,$$

and so $q^2$ is even which means $q$ is even. By definition, this means $q = 2n$ for some integer $n$. But now,

$$\frac{p}{q} = \frac{2m}{2n}$$

[7] There are some projects to prove all of mathematics directly from the axioms using computer assistance. They've made progress, but there are still theorems in calculus that have not been reduced to the axioms. We believe that they *could be* reduced to the axioms, but no one has taken the time to do so.

is not in lowest terms! This is a contradiction and so it cannot be that $(p/q)^2 = 2$. ■

This is nearly identical to the argument the ancient Greeks gave. It's elegant, beautiful, and convincing. But, if we look closer, it does skip some steps. For example, it relies on the fact that there is such a thing as *lowest terms*. This is something that would need to be proven—a priori, the conclusion of the proof could be that the assumption that $p/q$ could be in lowest terms is false.

You will not, over night, become a master at understanding what steps you can leave out and what steps you must show. However, with feedback, you'll get better. For a detailed guide about writing good proofs, please see Appendix A.

# Chapter 2

# Vectors

A *vector* is a quantity which is characterized by a *magnitude* and a *direction*. Many quantities are best described by vectors rather than numbers. For example, when driving a car, it may be sufficient to know your speed, which can be described by a single number, but the motion of an airplane must be described by a vector quantity—velocity—which takes into account its direction as well as its speed.

Ordinary numerical quantities are called *scalars* when we want to emphasize that they are not vectors.

Whereas numbers allow us to specify relationships between single quantities (put in twice as much flour as sugar), vectors will allow us to specify relationships between goemetric objects in space[1]. If we have two points, $P = (1, 1)$ and $Q = (3, 2)$, we specify the *displacement* from $P$ to $Q$ as a vector.

XXX Figure

We notate the displacement vector form $P$ to $Q$ by $\overrightarrow{PQ}$. The magnitude of $\overrightarrow{PQ}$ is given by the Pythagorean theorem to be $\sqrt{5}$ and its direction is specified by the line segment from $P$ to $Q$.

## 2.1 Vector Notation

There are many ways to represent vector quantities in writing. If we have two points, $P$ and $Q$, $\overrightarrow{PQ}$ represents the vector from $P$ to $Q$. Absent of points, bold-faced letters or a letter with an arrow over it are the most common typographical representations. For example, $\vec{a}$ or $\mathbf{a}$ may both be used to represent the vector

[1] Though in this book we will treat vectors as intertwined with Euclidean space, they are much more general. For instance, someone's internet browsing habits could be describe by a vector—the topics they find most interesting might be the "direction" and the amount of time they browse might be the "magnitude."

quantity named “$a$.” In this book we will use $\vec{a}$ to represent a vector. The notation $||\vec{a}||$ represents the magnitude of the vector $\vec{a}$, which is sometimes called the *norm* of $\vec{a}$.

Graphically vectors are represented as directed line segments (a line segment with an arrow at one end). The endpoints of the segment are called the *initial point* (the base) and the *terminal point* (the tip) of the vector.

Let $A=(1,1)$, $B=(2,3)$, $X=(0,1)$, and $Y=(1,3)$ and consider the vectors $\vec{a}=\overrightarrow{AB}$ and $\vec{x}=\overrightarrow{XY}$. Are these the same or different vectors? If we drew them as directed line segments, the drawing would be distinct. However, both $\vec{a}$ and $\vec{x}$ have equivalent magnitudes and directions. Thus, $\vec{a}$ and $\vec{x}$ are *equivalent*, and we would be justified writing $\vec{a}=\vec{x}$.

Alternatively, we could consider the *rooted vector* $\vec{a}$ rooted at the point $A$. In this terminology, $\vec{a}$ rooted at $A$ is *different* than $\vec{a}$ rooted at $X$. This idea of rooted vectors will occasionally be useful, but our primary study will be unrooted vectors.

### Vectors and Points

The distinction between vectors and points is sometimes nebulous because they are so closely related to each other. A *point* in Euclidean space specifies an absolute position whereas a vector specifies a magnitude and direction. However, given a point $P$, one can always specify the vector $\overrightarrow{OP}$, where $O$ is the origin. Similarly, given $\vec{v}$, we can specify the point $V$ to be the terminal point of $\vec{v}$ if it were rooted at the origin. Thus, we have a way to unambiguously go back and forth between vectors and points[2]. As such *we will treat vectors and points as interchangeable*.

