L-7

The 2D Discrete Fourier tranform

1D

Continuous time FT : $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \cdot dt$, $\omega \in (-\infty, \infty)$

Discrete-time FT : $X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$, $\omega \in [0, 2\pi)$

Discrete FT : $X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k n}{N}}$, $k = 0, 1, \ldots, N-1$

* 2D Discrete forier transform :—

(for X x Y img)

1D (along column)

$$F(u,v) = \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} F[x,y] e^{-\left(j\frac{2\pi u x}{N} + j\frac{2\pi v y}{M}\right)}$$

output img (M x N)

basis fn.

Input img (M x N)

1D along row

u : 0 … N-1

v : 0 … M-1

Inv

$$F(x,y) = \frac{1}{MN} \sum_{u=0}^{N-1} \sum_{v=0}^{M-1} F[u,v] e^{j\frac{2\pi u x}{N} + j\frac{2\pi v y}{M}}$$

forier Inv

(If M = N, $\frac{1}{N^2}$, so $F[u,v] = F(u,v)$)

(2D-DFT is seperable).

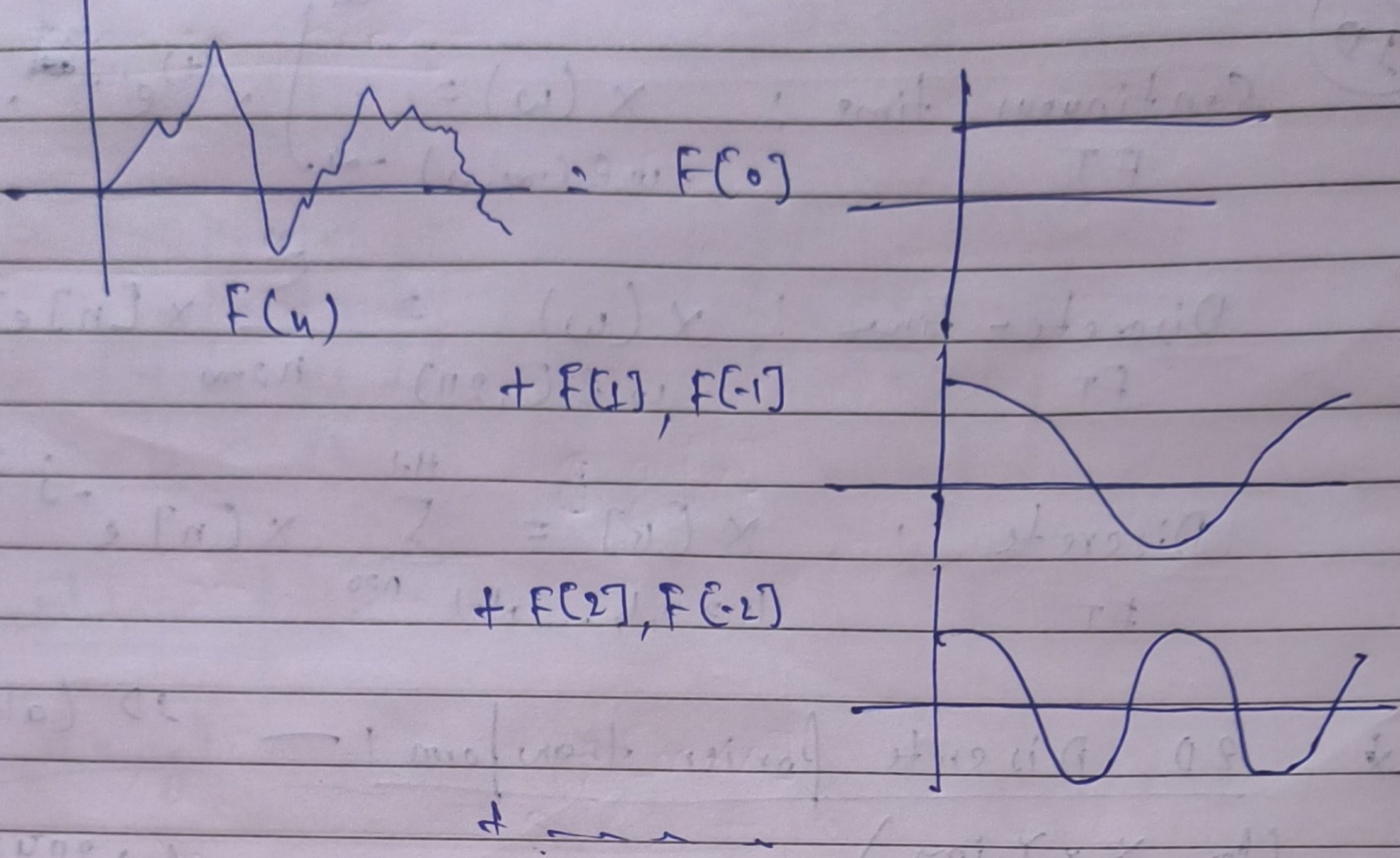
(FFT – fast forier transform)

