Appendix for "Instability in Generative Adversarial Imitation Learning with Deterministic Policy"

A Analysis of Exploding Gradients in DE-GAIL

A.1 Proof of Theorem 1

Theorem 1 *Let $\pi_h(\cdot|s)$ be the Gaussian stochastic policy with mean $h(s)$ and covariance $\mathbf{\Sigma}$. When the discriminator achieves its optimal $D^*(s,a)$ in Eq. (6), the gradient estimator of the policy loss with respect to the policy's parameter h satisfies $\|\hat{\nabla}_h D_{\mathrm{JS}}(\rho^{\pi_h}, \rho^{\pi_{\mathrm{E}}})\|_2 \to \infty$ with a probability of $\Pr(\|\mathbf{\Sigma}^{-1}(a_t - h(s_t))\|_2 \geq C$ for any $C > 0)$ as $\mathbf{\Sigma} \to \mathbf{0}$, where*

$$\hat{\nabla}_h D_{\mathrm{JS}}(\rho^{\pi_h}, \rho^{\pi_{\mathrm{E}}}) = \frac{d^{\pi_h}(s_t)\nabla_h \pi_h(a_t|s_t)}{2d^{\pi_{\mathrm{E}}}(s_t)\pi_{\mathrm{E}}(a_t|s_t)} \log \frac{2d^{\pi_h}(s_t)\pi_h(a_t|s_t)}{d^{\pi_h}(s_t)\pi_h(a_t|s_t) + d^{\pi_{\mathrm{E}}}(s_t)\pi_{\mathrm{E}}(a_t|s_t)},$$

and $\nabla_h \pi_h(a|s) = \pi_h(a|s)\kappa(s,\cdot)\mathbf{\Sigma}^{-1}(a - h(s))$.

Proof. Through importance sampling which transfers the learned state-action distribution to the expert demonstration distribution, the JS divergence can be rewritten from the definition in Eq. (4) as

$$\begin{aligned}
&D_{\mathrm{JS}}(\rho^{\pi_h}, \rho^{\pi_{\mathrm{E}}}) \\
&= \frac{1}{2} D_{\mathrm{KL}}(\rho^{\pi_h}, \frac{\rho^{\pi_h} + \rho^{\pi_{\mathrm{E}}}}{2}) + \frac{1}{2} D_{\mathrm{KL}}(\rho^{\pi_{\mathrm{E}}}, \frac{\rho^{\pi_h} + \rho^{\pi_{\mathrm{E}}}}{2}) \\
&= \frac{1}{2}\mathbb{E}_{(s,a)\sim\mathcal{D}}\left[\log \frac{2\rho^{\pi_h}(s,a)}{\rho^{\pi_h}(s,a) + \rho^{\pi_{\mathrm{E}}}(s,a)}\right] + \frac{1}{2}\mathbb{E}_{(s,a)\sim\mathcal{D}^\star}\left[\log \frac{2\rho^{\pi_{\mathrm{E}}}(s,a)}{\rho^{\pi_h}(s,a) + \rho^{\pi_{\mathrm{E}}}(s,a)}\right] \\
&= \frac{1}{2}\mathbb{E}_{(s,a)\sim\mathcal{D}^\star}\left[\frac{\rho^{\pi_h}(s,a)}{\rho^{\pi_{\mathrm{E}}}(s,a)} \log \frac{2\rho^{\pi_h}(s,a)}{\rho^{\pi_h}(s,a) + \rho^{\pi_{\mathrm{E}}}(s,a)} + \log \frac{2\rho^{\pi_{\mathrm{E}}}(s,a)}{\rho^{\pi_h}(s,a) + \rho^{\pi_{\mathrm{E}}}(s,a)}\right],
\end{aligned} \tag{9}$$

where $\mathcal{D}^\star$ and $\mathcal{D}$ denote the expert demonstration and the replay buffer of π_h respectively. Then we can approximate the gradient of Eq. (9) with respect to h with

