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Subject: Kinematics for Differential-drive Wheeled Mobile Robots

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1 Modeling

Based on the equation 13.6 on Modern Robotics[1], differential-drive robots can be modeled as a H matrix where

$$H(\theta) = \begin{bmatrix} h_1(\theta) \\ h_2(\theta) \end{bmatrix} \tag{1}$$

$$h_i(\theta) = \frac{1}{r_i \cos \gamma_i} \begin{bmatrix} x_i \sin(\beta_i + \gamma_i) - y_i \cos(\beta_i + \gamma_i) \\ \cos(\beta_i + \gamma_i + \theta) \\ \sin(\beta_i + \gamma_i + \theta) \end{bmatrix}^{\mathrm{T}} \tag{2}$$

For the differential-drive robots, the configuration parameters is defined as

$$x_1 = 0 \tag{3}$$
$$x_2 = 0 \tag{4}$$
$$y_1 = d \tag{5}$$
$$y_2 = -d \tag{6}$$
$$\beta_1 = 0 \tag{7}$$
$$\beta_2 = 0 \tag{8}$$
$$\gamma_1 = -\pi/2 \tag{9}$$
$$\gamma_2 = -\pi/2 \tag{10}$$
$$r_1 = r \tag{11}$$
$$r_2 = r \tag{12}$$

where d is the one half of the distance between two wheels, and r is the wheel radius.

So the F matrix of the differential-drive robot

$$F = H(0)^{\dagger} = r \begin{bmatrix} -\frac{1}{2d} & \frac{1}{2d} \\ \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \tag{13}$$

2 Kinematics

2.1 Forward Kinematics

The forward kinematics is to find the new configuration q' from the new wheel angle (ϕ'_l, ϕ'_r) and current configuration q.

1. Find the difference in wheel angles $(\Delta\phi_l, \Delta\phi_r)$
2. Find the twist from the current body frame to body frame of the new configuration

$$\mathcal{V}_b = F\Delta\phi = r \begin{bmatrix} -\frac{1}{2d} & \frac{1}{2d} \\ \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\phi_l \\ \Delta\phi_r \end{bmatrix} \tag{14}$$

3. Express the twist in 3d form.

$$\mathcal{V}_{b6} = \begin{bmatrix} 0 \\ 0 \\ \mathcal{V}_b \\ 0 \end{bmatrix} \tag{15}$$

4. Get the transform from frame $\{b\}$ to frame $\{b'\}$

$$T_{bb'} = e^{[\mathcal{V}_{b6}]} \tag{16}$$

5. Find the transform from space frame $\{s\}$ to new body frame $\{b'\}$

$$T_{sb} = T(\theta, x, y) \tag{17}$$
$$T_{sb'} = T_{sb}T_{bb'} = T(\theta', x', y') \tag{18}$$

6. Extract the new configuration $q'(\theta', x', y')$

2.2 Inverse Kinematics

The principle of the inverse kinematics is to find the wheel velocity required to follow the commanded body twist $\mathcal{V}_b\ (\omega_{bz}, v_{bx}, v_{by})$

Based on the equation 13.9 on Modern Robotics[1], the relationship between the wheel speed and the body twist can be modeled as

$$\begin{bmatrix} \dot{\phi}_l \\ \dot{\phi}_r \end{bmatrix} = H(0)\mathcal{V}_b = \frac{1}{r}\begin{bmatrix} -d & 1 & -\infty \\ d & 1 & -\infty \end{bmatrix}\begin{bmatrix} \omega_z \\ v_x \\ v_y \end{bmatrix} \tag{19}$$

Where $v_y = 0$. So that the left and right wheel velocity can be computed as

$$\dot{\phi}_l = \frac{1}{r}\left(-d\omega_z + v_x\right) \tag{20}$$
$$\dot{\phi}_r = \frac{1}{r}\left(d\omega_z + v_x\right) \tag{21}$$
$$\tag{22}$$

References

[1] K. M. Lynch and F. C. Park, Modern robotics : mechanics, planning, and control. Cambridge: University Press, 2017.