## 2.2 Vector Arithmetic

Vectors provide a natural way to give directions. For example, suppose $\hat{\mathbf{x}}$ points one mile eastwards and $\hat{\mathbf{y}}$ points one mile northwards. Now, if you were standing at the origin and wanted to move to a location 3 miles east and 2 miles north, you might say: “Walk 3 times the length of $\hat{\mathbf{x}}$ in the $\hat{\mathbf{x}}$ direction and 2 times the length of $\hat{\mathbf{y}}$ in the $\hat{\mathbf{y}}$ direction.” Mathematically we express this as

$$3\hat{\mathbf{x}}+2\hat{\mathbf{y}}.$$

[2] Mathematically, we say there is an *isomorphism* between vectors and points.

Of course, we've incidentally described a new vector. Namely, let $P$ be the point at 3-east and 2-north. Then

$$\overrightarrow{OP} = 3\hat{\mathbf{x}} + 2\hat{\mathbf{y}}.$$

If the vector $\vec{y}$ points north but has a length of 2 miles, we have a similar formula:

$$\overrightarrow{OP} = 3\hat{\mathbf{x}} + 1\vec{y},$$

and the relationship $\vec{y} = 2\hat{\mathbf{y}}$. Our notation here is very suggestive. Indeed, if we could make sense of what $\alpha\vec{v}$ is for any scalar $\alpha$ and vector $\vec{v}$, and we could make sense of what $\vec{v} + \vec{w}$ means for any vectors $\vec{v}$ and $\vec{w}$, we would be able to do algebra with vectors. We might even say we have *an algebra of vectors*.

Intuitively, for a vector $\vec{v}$, and a scalar $\alpha > 0$, the vector $\vec{w} = \alpha\vec{v}$ should point in the same direction as $\vec{v}$ but have magnitude scaled up by $\alpha$. That is, $||\vec{w}|| = \alpha||\vec{v}||$. Similarly, $-\vec{v}$ should be the vector of the same length as $\vec{v}$ but pointing in the exact opposite direction.

For two vectors $\vec{u}$ and $\vec{v}$, the sum $\vec{w} = \vec{u} + \vec{v}$ should be the displacement vector created by first displacing along $\vec{u}$ and then displacing along $\vec{v}$.

XXX Figure

Now, there is one snag. What should $\vec{v} + (-\vec{v})$ be? Well, first we displace along $\vec{v}$ and then we displace in the exact opposite direction by the same amount. So we have gone nowhere. This corresponds to a displacement with zero magnitude. But, what direction did we displace? Here we make a philosophical stand.

**Definition 2.2.1 — Zero Vector.** The *zero vector*, notated as $\vec{0}$, is the vector with no magnitude.

But what direction does $\vec{0}$ point? We will be pragmatic. The zero vector has no direction. Or maybe it has every direction. If we are to have an algebra of vectors, we must allow this special case where the direction of the zero vector is ill-defined[3]. Further along this line of thinking, we might hope that for any vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ and scalars $\alpha$ and $\beta$, the following conditions are satisfied.

$$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w}) \qquad \text{(Associativity)}$$
$$\vec{u} + \vec{v} = \vec{v} + \vec{u} \qquad \text{(Commutativity)}$$
$$\alpha(\vec{u} + \vec{v}) = \alpha\vec{u} + \alpha\vec{v} \qquad \text{(Distributivity)}$$

[3] In the mathematically precise definition of vector, the idea of "magnitude" and "direction" are dropped. Instead, a set of vectors is defined to be a set over which you can reasonably define addition and scalar multiplication.

and

$$(\alpha\beta)\vec{v} = \alpha(\beta\vec{v}) \qquad \text{(Associativity II)}$$
$$(\alpha+\beta)\vec{v} = \alpha\vec{v} + \beta\vec{v} \qquad \text{(Distributivity II)}$$

Indeed, if we intuitively think about vectors in flat, Euclidean, space all of these properties are satisfied[4]. From now on, these properties of vector operations will be considered the *laws (or axioms) of vector arithmetic*.

