Futura Pendulum
MECA 482: Control System Design

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I. Introduction

The furuta pendulum is a device that has a driven arm that rotates in the horizontal plane. This arm is attached to a pendulum which freely rotates in the vertical plane. The point of this device is to keep the pendulum balancing above the driven arm, and to swing the pendulum back into this position if it were to be knocked over.

This report will go over a prototype for the design of a furuta pendulum, including a mathematical model of the system, diagrams of the project, and a summary of the simulation results. The capabilities for the system include achieving self balancing through the method of correcting its alignment when experiencing external force. It also must return the pendulum to its upright position when experiencing up to 25 newtons of external force.

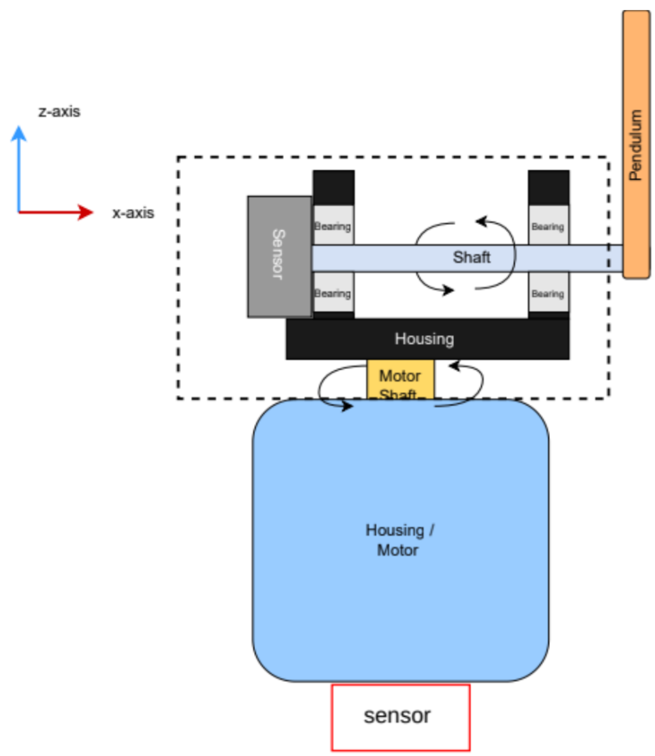

Figure 1. Front view of the operational diagram.

This is the front viewpoint for the design of the Furuta Pendulum. This includes a motor that can spin the driven shaft to swing the pendulum into its upright position if needed. The sensor connected to the driven shaft monitors the position of the pendulum and the sensor on the motor is to monitor the rotation of the motor.

