Measurable functions

A **measurable function** is a function that maps elements in one measure space to elements in another measure space such that the preimage of all measurable sets in the codomain are measurable sets in the domain (Definition 1). Figure 1.B illustrates a valid measurable function. Figure 1.C illustrates an invalid measurable function due to the fact that there exists a measurable set in the codomain whose preimage is not measurable.

Furthermore, it is notable that the status of a function of f as a measurable function is only determined by the measurability of preimages of measurable sets in the codomain. Figure 2.C illustrates a function

$$f : F \to H$$

that maps elements from the measurable space $(F, \mathcal{F})$ to the measurable set $(H, \mathcal{H})$. As seen in the figure, the image of a set $A \in \mathcal{F}$ may be unmeasurable in the codomain. That is $f(A) \notin \mathcal{H}$. This does not affect the status of f as a measurable function.

Definition 1 *Given measure spaces $(F, \mathcal{F})$ and $(H, \mathcal{H})$, a function*

$$f : F \to H$$

*is a **measurable function** if for all $A \in \mathcal{H}$, $f^{-1}(A) \in \mathcal{F}$.*

Intuition

A measurable function is a mapping between measurable spaces that allows us to utilize a measure defined in the domain in order to construct a measure in the codomain. Say we have some measure space $(F, \mathcal{F}, \mu_F)$ and a measurable space $(H, \mathcal{H})$ for which no measure is defined. A function f that maps elements in F to elements in H is measurable if it gives the ability to construct a measure μ_H on $(H, \mathcal{H})$. That is, if the preimage, via f, of each measurable set in $\mathcal{H}$ is measurable in $(F, \mathcal{F})$ then we can construct a measure μ_H on $(H, \mathcal{H})$ as

$$\mu_H(A) := \mu_F(f^{-1}(A))$$

where $A \in \mathcal{H}$. Thus, a measurable function "transfers" our ability of measuring one measure space to another measure space by utilizing the function's preimage.

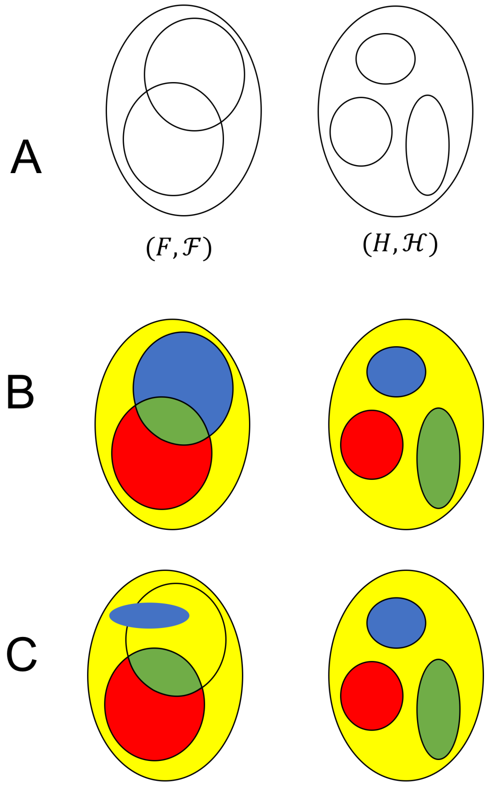

Figure 1: (A) Two measurable spaces $(F, \mathcal{F})$ and $(H, \mathcal{H})$. The σ-algebras are generated by the sets outlined in black lines. (B) A valid measurable function f mapping F to H. That is, the left set is the domain and the right set is the codomain. Colors illustrate image relations between subsets of F and H under f. For example, the image of the blue region in F is the blue region in H. We see that each member of $\mathcal{H}$ has a measurable preimage. (C) An invalid measurable function. The blue measurable set in $\mathcal{H}$ has a preimage that is not a measurable set in $\mathcal{F}$.

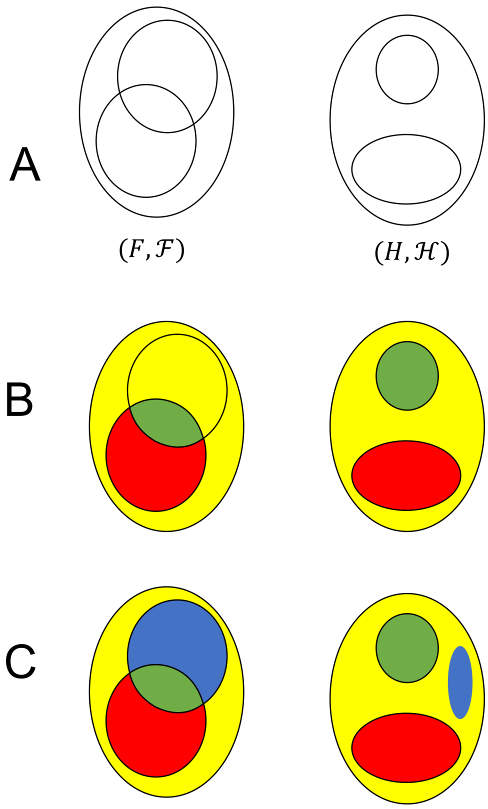

Figure 2: (A) Two measurable spaces $(F, \mathcal{F})$ and $(H, \mathcal{H})$. The σ-algebras are generated by the sets outlined in black lines. (B) A valid measurable function f mapping F to H. That is, the left set is the domain and the right set is the codomain. Colors illustrate image relations between subsets of F and H under f. For example, the image of the blue region in F is the blue region in H. We see that each member of $\mathcal{H}$ has a measurable preimage. (C) We note that the image of the blue region in $\mathcal{F}$ is not a measurable set in $\mathcal{H}$. Nonetheless, this does not effect the status of f as a measurable function. That is, we are only concerned with the measurability of preimages of members of $\mathcal{H}$.