We'll be talking about these vector operations (scalar multiplication and vector addition) a lot. So much so that it's worth naming.

**Definition 2.2.2 — Linear Combination.** A *linear combination* of the vectors $\vec{v}_1, \dots \vec{v}_n$ is any vector expressible as

$$\alpha_1\vec{v}_1 + \cdots + \alpha_n\vec{v}_n.$$

## 2.3 Coordinates

Recall that a coordinate system in the plane is specified by choosing an origin $O$ and then choosing two perpendicular axes meeting at the origin. These axes are chosen in some order so that we know which axis (usually the $x$-axis) comes first and which (usually the $y$-axis) second. Note that there are many different coordinate systems which could be used although we often draw pictures as if there were only one.

In physics, one often has to think carefully about the coordinate system because choosing it appropriately may greatly simplify the resulting analysis. Note that the axes are usually drawn with *right hand orientation* where the right angle from the positive $x$-axis to the positive $y$-axis is in the counter-clockwise direction. However, it would be equally valid to use the *left hand orientation* in which that angle is in the clockwise direction. One can easily switch the orientation of a coordinate system by reversing one of the axes. (The concept of orientation is quite fascinating and it arises in mathematics, physics, chemistry, and even biology in many interesting ways. Note that almost all of us base our intuitive concept of orientation on our inborn notion of "right" versus "left".)

XXX Figure

For any coordinate system, there are special vectors associated with it. For the plane, the vector point one unit along the positive $x$-axis is called $\hat{\mathbf{x}}$ and the

[4] If we deviate from flat space, some of these rules are no longer respected. Consider moving 100 miles north then 100 miles east on a sphere. Is this the same as moving 100 miles east and then 100 miles north?

vector pointing one unit along the positive $y$-axis is called $\hat{\mathbf{y}}$. The vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are called the *standard basis* vectors for $\mathbb{R}^2$.

Notice that every point (or vector) in the plane can be represented as a linear combination of $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, and the vector $\alpha\hat{\mathbf{x}}+\beta\hat{\mathbf{y}}$ is the vector $\overrightarrow{OP}$ where $P=(\alpha,\beta)$. Now, to state an intuitive fact. If $\vec{w}$ is a vector in the plane, *there is only one way to write a vector as a linear combination of* $\hat{\mathbf{x}}$ *and* $\hat{\mathbf{y}}$. This means, if $\vec{w}=\alpha\hat{\mathbf{x}}+\beta\hat{\mathbf{y}}$, the pair $(\alpha,\beta)$ captures all information[5] about $\vec{w}$.

For a vector $\vec{w}=\alpha\hat{\mathbf{x}}+\beta\hat{\mathbf{y}}$ we call the pair $(\alpha,\beta)$ the *components* of the vector $\vec{w}$. There are many equivalent notations used to represent components.

| | |
|---|---|
| $(\alpha,\beta)$ | parenthesis |
| $\langle\alpha,\beta\rangle$ | angle brackets |
| $\begin{bmatrix}\alpha & \beta\end{bmatrix}$ | square brackets in a row (a row matrix) |
| $\begin{bmatrix}\alpha \\ \beta\end{bmatrix}$ | square brackets in a column (a column matrix) |

Given what we now know about representing vectors and there equivalency with points, the notation $\mathbb{R}^2$ (pairs of real numbers) makes sense. Further, since vectors in $\mathbb{R}^2$ are equivalent to their representation in coordinates, we will often write

$$\vec{v}=(\alpha,\beta)$$

as a shorthand for $\vec{v}=\alpha\hat{\mathbf{x}}+\beta\hat{\mathbf{y}}$.

Breaking vectors into components, and in particular, viewing vectors as linear combinations of the standard basis vectors allows us to solve problems that were difficult before. For instance, suppose we have vectors $\vec{v}$ and $\vec{w}$. How can we compute $\|\vec{v}+\vec{w}\|$? With components, it's easy.

