Autumn Semester 2025
Master's Program in Quantitative Economics and Finance
University of St. Gallen

Quantitative Risk Management

Assignment

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1 Goals and conditions

The goal of this assignment is the estimation of value-at-risk and expected shortfall (see Chapter 3 of the lecture notes for the definition of these risk measures) for the profit and loss distribution of a given credit portfolio.

The following conditions must be satisfied:

(i) The assignment can be conducted in groups of maximum 3 students. The members of the groups must be announced to Ms. Barbara Langenegger (email: mathstat@unisg.ch) no later than **Friday 31 October 2025**. No further changes to the composition of the groups will be accepted, unless a member decides to leave her or his group. In this case, the change must announced by the leaving member via email to Ms. Barbara Langenegger, with all other members of the group in cc. The list of groups will be available on Canvas on 10 November 2025. The members of the group must be clearly indicated in the final solution of the assignment.

(ii) The **due time for the assignment is Tuesday 2 December 2025 at 9 am**. The solutions (pdf file in case of a single file or zip file if more than one file) have to be sent via email to Ms. Barbara Langenegger. She will confirm reception via email. **Solutions delivered after the due time will not be graded**. Please write the subject "7,320 Quantitative Risk Management: Assignment" in the email.

(iii) Present your analysis in a ordered way. Results must be reported (tables and figures) and **commented** in the main document. I will no run codes on my computer in order to obtain the results of your analysis! Formulas which are needed to estimate the parameters of the models or the risk measures must be justified and derived explicitly, also if you apply an existing package or function. The presentation of the results will be evaluated!

(iv) You can freely choose the software (MATLAB, S-Plus, R, etc.). Excel is not well-suited.

(v) Codes used for the analysis must be **commented** and attached to the solution.

(vi) The graded assignment will count for the final grade, according to the following rule:

$$\text{final grade G} = 0.70\,F + 0.30\,A$$

where A is the grade of the assignment and F is the grade of the oral exam.

2 Dataset

Please download from `https://learning.unisg.ch` the zip file `qrm25HSG_assignmentdata.zip`. It contains:

(i) `qrm25HSG_creditportfolio.xls`: description of the credit portfolio. For each counterparty k, the file contains the rating, the exposure E_k (in USD), the recovery rate R_k in case of default, the probability of default π_k, and the parameters of the model which will be explained below.

(ii) `qrm25HSG_indexes.xls`: daily data for the price (index level) of the Swiss Performance Index (SPI) and the price of Standard & Poor's 500 (SPX) from 1 January 2004 to 12 October 2025.

3 Questions

The credit portfolio contains 100 counterparties which are corporate bonds of companies with branches in Switzerland and the United States. We want to derive the portfolio loss distribution at time $T = 1$ week.

We denote by I_k the *default indicator* of counterparty $k = 1, \ldots, 100$, i.e., $I_k = 1$ if counterparty k defaults before or at time T, and zero else.

For each counterparty $k = 1, \ldots, 100$, we assume the following model:

$$I_k = 1 \Leftrightarrow Y_k \leq d_k$$

where

$$Y_k = \sqrt{\lambda_k}\,\mathbf{a}_k'\,\boldsymbol{\Theta} + \sqrt{1-\lambda_k}\,s_k\,\epsilon_k$$

and $\boldsymbol{\Theta}$ is the two-dimensional vector of *weekly log-returns* of the Swiss Performance Index (SPI) and of Standard & Poor's 500 (SPX). The parameters $\mathbf{a}_k = (a_{k1}, a_{k2})' \in \mathbb{R}^2$, $\lambda_k \in (0,1)$ are given in the excel sheet `qrm25HSG_creditportfolio.xls`, s_k is the standard deviation of Y_k, and the variables $\epsilon_k \sim N(0,1)$, $k = 1, \ldots, 100$, are independent and identically, standard normally distributed, *also independent* from $\boldsymbol{\Theta}$.