– As with 1-D DFT, 2D DFT is like a decomposition of an image into complex exponentials (sines & cosines).

$$\boxed{e^{-j\frac{2\pi u x}{N}} = \cos\frac{2\pi u x}{N} - j\sin\frac{2\pi u x}{N}}$$

* To reduce / remove noises (striping, spots, vibration)

* IFT : $$\boxed{f(x,y) = \frac{1}{N^2}\sum_{u=0}^{M-1}\sum_{v=0}^{M-1} F(u,v)\, e^{-i2\pi\left(\frac{xu}{N}+\frac{vy}{M}\right)}}$$



* <u>Fourier transform</u> :- (time → freq.)

It decompose an image into its sine and cosine components. Result of applying a Fourier Transform is an image represented in the frequency domain rather than the spatial domain.

* <u>Discrete FT</u> :-

Used for digital images to transform them from the spatial domain to frequency domain.

* <u>Inverse DFT</u> :- Freq. → spatial.

* <u>Spatial Domain</u> :- In which the original image is represented, (Pixel intensities at specific co-ordinates)

* <u>Frequency domain</u> :- where the image is represented in terms of its freq. components.

Low freq. - general shape & structure, High freq - edges & fine details.

abs - absolute value.

(symmetry)

{forier for edge representation in freq. domain}

* FT properties :-

SHIFT

$$g(x,y) = f(x-a, y-b)$$

$$g'(u,v) = f(u,v)\,\underbrace{e^{-2\pi j\left(\frac{au}{N}+\frac{bv}{M}\right)}}_{\text{complex phase shift}}$$

$$|g'(u,v)| = |f(u,v)|$$

Scale / flip :

$$g(x,y) = a \cdot F(x,y)$$
$$g'(u,v) = a \cdot F(u,v)$$

$$g(x,y) = f(ax, by)$$
$$g'(u,v) = \frac{1}{|ab|} \cdot f\left(\frac{u}{a}, \frac{v}{b}\right)$$

(∴ if a or $b = -1$; flip spatial = flip freq.)

Rotation :

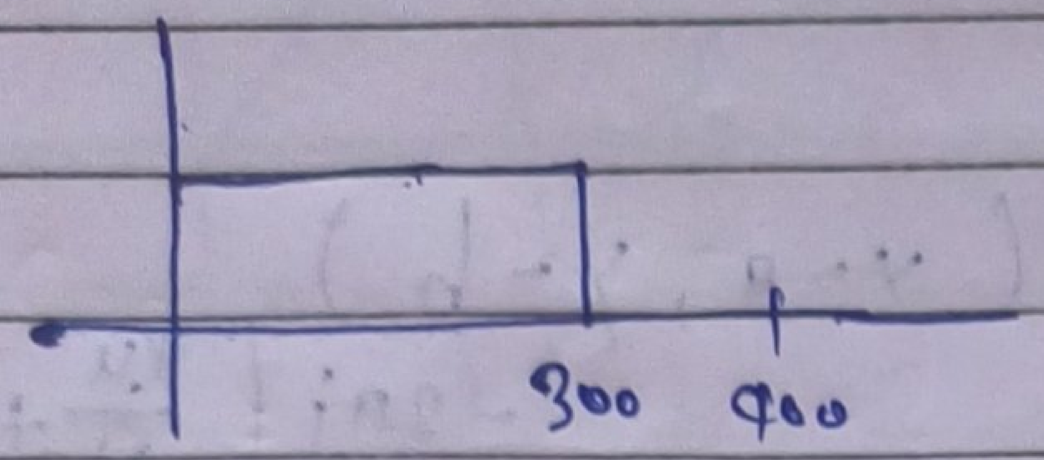
$$g(x,y) = f(x,y) \text{ ; ccw by } \theta^\circ$$
$$g'(u,v) = f(u,v) \text{ ; ccw by } \theta^\circ$$

Convolution :

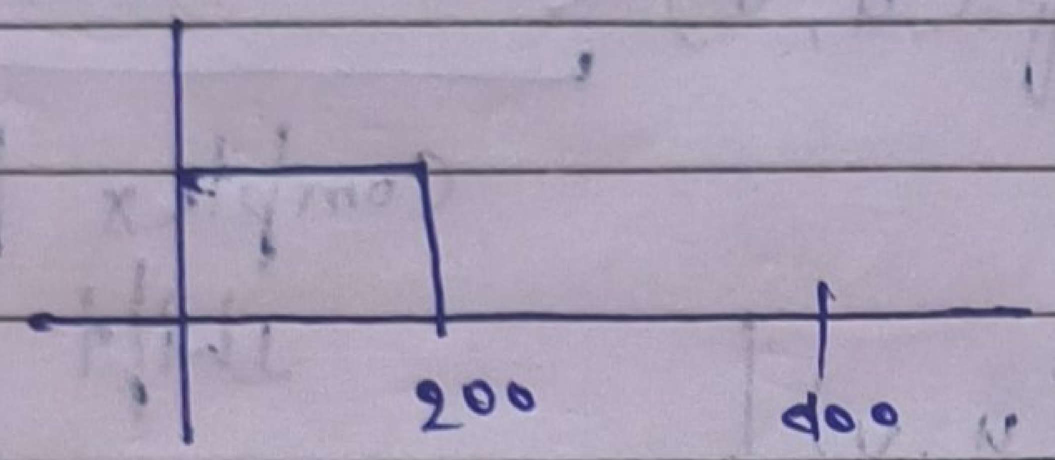
Circular conv.