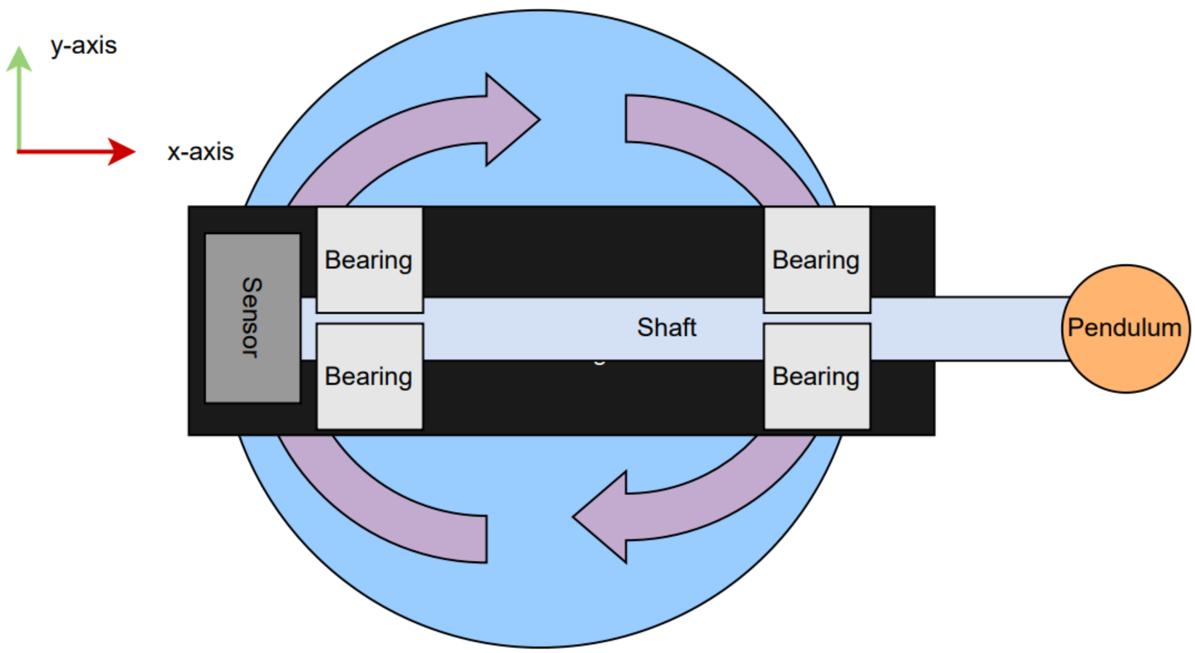

Figure 2. Top view of the operational diagram.

Figure 2 shows the top view for the design. This viewpoint gives a better view of the two bearings that support the driven shaft. It also shows the horizontal rotation of the driven shaft.

