Furuta Pendulum Project

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Figure 1. "Furuta Pendulum" – or rotational inverted pendulum by Quanser

Introduction:

The purpose of this project is to create a mathematical model of a Furuta Pendulum or rotational inverted pendulum, shown above in figure 1, which can be used along with a 3-D model and Simulink to simulate its response. A Furuta Pendulum is a system in where an inverted pendulum is held upright on a horizontal arm which is moved by the torque of a motor spinning the system about its vertical axis. Once the pendulum arm is brought up in the upright position the system must then stabilize it and hold it in this position. This is accomplished by a motor controlled by a programable controller with many know parameters. This parameters are important as the mathematics of the system are heavily dependent on the moment of inertia of the pendulum as well as the lengths of each arm and pendulum. Show below in figure 2 is an example of all of the parameters necessary in the modeling of the system. Where, θ_0 is the angular position of the spinning arm and θ_1 is the angular position of the pendulum measured with respect to the upright position. τ is the torque produced by the motor that is used to spin the system. I_0 is the moment of inertia of the spinning arm about its end and L_0 is the spinning arm's length. m_1, L_1, l_1, and J_1 are the mass, length of arm, the center of mass location, and the moment of inertia of the pendulum, respectively. Lastly, g is the acceleration due to gravity.

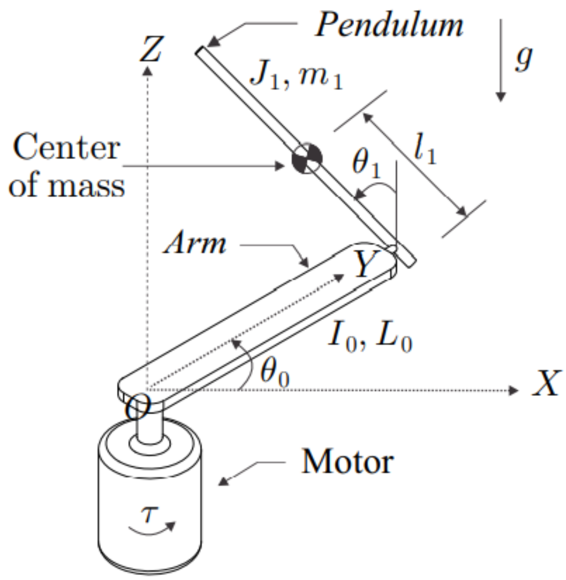

Figure 2. All known parameters of the Furuta Pendulum

This process of simulation is done in the real world because it is much cheaper to simulate a systems and its response than it is to build a real system. This is also done because the engineers are able to predict failures that might be unforeseen until the a real system is built, this as well also saves a great deal of money and time. In the following report you will be shown how the system was mathematical modeled and simulated. The MATLAB codes will be available in the appendix, as well as the functional viewpoint mapping used in the Simulink simulation. This report will also show you what type of hardware would be used and how the hardware would communicate with the controlling software.

Modeling:

The Lagrange equation of the system (L) is shown below in Eq.(1), where K is the kinetic energy and P potential energy of the pendulum arm,

$$L = K - P \tag{1}$$

Due to the systems two degrees of freedom the Lagrange equations of motion are,

$$\tau = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_0}\right) - \frac{\partial L}{\partial \theta_0} \tag{2}$$

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} \tag{3}$$

The kinetic energy of the arm (K_0) in shown below in Eq.(4), and the kinetic energy of the pendulum arm (K_1) is shown below in Eq.(5), where v_1 is the linear velocity of the center of mass for the pendulum arm,

$$K_0 = \frac{1}{2} I_0 \dot{\theta}_0^2 \tag{4}$$

$$K_1 = \frac{1}{2} J_1 \dot{\theta}_1^2 + \frac{1}{2} m_1 v_1^T v_1 \tag{5}$$

Because this is being represented in 3-D space, we must determine the location of the center of mass for the pendulum arm in state space form by using Eq.(6) below,

$$x = \left[x_x, x_y, x_z\right]^T \tag{6}$$

The values of x_x, x_y, x_z are defined below,

$$x_x = L_0 \cos(\theta_0) - l_1 \sin(\theta_1) \sin(\theta_0)$$

$$x_y = L_0 \sin(\theta_0) + l_1 \sin(\theta_1) cos(\theta_0)$$

$$x_z = l_1 \cos(\theta_1)$$

The same analysis must be done regarding the velocity of the center of mass of the pendulum arm, shown below in Eq.(7),

$$v_1 = \left[\dot{x}_x, \dot{x}_y, \dot{x}_z\right]^T \tag{7}$$

The values of $\dot{x}_x, \dot{x}_y, \dot{x}_z$ are defined below, which are just the derivatives of x_x, x_y, x_z above,

$$\dot{x}_x = -\dot{\theta}_0 L_0 \sin(\theta_0) - l_1\left[\dot{\theta}_0 \sin(\theta_1)\cos(\theta_0) + \dot{\theta}_1 \sin(\theta_0)\cos(\theta_1)\right]$$

