20170010(97)
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(A1.) a.) $\lim_{z \to 0} \dfrac{Im(z)^3}{Re(z)^3}$

for limit to not exist it must be independent of path

~~and lt~~ = $\lim_{z \to 0^+}$ $\Rightarrow \boxed{\left(\lim_{x \to 0}\left(\lim_{y \to 0} (f(z))\right)\right) = \left(\lim_{y \to 0}\left(\lim_{x \to 0} (f(z))\right)\right)}$

$\Rightarrow z = x + iy$

$z^3 = (x)^3 + 3(x)^2(iy) + 3(iy)^2(x) + (iy)^3$

$= x^3 - iy^3 + 3x^2 iy - 3y^2 x$

$= \underbrace{x^3 - 3y^2 x}_{Re(z^3)} + i\underbrace{(3x^2 y - y^3)}_{im(z^3)}$

$\Rightarrow \lim_{z \to 0} \dfrac{im(z^3)}{re(z^3)} = \lim_{x \to 0 || y \to 0} \left(\dfrac{3x^2 y - y^3}{x^3 - 3y^2 x}\right)$

Path 1)

$\left(\lim_{x \to 0}\left(\lim_{y \to 0} f(z)\right)\right)$

$\Rightarrow \lim_{x \to 0}\left(\lim_{y \to 0}\left(\dfrac{3x^2 y - y^3}{x^3 - 3y^2 x}\right)\right)$

$\lim_{x \to 0}\left(\dfrac{3x^2(0) - (0)^3}{x^3 - 3(0)(x)}\right) = \underline{\underline{0}}$

Path 2)

$\left(\lim_{y \to 0}\left(\lim_{x \to 0} f(z)\right)\right)$

$\Rightarrow \lim_{y \to 0}\left(\lim_{x \to 0}\left(\dfrac{3x^2 y - y^3}{x^3 - 3y^2 x}\right)\right)$

$\lim_{y \to 0} \dfrac{3(0)^2 y - y^3}{(0)^3 - 3y^2(0)} = \underline{\underline{\infty}}$

$\therefore$ Given ~~limit~~ function is path dependent $\Rightarrow$ Limit doesn't exist

(A1.(b)) $\lim_{z \to i} \frac{z^2}{z+i} = \frac{(x+iy)^2}{(x+i(y+1))} = \frac{x^2+2ixy-y^2}{x+i(y+1)}$

$\Rightarrow z = x+iy$

for $z = i \Rightarrow \boxed{\begin{array}{l} x = 0 \\ y = -1 \end{array}}$

**path 1**

$\lim_{x\to 0}\left(\lim_{y\to -1} f(z)\right)$

$\lim_{x\to 0}\left(\lim_{y\to -1}\left(\frac{x^2+2ixy-y^2}{x+i(y+1)}\right)\right)$

$\lim_{x\to 0}\left(\frac{x^2+2ix(-1)-(-1)^2}{x+i\cancel{((-1)+1)}^{\,0}}\right)$

$\lim_{x\to 0}\left(\frac{x^2-2ix-1}{x}\right) = \infty$

**path 2**

$\lim_{y\to -1}\left(\lim_{x\to 0}(f(z))\right)$

$\lim_{y\to -1}\left(\lim_{x\to 0}\left(\frac{x^2+2ixy-y^2}{x+i(y+1)}\right)\right)$

$\lim_{y\to -1}\left(\frac{(0)^2+2i(0)y-(y)^2}{0+i(y+1)}\right)$

$\lim_{y\to -1}\left(\frac{-y^2}{i(y+1)}\right) = \frac{-(-1)^2}{i(-1+1)} = -\infty$

$\therefore$ Since $\lim_{\text{path 1}} \neq \lim_{\text{path 2}}$, limit doesn't exist.

(A1.(c)) $\lim_{z\to 0} \frac{re(z)^2}{im(z)} = \lim_{z\to 0} \frac{x^2-y^2}{y}$

$z \to 0 = (x\to 0 \,||\, y \to 0)$

**path (1)**

$\lim_{x\to 0}\left(\lim_{y\to 0}\left(\frac{x^2-y^2}{y}\right)\right)$

$= \infty$

**path (2)**

$\lim_{y\to 0}\left(\lim_{x\to 0}\left(\frac{x^2-y^2}{y}\right)\right)$

$= 0$

(c)) Continuation :

$\therefore$ Since $\underset{path 1}{lt} \neq \underset{path 2}{lt} \Rightarrow$ limit doesnt exist.

(A1.(d))
$$\lim_{z\to 0} \frac{z}{(\bar{z})^2}$$

if $z = x+iy$

$\Rightarrow \bar{z} = x-iy$

$$\Rightarrow \frac{z}{(\bar{z})^2} = \frac{x+iy}{x^2-y^2-2ixy}$$

path ①

$$\lim_{x\to 0}\left(\lim_{y\to 0}\left(\frac{x+iy}{x^2-y^2-2ixy}\right)\right)$$

$$\lim_{x\to 0}\left(\frac{x+i(0)}{x^2-(0)^2-2ix(0)}\right)$$

$$= \infty$$

path ②

$$\lim_{y\to 0}\left(\lim_{x\to 0}\left(\frac{x+iy}{x^2-y^2-2ixy}\right)\right)$$

$$\lim_{y\to 0}\left(\frac{0+iy}{(0)^2-y^2-2i(0)y}\right)$$

$$= -\infty$$

$\therefore$ Since, $\underset{path 1}{lt} \neq \underset{path 2}{lt} \Rightarrow$ the limit doesn't exist.

(A2.(a))
$$\lim_{z\to 0} \frac{re(z)^2}{|z|}$$

if $z = x-iy$ and $z\to 0 \Rightarrow (x\to 0 \,||\, y\to 0)$

$|z| = \sqrt{x^2+y^2}$ and $re(z^2) = x^2-y^2$

$$\Rightarrow \cancel{\lim_{z\to 0}} \Rightarrow \lim_{z\to 0} \frac{x^2-y^2}{\sqrt{x^2+y^2}}$$

( $\because$ limit must be path independent if $y = mx$ the limit must be independ of "m")

$$\lim_{x\to 0} \frac{x^2-(mx)^2}{\sqrt{x^2+(mx)^2}} = \lim_{x\to 0} \frac{x^2(1-m^2)}{|x|\sqrt{1+m^2}}$$

$$= 0$$

$\therefore \lim_{z\to 0} \frac{Re(z^2)}{|z|} = 0$

(A2.(b)) $\lim_{z\to 1+i} \frac{z\, z^3}{im(z)^2}$

if $z = x+iy$

$\Rightarrow z^3 = x^3 + 3(x)(iy)^2 + 3x^2(iy) + (iy)^3$

$z^3 = x^3 - 3xy^2 + 3x^2iy - y^3$

$im\, z^2 = 2xy$

let $y = mx$

$\Rightarrow \lim_{\substack{x\to 1\\ y=mx}} \left(\frac{x^3-3xy^2+3x^2iy-y^3}{2xy}\right)$

$\lim_{x\to 1} \left(\frac{x^3-3x(mx)^2+3x^2i(mx)-(mx)^3}{2x(mx)}\right)$

$\lim_{x\to 1} \left(\frac{x^3-3mx^3+3ix^3m-m^3x^3}{2x^2m}\right)$

$\lim_{x\to 1} (x-3mx+3ixm-m^3x)$

$= 1-3m+3im-m^3$

$\therefore$ lt doesn't exist since it is path ~~dependend~~ dependent

2.(c))

$$\lim_{z \to 0} \frac{z^2+6z+3}{z^2+2z+2}$$

$$\lim_{z \to 0} \frac{z^2+6z+3}{z^2+2z+2}$$

$$\Rightarrow \frac{(0)^2+6(0)+3}{(0)^2+2(0)+2} = \frac{3}{2}$$

(A3.(a))

Method 1: if $f(z)$ is analytic

$\Rightarrow$ $f(z)$ must be continuous everywhere

$\Rightarrow$ To prove $\cos(z)$ is analytic

$$\cos(x+iy) = \cos(x)\cos(iy) + \sin(x).\sin(iy)$$

since $\cos(ix) = \cosh(x)$

and $\sin(ix) = i\sinh(x)$

$$\Rightarrow \cos(z) = \cos(x).\cosh(y) - i\sin(x).\sinh(y)$$

apply C.R conditions

$$\frac{\partial u}{\partial x} = -\sin x \cosh(y) \qquad \frac{\partial v}{\partial x} = -(\cos x.\sinh y)$$

$$\frac{\partial u}{\partial y} = \cos x.\sinh y \qquad \frac{\partial v}{\partial y} = -(\sin x.\cosh y)$$

as $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$ and all are continuous the $f(z) = \cos(z)$ is analytic $\Rightarrow$ it must be continuous.

(A2(b)) $\lim_{z\to 1+i} \frac{2z^3}{\text{im}(z^2)}$

$z \to i+1 \therefore x \to 1$ and $y \to 1$

$$\lim_{z\to 1+i} \frac{2z^3}{\text{im}(z^2)} = \lim_{\substack{x\to 1\\ y\to 1}} \frac{2(x+iy)^3}{\text{im}(x+iy)^2}$$

$$\Rightarrow \lim_{\substack{x\to 1\\ y\to 1}} \frac{2(x^3 - iy^3 + 3x^2 iy - 3y^2 x)}{\text{im}(x^2 - y^2 + 2yx\,i)}$$

1 $\Rightarrow \lim_{x\to 1}\left(\lim_{y\to 1}\right. = \text{path } 1$

$$\Rightarrow \lim_{x\to 1}\left(\lim_{y\to 1} \frac{2(x^3 - 3y^2 x + i(3x^2y - y^3))}{2xy}\right)$$

$$\Rightarrow \lim_{x\to 1}\left(\frac{2(x^3 - 3x + i(3x^2 - 1))}{2x}\right)$$

$$\Rightarrow \frac{2(1 - 3(1) + i(3(1) - 1)))}{1}$$

$$\Rightarrow 2i - 2$$

2 $\Rightarrow \lim_{y\to 1}\left(\lim_{x\to 1}\right. = \text{path } 2$

$$\lim_{y\to 1}\left(\lim_{x\to 1} \frac{2(x^3 - 3xy^2 + i(3x^2y - y^3))}{2xy}\right)$$

$$\lim_{y\to 1}\left(\frac{2(1 - 3y^2 + i(3y - y^3))}{2y}\right)$$

$$\Rightarrow \frac{2(1 - 3(1)^2 + i(3(1) - (1)^3))}{2(1)} = 2i - 2$$

$\therefore \lim_{\text{path } 1} = \lim_{\text{path } 2} \Rightarrow \boxed{\text{limit} = 2i - 2}$

Method 2

$\cos(z)$

$\Rightarrow \cos(x+iy) = \cos x \cdot \cosh(y) - i \sin x \cdot \sinh y$

Continuous

Since all functions are continuous everywhere the linear combination of them in any order must be continuous.

(A.3 (b))

$$f(z) = \exp(2z) = e^{2z} = e^{(x+iy)\cdot 2}$$

$$= e^{2x} \cdot e^{2iy}$$

$$\lim_{z\to z_0} e^{2z} \Rightarrow \lim_{\substack{x\to x_0 \\ y \to y_0}} e^{2x} \cdot e^{2iy} \quad \left(\begin{array}{l}\text{where } x_0, y_0 \\ \text{are } \in \mathbb{R}\end{array}\right)$$

Prove that they are independent of path.

$\Rightarrow$

| Path 1 | Path 2 |
|---|---|
| $\lim_{x\to x_0}\left(\lim_{y\to y_0} (e^{2x} \cdot e^{2iy})\right)$ | $\lim_{y\to y_0}\left(\lim_{x\to x_0} (e^{2x} \cdot e^{2iy})\right)$ |
| $\Rightarrow e^{2x_0} \cdot e^{2iy_0}$ | $\Rightarrow e^{2x_0} \cdot e^{2iy_0}$ |

$\therefore$ Since $\lim_{\text{path 1}} = \lim_{\text{path 2}}$ $\left[\begin{array}{l}\text{the function is continuous} \\ \text{everywhere for } x_0, y_0 \in \mathbb{R}\end{array}\right]$

(A4.) Given:
$\sin(x)\cosh(y) + i\cos x \cdot \sinh(y)$

In the sum of $u+iv$ ($\because$ compare the co-efficients)

$\therefore u = \sin x \cosh(y)$
$v = \cos x \cdot \sinh(y)$

Check for C-R i.e $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

where all are continuous functions.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial(\sin x \cdot \cosh y)}{\partial x} \Rightarrow \cos x \cdot \cosh y$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(\sin x \cosh y) \Rightarrow \sin x \cdot \sinh y$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(\cos x \cdot \sinh y) \Rightarrow \cos x \cdot \cosh y$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(\cos x \cdot \sinh y) \Rightarrow -\sin x \cdot \sinh y$$

$\therefore \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$ is satisfied also

$\boxed{\frac{\partial v}{\partial y} = -\frac{\partial v}{\partial x}}$ is also satisfied

$\therefore$ above function is analytic.

$$\oint_C \frac{\cos \pi z^2}{(z-1)(z-2)} \cdot dz \quad , \quad C \to |z| = 3$$

The integrand $\oint_C \frac{\cos \pi z^2}{(z-1)(z-2)}$ has poles at $z=1, z=2$

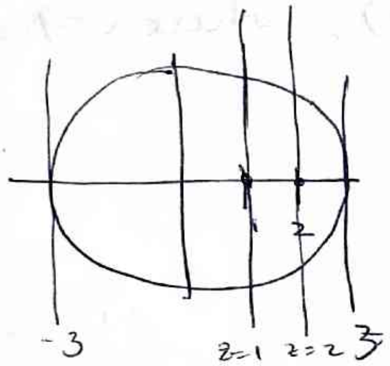
$(\because (z-1)(z-2)=0)$

The circle $|z|=3$ encloses $z=1$ and $z=2$

let $C = C_1 + C_2$

$$\oint_C \frac{\cos \pi z^2}{(z-1)(z-2)} \cdot dz = \int_{C_1} \frac{\cos \pi z^2 . dz}{(z-1)(z-2)} + \int_{C_2} \frac{\cos \pi z^2}{(z-1)(z-2)} \cdot dz$$

— (1) — (2)




Cauchy - Integral formula

$$f^n(a) = \frac{n!}{2\pi i} \int_C \frac{f(z) \cdot dz}{(z-a)^{n+1}}$$

$$\oint_C \frac{\cos \pi z^2}{(z-1)(z-2)} = \int_{C_1} \frac{\cos \pi z^2}{(z-1)(z-2)} + \int_{C_2} \frac{\cos \pi z^2}{(z-2)(z-1)}$$

$$\Rightarrow 2\pi i \left[\frac{\cos \pi z^2}{(z-2)}\right]_{z=1} + 2\pi i \left[\frac{\cos \pi z^2}{z-1}\right]_{z=2}$$

$$\Rightarrow 2\pi i \left[\frac{\cos \pi}{-1}\right] + 2\pi i \left[\frac{\cos 2\pi}{1}\right]$$

$$\Rightarrow \underline{\underline{4\pi i}}$$

$$\therefore \int_C \frac{f(z)\cdot dz}{(z-a)^{n+1}} = \frac{2\pi i}{n!} f^{n}(a)$$

for $\displaystyle\int_{C_1} \frac{\cos \pi z^2}{(z-1)\cdot(z-2)}$ $\displaystyle\int_C \frac{f(z)\cdot dz}{(z-a)^{n+1}}$

$f(z) = \dfrac{\cos \pi z^2}{z-2}$ and $n=0,\ a=1$

$\therefore \displaystyle\int \frac{\cos \pi z^2}{(z-1)(z-2)} \Rightarrow \frac{2\pi i}{1} f(a)$, where $a=1$

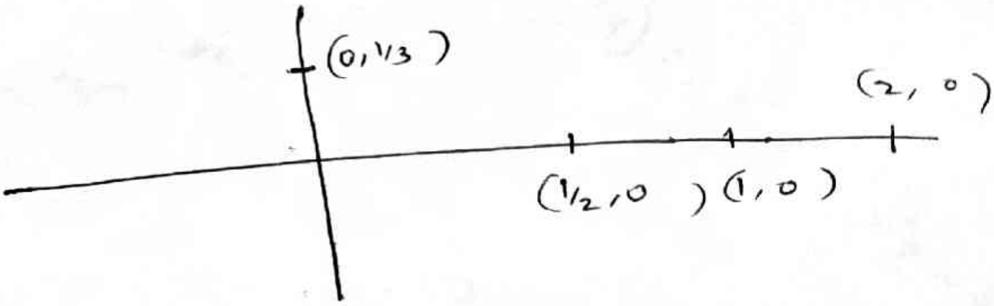
similarly for eq 2

$$\int_{C_2} \frac{\cos \pi z^2}{(z-2)} \cong \int \frac{f(z)}{(z-a)^{n+1}}$$

$f(z) = \dfrac{\cos \pi z^2}{z-1}$ & $n=0,\ a=2$

$\therefore \displaystyle\int_{C_2} \frac{\cos \pi z^2}{(z-1)(z-2)} \Rightarrow 2\pi i\ f(a)$ where $a=2$.

$4x^2 + 9y^2 = 1$




$$\frac{x^2}{(1/4)} + \frac{y^2}{(1/9)} = 1$$

$$\frac{x^2}{(1/2)^2} + \frac{y^2}{(1/3)^2} = 1$$

Poles are determine by putting the denominator equal to zero

$$z^2 - 3z + 2 \Rightarrow z^2 - 2z - z + 2$$

$$\Rightarrow z(z-2) - 1(z-2) = 0$$

$$z = 1, z = 2$$

They lie i.e poles outside the ecllipce So by Cauchy Integral theorem value is zero

$$\therefore \oint_C \frac{z^3+z+1}{z^3-3z+2} \, dz = 0$$