Time Series Econometrics
Assignment 2 – Spring 2025

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1 ARMA models

Let $z = (z_t)_{t\in\mathbb{Z}} \subset \mathbb{R}$, be a stochastic process satisfying

$$z_t = z_{t-1} - 0.5z_{t-2} + u_t - 4.75u_{t-1} + 7.375u_{t-2} - 5u_{t-3} + 1.5625u_{t-4} \quad (1)$$

where $(u_t)_{t\in\mathbb{Z}} \sim WN(0,1)$. At first glance, this is an ARMA(2, 4) process, however, is it true? In this exercise, we will check if this is a well-defined ARMA(2, 4) process.

a) Write the model in lag polynomial form with an AR polynomial $\Phi(L)$ and an MA polynomial $\Theta(L)$.

b) Check for common roots of the polynomials to show that the minimal representation of the original ARMA(2,4) is an MA(2) model. You can use the R command polyroot to compute the roots of the lag polynomials.

c) Write the reduced MA(2) model in explicit form, i.e. compute the coefficients θ_1 and θ_2 in the following expression

$$z_t = u_t + \theta_1 u_{t-1} + \theta_2 u_{t-2} \quad (2)$$

where $(u_t)_{t\in\mathbb{Z}} \sim WN(0,1)$.

d) Is the reduced MA(2) model in equation (2) covariance stationary? Is it causal? Is it invertible? Justify your answer.

e) Find an invertible model that has the same second order properties as the original reduced MA(2) model in equation (2). You can proceed as follows: consider a new MA(2) model

$$y_t = \left(1 - \frac{1}{y_1^*}L\right)\left(1 - \frac{1}{y_2^*}L\right)\varepsilon_t \quad (3)$$

such that $(\varepsilon_t)_{t\in\mathbb{Z}} \sim WN(0, \sigma^2)$ and

$$y_i^* = \begin{cases} z_i^*, & \text{if } z_i^* \geq 1 \\ \frac{1}{z_i^*}, & \text{if } z_i^* < 1 \end{cases} \quad , i = 1, 2 \tag{4}$$

where $z_i^*, i = 1, 2$ are the roots of the lag polynomial of the reduced MA(2) model in equation (2). Finally, specify the variance σ^2 of the WN process $(\varepsilon_t)_{t\in\mathbb{Z}}$ by imposing $\mathrm{Var}[y_t] = \mathrm{Var}[z_t]$.

f) Check that y_t is indeed invertible and that it has the same second order properties as the original reduced form process by calculating the mean, variance and autocovariances of both processes.

g) What does it mean if a process is not invertible?

2 Model Selection and Estimation

The file eurodata70Q1-14Q4.dat contains data on quarterly, seasonally adjusted real Euro GDP for the period of 1970Q1-2014Q4. In this problem you explore the properties of the time series, select, estimate and check a suitable model. All the parts should be implemented in R.

a) Provide a time series plot and briefly comment on the properties of the real GDP series. Compute and plot the quarterly growth rate of real GDP. For the real GDP series determine the order of integration of y_t using an Augmented Dickey-Fuller (ADF) test. Provide the same analysis for the transformed data (growth rate of real GDP). What do you conclude?

b) Use the Box-Jenkins approach for model selection and suggest a candidate model for the quarterly growth rate of real GDP. Briefly explain your choice. Also apply the Akaike and the Bayesian information criterion (AIC and BIC) to select a pure AR model (use models with an intercept, $p_{max} = 9$ and make sure that you use the same number of observations for estimating alternative models). What are the suggested lag orders?

c) Estimate the pure AR models suggested in 2b). You can use the Arima command of the R package "forecast" to fit the different models. Provide plots of standardized residuals and residual ACFs and PACFs. Interpret your results. Conduct diagnostic tests for remaining residual autocorrelation and heteroscedasticity for each of the estimated models. Which model would you prefer? Is your model adequately specified? Explain your answer. Explain why a researcher may not want to use the model with the highest suggested lag order in 2b).

d) Use the growth rate of real Euro GDP ranging from 1972Q2-2007Q4 and the Arima command of the R package "forecast" to estimate an AR(2) model.