$$h(x,y) = f(x,y) * g(x,y)$$
$$h'(u,v) = f(u,v) \times g'(u,v)$$

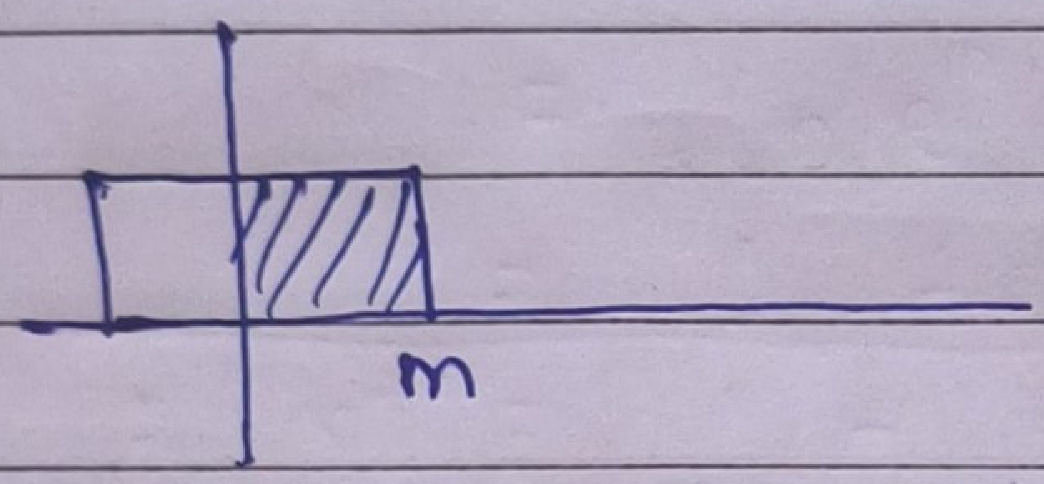
* Regular conv:-



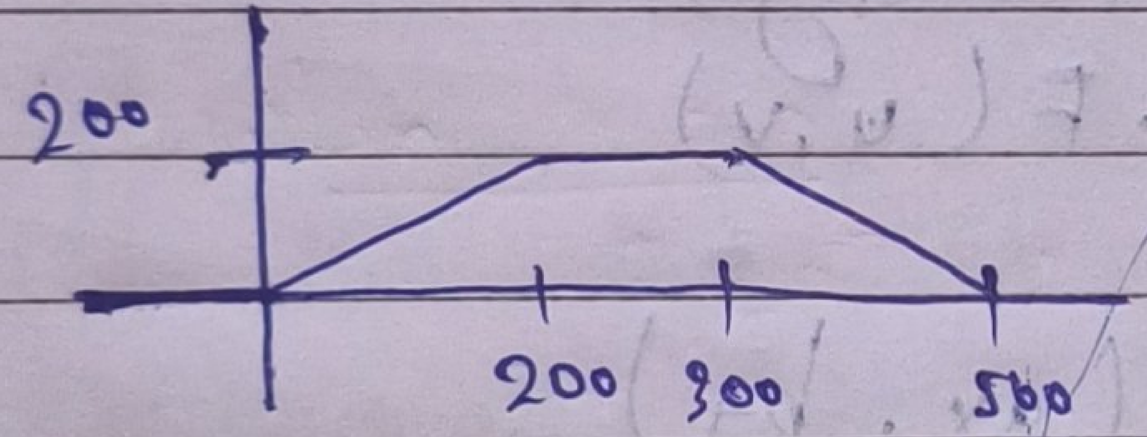
F[n]



g[n]