The couple $(Y_k, d_k)_{k=1,\ldots,100}$ is the latent variable model for $(I_k)_{k=1,\ldots,100}$. The variable Y_k is interpreted as the *asset value log-return of company* k at time T and d_k is the default threshold (see Chapter 4, Merton's model). The model assumes that the asset value log-return of company k can be described as a linear combination of two common risk factors (SPI and SPX) and an idiosyncratic risk factor ϵ_k.

(i) Derive the formula for the *conditional* probability of default given the risk factors $\boldsymbol{\Theta}$. How is the random variable I_k distributed conditionally on $\boldsymbol{\Theta}$?

(ii) Derive the formula for the portfolio loss as function of I_k, E_k (exposure) and R_k (recovery rate).

(iii) Derive a formula for standard deviation of Y_k, as function of the covariance matrix of $\boldsymbol{\Theta}$.

(iv) We consider three different models for the risk factors $\boldsymbol{\Theta}$.

M1: $\mathbf{\Theta}$ is distributed according to the empirical distribution.

M2: $\mathbf{\Theta}$ is bivariate Gaussian distributed with mean vector $\boldsymbol{\mu}$ and covariance matrix $\mathbf{\Sigma}$ (see Section 4 below).

M3: Θ_i is Gaussian distributed with mean μ_i and variance σ_i^2, for $i = 1, 2$. The vector $\left(\Phi\left(\frac{\Theta_1-\mu_1}{\sigma_1}\right), \Phi\left(\frac{\Theta_2-\mu_2}{\sigma_2}\right)\right)'$ possesses a t-copula with ν degrees of freedom and correlation parameter ρ.

Using maximum-likelihood estimation, estimate models M2 and M3 on historical data.

Hint: the file `qrm25HSG_indexes.xls` contains price levels for SPI and SPX.

(v) Using model M1 for $\mathbf{\Theta}$, simulate for each counterparty k the distribution of Y_k (10,000 simulations). Moreover, derive the threshold d_k using the *unconditional* probability of default π_k given in the excel sheet `qrm25HSG_creditportfolio.xls`.

Hint:
d_k satisfies $\mathbb{P}\big[Y_k \leq d_k\big] = \pi_k$.

(vi) Repeat the same exercise as in point (v) using models M2 and M3, respectively.

(vii) Using the simulated distributions for the Y_k, derive the portfolio loss distribution under the three models M1, M2 and M3 for $\mathbf{\Theta}$.

(viii) Estimate value-at-risk and expected shortfall for the portfolio loss distribution under the three models M1, M2 and M3 for $\mathbf{\Theta}$. Compare and discuss the results.

(ix) Calculate how value-at-risk and expected shortfall dynamically change over time under models M1, M2 and M3, respectively, if the last 500 observations of daily returns are used to estimate the models.

4 Formulae

4.1 Bivariate Gaussian distribution

The random vector $\mathbf{X} = (X_1, X_2)'$ possesses a bivariate Gaussian distribution with mean vector $\boldsymbol{\mu} = (\mu_1, \mu_2)'$ and covariance matrix

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \rho\,\sigma_1\,\sigma_2 \\ \rho\,\sigma_1\,\sigma_2 & \sigma_2^2 \end{pmatrix},$$

when its density function is

$$f(x_1, x_2; \boldsymbol{\mu}, \mathbf{\Sigma}) = \frac{1}{2\pi\,\sigma_1\,\sigma_2\,\sqrt{1-\rho^2}}\, e^{-\frac{1}{2\,(1-\rho^2)}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2\,\rho\,(x_1-\mu_1)\,(x_2-\mu_2)}{\sigma_1\,\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)}.$$

4.2 Standard t-distribution

A random variable X is (univariate) standard t-distributed with ν degrees of freedom when its density function is

$$f_{t,1}(x;\nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\,\pi}\,\Gamma(\frac{\nu}{2})}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}.$$

The random vector $\mathbf{X} = (X_1, X_2)'$ is bivariate standard t-distributed with ν degrees of freedom and correlation parameter ρ when its density function is

$$f_{t,2}(x_1, x_2; \nu, \rho) = \frac{\Gamma(\frac{\nu+2}{2})}{\Gamma(\frac{\nu}{2})\,\pi\,\nu\,\sqrt{1-\rho^2}}\left(1+\frac{x_1^2 - 2\,\rho\,x_1\,x_2 + x_2^2}{\nu\,(1-\rho^2)}\right)^{-\frac{\nu+2}{2}}.$$

Γ is the gamma-function, $\Gamma(z) = \int_0^\infty t^{z-1}\,e^{-t}\,dt$.

