SVCDE by Yiding 2024.11.17

Partial Derivatives :

1. **Level curves** are the curves with equations $f(x, y) = k$ where k is a constant
2. **METHOD - Limit of $f(x, y)$**: The limit $\lim_{(x,y)\to(a,b)} f(x, y) = L$ exist when all pass to (a, b) give the same result. some useful path: $x = 0$, $y = 0$, $y = x$, $y = x^2$
3. **Clairaut's Theorem:** Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$
4. **METHOD - Tangent Plane**: $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.
5. **Chain Rule 2**: Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then $\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$ and $\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$
6. **Chain Rule 1**: $\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$
7. **The directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = (a, b)$ is $D_{\mathbf{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u} = f_x(x_0, y_0)a + f_y(x_0, y_0)b$ if this limit exists.
8. **Gradient vector** $\nabla f(x, y) = f_x\mathbf{i} + f_y\mathbf{j}$
9. **METHOD - Second derivative test to find the property of a critical point ($\mathbb{R}_2$)**:

$$D_{2,2} = \begin{vmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2$$

if at point (a, b), $f_x(a, b) = 0$ and $f_y(a, b) = 0$ then (a, b) is a critical point that:
- If $D_{2,2} > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
- If $D_{2,2} > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
- If $D_{2,2} < 0$ then $f(a, b)$ is not a local maximum or minimum. (Saddle Point)

10. **Critical Point($\mathbb{R}_3$)**
-If $D_{1,1} > 0$, $D_{2,2} > 0$, $D_{3,3} > 0$, then $f(a, b, c)$ is a local minimum.
-If $D_{1,1} < 0$, $D_{2,2} > 0$, $D_{3,3} < 0$, then $f(a, b, c)$ is a local maximum.
-If $D_{3,3} \neq 0$ and (a, b, c) is neither a local minimum nor a local maximum, then (a, b, c) is a saddle point.
-If $D_{3,3} = 0$, then the test is inconclusive; $f(a, b, c)$ could be a local maximum or minimum, or (a, b, c) could be a saddle point.
11. **METHOD : Find ABSOLUTE maximum or minimum**:
- find all the value of critical point
- find extreme value on boundary
- compare them

12. **METHOD: Lagrange multipliers**:
$\nabla f(x, y, z) = \lambda \nabla g(x, y, z))$
$g(x, y, z) = k$ (constraint)

Multiple integral :

1. **Polar Coordinates**:
$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$
2. **Cylinder Coordinates**:
$$\iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$
3. **Spherical Coordinates**:
$$\iiint_E f(x, y, z)\, dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$
4. **Jacobian**:
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$
5. **Change of variables**:
$$\iint_R f(x, y)\, dA = \iint_S f(x(u, v), y(u, v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$$

Vector Calculus

Line Integral :

1. **Line Integral with respect to Arc length**:
$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$
2. **Line Integral with respect to x or y**:
$\int_C f(x, y)\, dx = \int_a^b f(x(t), y(t)) \frac{dx}{dt}\, dt$
$\int_C f(x, y)\, dy = \int_a^b f(x(t), y(t)) \frac{dy}{dt}\, dt$
3. **Line segment start at $\mathbf{r}_0$ and end at $\mathbf{r}_1$**:
$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1$,
4. **Line segment of Vector Field**:
$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$
5. **Green's Theorem**:
$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

surface integral :

1. **Surface Integralf of f over the surface S**:
$$\iint_S f(x, y, z)\, dS = \iint_D f(\mathbf{r}(u, v))\, |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$
2. **Surface Integral of Vector Field**:
$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$
$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA,\ g(x, y) = z$$
3. **Stoke's Theorem**:
$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$
4. **Divergence Theorem**:
$$\iiint_E \nabla \cdot \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$
Where $\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$

Parametric Surface :

