INTEGRAL CALCULUS

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1. Definite Integral

Lemma 1.1. Let $P = \{x_0, x_1, ..., x_n\} \in P[a, b]$ and $c_1, c_2, ..., c_n \in \mathbb{R}$ such that $x_{k-1} < c_k < x_k$

- $\inf(f) \cdot \Delta x_k \leq \inf(f) \cdot (c_k - x_{k-1}) + \inf(f) \cdot (x_k - c_k)$.
- $\sup(f) \cdot \Delta x_k \leq \sup(f) \cdot (c_k - x_{k-1}) + \sup(f) \cdot (x_k - c_k)$.

Lemma 1.2. Let $P, Q \in P[a, b]$. If $P \subset Q$ then

- $L(f, P) \leq L(f, Q)$.
- $U(f, P) \geq U(f, Q)$.

Proof. Let $P = \{x_0, x_1, ..., x_n\}$ suppose for Q the following

$$a = x_0 < c_1 < x_1 < c_2 < x_2 < ... < x_{n-1} < c_n < x_n = b.$$

For each $k \in I_n$ we have

$$\inf(f) \cdot \Delta x_k \leq \inf(f) \cdot (c_k - x_{k-1}) + \inf(f) \cdot (x_k - c_k)$$

then

$$\sum_{k=1}^{n} \inf(f) \cdot \Delta x_k \leq \sum_{k=1}^{n} \inf(f) \cdot (c_k - x_{k-1}) + \sum_{k=1}^{n} \inf(f) \cdot (x_k - c_k)$$

$$L(f, P) \leq L(f, Q).$$

□

Remark. In a refinement of a partition, the lower sum is crescent and the upper sum is decrescent.

Remark. The following

$$L(f, P) \leq M \cdot (b - a), \quad \forall P \in P[a, b]$$

implies $\{L(f, P) \mid P \in P[a, b]\}$ is a set with an upper bound. Therefore it has a supremum.

Remark. The following

$$m \cdot (b - a) \leq U(f, P), \quad \forall P \in P[a, b]$$

implies $\{U(f, P) \mid P \in P[a, b]\}$ is a set with a lower bound. Therefore it has an infimum.

Definition 1.3. The lower integral of a function f is defined as

$$\underline{\int_a^b} f = \sup L(f,P), \quad P \in P[a,b].$$

Definition 1.4. The upper integral of a function f is defined as

$$\overline{\int_a^b} f = \inf U(f,P), \quad P \in P[a,b].$$

Lemma 1.5. $\forall P, Q \in P[a,b] : L(f,P) \leq U(f,Q)$.

Proof. Let $P, Q \in P[a,b]$ we know $P \cup Q \in P[a,b]$, then

$$P \subset (P \cup Q) \Rightarrow L(f,P) \leq L(f, P \cup Q)$$
$$Q \subset (P \cup Q) \Rightarrow U(f, P \cup Q) \leq U(f,Q)$$

where

$$L(f,P) \leq L(f,P \cup Q) \leq U(f, P \cup Q) \leq U(f,Q)$$

therefore

$$L(f,P) \leq U(f,Q), \quad \forall P, Q \in P[a,b].$$

□

Lemma 1.6.

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f$$

Proof. Let $Q \in P[a,b]$ then

$$L(f,P) \leq U(f,Q), \quad \forall P \in P[a,b].$$

evaluate the supremum on both sides

$$\sup L(f,P) \leq \sup U(f,Q)$$
$$\underline{\int_a^b} f \leq U(f,Q)$$

evaluate the infimum on both sides

$$\inf \underline{\int_a^b} f \leq \inf U(f,Q)$$
$$\underline{\int_a^b} f \leq \overline{\int_a^b} f$$

□

Definition 1.7. Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a,b] \to \mathbb{R}$ a bounded function. The function f is said to be Riemann integrable if and only if

$$\underline{\int_a^b} f = \overline{\int_a^b} f$$

in this case, the definite integral of f over the interval $[a, b]$ is

$$\int_a^b f = \underline{\int_a^b} f = \overline{\int_a^b} f.$$

Lemma 1.8. Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ an integrable function. Then, given $\varepsilon > 0$,

$$\exists P_1, P_2 \in P[a, b] : \int_a^b f - \varepsilon < L(f, P_1) \leq \int_a^b f \leq U(f, P_2) < \int_a^b f + \varepsilon.$$

Proof. The function f is integrable over $[a, b]$, therefore

$$\int_a^b f = \underline{\int_a^b} f = \overline{\int_a^b} f.$$

and

$$\int_a^b f = \underline{\int_a^b} f = \sup\{L(f, P_1) \mid P_1 \in P[a, b]\}$$

then

$$\forall \varepsilon > 0 : \exists P_1 \in P[a, b] \text{ such that } \int_a^b f - \varepsilon < I(f, P_1) \leq \int_a^b f.$$

Furthermore

$$\int_a^b f = \overline{\int_a^b} f = \inf\{U(f, P_2) \mid P_2 \in P[a, b]\}$$

then

$$\forall \varepsilon > 0 : \exists P_2 \in P[a, b] \text{ such that } \int_a^b f \leq I(f, P_1) < \int_a^b f + \varepsilon.$$

Therefore

$$\int_a^b f - \varepsilon < L(f, P_1) \leq \int_a^b f \leq U(f, P_2) < \int_a^b f + \varepsilon.$$

□

Lemma 1.9. Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ an integrable function. Then, given $\varepsilon > 0$,

$$\exists P \in P[a, b] : \int_a^b f - \varepsilon < L(f, P) \leq \int_a^b f \leq U(f, P) < \int_a^b f + \varepsilon.$$

Lemma 1.10. Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ an integrable function in $[a, b]$. Then, given $\varepsilon > 0$,

$$\exists P \in P[a, b] : U(f, P) - L(f, P) < \varepsilon.$$

Lemma 1.11. Let $a \in \mathbb{R}$. If $\forall \varepsilon > 0 : 0 \leq a < \varepsilon$ then $a = 0$.

Lemma 1.12. Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ a bounded function function in $[a, b]$. If `Lemma 1.10` holds for a given $\varepsilon > 0$, then f is Riemann integrable in $[a, b]$.

Proof. Given $\varepsilon > 0$, `Lemma 1.10` holds,

$$L(f, P) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq U(f, P),$$

then

$$\overline{\int_a^b} - \underline{\int_a^b} f \leq U(f, P) - L(f, P) < \varepsilon.$$

For any given $\varepsilon > 0$,

$$0 \leq \overline{\int_a^b} - \underline{\int_a^b} < \varepsilon \Rightarrow \overline{\int_a^b} = \underline{\int_a^b}.$$

$\square$

Lemma 1.13. Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ a bounded function. The following is equivalent.

- f is Riemann integrable in $[a, b]$.
- Given $\varepsilon > 0$, `Lemma 1.10` holds.

Lemma 1.14. Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ a bounded function

$$L(f, P) \leq \sum_{k=1}^{n} f(\overline{x_k}) \cdot \Delta x_k \leq U(f, P), \quad \overline{x} \in [x_{k-1}, x_k].$$

If f is integrable, then

$$\left| \int_a^b f - \sum_{k=1}^{n} f(\overline{x}) \cdot \Delta x_k \right| \leq U(f, P) - L(f, P).$$

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