■ **Example 2.1** Suppose $\vec{v}=\alpha_1\hat{\mathbf{x}}+\beta_1\hat{\mathbf{y}}$ and $\vec{w}=\alpha_2\hat{\mathbf{x}}+\beta_2\hat{\mathbf{y}}$. By the laws of vector arithmetic we have

$$\vec{v}+\vec{w}=(\alpha_1\hat{\mathbf{x}}+\beta_1\hat{\mathbf{y}})+(\alpha_2\hat{\mathbf{x}}+\beta_2\hat{\mathbf{y}})=(\alpha_1+\alpha_2)\hat{\mathbf{x}}+(\beta_1+\beta_2)\hat{\mathbf{y}}.$$

Now, since $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are orthogonal to each other, the Pythagorean theorem gives

$$\|\vec{v}+\vec{w}\|=\sqrt{(\alpha_1+\alpha_2)^2+(\beta_1+\beta_2)^2}.$$

■

[5] Maybe you already knew this because the point $(\alpha,\beta)$ is described by the pair of numbers $(\alpha,\beta)$, duh! But consider, what would we do if we didn't know about coordinates at all? One approach is to *define* coordinates in terms of vectors, which is really what we're doing.

Writing things in terms of the standard basis allowed us to make easy work of computing $\|\vec{v}+\vec{w}\|$ in Example 2.1. We can use the laws of vector arithmetic to produce rules for working with components.

The rules are are likely familiar:

$$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} a+\alpha \\ b+\beta \end{bmatrix} \qquad \text{and} \qquad \alpha \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \alpha a \\ \alpha b \end{bmatrix}.$$

**Exercise 2.1** Prove the rules for adding the component representation of vectors and multiplying the component representation of vectors directly from the laws of vector arithmetic. ■

Armed with these rules, we will be able to tackle sophisticated vector problems.

## Three-dimensional Coordinates

In three-dimensional space, the story is very similar. Again, we imagine three perpendicular axes, the $x$, $y$, and $z$ axes. To draw consistent pictures, we have an notion of a right-handed three-dimensional coordinate system given by the *right hand rule*.

XXX Figure

We now add a third standard basis vector $\hat{\mathbf{z}}$ which points one unit along the positive $z$-axis, and any vector in three-dimensional space can be represented in exactly one way as a linear combination $\alpha\hat{\mathbf{x}}+\beta\hat{\mathbf{y}}+\gamma\hat{\mathbf{z}}$. Thus, vectors in three-dimensional space, notated $\mathbb{R}^3$, are synonymous with triplets $(\alpha,\beta,\gamma)$ of real numbers. With some clever geometry, we deduce

$$\|\alpha\hat{\mathbf{x}}+\beta\hat{\mathbf{y}}+\gamma\hat{\mathbf{z}}\| = \sqrt{\alpha^2+\beta^2+\gamma^2}.$$

Historically, three-dimensional space has been studied a lot and there are several notations for the standard basis vectors still in use.

The following is a non-exhaustive list.

$$\begin{array}{ccc} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \hat{\imath} & \hat{\jmath} & \hat{k} \\ \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{array}$$

### Higher dimensions

One can't progress very far in the study of science and mathematics without encountering a need for higher dimensional "vectors". For example, physicists have known since Einstein that the physical universe is best thought of as a 4-dimensional entity called spacetime in which time plays a role close to that of the 3 spatial coordinates. Since, we don't have any way to deal with $\mathbb{R}^n$ intuitively, we must proceed by analogy with two and three dimensions. The easiest way to proceed is to generalize the idea of a standard basis. From there, we can represent vectors in $\mathbb{R}^n$ as $n$-tuples of real numbers. We then define

$$||(x_1, x_2, \ldots, x_n)|| = \sqrt{x_1 2 + x_2^2 + \cdots + x_n^2}.$$

We've now unified our theory of vectors across all integer dimensions $n > 0$. The case $n = 1$ yields "geometry" on a line, the cases $n = 2$ and $n = 3$ geometry in the plane and in space, and the case $n = 4$ yields the geometry of "4-vectors" which are used in the special theory of relativity. Larger values of $n$ are used in a variety of contexts, some of which we shall encounter later.

### Exercises for 2.3

1. Find $||a||$, $5\vec{a} - 2\vec{b}$, and $-3\vec{b}$ for each of the following vector pairs.

   a) $\vec{a} = 2\hat{\mathbf{x}} + 3\hat{\mathbf{y}}$, $\vec{b} = 4\hat{\mathbf{x}} - 9\hat{\mathbf{y}}$

   b) $\vec{a} = (1, 2, -1)$, $\vec{b} = (2, -1, 0)$

2. Let $P = (7, 2, 9)$ and $Q = (-2, 1, 4)$. Find $\overrightarrow{PQ}$ as a linear combination of $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$.

## 2.4 Dot Products & Projections

Let $\vec{a}$ and $\vec{b}$ be vectors. We assume they are placed so their tails coincide. Let $\theta$ denote the *smaller* of the two angles between them, so $0 \leq \theta \leq \pi$. Their *dot product* is defined to be

$$\vec{a} \cdot \vec{b} = ||\vec{a}|| ||\vec{b}|| \cos\theta.$$

We will call this the *geometric definition of the dot product*. This is also sometimes called the *scalar product* because the result is a scalar. Note that $\vec{a} \cdot \vec{b} = 0$ when either $\vec{a}$ or $\vec{b}$ is zero or, more interestingly, if their directions are perpendicular. If the two vectors have parallel directions, $\vec{a} \cdot \vec{b}$ is the product of their magnitudes if they point the same way or the negative of that product if they point in opposite directions.

XXX Figure showing angle theta

Algebraically, we can define the dot product in terms of components:

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

We will call this the *algebraic definition of the dot product*[6].

By switching between algebraic and geometric definitions, we can use the dot product to find quantities that are otherwise difficult to find.

■ **Example 2.2** Find the angle between the vectors $\vec{v} = (1, 2, 3)$ and $\vec{w} = (1, 1, -2)$.

From the algebraic definition of the dot product, we know

$$\vec{v} \cdot \vec{w} = 1(1) + 2(1) + 3(-2) = -3.$$

From the geometric definition, we know

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos\theta = \sqrt{14}\sqrt{6} \cos\theta = \sqrt{21} \cos\theta.$$

Equating the equations coming from each definition we see

$$\cos\theta = \frac{-3}{\sqrt{21}}$$

and so $\theta = \arccos(-3/\sqrt{21})$. ■

The dot product is a useful concept when one needs to find the component of one vector in the direction of another. For example, in a typical inclined plane problem in elementary mechanics, one needs to resolve the vertical gravitational force into components, one parallel to the inclined plane, and one perpendicular to it. To see how the dot product enters into that, note that

[6] Philosophically, every object should have only one definition from which equivalent characterizations can be deduced as theorems. If you're bothered, pick your favorite definition to be the "true" definition and consider the other definition a theorem.

# Appendix A

# Proofs

Below are some guidelines to help you write proofs. The following rules apply whenever you write a proof[1].

1. **The burden of communication lies on you, not on your reader.** It is your job to explain your thoughts; it is not your reader's job to guess them from a few hints. You are trying to convince a skeptical reader who doesn't believe you, so you need to argue with airtight logic in crystal clear language; otherwise the reader will continue to doubt. If you didn't write something on the paper, then (a) you didn't communicate it, (b) the reader didn't learn it, and (c) the grader has to assume you didn't know it in the first place.

2. **Tell the reader what you're proving.** The reader doesn't necessarily know or remember what "Theorem 2.13" is. Even a professor grading a stack of papers might lose track from time to time. Therefore, the statement you are proving should be on the same page as the beginning of your proof. For an exam this won't be a problem, of course, but on your homework, recopy the claim you are proving. This has the additional advantage that when you study for exams by reviewing your homework, you won't have to flip back in the notes/textbook to know what you were proving.

3. **Use English words.** Although there will usually be equations or mathematical statements in your proofs, use English sentences to connect them and display their logical relationships. If you look in your notes/textbook, you'll see that each proof consists mostly of English words.

[1] This list is an adaptation of *The Elements of Style for Proofs* written by Anders Hendrickson of St. Norbert College and modified by Dana Ernst of Northern Arizona University.