$$\begin{aligned}
&\hat{\nabla}_h D_{\mathrm{JS}}(\rho^{\pi_h}, \rho^{\pi_{\mathrm{E}}}) \\
&\overset{(i)}{=} \frac{1}{2}\nabla_h\left(\frac{\rho^{\pi_h}(s_t,a_t)}{\rho^{\pi_{\mathrm{E}}}(s_t,a_t)} \log \frac{2\rho^{\pi_h}(s_t,a_t)}{\rho^{\pi_h}(s_t,a_t) + \rho^{\pi_{\mathrm{E}}}(s_t,a_t)} + \log \frac{2\rho^{\pi_{\mathrm{E}}}(s_t,a_t)}{\rho^{\pi_h}(s_t,a_t) + \rho^{\pi_{\mathrm{E}}}(s_t,a_t)}\right) \\
&\overset{(ii)}{=} \frac{1}{2}\Bigg(\frac{d^{\pi_h}(s_t)\nabla_h \pi_h(a_t|s_t)}{d^{\pi_{\mathrm{E}}}(s_t)\pi_{\mathrm{E}}(a_t|s_t)} \log \frac{2d^{\pi_h}(s_t)\pi_h(a_t|s_t)}{d^{\pi_h}(s_t)\pi_h(a_t|s_t) + d^{\pi_{\mathrm{E}}}(s_t)\pi_{\mathrm{E}}(a_t|s_t)} \\
&\quad + \frac{\rho^{\pi_h}(s_t,a_t)}{\rho^{\pi_{\mathrm{E}}}(s_t,a_t)} \cdot \frac{\rho^{\pi_h}(s_t,a_t) + \rho^{\pi_{\mathrm{E}}}(s_t,a_t)}{2\rho^{\pi_h}(s_t,a_t)} \\
&\qquad \cdot \frac{2d^{\pi_h}(s_t)\nabla_h \pi_h(a_t|s_t)\big(\rho^{\pi_h}(s_t,a_t) + \rho^{\pi_{\mathrm{E}}}(s_t,a_t)\big) - 2\rho^{\pi_h}(s_t,a_t)d^{\pi_h}(s_t)\nabla_h \pi_h(a_t|s_t)}{\big(\rho^{\pi_h}(s_t,a_t) + \rho^{\pi_{\mathrm{E}}}(s_t,a_t)\big)^2} \\
&\quad - \frac{\rho^{\pi_h}(s_t,a_t) + \rho^{\pi_{\mathrm{E}}}(s_t,a_t)}{2\rho^{\pi_{\mathrm{E}}}(s_t,a_t)} \cdot \frac{2\rho^{\pi_{\mathrm{E}}}(s_t,a_t)d^{\pi_h}(s_t)\nabla_h \pi_h(a_t|s_t)}{\big(\rho^{\pi_h}(s_t,a_t) + \rho^{\pi_{\mathrm{E}}}(s_t,a_t)\big)^2}\Bigg) \\
&\overset{(iii)}{=} \frac{1}{2}\Bigg(\frac{d^{\pi_h}(s_t)\nabla_h \pi_h(a_t|s_t)}{d^{\pi_{\mathrm{E}}}(s_t)\pi_{\mathrm{E}}(a_t|s_t)} \log \frac{2d^{\pi_h}(s_t)\pi_h(a_t|s_t)}{d^{\pi_h}(s_t)\pi_h(a_t|s_t) + d^{\pi_{\mathrm{E}}}(s_t)\pi_{\mathrm{E}}(a_t|s_t)} \\
&\quad + \frac{d^{\pi_h}(s_t)\nabla_h \pi_h(a_t|s_t)}{\rho^{\pi_h}(s_t,a_t) + \rho^{\pi_{\mathrm{E}}}(s_t,a_t)} - \frac{d^{\pi_h}(s_t)\nabla_h \pi_h(a_t|s_t)}{\rho^{\pi_h}(s_t,a_t) + \rho^{\pi_{\mathrm{E}}}(s_t,a_t)}\Bigg) \\
&\overset{(iv)}{=} \frac{d^{\pi_h}(s_t)\nabla_h \pi_h(a_t|s_t)}{2d^{\pi_{\mathrm{E}}}(s_t)\pi_{\mathrm{E}}(a_t|s_t)} \log \frac{2d^{\pi_h}(s_t)\pi_h(a_t|s_t)}{d^{\pi_h}(s_t)\pi_h(a_t|s_t) + d^{\pi_{\mathrm{E}}}(s_t)\pi_{\mathrm{E}}(a_t|s_t)},
\end{aligned} \tag{10}$$

where (ii) comes from Eq. (1). By the fact that

$$\nabla_h \pi_h(a|s) = \pi_h(a|s) \nabla_h \log \pi_h(a|s) = \pi_h(a|s) \kappa(s, \cdot) \mathbf{\Sigma}^{-1}(a - h(s)), \tag{11}$$

