Solutions for Problem set - 2

1. Let $Z = \max(X, Y)$ and $W = \min(X, Y)$, where X and Y are arbitrary random variables. Express the joint pdf of Z and W in terms of the joint pdf of X and Y. If X and Y are independent and identically distributed (i.i.d) with

$$f_X(x) = \begin{cases} \lambda \exp(-\lambda x) & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

determine the joint pdf of Z and W.

Sol.

$$F_{Z,W}(z, w) = P[\max(X, Y) \leq z, \min(X, Y) \leq w]$$

for $z \geq w$

$$F_{Z,W}(z, w) = F_{X,Y}(z, z) - (F_{X,Y}(z, z) - F_{X,Y}(w, z) - F_{X,Y}(z, w) + F_{X,Y}(w, w))$$

$$F_{Z,W}(z, w) = \begin{cases} F_{X,Y}(w, z) + F_{X,Y}(z, w) - F_{X,Y}(w, w) & z \geq w \\ 0 & z < w \end{cases}$$

Therefore,

$$f_{Z,W}(z, w) = \begin{cases} f_{X,Y}(w, z) + f_{X,Y}(z, w) & z \geq w \\ 0 & z < w \end{cases}$$

When X and Y are i.i.d. and

$$f_X(x) = \begin{cases} \lambda \exp(-\lambda x) & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

we have

$$f_{X,Y}(x, y) = \lambda^2 \exp(-\lambda(x + y))$$

and

$$f_{Z,W}(z, w) = \begin{cases} 2\lambda^2 \exp(-\lambda(w + z)) & z \geq w \\ 0 & z < w \end{cases}$$

2. X and Y are i.i.d Gaussian random variables with zero mean and σ^2 variance. Let $Z = \sqrt{X^2 + Y^2}$ and $W = \tan^{-1}(Y/X)$. Determine the joint pdf of Z and W.

Sol.

Since X and Y are i.i.d. zero mean Gaussian random variables with variance σ^2, we have

$$f_{X,Y}(x, y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right)$$

The Jacobian for the transformation $Z = \sqrt{X^2 + Y^2}$ and $W = \tan^{-1}(Y/X)$ is

$$|J| = \begin{vmatrix} \frac{\partial}{\partial x}(z) & \frac{\partial}{\partial y}(z) \\ \frac{\partial}{\partial x}(w) & \frac{\partial}{\partial y}(w) \end{vmatrix} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} = \frac{1}{\sqrt{x^2 + y^2}}$$

The inverse transformation is $x = z\cos w$ and $y = z\sin w$

$$f_{Z,W}(z,w) = \frac{f_{X,Y}(z\cos w, z\sin w)}{|J(z\cos w, z\sin w)|}$$

Therefore, we have

$$f_{Z,W}(z,w) = \frac{z}{2\pi\sigma^2}\exp\Big(-\frac{z^2}{2\sigma^2}\Big)$$

3. An exponential random variable X has a probability density function of the form:

$$f_X(x) = \begin{cases} \lambda\exp(-\lambda x) & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the mean and variance using moment generation function (MGF).

Sol.

The moment generating function (MGF):

$$\begin{aligned} M_X(t) &= \mathbf{E}[\exp(tx)] \\ &= \int_{-\infty}^{\infty} f_X(x)\exp(tx)dx \\ &= \lambda\int_0^{\infty}\exp((t-\lambda)x)dx \\ &= \frac{\lambda}{\lambda - t} \end{aligned}$$

Mean:

$$M_X^{'}(t) = \Big[\frac{\lambda}{(\lambda-t)^2}\Big]_{t=0} = \frac{1}{\lambda}$$

Variance:

$$M_X^{"}(t) = \Big[\frac{2\lambda}{(\lambda-t)^3}\Big]_{t=0} = \frac{2}{\lambda^2}$$

$$Var[x] = \mathbf{E}[x^2] - \mathbf{E}[x]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

4. The pdf of a random variable X is given by:

$$f_X(x) = \begin{cases} \frac{1}{2^{n/2}\Gamma(n/2)}x^{\frac{n}{2}-1}\exp\left(\frac{-x}{2}\right) & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

a. Find MGF.
b. Find mean of X using MGF.
c. Find variance of X using MGF.

Sol.