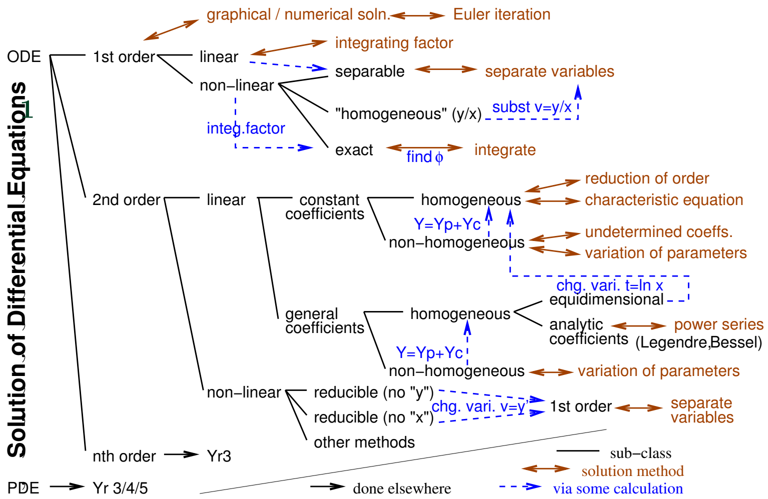
1. **Sphere(a as a constant)**:
$x = a\sin\phi\cos\theta, y = a\sin\phi\sin\theta, z = a\cos\phi$
$\mathbf{r}_\phi \times \mathbf{r}_\theta = a^2\sin^2\phi\cos\theta\,\mathbf{i} + a^2\sin^2\phi\sin\theta\,\mathbf{j} + a^2\sin\phi\cos\phi\,\mathbf{k}$
$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = a^2\sin\phi$
2. **surface area**:
$\iint_S dS = \iint_D |\mathbf{r}_x \times \mathbf{r}_y|\, dA$; $\iint_S dS = \iint_D \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2}\, dA$, when $z = g(x, y)$

Definitions :

$F = \mathbf{P}\,\mathbf{i} + \mathbf{Q}\,\mathbf{j} + \mathbf{R}\,\mathbf{k}$
1. Curl of F:
curl $\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$

If F is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $F = 0$, then F is a conservative vector field

2. Div of F: div $\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$
3. F, Vector Field on$\mathbb{R}^3$, has continuous second order derivative, then: div curl $F = 0$
4. Laplace operator: $\nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$



,
,
,

Figure 1: An example figure.

Classification of ODE

First Order ODE :

Standard Form : $\frac{dy}{dx} + P(x)y = Q(x)$

1. **graphic method is plot the direction field**
2. **numerical method/Euler's Method**:

$y_1 = y_0 + f(t_0, y-0)h$, h is the step

3. **If Linear?**

- Separable?

- integrating factor:

$e^{\int P(x)dx}[\frac{dy}{dx} + P(x)y] = e^{\int P(x)dx}Q(x)$

4 **non-Linear?**

- Separable?

- homogeneous

Can be express as $\frac{y}{x}$ only, let $v(x) = \frac{y}{x}$,
$y = v(x)x$ sub in y' and separate the variable

- Exact equation : $M(x,y) + N(x,y)\frac{dy}{dx} = 0$

ODE is exact on R if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

there exist $\psi(x,y)$ such that $\frac{\partial \psi}{\partial x} = M$ and $\frac{\partial \psi}{\partial y} = N$

Solve it and get $\psi(x,y) = C$ as solution

From $M = \psi(x,y)_x$, we get $\psi(x,y) = some(x,y) + h(y)$ and compare it with $\psi(x,y)_y = some(x,y)_y + h(y)'$ with N

- Integrating factor to make equation exact:

For $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ if it is not exact:

Find some $\mu(x)$ that $\mu M(x,y) + \mu N(x,y)\frac{dy}{dx} = 0$ is exact:

$\frac{d\mu}{dx} = \frac{M_y - N_x}{N \cdot \mu}$

solve the separable ODE and solve the original equation with Exact method.

Uniqueness IVP(initial value problem)

1. **non Linear**

- the range should include the initial value
- not a general solution

Second Order ODE :

CC := Constant Coefficient

NCC := Non Constant Coefficient

Standard Form:

$P(t)\frac{d^2y}{dt^2} + Q(t)\frac{dy}{dt} + R(t)y = G(t)$;

provided that $P(t) \neq 0$, it may be expressed as

$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t)$.