Eq. (10) can be shown that

$$\begin{aligned} &\|\hat{\nabla}_h D_{\mathrm{JS}}(\rho^{\pi_h}, \rho^{\pi_{\mathrm{E}}})\|_2 \\ =& \left\| \frac{d^{\pi_h}(s_t)\pi_h(a_t|s_t)\kappa(s_t, \cdot)\mathbf{\Sigma}^{-1}(a_t - h(s_t))}{2d^{\pi_{\mathrm{E}}}(s_t)\pi_{\mathrm{E}}(a_t|s_t)} \log \frac{2d^{\pi_h}(s_t)\pi_h(a_t|s_t)}{d^{\pi_h}(s_t)\pi_h(a_t|s_t) + d^{\pi_{\mathrm{E}}}(s_t)\pi_{\mathrm{E}}(a_t|s_t)} \right\|_2 . \end{aligned}$$

Then it follows that $\|\hat{\nabla}_h D_{\mathrm{JS}}(\rho^{\pi_h}, \rho^{\pi_{\mathrm{E}}})\|_2 \to \infty$ with a probability of $\Pr(\|\mathbf{\Sigma}^{-1}(a_t - h(s_t))\|_2 \geq C$ for any $C > 0)$ as $\mathbf{\Sigma} \to \mathbf{0}$. □

A.2 Proof of Corollary 1

Corollary 1 *Let $\pi_h(\cdot|s)$ be the Gaussian stochastic policy with mean $h(s)$ and covariance $\mathbf{\Sigma}$. When the discriminator achieves its regular $\tilde{D}(s,a)$ in Eq. (7), i.e., $\tilde{D}(s,a) \in (0,1)$, the gradient estimator of the policy loss with respect to the policy's parameter h satisfies*

$$\left\| \hat{\nabla}_h \left(\mathbb{E}_{\mathcal{D}^\star}[\log \tilde{D}(s,a)] + \mathbb{E}_{\mathcal{D}}[\log(1 - \tilde{D}(s,a))] \right) \right\|_2 \to \infty$$

with a probability of $\Pr(\|\mathbf{\Sigma}^{-1}(a_t - h(s_t))\|_2 \geq C$ for any $C > 0)$ as $\mathbf{\Sigma} \to \mathbf{0}$, where $\mathcal{D}^\star$ and $\mathcal{D}$ denote the expert demonstration and the replay buffer of π_h respectively,

$$\begin{aligned} &\hat{\nabla}_h \left(\mathbb{E}_{\mathcal{D}^\star}[\log(\tilde{D}(s,a))] + \mathbb{E}_{\mathcal{D}}[\log(1 - \tilde{D}(s,a))] \right) \\ =& \frac{d^{\pi_h}(s_t)\nabla_h \pi_h(a_t|s_t)}{d^{\pi_{\mathrm{E}}}(s_t)\pi_{\mathrm{E}}(a_t|s_t)} \log \frac{(1-\epsilon_2)d^{\pi_h}(s_t)\pi_h(a_t|s_t)}{(1+\epsilon_1)d^{\pi_{\mathrm{E}}}(s_t)\pi_{\mathrm{E}}(a_t|s_t) + (1-\epsilon_2)d^{\pi_h}(s_t)\pi_h(a_t|s_t)} \\ &+ \frac{(\epsilon_1+\epsilon_2)d^{\pi_h}(s_t)\nabla_h \pi_h(a_t|s_t)}{(1+\epsilon_1)\rho^{\pi_{\mathrm{E}}}(s_t,a_t) + (1-\epsilon_2)\rho^{\pi_h}(s_t,a_t)}, \end{aligned}$$

and $\nabla_h \pi_h(a|s) = \pi_h(a|s)\kappa(s,\cdot)\mathbf{\Sigma}^{-1}(a - h(s))$.

Proof. Referring to the proof strategy of Theorem 1, the learned state-action distribution can be transferred to the expert demonstration distribution by importance sampling. Thus when the discriminator achieves its regular $\tilde{D}(s,a)$, we can write the policy objective from the optimization problem in Eq. (2) as