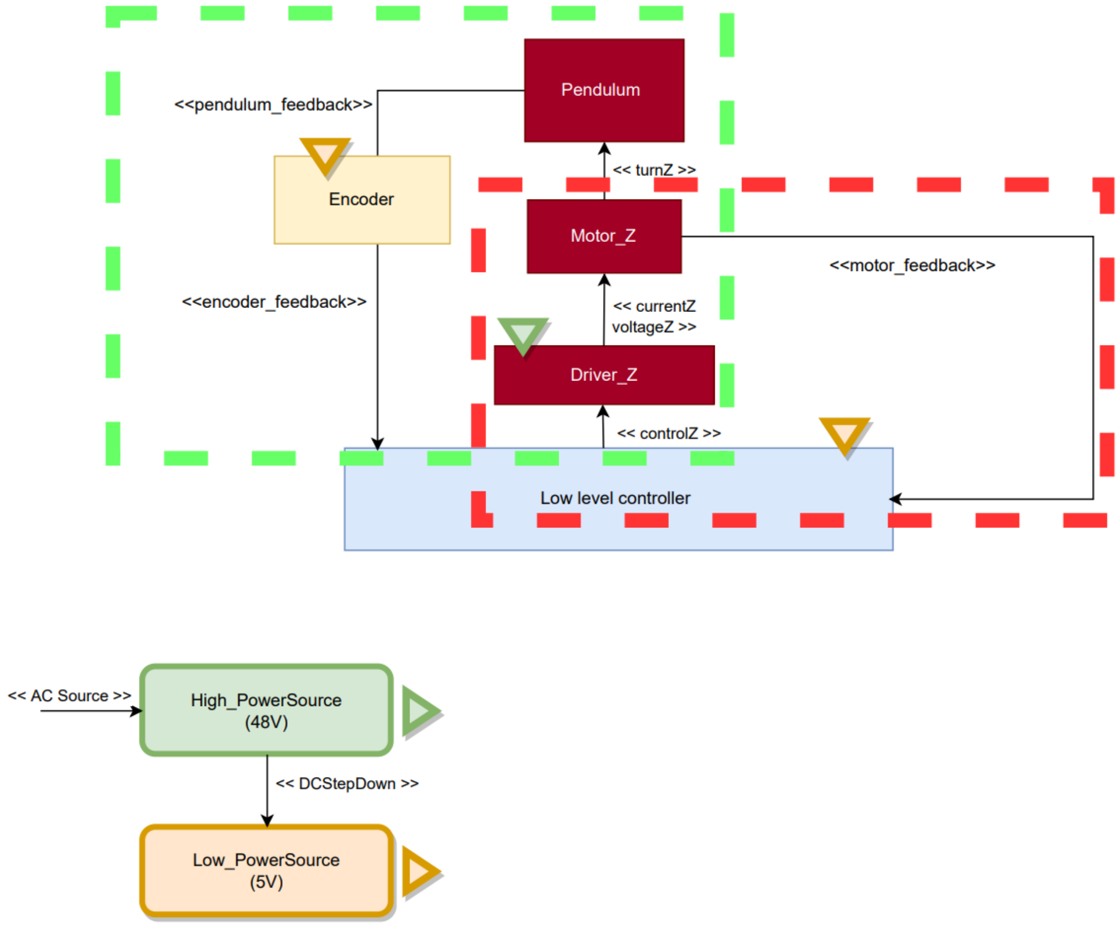

Figure 3. Logical viewpoint of the Furuta Pendulum.

The figure above shows the logical viewpoint of the Furuta Pendulum. The logical viewpoint shows the network of communication between the logical components responsible for keeping the pendulum upright. The encoder is responsible for tracking the position of the pendulum which is then interpreted by the controller.

II. Model

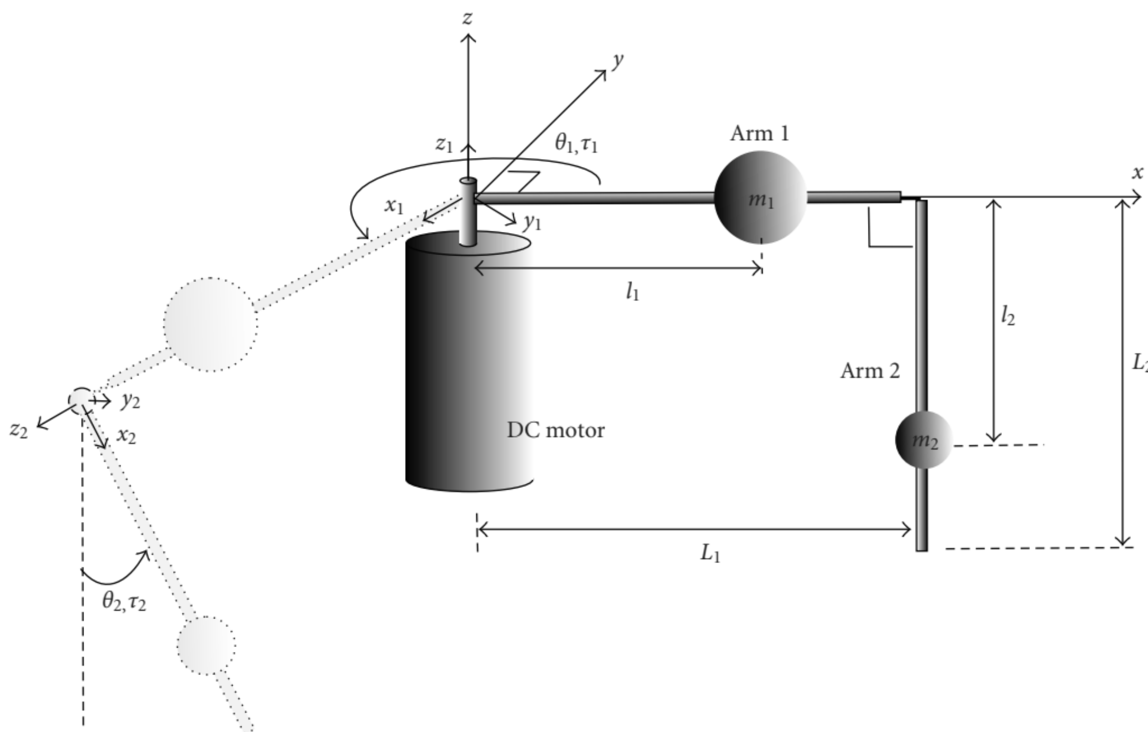

Figure 4. Furuta Pendulum model (Cazzolato and Prime, 2011).