The moment generating function (MGF):

$$\begin{aligned} M_X(t) &= \mathbf{E}[\exp(tx)] \\ &= \frac{1}{2^{n/2}\Gamma(n/2)}\int_0^{\infty} x^{\frac{n}{2}-1}\exp\big(-(\frac{1}{2}-t)x\big)dx \end{aligned}$$

Let $s = (\frac{1}{2}-t)x \to ds = (\frac{1}{2}-t)dx$

$$\begin{aligned} M_X(t) &= \frac{1}{2^{n/2}\Gamma(n/2)}\big(\frac{2}{1-2t}\big)^{\frac{n}{2}}\int_0^{\infty} s^{\frac{n}{2}-1}\exp\big(-s\big)ds \\ &= \frac{1}{2^{n/2}\Gamma(n/2)}\big(\frac{2}{1-2t}\big)^{\frac{n}{2}}\Gamma(n/2) \\ &= \frac{1}{(1-2t)^{\frac{n}{2}}} \end{aligned}$$

Mean:

$$M_X^{'}(t) = \Big[\frac{n}{(1-2t)^{\frac{n}{2}+1}}\Big]_{t=0} = n$$

Variance:

$$M_X^{"}(t) = \Big[\frac{n(\frac{n}{2}+1)2}{(1-2t)^{\frac{n}{2}+2}}\Big]_{t=0} = n(n+2)$$

$$Var[x] = \mathbf{E}[x^2] - \mathbf{E}[x]^2 = n(n+2) - n^2 = 2n$$

5. Determine the MGF if the pmf of a discrete random variable X is given as:

$$p_X(x) = \frac{\exp(-\lambda)\lambda^x}{x!}$$

Sol.

The moment generating function (MGF):

$$\begin{aligned} M_X(t) &= \mathbf{E}[\exp(tx)] \\ &= \exp(-\lambda)\Sigma_{x=0}\infty\frac{1}{x!}(\lambda\exp(t))^x \\ &= \exp(-\lambda)\exp(\lambda\exp(t)) \\ &= \exp(\lambda[\exp(t)-1]) \end{aligned}$$

Mean:

$$M_X^{'}(t) = \Big[\exp(\lambda[\exp(t)-1])\lambda(\exp(t))\Big]_{t=0} = \lambda$$

Variance:

$$M_X^{"}(t) = \lambda\Big[(\exp(t))\exp(\lambda[\exp(t)-1])\lambda(\exp(t)) + \exp(\lambda[\exp(t)-1])(\exp(t))\Big]_{t=0} = \lambda(\lambda+1)$$

$$Var[x] = \mathbf{E}[x^2] - \mathbf{E}[x]^2 = \lambda(\lambda+1) - \lambda^2 = \lambda$$

6. Use MGFs to determine whether $X + 2Y$ is Poisson if X and Y are i.i.d. Poisson(λ).

Sol.

The moment generating function (MGF) of Z:

$$\begin{aligned} M_Z(t) &= \mathbf{E}[\exp(tz)] \\ &= \mathbf{E}[\exp(tx)]\mathbf{E}[\exp(2ty)] \\ &= \exp(\lambda[\exp(t)-1])\exp(\lambda[\exp(2t)-1]) \\ &= \exp(\lambda[\exp(2t)+\exp(t)-2]) \end{aligned}$$

For Z to be Poisson, the MGF:

$$M_Z(t) = \exp(\mu[\exp(t)-1])$$

where $\mu > 0$ (constant, Therefore:

$$\begin{aligned} \exp(\mu[\exp(t)-1]) &= \exp(\lambda[\exp(2t)+\exp(t)-2]) \\ \mu &= \frac{\lambda[\exp(2t)+\exp(t)-2]}{\exp(t)-1} \\ \mu &= \lambda[\exp(t)+2] \end{aligned}$$

where μ depends on t. Therefore, Z is not a Poisson

7. Let X and Y be the jointly Gaussian random variables with the pdf

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}}\exp\big(-(x^2-2\rho xy+y^2)/2(1-\rho^2)\big) \qquad -\infty < x, y < \infty$$

Let $Z = \sqrt{2}\sqrt{X+Y}$ and $W = \sqrt{2}\sqrt{Y-X}$. Determine the joint pdf of Z and W.

Sol.

The joint pdf of Z and W is given as:

$$f_{Z,W}(z,w) = \frac{f_{X,Y}(\frac{z^2-w^2}{2}, \frac{z^2+w^2}{2})}{|J(\frac{z^2-w^2}{2}, \frac{z^2+w^2}{2})|}$$

where $x = \frac{z^2-w^2}{2}$ and $x = \frac{z^2+w^2}{2}$ Therefore,

$$\begin{aligned} f_{Z,W}(z,w) &= \frac{zw}{4\pi\sqrt{1-\rho^2}} \exp\left(-\frac{((\frac{z^2-w^2}{2})^2 - 2\rho(\frac{z^2-w^2}{2})(\frac{z^2+w^2}{2}) + (\frac{z^2+w^2}{2})^2)}{2(1-\rho^2)} \right) \\ &= \frac{zw}{4\pi\sqrt{1-\rho^2}} \exp\left(-(\frac{(z^2)^2}{4(1+\rho)} + \frac{(w^2)^2}{4(1-\rho)}) \right) \end{aligned}$$

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8. Suppose that X and Y are independent normal random variables and define $W = X+Y$ and $Z = X-Y$. Find the MGF of W and Z.

