Measure

A **measure** is a mathematical abstraction of length. There are three fundamental properties of length that a mathematical measure abstracts. First, length is always positive; we cannot have negative length. Second, the length of nothing is zero. Third, if we break up a length into disjoint segments, the sum of each segment's length will be equal to the original length.

Many real-world concepts have this property. For example area, volume, or mass are all similar to length in these regards. Mass is analogous to length in that we cannot have negative mass, the mass of nothing is zero, and if we break up an object into two pieces, the sums of their masses will be equal to the original object's mass. We will use the term "length" as the general name for this property, but as previously described it can be substituted with "area", "volume", "mass", and many other properties.

The idea of assigning length to pieces of an object presumes that we have a way of defining what constitutes a "piece" of the object. The way in which an object "breaks apart" into pieces is captured by the notion of a **σ-algebra** over the object. If we let the object be F, then a σ-algebra $\mathcal{F}$ is the collection of pieces. That is, each $A \in \mathcal{F}$ is one piece of the object.

Our goal now is to assign lengths to the pieces such that the sum of the lengths of any two pieces is equal to the length of the fragment formed by gluing the two pieces together. This assignment should also not assign any negative lengths to the pieces and should assign a length of zero to the null piece (i.e. the empty set). A function μ that assigns lengths to each piece of the object and satisfies these conditions is called a **measure** as defined in Definition 1. An object F together with a σ-algebra $\mathcal{F}$ and measure μ form a **measure space** as defined in Definition 2.

Without the measure, we refer to the object F and its σ-algebra $\mathcal{F}$ as a **measurable space**. That is, we can imagine several ways that we might wish to assign lengths to the pieces of the object using different measures. Thus, the object and its σ-algebra are "measurable" because it is possible to equip the pair with an arbitrary measure.

Definition 1 *Given a set F and a σ-algebra on F, denoted $\mathcal{F}$, the function*

$$\mu : \mathcal{F} \to \mathbb{R}$$

*is a **measure** if the following properties of μ hold:*

1. *Nonnegative:*
$$\forall A \in \mathcal{F},\ \mu(A) \geq 0$$

2. *Empty set has zero measure:*
$$\mu(\emptyset) = 0$$

3. *Countable additivity: Given $A, B \in \mathcal{F}$ where $A \cap B = \emptyset$,*

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

Definition 2 *The tuple $(F, \mathcal{F}, \mu)$ where F is a set, $\mathcal{F}$ is a σ-algebra on F, and μ is a measure on $\mathcal{F}$ is a **measure space**.*

Definition 3 *The tuple $(F, \mathcal{F})$ where F is a set and $\mathcal{F}$ is a σ-algebra on F is a **measureable space**.*

Definition 4 *Given a measurable space $(F, \mathcal{F}, \mu)$, any set $A \subseteq \mathcal{F}$ is a **measurable set**.*

Figure 1 illustrates schematically the third axiom of a measure for a given measure space $(F, \mathcal{F}, \mu)$.

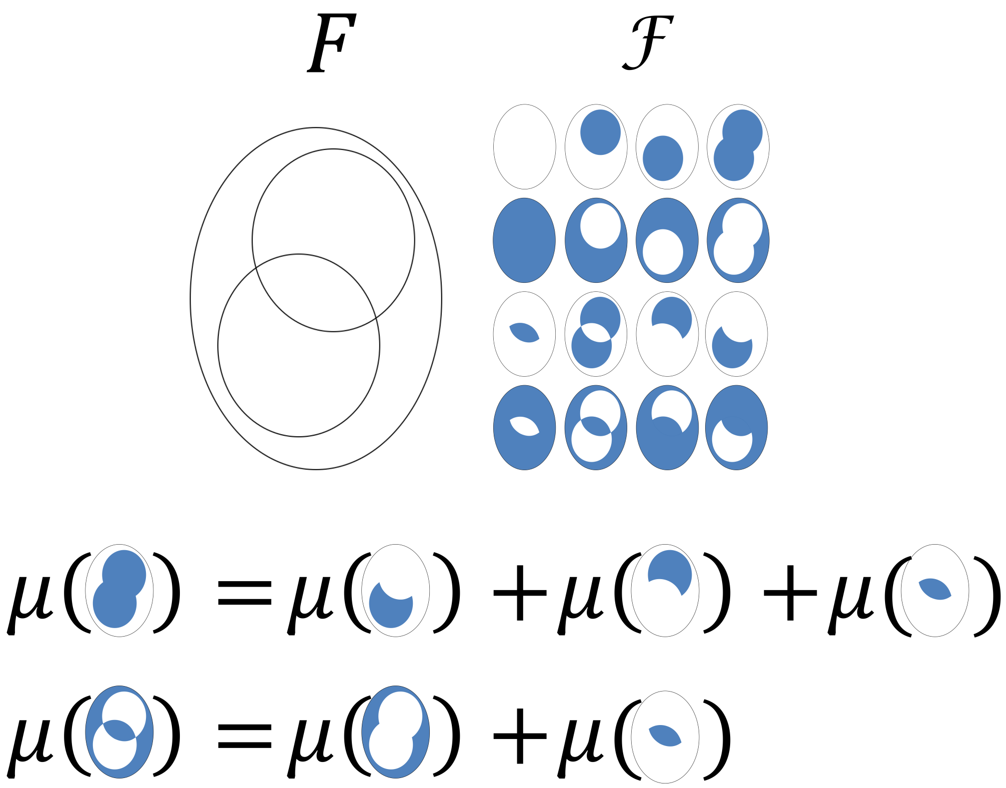

Figure 1: The large oval represents the set F and a collection of subsets. $\mathcal{F}$ is the sigma algebra on F induced by these subsets. Below are two illustrations of Axiom 3 for the measure μ.