The parameters for this project are:

x_1 - Rotational position of Arm 1 in the x-axis
x_2 - Rotational position of Arm 2 in the x-axis
y_1 - Rotational position of Arm 1 in the y-axis
y_2 - Rotational position of Arm 2 in the y-axis
z_1 - Rotational position of Arm 1 in the z-axis
z_2 - Rotational position of Arm 2 in the z-axis
l_1 - Distance from the center of mass of Arm 1 to the z-axis
l_2 - Distance from the center of mass of Arm 2 to the x-axis
L_1 - Length of Arm 1
L_2 - Length of Arm 2
m_1 - Mass for Arm 1
m_2 - Mass for Arm 2
θ_1 - Angular position of Arm 1 in radians
θ_2 - Angular position of Arm 2 in radians
τ_1 - Torque from DC motor
τ_2 - Torque on pendulum

The potential and kinetic energy can be shown with the equations from Cazzolato and Prime below.

$$E_p = E_{p1} + E_{p2}$$
$$E_k = E_{k1} + E_{k2}.$$

With the energies defined, the lagrangian can be described with the following equation.

$$L = E_k - E_p$$

Using this, the Euler-Lagrange is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + b_i\dot{q}_i - \frac{\partial L}{\partial q_i} = Q_i$$

Where,
$q_i = [\theta_1, \theta_2]^T$: generalized coordinate
$b_i = [b_1, b_2]^T$: generalized viscous damping coefficient
$Q_i = [\tau_1, \tau_2]^T$: generalized torque

Evaluating for $Q_i = \theta_1$ gives:

$$-\frac{\partial L}{\partial \theta_1} = 0$$

And for $Q_i = \theta_2$ gives:

$$-\frac{\partial L}{\partial \theta_2} = -\frac{1}{2}\dot{\theta}_1^2 \sin(2\theta_2)\left(m_2 l_2^2 + J_{2yy} - J_{2xx}\right) + \dot{\theta}_1\dot{\theta}_2 m_2 L_1 l_2 \sin(\theta_2) + g m_2 l_2 \sin(\theta_2)$$

The linear velocities for the center of masses in arm one and two are shown below.

$$\mathbf{v}_{1c} = \mathbf{v}_1 + \boldsymbol{\omega}_1 \times \begin{bmatrix} l_1 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & \dot{\theta}_1 l_1 & 0 \end{bmatrix}^T$$

$$\mathbf{v}_{2c} = \mathbf{v}_2 + \boldsymbol{\omega}_2 \times \begin{bmatrix} l_2 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} \dot{\theta}_1 L_1 \sin(\theta_2) \\ \dot{\theta}_1 L_1 \cos(\theta_2) + \dot{\theta}_2 l_2 \\ -\dot{\theta}_1 l_2 \sin(\theta_2) \end{bmatrix}$$

And the linear acceleration for both arms through the center of mass is represented by:

$$\dot{\mathbf{v}}_{1c} = \dot{\boldsymbol{\omega}}_1 \times \begin{bmatrix} l_1 & 0 & 0 \end{bmatrix}^T + \boldsymbol{\omega}_1 \times \left(\boldsymbol{\omega}_1 \times \begin{bmatrix} l_1 & 0 & 0 \end{bmatrix}^T\right) + \dot{\mathbf{v}}_1 = \begin{bmatrix} -l_1\dot{\theta}_1^2 & l_1\ddot{\theta}_1 & g \end{bmatrix}^T$$