Sol.

The joint moment generating function (MGF):

$$\begin{aligned} M_{W,Z}(t_1,t_2) &= \mathbf{E}[\exp(t_1 w + t_2 z)] \\ &= \mathbf{E}[\exp((t_1+t_2)x)\mathbf{E}[\exp((t_1-t_2)y) \end{aligned}$$

MGF of Gaussian random variable with mean μ and variance σ^2 is given as:

$$M_X(t) = \exp\left(\mu t + \frac{t^2\sigma^2}{2}\right)$$

Therefore,

$$\begin{aligned} M_{W,Z}(t_1,t_2) &= \exp\left(\frac{(t_1+t_2)^2}{2}\right) \exp\left(\frac{(t_1-t_2)^2}{2}\right) \\ &= \exp(t_1^2)\exp(t_2^2) \end{aligned}$$

Therefore, the MGF of W and Z indicates that both Z and W are gaussian random variables 9. Let X and Y be two independent Poisson random variables with

$$\begin{aligned} P_X(k) &= \frac{1}{k!}\exp(-2)2^k \\ P_Y(k) &= \frac{1}{k!}\exp(-3)3^k \end{aligned}$$

Compute the pmf of $Z = X+Y$ usig MGF.

Sol.

The moment generating function of $Z = X+Y$ is given as:

$$M_Z(t) = M_X(t)M_Y(t)$$

Therefore,

$$\begin{aligned} M_X(t) &= \mathbf{E}[\exp(tk)] \\ &= \exp(-2)\sum_{k=0}^{\infty}\frac{1}{k!}2^k\exp(tk) \\ &= \exp(-2)\exp(2\exp(t)) \\ &= \exp(2(\exp(t-1)) \end{aligned}$$

Similarly,

$$M_Y(t) = \exp(3(\exp(t-1))$$

and

$$M_Z(t) = \exp(5(\exp(t-1))$$

Therefore pmf of Z is:

$$P_Z(k) = \frac{1}{k!}\exp(-5)5^k$$

10. Let X be a discrete random variable with the MGF given by

$$M_X(t) = \frac{1}{1-\frac{e^t}{2}} - 1$$

Find the pmf of X.

Sol.

The MGF of geometric distribution is given as:

$$M_X(t) = \frac{p\exp(t)}{1-(1-p)\exp(t)}$$

The MGF given for X discrete random variable can be written as:

$$M_X(t) = \frac{(1/2)\exp(t)}{1-(1-(1/2))\exp(t)}$$

Therefore the pmf of discrete random variable X is:

$$p_X(x) = (1-(1/2))^{x-1}(1/2)$$

11. Let $X = N(0,\sigma^2)$ and $Y = N(0,\sigma^2)$ be two random variables with pdfs:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{x^2}{2\sigma^2}\right\}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{y^2}{2\sigma^2}\right\}$$

Find the pdf for $Z = \sqrt{X^2+Y^2}$.

Sol. The joint pdf of X and Y

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)$$

The Jacobian for the transformation $Z = \sqrt{X^2+Y^2}$ and $W = \tan^{-1}(Y/X)$ is

$$|J| = \begin{vmatrix} \frac{\partial}{\partial x}(z) & \frac{\partial}{\partial y}(z) \\ \frac{\partial}{\partial x}(w) & \frac{\partial}{\partial y}(w) \end{vmatrix} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} = \frac{1}{\sqrt{x^2+y^2}}$$

The inverse transformation is $x = z\cos w$ and $y = z\sin w$

$$f_{Z,W}(z,w) = \frac{f_{X,Y}(z\cos w, z\sin w)}{|J(z\cos w, z\sin w)|}$$

Therefore, we have

$$f_{Z,W}(z,w) = \frac{z}{2\pi\sigma^2}\exp\left(-\frac{z^2}{2\sigma^2}\right)$$

Now, the marginal pdf $f_Z(z)$ is:

$$\begin{aligned} f_Z(z) &= \int_0^{2\pi} \frac{z}{2\pi\sigma^2} \exp\left(-\frac{z^2}{2\sigma^2}\right) dw \\ &= \frac{z}{\sigma^2} \exp\left(-\frac{z^2}{2\sigma^2}\right) \end{aligned}$$

12. Let $X = N(\mu_1, \sigma^2)$ and $Y = N(\mu_2, \sigma^2)$ be two random variables with pdfs:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu_1)^2}{2\sigma^2}\right\}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(y-\mu_2)^2}{2\sigma^2}\right\}$$

Find the pdf for $Z = \sqrt{X^2 + Y^2}$.