4.3 Simulations

4.3.1 Univariate distribution

Some softwares (like Excel, see the RAND() function) only allow to simulate a uniform distribution on $[0,1]$. How do we simulate a general (continuous) distribution?

Let X be a random variable with a continuous cumulative distribution function F and F^{-1} its quantile function. Let U be a uniformly distributed random variable on $[0,1]$ and define

$$Y = F^{-1}(U).$$

Then Y has the cumulative distribution function F:

$$\mathbb{P}\big[Y \le y\big] = \mathbb{P}\big[F^{-1}(U) \le y\big] = \mathbb{P}\big[U \le F(y)\big] = F(y)$$

since U is uniformly distributed on $[0,1]$.

4.3.2 Bivariate Gaussian distribution

Some softwares (like Excel) only allow to simulate a uniform distribution on $[0,1]$. Using the approach explained in Subsection 4.3.1, we can easily simulate any univariate distribution, as long as we know its quantile function. However, how do we simulate bivariate distributions?

Let $\mathbf{X} = (X_1, X_2)'$ be a bivariate Gaussian distributed random vector with mean vector $\boldsymbol{\mu} = (\mu_1, \mu_2)'$ and covariance matrix $\boldsymbol{\Sigma}$. Since $\boldsymbol{\Sigma}$ is symmetric and positive definite, we find a matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$$

with $a_{11}, a_{22} > 0$ such that

$$\boldsymbol{\Sigma} = \mathbf{A}'\,\mathbf{A}.$$

$\mathbf{A}$ is a upper triangular matrix with strictly positive diagonal entries, and the last equation is called the *Cholesky decomposition* of $\boldsymbol{\Sigma}$.

Let $\mathbf{Z} = (Z_1, Z_2)'$ be a bivariate Gaussian distributed random vector with mean vector $\mathbf{0} = (0, 0)'$ and covariance matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

i.e., Z_1, Z_2 are independent and identically, standard normally distributed. Let $\mathbf{A}$ such that $\mathbf{A}'\,\mathbf{A} = \mathbf{\Sigma}$ (Cholesky decomposition of $\mathbf{\Sigma}$), then

$$\mathbf{Y} = \boldsymbol{\mu} + \mathbf{A}'\,\mathbf{Z}$$

is a bivariate Gaussian distributed random vector with mean vector $\boldsymbol{\mu} = (\mu_1, \mu_2)'$ and covariance matrix $\mathbf{\Sigma}$.

In words: to simulate a bivariate Gaussian distributed random vector $\mathbf{Y}$ with mean vector $\boldsymbol{\mu} = (\mu_1, \mu_2)'$ and covariance matrix $\mathbf{\Sigma}$, one can simulate *two independent standard normally distributed random variables* Z_1, Z_2, compute the Cholesky decomposition $\mathbf{A}$ of $\mathbf{\Sigma}$, and set

$$\mathbf{Y} = \boldsymbol{\mu} + \mathbf{A}'\,\mathbf{Z},$$

where $\mathbf{Z} = (Z_1, Z_2)'$.

4.3.3 Bivariate t-distribution

Let $\mathbf{Z} = (Z_1, Z_2)'$ be a bivariate Gaussian distributed random vector with mean vector $\mathbf{0} = (0, 0)'$ and covariance matrix

$$\mathbf{\Sigma} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

Let W be a Chi-squared distributed random variable with ν degrees of freedom random variable, *and* independent from $\mathbf{Z}$.

The random vector

$$\mathbf{X} = \sqrt{\frac{\nu}{W}}\,\mathbf{Z}$$

is standard t-distributed with ν degrees of freedom and correlation parameter ρ.