Linear transformations

A **linear transformation** (also called a **linear function** or **linear operator**) is a function between vector spaces that preserves the operations of vector addition and scalar multiplication (Definition 1).

Definition 1 *Given vector spaces* $(\mathcal{V}, \mathcal{F})$ *and* $(\mathcal{U}, \mathcal{F})$*, a function* $T : \mathcal{V} \rightarrow \mathcal{U}$ *is a* ***linear transformation****, or* ***linear****, if for all* $\mathbf{u}, \mathbf{v} \in \mathcal{V}$ *and all scalars* $c \in \mathcal{F}$*,*

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

and,

$$T(c\mathbf{u}) = cT(\mathbf{u})$$

.

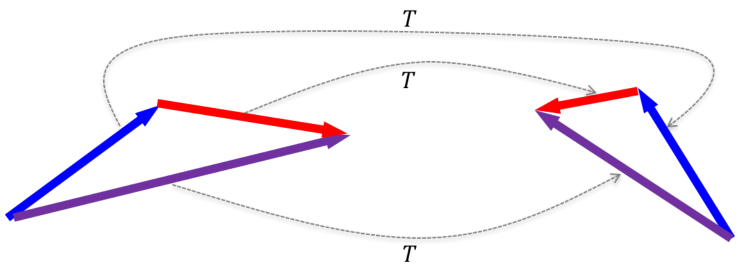

Figure 1: A schematic of a linear transformation T applied to three vectors (red, blue, and purple). The vectors on the left are in T's domain and the vectors on the right are in T's range. The dotted lines connect each vector in the domain to its vector in the range as mapped by T. Notice that the $T(\text{red}) + T(\text{blue}) = T(\text{red} + \text{blue})$.

Properties of linear transformations

Linear transformations on the zero-vector

A linear transformation on the zero vector in the domain will yield the zero vector in the range of the transformation.

Theorem 1 *Given a linear transformation T over vector space $(\mathcal{V}, \mathcal{F})$,*

$$T(\mathbf{0}) = \mathbf{0}$$

.

Proof:

$$\begin{aligned} T(\mathbf{0}) &= T(0\mathbf{u}) && \text{for any } \mathbf{u} \in \mathcal{V} \\ &= 0T(\mathbf{u}) \\ &= \mathbf{0} \end{aligned}$$

□