Time Series Econometrics
Assignment 1, Part II – Spring 2025

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1 Forecasting MA models

In this problem you explore the properties of the two forecasting methods discussed in class. Approach 1 approximates the optimal mean squared error (MSE) forecast, i.e. the conditional mean, by reconstructing an approximate error sequence, imposing assumptions on the initial values of the theoretical (and infeasible) error sequence. Approach 2 uses the projection method to deliver the optimal linear forecast as a linear combination of the previous m observations. Although infeasible in practice, the optimal MSE forecasts are also computed for comparison purposes.

a) Simulate $K = 1000$ trajectories from the following MA(3) model, each of length $N + H = 103$, where $N = 100$ is the number of in sample observations and $H = 3$ is the number of out of sample periods to forecast,

$$x_t = \varepsilon_t - 0.5\varepsilon_{t-1} + 0.3\varepsilon_{t-2} - 0.2\varepsilon_{t-3} \tag{1}$$

where $(\varepsilon_t)_{t\in\mathbb{Z}^+} \sim iidN(0, 1)$. Use the R command arima.sim to generate the trajectories and set.seed(2025) to ensure reproducibility of your results.

b) **Optimal MSE Forecasts**
For each trajectory $k = 1, ..., K$, generate the optimal MSE forecasts by means of the following steps:

i. Show that for an MA(3) process the h step ahead optimal MSE forecast is given by

$$\hat{X}^*_{t+h|t} = \begin{cases} \theta_1\varepsilon_t + \theta_2\varepsilon_{t-1} + \theta_3\varepsilon_{t-2} & h = 1 \\ \theta_2\varepsilon_t + \theta_3\varepsilon_{t-1} & h = 2 \\ \theta_3\varepsilon_t & h = 3 \\ 0 & h \geq 4 \end{cases} \tag{2}$$

where θ_j, $j = 1, ..., 3$ are the coefficients of the MA(3) process.

ii. Plug the (infeasible) error sequence, used to simulate the k-th trajectory of the MA process in point (a), into Equation (2) to compute the optimal MSE forecasts for horizons $h = 1, ..., H$.

For each forecast horizon $h = 1, ..., H$, compute the MSE by averaging the squared forecast errors over the K simulations. In the following these MSEs will serve as a theoretical benchmark to which compare the feasible forecasting methods.

c) **Approach 1 - Naive Error Sequence Reconstruction**
For each trajectory $k = 1, ..., K$, generate forecasts using Approach 1 by means of the following steps:

i. In R, assuming $\hat{\varepsilon}_0 = \hat{\varepsilon}_{-1} = \hat{\varepsilon}_{-2} = 0$, reconstruct an approximation $(\hat{\varepsilon}_t)_{t\in\mathbb{Z}^+}$ of the infeasible error sequence $(\varepsilon_t)_{t\in\mathbb{Z}^+}$.

ii. Plug the approximated error sequence $(\hat{\varepsilon}_t)_{t\in\mathbb{Z}^+}$ into Equation (2) to obtain the forecasts for horizons $h = 1, ..., H$.

For each forecast horizon $h = 1, ..., H$, compute the MSE by averaging the squared forecast errors over the K simulations.

d) **Approach 2 - Projection Method**
For each trajectory $k = 1, ..., K$, generate forecasts using Approach 2 by means of the following steps:

i. Show that the autocovariance function of an MA(3) model is given by

$$\gamma(h) = \begin{cases} \sigma_w^2 \sum_{j=0}^{3-h} \theta_j \theta_{j+h}, & 0 \leq h \leq 3 \\ 0, & h > 3 \end{cases} \tag{3}$$

where $\theta_0 = 1$, θ_j, $j = 1, ..., 3$ are the coefficients of the MA(3) process and σ_w^2 is the variance of the WN process.

ii. Let $\mathbf{X}_t^{(m)} = (X_t, X_{t-1}, ..., X_{t-m+1})$, where $m = 1, ..., M$ is the number of past values of X used to forecast in the projection method and $M = 5$. Compute the forecasts for horizons $h = 1, ..., H$ as

$$\hat{X}_{t+h|t}^{(m)} = (\alpha_h^{(m)})^T \mathbf{X}_t^{(m)} \tag{4}$$

where $\alpha_h^{(m)} = \Gamma_m^{-1} \gamma_h^{(m)}$ is the $m \times 1$ vector of optimal coefficients, $\Gamma_m = [\gamma_{i-j}]_{i,j=1}^m$ is a symmetric $m \times m$ matrix,
$\gamma_h^{(m)} = (\gamma_h, \gamma_{h+1}, ..., \gamma_{h+m-1})^T$ is a $m \times 1$ vector and T denotes the transpose.

For each forecast horizon $h = 1, ..., H$, compute the MSE by averaging the squared forecast errors over the K simulations.

e) Compare the forecasting methods in terms of MSE.
In particular, which method performs the best? Plot the MSEs of approach 1 and approach 2 with $m = 5$ as a function of the forecast horizon $h = 1, ..., H$.
How does the accuracy of the projection method change as m increases? Plot the MSEs of the projection method for $m = 1, ..., M$ as a function of the forecast horizon.
In both plots, also include the MSE of the optimal (but infeasible) forecasts, obtained in point (b), for comparison.