Linear :

1. Suppose that y_1, y_2 are 2 linearly independent solutions to the homogeneous ODE $y''(t) + P(t)y'(t) + Q(t)y(t) = 0$, where p, q are continuous, Then every solutions to the ODE has the form(general solution) $y = C_1y_1(t) + C_2y_2(t)$, where C_1, C_2 are constants.
2. $y_1(t), y_2(t)$ are linearly independent iff
$W(y_1(t), y_2(t)) = y_1 \cdot y_2' - y_1' \cdot y_2$
3. y_1, y_2: fundamental set of solutions

- Caracteristic Equation (CC, Homogeneous):

$ay''(t) + by'(t) + cy(t) = 0$, $ar^2 + br + c = 0$

if $\Delta > 0$: $y(t) = C_1exp(r_1t) + C_2exp(r_2t)$

if $\Delta = 0$: **- Method of Reduction of Order**:

When r has two same solutions, let $y_2(t) = v(t) \cdot y_1(t)$, find y_2' and y_2'' sub in

$y(t) = C_1exp(r_1t) + C_2t \cdot exp(r_1t)$

if $\Delta < 0$: $r = \alpha \pm i\beta$:

$y(t) = e^{\alpha t}(C_m \cos(\beta t) + C_N \sin(\beta t))$

- Complementary Equation (CC, Non homogeneous)

$ay''(t) + by'(t) + cy(t) = g(t)$

– Find the general soln y_c of the corresponding homogeneous ODE

– find any particular soln y_p to the non homogeneous ODE;

– then add y_c and y_p together.

- How to find y_p particular solution

- Method of Undetermined Coefficients(CC, Special Cases):

Suppose $r_1, andr_2$ are two solution to the characteristic equation

- Case1: $g(t) = Me^{kt}$

if $k \neq r_1 andk \neq r_2$, $y_p = Ce^{kt}$

if k equals one of them, $y_p = Cte^{kt}$

if $k = r_1 = r_2$, $y_p = Ct^2e^{kt}$

- Case2: $M\cos kt + N\sin kt$:

compare k and β: $y_p(t) = C\cos kt + D\sin kt$ or

$y_p(t) = t(C\cos kt + D\sin kt)$

- Case3: $g(t) = a_nt^n + ...a_0$ if 0 is not, one, two of solution of characteristic equation:

$y_p(t) = b_nt^n + ... + a_0$ or

$y_p(t) = t(b_nt^n + ... + a_0)$ or

$y_p(t) = t^2(b_nt^n + ... + a_0)$

- Combination of Case 1,2,3:

P, A, B are polynomial with different coefficients

if $g(t) = P_n(t)$, try $y_p(t) = t^sA_n(t)$

if $g(t) = P_n(t)e^{mt}$, try $y_p(t) = t^sA_n(t)e^{mt}$

if $g(t) = P_n(t)e^{mt}\cos kt$, try $y_p(t) = t^se^{mt}\big[A_n(t)\cos kt + B_n(t)\sin kt\big]$

if $g(t) = P_n(t)e^{mt}\sin kt$, try $y_p(t) = t^se^{mt}\big[A_n(t)\cos kt + B_n(t)\sin kt\big]$

- Method of variation of parameters(CC, General Method)

$g(t)$: see the standard form

$y_p(t) = y_1(t)u_1(t) + y_2(t)u_2(t)$

$y_p(t) = -y_1(t)\int \frac{y_2(t)g(t)}{aW(y_1,y_2)}\,dt + y_2(t)\int \frac{y_1(t)g(t)}{aW(y_1,y_2)}\,dt$

- Reducible ODEs(NCC, non-linear)

Normally we have $F(y'', y', y, x)$

F in the form of $F(y'', y', x)$ we let $v(x) = \frac{dy}{dx}$, $\frac{dv}{dx} = y''$

F in the form of $F(y'', y', y)$ we let $v(y) = \frac{dy}{dx}$, $\frac{dv}{dy} = y''$

- Euler's method(NCC)

In the form of $ax^2y'' + bxy' + cy = 0$, let $t = lnx$ and change variable

Series Solution :
$n = k$ then for all n at left$n = 0$ at right $n = n + k$
$y = \sum_{n=0}^{\infty} a_n x^n$
$y' = \sum_{n=0}^{\infty} a_{n+1}(n+1)x^n$
$y'' = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n$
Legendre equation :
$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0, \alpha > -1$
Method of Frobenius: near a regular singular point :
$y = x^r \sum_{n=0}^{\infty} a_n x^n$

Other Tricks :
1. $\frac{d}{dx}\sinh(x) = \cosh(x)$, $\frac{d}{dx}\cosh(x) = \sinh(x)$,$\sinh(2x) = 2\sinh(x)\cosh(x)$,
2. d is the shortest distance between a point q and a line l with directional vector v, p is also a point on the line: $d = \frac{|(\mathbf{p}-\mathbf{q})\times\mathbf{v}|}{|\mathbf{v}|}$.
3. Use symmetry when calculating ∂f
4. Contour curve: $\nabla f = 0$ would be the function of some line on the graph,
Check for symmetry,Check for x = 0, y = 0