$$\dot{x}_y = \dot{\theta}_0 L_0 \cos(\theta_0) + l_1\left[\dot{\theta}_0 \cos(\theta_0)\cos(\theta_1) - \dot{\theta}_0 \sin(\theta_0)\sin(\theta_1)\right]$$

$$\dot{x}_z = -\dot{\theta}_0 l_1 \sin(\theta_1)$$

To find the real value of the kinetic energy of the pendulum arm (K_1), Eq.(7) must be plugged into Eq.(5) giving us,

$$K_1 = \frac{1}{2}J_1\dot{\theta}_1^2 + \frac{1}{2}m_1\left[\left(\dot{\theta}_0 L_0\right)^2 + \left(l_1\dot{\theta}_0 sin(\theta_1)\right)^2 + \left(l_1\dot{\theta}_1\right)^2 + 2\dot{\theta}_0\dot{\theta}_1 L_0 l_1 \cos(\theta_1)\right]$$

To find the kinetic energy of the Lagrange equation of the system (L) we use Eq.(8) below, which adds our new kinetic energy of the pendulum arm (K_1) with the already known kinetic energy of the arm (K_0),

$$K = K_0 + K_1 \tag{8}$$

The kinetic energy of the Lagrange equation of the system (L) is defined below,

$$K = \frac{1}{2}I_0\dot{\theta}_0^2 + \frac{1}{2}J_1\dot{\theta}_1^2 + \frac{1}{2}m_1\left[\left(\dot{\theta}_0 L_0\right)^2 + \left(l_1\dot{\theta}_0 sin(\theta_1)\right)^2 + \left(l_1\dot{\theta}_1\right)^2 + 2\dot{\theta}_0\dot{\theta}_1 L_0 l_1 \cos(\theta_1)\right]$$

Because the arm is only acting in the horizontal plane we only need to consider the potential energy of the pendulum arm, which is defined in Eq.(9) below,

$$P = -hm_1 g = m_1 g l_1(\cos(\theta_1) - 1) \tag{9}$$

Now that the kinetic energy (K) and potential energy (P) have been found, these values can be plugged into the Lagrange equation of the system. When the Lagrange equation of the system is plugged into the Lagrange equations of motion and derived we find the dynamics of the Furuta Pendulum, shown below in Eq.(10) & Eq.(11),

$$\tau = \alpha\ddot{\theta}_0 + \beta\dot{\theta}_0\dot{\theta}_1 + \gamma\ddot{\theta}_1 - \sigma\dot{\theta}_1^2 \tag{10}$$

$$0 = \gamma\ddot{\theta}_0 + (m_1 l_1^2 + J_1)\ddot{\theta}_1 - \tfrac{1}{2}\beta\dot{\theta}_0^2 - m_1 g l_1 \sin(\theta_1) \tag{11}$$

Where alpha, beta, gamma, and sigma are defined in Eqs.(12-15) respectively,

$$\alpha = I_0 + m_1 L_0^2 + m_1 l_1^2 sin^2(\theta_1) \tag{12}$$

$$\beta = m_1 l_1^2 \sin(2\theta_1) \tag{13}$$

$$\gamma = m_1 L_0 l_1 \cos(\theta_1) \tag{14}$$

$$\sigma = m_1 L_0 l_1 \sin(\theta_1) \tag{15}$$

Table 1. Assumed values for the pendulum system.

Parameter	Value	Parameter	Value
Spinning Arm Length (L_0)	6.6 (*cm*)	Spinning Arm Mass (M_0)	.380 (*kg*)
Pendulum Arm Length (L_1)	14.6 (*cm*)	Pendulum Arm Mass (M_1)	.054 (*kg*)
Center of Mass Location (l_1)	7.3 (*cm*)	Pendulum Inertia (J_1)	3.53×10^4 ($kg\text{-}m^2$)

Sensor Calibration:

Hello

Controller Design and Simulation:

Hello

Optional: Controller Implementation

Appendix A

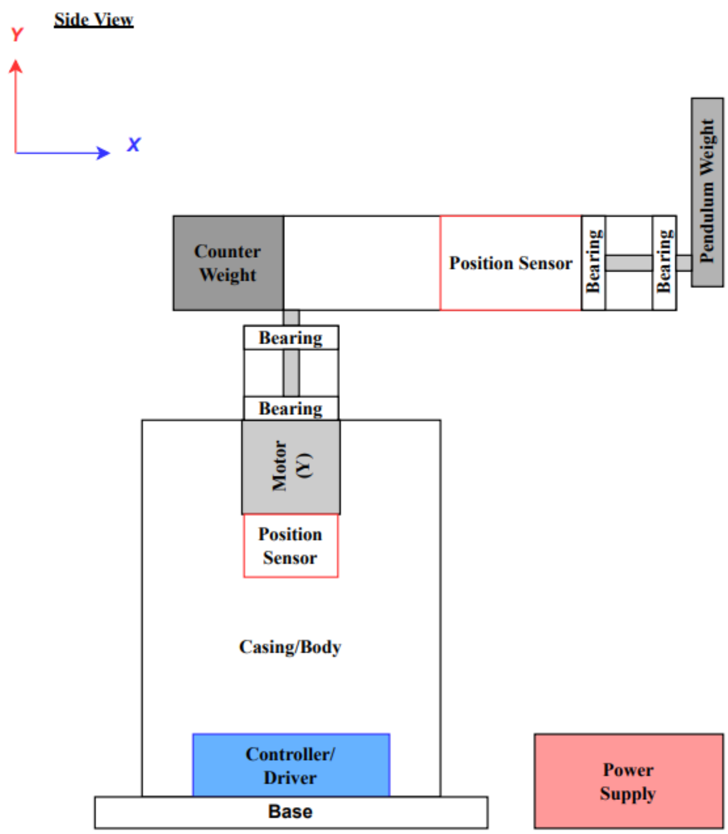

Figure X. Side View of the Operational Viewpoint

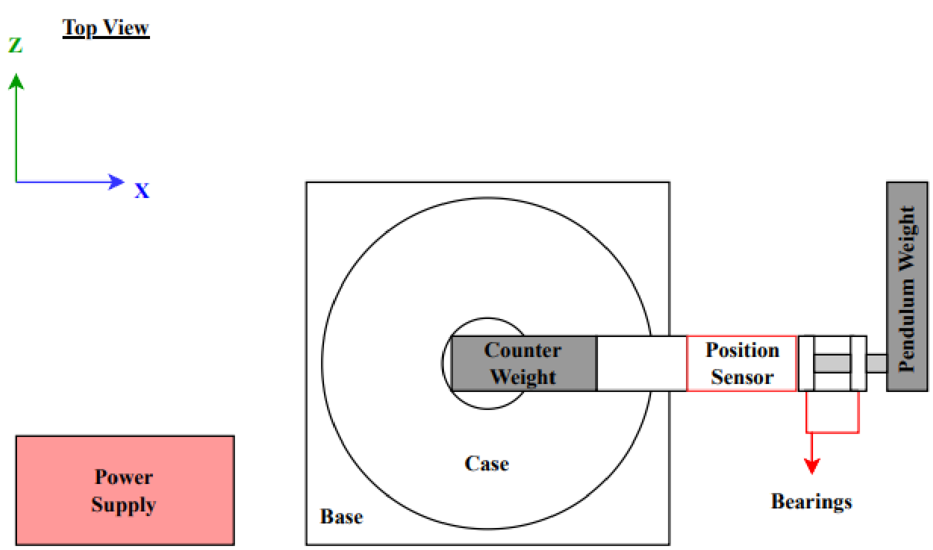

Figure X. Top View of the Operational Viewpoint

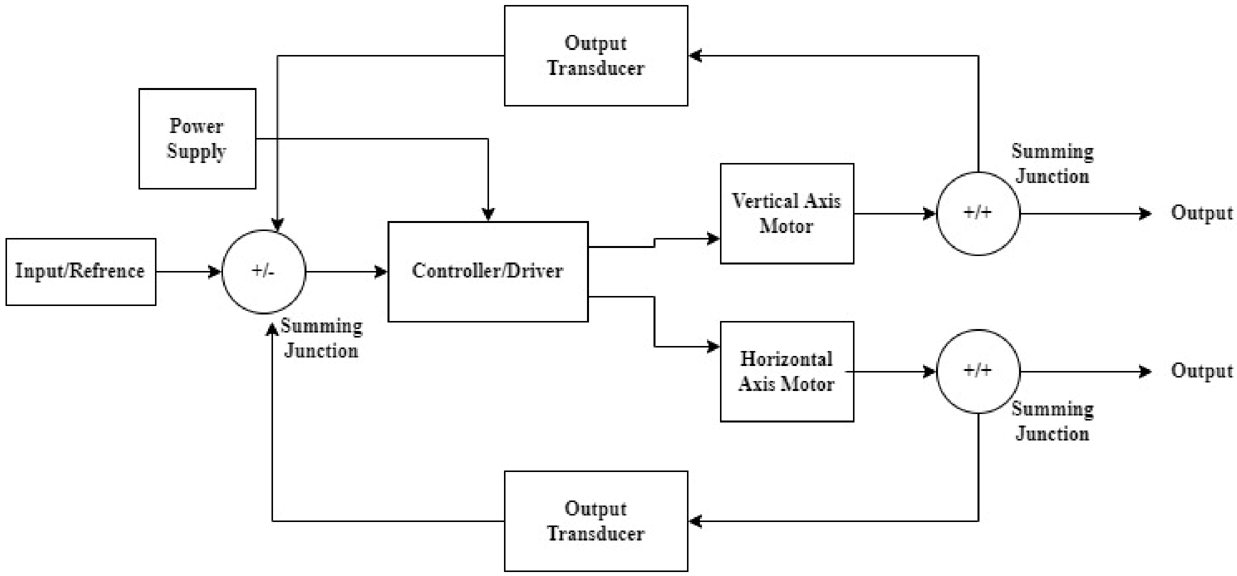

Figure X. Functional Viewpoint of Furuta Pendulum

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