4. **Use complete sentences.** If you wrote a history essay in sentence fragments, the reader would not understand what you meant; likewise in mathematics you must use complete sentences, with verbs, to convey your logical train of thought.

   Some complete sentences can be written purely in mathematical symbols, such as equations (e.g., $a^3 = b^{-1}$), inequalities (e.g., $x < 5$), and other relations (like $5\big|10$ or $7 \in \mathbb{Z}$). These statements usually express a relationship between two mathematical *objects*, like numbers or sets. However, it is considered bad style to begin a sentence with symbols. A common phrase to use to avoid starting a sentence with mathematical symbols is "We see that..."

5. **Show the logical connections among your sentences.** Use phrases like "Therefore" or "because" or "if…, then…" or "if and only if" to connect your sentences.

6. **Know the difference between statements and objects.** A mathematical object is a *thing*, a noun, such as a group, an element, a vector space, a number, an ordered pair, etc. Objects either exist or don't exist. Statements, on the other hand, are mathematical *sentences*: they can be true or false.

   When you see or write a cluster of math symbols, be sure you know whether it's an object (e.g., "$x^2 + 3$") or a statement (e.g., "$x^2 + 3 < 7$"). One way to tell is that every mathematical statement includes a verb, such as $=$, $\leq$, "divides", etc.

7. **"=" means equals.** Don't write $A = B$ unless you mean that $A$ actually equals $B$. This rule seems obvious, but there is a great temptation to be sloppy. In calculus, for example, some people might write $f(x) = x^2 = 2x$ (which is false), when they really mean that "if $f(x) = x^2$, then $f'(x) = 2x$."

8. **Don't interchange $=$ and $\implies$.** The equals sign connects two *objects*, as in "$x^2 = b$"; the symbol "$\implies$" is an abbreviation for "implies" and connects two *statements*, as in "$ab = a \implies b = 1$." You should avoid using $\implies$ in your formal write-ups.

9. **Say exactly what you mean.** Just as the $=$ is sometimes abused, so too people sometimes write $A \in B$ when they mean $A \subseteq B$, or write $a_{ij} \in A$ when they mean that $a_{ij}$ is an entry in matrix $A$. Mathematics is a very precise language, and there is a way to say exactly what you mean; find it and use it.

10. **Don't write anything unproven.** Every statement on your paper should be something you *know* to be true. The reader expects your proof to be a series of statements, each proven by the statements that came before it. If you ever need to write something you don't yet know is true, you *must* preface it with words like "assume," "suppose," or "if" (if you are temporarily assuming it), or with words like "we need to show that" or "we claim that" (if it is your goal). Otherwise the reader will think they have missed part of your proof.

11. **Write strings of equalities (or inequalities) in the proper order.** When your reader sees something like

$$A = B \leq C = D,$$

he/she expects to understand easily why $A = B$, why $B \leq C$, and why $C = D$, and he/she expects the *point* of the entire line to be the more complicated fact that $A \leq D$. For example, if you were computing the distance $d$ of the point $(12, 5)$ from the origin, you could write

$$d = \sqrt{12^2 + 5^2} = 13.$$

In this string of equalities, the first equals sign is true by the Pythagorean theorem, the second is just arithmetic, and the *point* is that the first item equals the last item: $d = 13$.

A common error is to write strings of equations in the wrong order. For example, if you were to write "$\sqrt{12^2 + 5^2} = 13 = d$", your reader would understand the first equals sign, would be baffled as to how we know $d = 13$, and would be utterly perplexed as to why you wanted or needed to go through 13 to prove that $\sqrt{12^2 + 5^2} = d$.

12. **Avoid circularity.** Be sure that no step in your proof makes use of the conclusion!

13. **Don't write the proof backwards.** Beginning students often attempt to write "proofs" like the following, which attempts to prove that $\tan^2(x) =$

$\sec^2(x) - 1$:

$$\begin{aligned}
\tan^2(x) &= \sec^2(x) - 1 \\
\left(\frac{\sin(x)}{\cos(x)}\right)^2 &= \frac{1}{\cos^2(x)} - 1 \\
\frac{\sin^2(x)}{\cos^2(x)} &= \frac{1 - \cos^2(x)}{\cos^2(x)} \\
\sin^2(x) &= 1 - \cos^2(x) \\
\sin^2(x) + \cos^2(x) &= 1 \\
1 &= 1
\end{aligned}$$

Notice what has happened here: the student *started* with the conclusion, and deduced the true statement "$1 = 1$." In other words, he/she has proved "If $\tan^2(x) = \sec^2(x) - 1$, then $1 = 1$," which is true but highly uninteresting.

Now this isn't a bad way of *finding* a proof. Working backwards from your goal often is a good strategy *on your scratch paper*, but when it's time to *write* your proof, you have to start with the hypotheses and work to the conclusion.

14. **Be concise.** Most students err by writing their proofs too short, so that the reader can't understand their logic. It is nevertheless quite possible to be too wordy, and if you find yourself writing a full-page essay, it's probably because you don't really have a proof, but just an intuition. When you find a way to turn that intuition into a formal proof, it will be much shorter.

15. **Introduce every symbol you use.** If you use the letter "$k$," the reader should know exactly what $k$ is. Good phrases for introducing symbols include "Let $n \in \mathbb{N}$," "Let $k$ be the least integer such that...," "For every real number $a$...," and "Suppose that $X$ is a counterexample."

16. **Use appropriate quantifiers (once).** When you introduce a variable $x \in S$, it must be clear to your reader whether you mean "for all $x \in S$" or just "for some $x \in S$." If you just say something like "$y = x^2$ where $x \in S$," the word "where" doesn't indicate whether you mean "for all" or "some".

    Phrases indicating the quantifier "for all" include "Let $x \in S$"; "for all $x \in S$"; "for every $x \in S$"; "for each $x \in S$"; etc. Phrases indicating the quantifier "some" (or "there exists") include "for some $x \in S$"; "there exists an $x \in S$"; "for a suitable choice of $x \in S$"; etc.

On the other hand, don't introduce a variable more than once! Once you have said "Let $x \in S$," the letter $x$ has its meaning defined. You don't *need* to say "for all $x \in S$" again, and you definitely should *not* say "let $x \in S$" again.

17. **Use a symbol to mean only one thing.** Once you use the letter $x$ once, its meaning is fixed for the duration of your proof. You cannot use $x$ to mean anything else.

18. **Don't "prove by example."** Most problems ask you to prove that something is true "for all"—You *cannot* prove this by giving a single example, or even a hundred. Your answer will need to be a logical argument that holds for *every example there possibly could be*.

19. **Write "Let $x = \ldots$," not "Let $\cdots = x$."** When you have an existing expression, say $a^2$, and you want to give it a new, simpler name like $b$, you should write "Let $b = a^2$," which means, "Let the new symbol $b$ mean $a^2$." This convention makes it clear to the reader that $b$ is the brand-new symbol and $a^2$ is the old expression he/she already understands.

    If you were to write it backwards, saying "Let $a^2 = b$," then your startled reader would ask, "What if $a^2 \neq b$?"

20. **Make your counterexamples concrete and specific.** Proofs need to be entirely general, but counterexamples should be absolutely concrete. When you provide an example or counterexample, make it as specific as possible. For a set, for example, you must name its elements, and for a function you must give its rule. Do not say things like "$\theta$ could be one-to-one but not onto"; instead, provide an actual function $\theta$ that *is* one-to-one but not onto.

21. **Don't include examples in proofs.** Including an example very rarely adds anything to your proof. If your logic is sound, then it doesn't need an example to back it up. If your logic is bad, a dozen examples won't help it (see rule 18). There are only two valid reasons to include an example in a proof: if it is a *counterexample* disproving something, or if you are performing complicated manipulations in a general setting and the example is just to help the reader understand what you are saying.

22. **Use scratch paper.** Finding your proof will be a long, potentially messy process, full of false starts and dead ends. Do all that on scratch paper until you find a real proof, and only then break out your clean paper to write your final proof carefully. *Do not hand in your scratch work!*

Only sentences that actually contribute to your proof should be part of the proof. Do not just perform a “brain dump,” throwing everything you know onto the paper before showing the logical steps that prove the conclusion. *That is what scratch paper is for.*

# Appendix B

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