Derivative :
1. $\frac{d}{dx}\sinh(x) = \cosh(x)$, $\frac{d}{dx}\cosh(x) = \sinh(x)$,
Integral :
1. $\int xe^x \, dx = (x-1)e^x + C$
2. $\int x^2 e^x \, dx = (x^2 - 2x + 2)e^x + C$

Transformation :
$\sinh(2x) = 2\sinh(x)\cosh(x)$

Vector Calculus :
1. $\underline{r}(t) = f(t)\underline{i} + g(t)\underline{j}$
2. **Arc length of Vector** $= \int_b^a \sqrt{f'(t)^2 + g'(t)^2}dt$
3. The forms of Line
vector form : $\underline{r} = \underline{r_0} + t\underline{v}$
parametric form : $x = x_0 + at, y = y_0 + bt$
symmetric form : $\frac{x-x_0}{a} = \frac{y-y_0}{b} = t$
4. **Unit Tangent vector** : $\underline{T}(t) = \frac{\underline{r}'(t)}{|\underline{r}'(t)|}$
5. **Cuvature** : $\kappa = |\frac{dT}{ds}| = \frac{|\underline{r}'(t)\times\underline{r}''(t)|}{|\underline{r}'(t)|^3}$
6. **Normal vector** : $\underline{N}(t) = \frac{\underline{T}'(t)}{|\underline{T}'(t)|}$
7. **Binormal vector** : $\underline{B}(t) = \underline{T}(t) \times \underline{N}(t)$

Important Summations :
1. $\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
2. $\sum_{k=0}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$
3. $e^\lambda = \sum_{k=0}^{n} \lambda^k/k!$
4. $sin(x) = \sum_{k=0}^{n}(-1)^k x^{2k+1}/(2k+1)!$
5. $cos(x) = \sum_{k=0}^{n}(-1)^k x^{2k}/(2k)!$

This requires some puzzling out; no strategy 'recipes'. Use a combination of following insights:

- check symmetries, e.g. $x \to -x$ or $y \to -y$ or $x \to y, y \to x$, ...
- observe monotonicity of functions, e.g. $\log x$, e^x, $\sqrt{x}$, ...
- observe boundedness of functions, e.g. $\log x$, e^x, $\sin x$, ...
- observe what happens at $x=0$ or $y=0$ or $x=y$ or $x=-y$ or other particular values.
- possibly find level curves $F(x,y) = C$ explicitly (i.e. $x = f(y), y = g(x)$) for some functions.
- examine the implict derivative ($\underline{\nabla} F(x,y) = \underline{0}$) rather than the function itself.
- by elimination.

Figure 2:

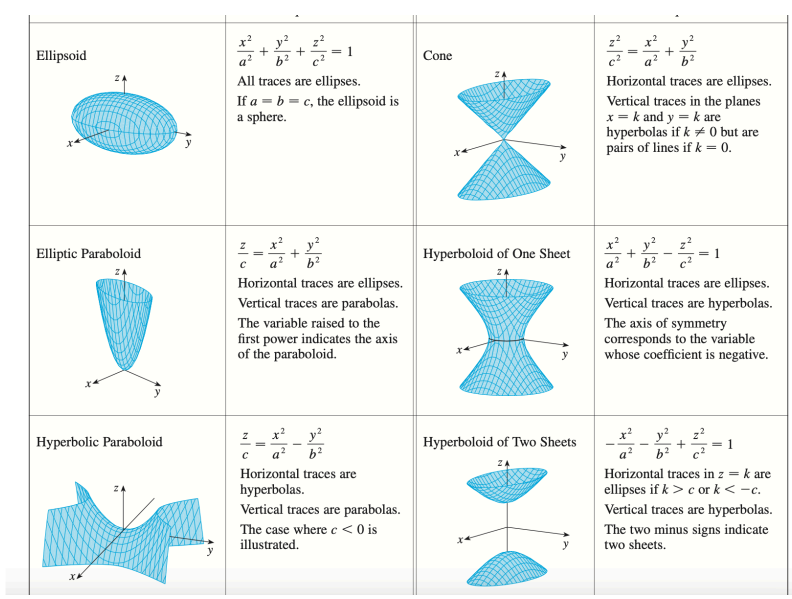


Figure 3: