Vector spaces

A **vector space** consists of an abstract set of elements, called **vectors** that can be added together and scaled. The notion of "scaling" is modeled by a **field**. That is, a vector space involves a set of vectors $\mathcal{V}$ and a field of scalars $\mathcal{F}$ for which one can add together vectors in $\mathcal{V}$ as well as scale these vectors by elements in the field $\mathcal{F}$ according to the set of rules outlined in Definition 1. Axioms 1-5 of the definition describe how vectors can be added together. Axioms 6-10 describe how these vectors can be scaled using the field of scalars.

Definition 1 *Given a set of objects $\mathcal{V}$ called vectors and a field $\mathcal{F} := (C, +, \cdot, -, ^{-1}, 0, 1)$ where C is the set of elements in the field, called scalars, the tuple $(\mathcal{V}, \mathcal{F})$ is a **vector space** if for all $\mathbf{v}, \mathbf{u}, \mathbf{w} \in \mathcal{V}$ and $c, d \in C$, it obeys the following ten axioms:*

1. $\mathbf{u} + \mathbf{v} \in \mathcal{V}$
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. *There exists a zero vector* $\mathbf{0} \in \mathcal{V}$ *such that* $\mathbf{u} + \mathbf{0} = \mathbf{u}$
5. *For each* $\mathbf{u} \in \mathcal{V}$ *there exists a* $\mathbf{u}' \in \mathcal{V}$ *such that* $\mathbf{u} + \mathbf{u}' = \mathbf{0}$. *We call* $\mathbf{u}'$ *the negative of* $\mathbf{u}$ *and denote it as* $-\mathbf{u}$
6. *The scalar multiple of* $\mathbf{u}$ *by* c*, denoted by* $c\mathbf{u}$ *is in* $\mathcal{V}$
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. $1\mathbf{u} = \mathbf{u}$

Vector subspaces

A vector space can be induced by an appropriate subset of vectors from some larger vector space. We call such a subspace a **vector subspace** (Definition 2). By merits of the original vector space, seven out of 10 axioms will always hold; however, there are three axioms that may not hold that must be verified whenever a subset of vectors from a vector space are to considered as a vector space in their own right.

Definition 2 *A subset of vectors $\mathcal{H} \subseteq \mathcal{V}$ from a vector space $(\mathcal{V}, \mathcal{F})$ forms a* ***vector subspace*** *$(\mathcal{H}, \mathcal{F})$ if the following three properties hold:*

1. *The zero vector of $\mathcal{V}$ is in $\mathcal{H}$.*
2. *$\mathcal{H}$ is closed under vector addition. That is, for each $\mathbf{u}, \mathbf{v} \in \mathcal{H}$, the vector $\mathbf{u} + \mathbf{v}$ is also in $\mathcal{H}$.*
3. *$\mathcal{H}$ is closed under scalar multiplication. That is, for each $\mathbf{u} \in \mathcal{H}$ and each scalar $c \in \mathcal{F}$, the vector $c\mathbf{u}$ is in $\mathcal{H}$.*

Properties

1. **The zero vector is unique** (Theorem 1). There is only one distinct zero vector in a vector space.
2. **Any vector multiplied by the zero scalar is the zero vector** (Theorem 2).The zero scalar converts any vector into the zero vector. That is, given a vector $\mathbf{v}$, it holds that
$$0\mathbf{v} = \mathbf{0}$$
This is analogous to how any number multiplied by zero becomes zero.
3. **The negative of a vector is unique** (Theorem 3). Given a vector $\mathbf{v}$, we denote its negative vector as $-\mathbf{v}$. This is analogous to each number x having a matching negative number $-x$ that lies $|x|$ distance from 0 on the opposite side of 0.
4. **Multiplying a negative vector by the scalar -1 produces its negative vector** (Theorem 4). That is, given a vector $\mathbf{v}$
$$-1\mathbf{v} = -\mathbf{v}$$
This is analogous to the fact that if you multiply any number x by -1 you get the number $-x$ that lies $|x|$ distance from 0 on the opposite side of 0.
5. **The zero vector multiplied by any scalar is the zero vector** (Theorem 5). The zero vector remains the zero vector despite being multiplied by any scalar. That is,
$$c\mathbf{0} = \mathbf{0}$$

for any $c \in \mathcal{F}$. This is analogous to the fact that zero multiplied by any number remains zero.

6. **The only vector whose negative is not distinct from itself is the zero vector** (Theorem 6). For every vector other than the zero vector, its negative vector is a distinct vector in the vector space. For the zero vector, its negative is itself. This is analogous to the fact that for any number $x \neq 0$, the number $-x$ is a distinct number from x that lies on the opposite side of 0. However, for $x = 0$, $-x = x$.

Theorem 1 *Given vector space $(\mathcal{V}, \mathcal{F})$, the zero vector is unique.*

Proof: Assume for the sake of contradiction that there exists a vector $\mathbf{a}$ such that $\mathbf{a} \neq \mathbf{0}$ and that $\forall \mathbf{v} \in \mathcal{V}$

$$\mathbf{a} + \mathbf{v} = \mathbf{v}$$

Then, this implies that :

$$\mathbf{a} + \mathbf{0} = \mathbf{0}$$

However, axiom 4 states that for the zero-vector

$$\mathbf{a} + \mathbf{0} = \mathbf{a}$$

Since $\mathbf{a} \neq \mathbf{0}$, we reach a contradiction. Therefore, there does not exist a vector $\mathbf{a} \neq \mathbf{0}$ for which $\forall \mathbf{v} \in \mathcal{V} \quad \mathbf{a} + \mathbf{v} = \mathbf{v}$. Thus, the zero-vector is unique.

□

Theorem 2 *Given a vector space $(\mathcal{V}, \mathcal{F})$*

$$\forall \mathbf{v} \in \mathcal{V}, 0\mathbf{v} = \mathbf{0}$$

Proof: Assume for the sake of contradiction that there exists a vector $\mathbf{a} \neq \mathbf{0}$ such that

$$0\mathbf{v} = \mathbf{a}$$

Now, for any scalar $c \neq 0$, we have

$$\begin{aligned} c\mathbf{v} &= (c + 0)\mathbf{v} \\ &= c\mathbf{v} + 0\mathbf{v} && \text{by axiom 8} \\ &= c\mathbf{v} + \mathbf{a} \end{aligned}$$

Our assumption assumed that $\mathbf{a} \neq \mathbf{0}$ must be false because by Theorem 1 the only vector $\mathbf{a}$ for which $c\mathbf{v} + \mathbf{a} = c\mathbf{v}$ would be true is the zero-vector.

□

Theorem 3 *Given a vector space* $(\mathcal{V}, \mathcal{F})$ *and vector* $\mathbf{v} \in \mathcal{V}$*, its negative,* $-\mathbf{v}$*, is unique. That is,*

$$\mathbf{v} + \mathbf{a} = \mathbf{0} \iff \mathbf{a} = -\mathbf{v}$$

Proof:

We need only prove $\mathbf{v} + \mathbf{a} = \mathbf{0} \implies \mathbf{a} = -\mathbf{v}$. The other direction is stated in the axioms.

$$\begin{aligned}
&\mathbf{v} + \mathbf{a} = \mathbf{0} & \\
\implies &-\mathbf{v} + \mathbf{v} + \mathbf{a} = -\mathbf{v} + \mathbf{0} & \\
\implies &[-\mathbf{v} + \mathbf{v}] + \mathbf{a} = -\mathbf{v} & \\
\implies &\mathbf{0} + \mathbf{a} = -\mathbf{v} & \text{by axiom 5} \\
\implies &\mathbf{a} = -\mathbf{v} & \text{by axiom 4}
\end{aligned}$$

□

Theorem 4 *Given a vector* $\mathbf{v} \in \mathcal{V}$*, it's negative is* $(-1)\mathbf{v}$*. That is,*

$$-\mathbf{v} = (-1)\mathbf{v}$$

Proof:

$$\begin{aligned}
\mathbf{v} + (-1)\mathbf{v} &= (1)\mathbf{v} + (-1)\mathbf{v} && \text{by axiom 10} \\
&= (1 - 1)\mathbf{v} && \text{by axiom 8} \\
&= 0\mathbf{v} && \\
&= \mathbf{0} && \text{by Theorem 2}
\end{aligned}$$

Then, by axiom 5, it must be that $(-1)\mathbf{v} = -\mathbf{v}$.

□

Theorem 5 *Given a vector space* $(\mathcal{V}, \mathcal{F})$

$$c\mathbf{0} = \mathbf{0} \iff \mathbf{a} = \mathbf{0}$$

Proof:

$$\begin{aligned} \mathbf{0} + \mathbf{0} &= \mathbf{0} && \text{by axiom 4} \\ c(\mathbf{0} + \mathbf{0}) &= c\mathbf{0} \\ c\mathbf{0} + c\mathbf{0} &= c\mathbf{0} && \text{by axiom 8} \end{aligned}$$

By Theorem 1, the only vector $\mathbf{a}$ in $\mathcal{V}$ for which $\mathbf{a} + \mathbf{v} = \mathbf{v}$ for all vectors $\mathbf{v} \in \mathcal{V}$ is the zero vector $\mathbf{0}$. Thus, $c\mathbf{0} = \mathbf{0}$.

□

Theorem 6 *Given a vector space* $(\mathcal{V}, \mathcal{F})$

$$-\mathbf{0} = \mathbf{0}$$

Proof:

$$\begin{aligned} \mathbf{a} + -\mathbf{a} &= \mathbf{0} && \text{by axiom 5} \\ \mathbf{a} + \mathbf{a} &= \mathbf{0} && \text{assume } \mathbf{a} = -\mathbf{a} \\ \implies 2\mathbf{a} &= \mathbf{0} \\ \implies \mathbf{a} &= \mathbf{0} && \text{by Theorem 5} \end{aligned}$$

Thus, if we assume $\mathbf{a} = -\mathbf{a}$, then $\mathbf{a}$ must be the zero vector.

□