$$\begin{aligned} &\mathbb{E}_{(s,a)\sim\mathcal{D}^\star}[\log(\tilde{D}(s,a))] + \mathbb{E}_{(s,a)\sim\mathcal{D}}[\log(1 - \tilde{D}(s,a))] \\ =& \mathbb{E}_{(s,a)\sim\mathcal{D}^\star}\left[\log \frac{(1+\epsilon_1)\rho^{\pi_{\mathrm{E}}}(s_t,a_t)}{(1+\epsilon_1)\rho^{\pi_{\mathrm{E}}}(s_t,a_t) + (1-\epsilon_2)\rho^{\pi}(s_t,a_t)}\right] \\ &+ \mathbb{E}_{(s,a)\sim\mathcal{D}}\left[\log \frac{(1-\epsilon_2)\rho^{\pi}(s_t,a_t)}{(1+\epsilon_1)\rho^{\pi_{\mathrm{E}}}(s_t,a_t) + (1-\epsilon_2)\rho^{\pi}(s_t,a_t)}\right] \\ =& \mathbb{E}_{(s,a)\sim\mathcal{D}^\star}\Bigg[\log \frac{(1+\epsilon_1)\rho^{\pi_{\mathrm{E}}}(s,a)}{(1+\epsilon_1)\rho^{\pi_{\mathrm{E}}}(s,a) + (1-\epsilon_2)\rho^{\pi_h}(s,a)} \\ &+ \frac{\rho^{\pi_h}(s,a)}{\rho^{\pi_{\mathrm{E}}}(s,a)} \log \frac{(1-\epsilon_2)\rho^{\pi_h}(s,a)}{(1+\epsilon_1)\rho^{\pi_{\mathrm{E}}}(s,a) + (1-\epsilon_2)\rho^{\pi_h}(s,a)}\Bigg]. \end{aligned} \tag{12}$$

Then the gradient of Eq. (12) can be approximated with

$$
\begin{aligned}
&\hat{\nabla}_h \left(\mathbb{E}_{(s,a)\sim\mathcal{D}^\star}[\log(\tilde{D}(s,a))] + \mathbb{E}_{\mathcal{D}}[\log(1-\tilde{D}(s,a))]\right) \\
&= \nabla_h \Bigg(\log \frac{(1+\epsilon_1)\rho^{\pi_{\mathrm{E}}}(s,a)}{(1+\epsilon_1)\rho^{\pi_{\mathrm{E}}}(s,a) + (1-\epsilon_2)\rho^{\pi_h}(s,a)} \\
&\quad + \frac{\rho^{\pi_h}(s,a)}{\rho^{\pi_{\mathrm{E}}}(s,a)} \log \frac{(1-\epsilon_2)\rho^{\pi_h}(s,a)}{(1+\epsilon_1)\rho^{\pi_{\mathrm{E}}}(s,a) + (1-\epsilon_2)\rho^{\pi_h}(s,a)} \Bigg) \\
&= -\frac{(1+\epsilon_1)\rho^{\pi_{\mathrm{E}}}(s,a) + (1-\epsilon_2)\rho^{\pi_h}(s,a)}{(1+\epsilon_1)\rho^{\pi_{\mathrm{E}}}(s_t,a_t)} \cdot \frac{(1+\epsilon_1)(1-\epsilon_2)\rho^{\pi_{\mathrm{E}}}(s_t,a_t) d^{\pi_h}(s_t)\nabla_h \pi_h(a_t|s_t)}{\big((1+\epsilon_1)\rho^{\pi_{\mathrm{E}}}(s,a) + (1-\epsilon_2)\rho^{\pi_h}(s,a)\big)^2} \\
&\quad + \frac{d^{\pi_h}(s_t)\nabla_h \pi_h(a_t|s_t)}{d^{\pi_{\mathrm{E}}}(s_t)\pi_{\mathrm{E}}(a_t|s_t)} \log \frac{(1-\epsilon_2)d^{\pi_h}(s_t)\pi_h(a_t|s_t)}{(1+\epsilon_1)d^{\pi_{\mathrm{E}}}(s_t)\pi_{\mathrm{E}}(a_t|s_t) + (1-\epsilon_2)d^{\pi_h}(s_t)\pi_h(a_t|s_t)} \\
&\quad + \frac{\rho^{\pi_h}(s_t,a_t)}{\rho^{\pi_{\mathrm{E}}}(s_t,a_t)} \cdot \frac{(1+\epsilon_1)\rho^{\pi_{\mathrm{E}}}(s_t,a_t) + (1-\epsilon_2)\rho^{\pi_h}(s_t,a_t)}{(1-\epsilon_2)\rho^{\pi_h}(s_t,a_t)} \\
&\qquad\qquad \cdot \frac{(1-\epsilon_2)(1+\epsilon_1)\rho^{\pi_{\mathrm{E}}}(s_t,a_t) d^{\pi_h}(s_t)\nabla_h \pi_h(a_t|s_t)}{\big((1+\epsilon_1)\rho^{\pi_{\mathrm{E}}}(s_t,a_t) + (1-\epsilon_2)\rho^{\pi_h}(s_t,a_t)\big)^2} \\
&= \frac{d^{\pi_h}(s_t)\nabla_h \pi_h(a_t|s_t)}{d^{\pi_{\mathrm{E}}}(s_t)\pi_{\mathrm{E}}(a_t|s_t)} \log \frac{(1-\epsilon_2)d^{\pi_h}(s_t)\pi_h(a_t|s_t)}{(1+\epsilon_1)d^{\pi_{\mathrm{E}}}(s_t)\pi_{\mathrm{E}}(a_t|s_t) + (1-\epsilon_2)d^{\pi_h}(s_t)\pi_h(a_t|s_t)} \\
&\quad - \frac{(1-\epsilon_2)d^{\pi_h}(s_t)\nabla_h \pi_h(a_t|s_t)}{(1+\epsilon_1)\rho^{\pi_{\mathrm{E}}}(s_t,a_t) + (1-\epsilon_2)\rho^{\pi_h}(s_t,a_t)} + \frac{(1+\epsilon_1)d^{\pi_h}(s_t)\nabla_h \pi_h(a_t|s_t)}{(1+\epsilon_1)\rho^{\pi_{\mathrm{E}}}(s_t,a_t) + (1-\epsilon_2)\rho^{\pi_h}(s_t,a_t)} \\
&= \frac{d^{\pi_h}(s_t)\nabla_h \pi_h(a_t|s_t)}{d^{\pi_{\mathrm{E}}}(s_t)\pi_{\mathrm{E}}(a_t|s_t)} \log \frac{(1-\epsilon_2)d^{\pi_h}(s_t)\pi_h(a_t|s_t)}{(1+\epsilon_1)d^{\pi_{\mathrm{E}}}(s_t)\pi_{\mathrm{E}}(a_t|s_t) + (1-\epsilon_2)d^{\pi_h}(s_t)\pi_h(a_t|s_t)} \\
&\quad + \frac{(\epsilon_1+\epsilon_2)d^{\pi_h}(s_t)\nabla_h \pi_h(a_t|s_t)}{(1+\epsilon_1)\rho^{\pi_{\mathrm{E}}}(s_t,a_t) + (1-\epsilon_2)\rho^{\pi_h}(s_t,a_t)}.
\end{aligned} \tag{13}
$$

