σ-algebras

Given a set F, a collection of subsets $\mathcal{F}$ (i.e. $A \in \mathcal{F} \implies A \subseteq F$) is a **$\sigma$-algebra** if that collection is closed under a countable number of intersection, union, compliment, subtraction, and symmetric difference operations. The concept of a σ-algebra is central to the definition of a **measure** and is at the foundation of the definition of **probability**.

Definition 1 *Given a set F and a collection of subsets $\mathcal{F}$, the collection $\mathcal{F}$ is called a **σ-algebra** if it satisfies the following conditions:*

1. $\emptyset \in \mathcal{F}$
2. $A \in \mathcal{F} \implies A^c \in \mathcal{F}$
3. $A_1, A_2, \dots \in \mathcal{F} \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

Example: Given a set F, the smallest possible σ-algebra on F is

$$\{\emptyset, F\}$$

Example: Given a set F, and set $A \subseteq F$, the following is a valid σ-algebra

$$\mathcal{F} = \{\emptyset, A, \ A^c, F\}$$

Intuition

A σ-algebra can be understood as an abstraction of the concept of "breaking something." Say we have plate with cracks along its surface. We imagine that we can break this plate apart only along these cracks. That is, we can remove chunks of the plate if the chunk is surrounded by cracks. A σ-algebra on the plate is then the set of all possible ways we can take out pieces of the plate along these cracks. Figure 1 provides an illustration. The top portion of the diagram shows the plate with its cracks. If we let the plate be a set F where each location (i.e. point) on the plate is a member of F, then a subset of points A is in the σ-algebra $\mathcal{F}$ if A represents all the points of the plate that you get when removing a chunk of the plate along the cracks. That is, a member $A \in \mathcal{F}$ is the set of points remaining after some combination of the chunks of the plate are removed. Below the plate in Figure 1, we illustrate all of the combinations of chunks of the plate that you can be removed. The blue region in each diagram illustrates one member of $\mathcal{F}$. The entire set of blue regions (i.e. one blue region per figure) represents the σ-algebra.

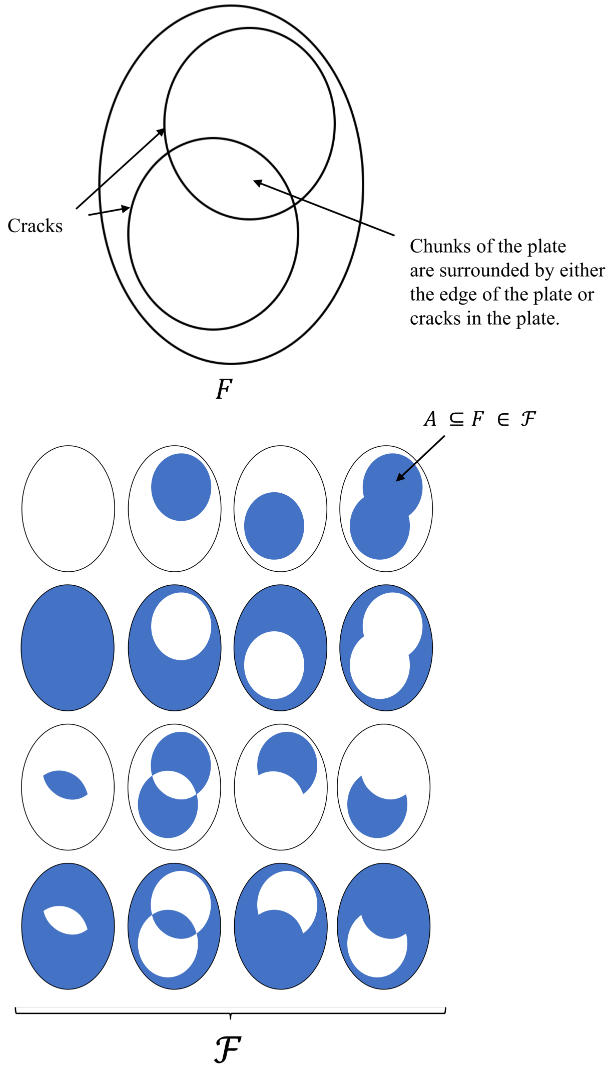

Figure 1: The large oval above is a ceramic plate with cracks along its surface. Let the points on the plate be a set F. Below, we illustrate every subset $A \subseteq F$ where A represents the plate with some combination of chunks of the plate removed. We are only allowed to remove a chunk if it is surrounded by cracks. The entire set of configurations of ways we can removing chunks of the plate is a σ-algebra $\mathcal{F}$ over F.

Properties

As described in the introductory paragraph, the sets in a σ-algebra are closed under a countable number of set intersection, union, compliment, subtraction, and symmetric difference operations. The closure under union and compliment is required in the axioms, however the closure under intersection, subtraction, and symmetric difference can be proven from the axioms.

Closure under set intersection

Theorem 1 *Let F be a set and $\mathcal{F}$ be a σ-algebra on F with $A, B \in \mathcal{F}$. Then*

$$A, B \in \mathcal{F} \implies A \cap B \in \mathcal{F}$$

Proof: By Axiom 2, we know that $A^c, B^c \in \mathcal{F}$. Then by Axiom 3, we have $A^c \cup B^c \in \mathcal{F}$. By De Morgan's Laws, we know that $A^c \cup B^c = (A \cap B)^c$ and thus, $(A \cap B)^c \in \mathcal{F}$. Finally, by Axiom 2, $((A \cap B)^c)^c \in \mathcal{F}$ and we know that $((A \cap B)^c)^c = A \cap B$.

□

Closure under set subtraction

Theorem 2 *Let F be a set and $\mathcal{F}$ be a σ-algebra on F with $A, B \in \mathcal{F}$. Then*

$$A \setminus B \in \mathcal{F}$$

Proof: First, by Axiom 2, we have $A^c \in \mathcal{F}$. By Theorem 1, $A \cap B^c \in \mathcal{F}$. Since $A \cap B^c = A \setminus B$, it follows that $A \setminus B \in \mathcal{F}$.

□

Closure under symmetric difference

Theorem 3 *Let F be a set and $\mathcal{F}$ be a σ-algebra on F with $A, B \in \mathcal{F}$. Then*

$$A \triangle B \in \mathcal{F}$$

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By Theorem 2 we know that $A \setminus B \in \mathcal{F}$ and that $B \setminus A \in \mathcal{F}$. By Axiom 3, we have $(A \setminus B) \cup (B \setminus A) \in \mathcal{F}$ and we know that $(A \setminus B) \cup (B \setminus A) = A \triangle B$.

$\square$