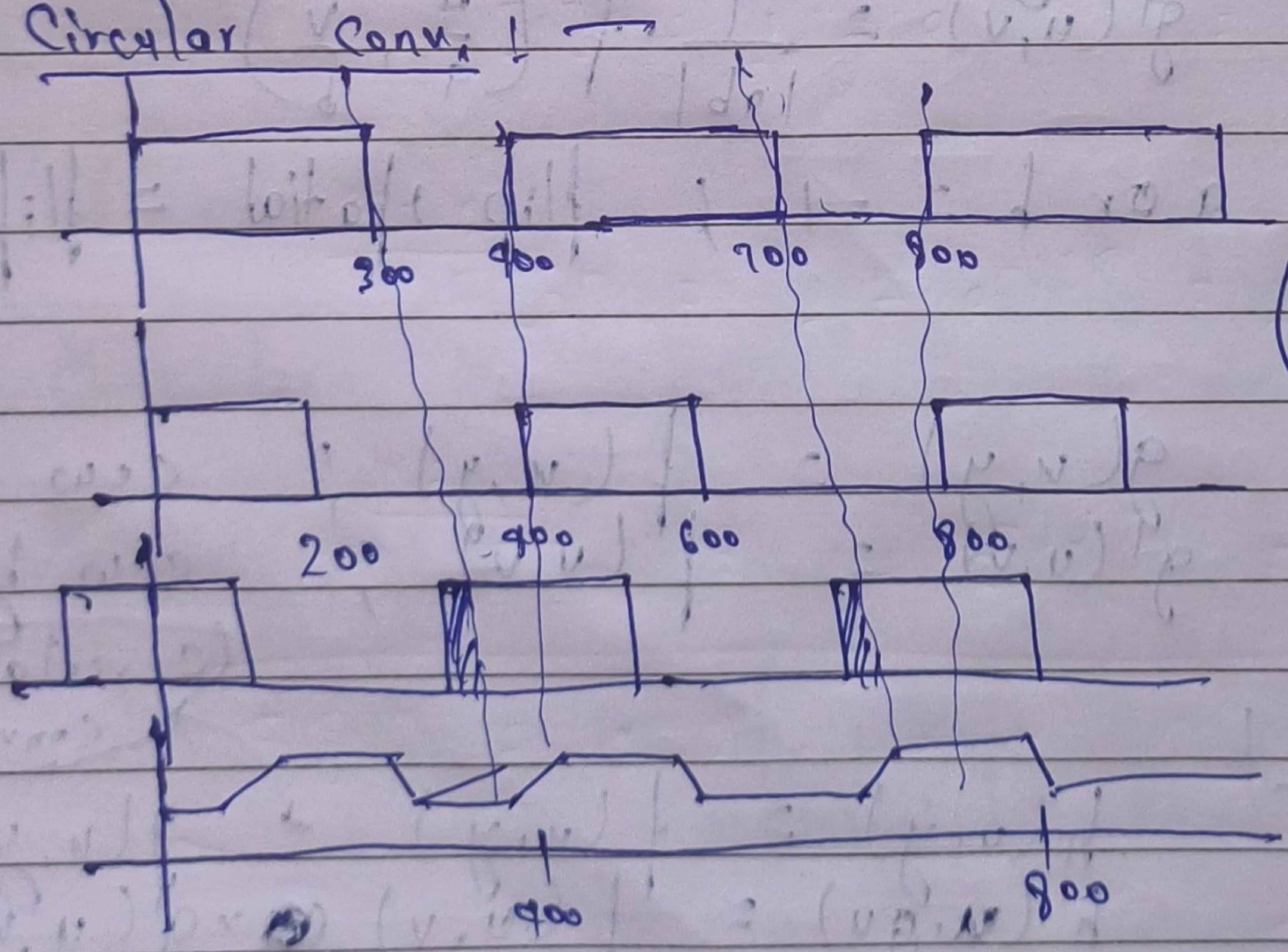


g[m-n]



f * g

* Circular conv:-



f(n) (Periodically extended)

g(n)

g(m-n)

F ⊛ g is a periodic signal itself, this may not be what we want since copies may intrude into regions we don't expect.

Important formula

$e^{j\pi} = -1$, $e^{-j\pi} = -1$, $e^{j\pi/2} = j$, $e^{-j\pi/2} = -j$, $e^{-j2\pi} = 1$

$e^{-j3\pi} = -1$, $e^{-j3\pi/2} = j$, $e^{-j9\pi/2} = -j$, $\boxed{e^{-j\theta} = \cos\theta - j\sin\theta}$

$\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$

Q1 - Compute DFT of the sequence ~~f(n) = {0, 1,~~
$f(n) = \{1, 0, 0, 1\}$; $F[k] = \sum_{n=0}^{N-1} f(n)\, e^{-j2\pi kn/N}$,
where; $k = 0, 1, \ldots, N-1$,
& $N = 4$.

2)
$$F[k] = \sum_{n=0}^{3} f(n)\, e^{-j2\pi kn/4}$$

$$= f(0)e^{0} + f(1)e^{-j2\pi k/4} + f(2)e^{-j\pi k} + f(3)e^{-j3\pi k/2}$$

$$= 1 + 0 + 0 + e^{-j3\pi k/2}$$

$$\boxed{F[k] = 1 + e^{-j3\pi k/2}}$$

when $k = 0$:
$$F[0] = 1 + e^{0} = 2$$

$k = 1$:
$$F[1] = 1 + e^{-j3\pi/2} = 1 + j$$

$k = 2$:
$$F[2] = 1 + e^{-j3\pi} = 1 - 1 = 0$$

$k = 3$:
$$F[3] = 1 + e^{-j9\pi/2} = 1 - j$$

$$\therefore \boxed{F[k] = \{2, 1+j, 0, 1-j\}}$$

* Compute 2D DFT of 4x4 gray scale img.

∴ a)

$$f(x,y) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

formula = $\boxed{F(u,v) = \text{kernel} * f(x,y) * [\text{kernel}]^T}$

DFT basis function (kernel) for N = 4. is :-

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

∴ $F(u,v) = \underbrace{\begin{bmatrix} \text{Kernel} \end{bmatrix} \begin{bmatrix} \text{img} \end{bmatrix}} \begin{bmatrix} \text{Kernel}^T \end{bmatrix}$

$$= \begin{bmatrix} 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$\boxed{F(u,v) = \begin{bmatrix} 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$