Plugging Eq. (11) into Eq. (13), when $\|\mathbf{\Sigma}^{-1}(a_t - h(s_t))\|_2 \geq C$ for any $C > 0$, we have

$$\left\|\hat{\nabla}_h \left(\mathbb{E}_{(s,a)\sim\mathcal{D}^\star}[\log(\tilde{D}(s,a))] + \mathbb{E}_{\mathcal{D}}[\log(1-\tilde{D}(s,a))]\right)\right\|_2 \to \infty.$$

□

B Analysis of Relieving Exploding Gradients

B.1 Proof of Proposition 1

Proposition 1 *When the discriminator achieves its optimal $D^*(s,a)$ in Eq. (6), we have*

$$D^*(s_t,a_t) \approx 1 \Leftrightarrow h(s_t) \text{ mismatches } a_t.$$

Proof. The optimal discriminator of (s_t,a_t) can be denoted by

$$D^*(s_t,a_t) = \frac{\rho^{\pi_{\mathrm{E}}}(s_t,a_t)}{\rho^{\pi_{\mathrm{E}}}(s_t,a_t) + \rho^{\pi_h}(s_t,a_t)}.$$

We can derive that the necessary and sufficient condition of $D^*(s_t,a_t) \approx 1$ is that $\rho^{\pi_h}(s_t,a_t) \approx 0$, i.e., $(s_t,h(s_t))$ mismatches (s_t,a_t). □

B.2 Proof of Proposition 2

Proposition 2 *When the discriminator achieves its optimal $D^*(s,a)$ in Eq. (6), we have $\beta \geq \alpha$.*

Proof. When $r_i(s_t,a_t) = c,\ i = 1,2$, we obtain $\log\beta - \log(1-\beta) = -\log(1-\alpha)$, which is followed by

$$\beta - \alpha = \frac{\alpha^2 - 2\alpha + 1}{2-\alpha} \geq 0.$$

□

C Representative Reward Functions

Table 1: Representative Reward Functions.

Reward	Function Shape
$r_1(s,a)$	$-\log(1-D(s,a))$
$r_2(s,a)$	$\log D(s,a) - \log(1-D(s,a))$
$r_3(s,a)$	$\log D(s,a)$
$r_4(s,a)$	$D(s,a)$
$r_5(s,a)$	$e^{D(s,a)}$
$r_6(s,a)$	$-1/D(s,a)$
$r_7(s,a)$	$D(s,a)^2$
$r_8(s,a)$	$\sqrt{D(s,a)}$