Sol. The joint pdf of X and Y

$$f_{X,Y}(x, y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{(x-\mu_1)^2 + (y-\mu_2)^2}{2\sigma^2}\right)$$

The Jacobian for the transformation $Z = \sqrt{X^2 + Y^2}$ and $W = \tan^{-1}(Y/X)$ is

$$|J| = \begin{vmatrix} \frac{\partial}{\partial x}(z) & \frac{\partial}{\partial y}(z) \\ \frac{\partial}{\partial x}(w) & \frac{\partial}{\partial y}(w) \end{vmatrix} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} = \frac{1}{\sqrt{x^2 + y^2}}$$

The inverse transformation is $x = z\cos w$ and $y = z\sin w$

$$f_{Z,W}(z, w) = \frac{f_{X,Y}(z\cos w, z\sin w)}{|J(z\cos w, z\sin w)|}$$

Therefore, we have

$$f_{Z,W}(z, w) = \frac{z}{2\pi\sigma^2} \exp\left(-\frac{\mu_1^2 + \mu_2^2}{2\sigma^2}\right) \exp\left(-\frac{z^2}{2\sigma^2}\right) \exp\left(-\frac{z(\mu_1 \cos w + \mu_2 \sin w)}{\sigma^2}\right)$$

Now, the marginal pdf $f_Z(z)$ is:

$$\begin{aligned} f_Z(z) &= \frac{z}{2\pi\sigma^2} \exp\left(-\frac{\mu_1^2 + \mu_2^2}{2\sigma^2}\right) \exp\left(-\frac{z^2}{2\sigma^2}\right) \int_0^{2\pi} \exp\left(-\frac{z(\mu_1 \cos w + \mu_2 \sin w)}{\sigma^2}\right) dw \\ &= \frac{z}{\sigma^2} \exp\left(-\frac{\mu_1^2 + \mu_2^2}{2\sigma^2}\right) \exp\left(-\frac{z^2}{2\sigma^2}\right) I_0(\frac{z\mu}{\sigma^2}) u(z) \\ &= \frac{z}{\sigma^2} \exp\left(-\frac{z^2 + \mu^2}{2\sigma^2}\right) I_0(\frac{z\mu}{\sigma^2}) u(z) \end{aligned}$$

where $I_0(\frac{z\mu}{\sigma^2})$ is the modified Bessel function of zeroth kind, and $\mu = \sqrt{\mu_1^2 + \mu_2^2}$

13. Let X_i for $i = 1, 2, \cdots, n$ be a sequence of i.i.d Gaussian random variables with zero mean and σ^2 variance. The pdf of X_i is given as:

$$f_{X_i}(x_i) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x_i^2}{2\sigma^2}\right)$$

Find the pdf for

$$Z = \sqrt{\Sigma_{i=1}^n X_i^2}$$

Solution: Please find the solution in attached steps:

- If $x\ \mathcal{N}(0,\sigma^2)$, the p.d.f of $y = x^2$, is given by $p_y(y) = \frac{1}{\sqrt{2\pi y}\sigma}\exp\left(-\frac{y}{2\sigma^2}\right)$, and the MGF of y, denoted by $M_y(t) = \frac{1}{(1-2\sigma^2 t)^{\frac{1}{2}}}$.
- Sum of k such independent r.v., i.e. $z = \sum_{\forall i} x_i^2$, MGF is given by $M_z(t) = \frac{1}{(1-2\sigma^2 t)^{\frac{k}{2}}}$. This is the MGF of a Gamma distribution with shape $\frac{k}{2}$ and scale $2\sigma^2$. The p.d.f can be written as

$$p_z(z) = \frac{1}{\Gamma(\frac{k}{2})(2\sigma^2)^{\frac{k}{2}}} z^{\frac{k}{2}-1} \exp\left(-\frac{z}{2\sigma^2}\right) u(z)$$

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- Final step is to find the p.d.f of

$$w = \sqrt{z} \implies \frac{dw}{dz} = \frac{1}{2\sqrt{z}}$$

which gives us the following p.d.f for w:

$$p_w(w) = \frac{w^{k-1} \exp\left(-\frac{w^2}{2\sigma^2}\right)}{2^{\frac{k}{2}-1}\Gamma(\frac{k}{2})\sigma^k} u(w)$$

This is a scaled chi-distribution (mind don't confuse with chi-squared; its squared root of chi-squared).

- $$https://en.wikipedia.org/wiki/Chi\_distribution$$
- $$https://en.wikipedia.org/wiki/Gamma\_distribution$$