$$\dot{\mathbf{v}}_{2c} = \dot{\boldsymbol{\omega}}_2 \times \begin{bmatrix} l_2 & 0 & 0 \end{bmatrix}^T + \boldsymbol{\omega}_2 \times \left(\boldsymbol{\omega}_2 \times \begin{bmatrix} l_2 & 0 & 0 \end{bmatrix}^T \right) + \dot{\mathbf{v}}_2 = \begin{bmatrix} -\dot{\theta}_1^2 \sin^2(\theta_2) l_2 - \dot{\theta}_2^2 l_2 + \ddot{\theta}_1 \sin(\theta_2) L_1 - \cos(\theta_2) g \\ \ddot{\theta}_2 l_2 - \frac{1}{2} \dot{\theta}_1^2 l_2 \sin(2\theta_2) + \cos(\theta_2) \ddot{\theta}_1 L_1 + \sin(\theta_2) g \\ -l_2 \sin(\theta_2) \ddot{\theta}_1 - 2\dot{\theta}_1 \dot{\theta}_2 l_2 \cos(\theta_2) - \dot{\theta}_1^2 L_1 \end{bmatrix}$$

Due to the geometry of the two arms, the moment inertia in the axis of the arm is negligible. The rotational symmetry of the arms also allows the assumption that the moments of inertia in two of the principal axes are equal. The simplified inertia matrices are shown below.

$$\mathbf{J}_1 = \begin{bmatrix} J_{1xx} & 0 & 0 \\ 0 & J_{1yy} & 0 \\ 0 & 0 & J_{1zz} \end{bmatrix} \approx \begin{bmatrix} 0 & 0 & 0 \\ 0 & J_1 & 0 \\ 0 & 0 & J_1 \end{bmatrix}$$

$$\mathbf{J}_2 = \begin{bmatrix} J_{2xx} & 0 & 0 \\ 0 & J_{2yy} & 0 \\ 0 & 0 & J_{2zz} \end{bmatrix} \approx \begin{bmatrix} 0 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_2 \end{bmatrix}$$

Through further simplification and manipulation, the two angular velocities of the arms are shown below.

$$\ddot{\theta}_1 = \frac{\left(\begin{bmatrix} -\hat{J}_2 b_1 \\ m_2 L_1 l_2 \cos(\theta_2) b_2 \\ -\hat{J}_2^2 \sin(2\theta_2) \\ -(1/2)\hat{J}_2 m_2 L_1 l_2 \cos(\theta_2) \sin(2\theta_2) \\ \hat{J}_2 m_2 L_1 l_2 \sin(\theta_2) \end{bmatrix}^T \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_1 \dot{\theta}_2 \\ \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \end{bmatrix} + \begin{bmatrix} \hat{J}_2 \\ -m_2 L_1 l_2 \cos(\theta_2) \\ (1/2) m_2^2 l_2^2 L_1 \sin(2\theta_2) \end{bmatrix}^T \begin{bmatrix} \tau_1 \\ \tau_2 \\ g \end{bmatrix} \right)}{\left(\hat{J}_0 \hat{J}_2 + \hat{J}_2^2 \sin^2(\theta_2) - m_2^2 L_1^2 l_2^2 \cos^2(\theta_2) \right)}$$

$$\ddot{\theta}_2 = \frac{\left(\begin{bmatrix} m_2 L_1 l_2 \cos(\theta_2) b_1 \\ -b_2 \left(\hat{J}_0 + \hat{J}_2 \sin^2(\theta_2) \right) \\ m_2 L_1 l_2 \hat{J}_2 \cos(\theta_2) \sin(2\theta_2) \\ -(1/2) \sin(2\theta_2) \left[\hat{J}_0 \hat{J}_2 + \hat{J}_2^2 \sin^2(\theta_2) \right] \\ -(1/2) m_2^2 L_1^2 l_2^2 \sin(2\theta_2) \end{bmatrix}^T \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_1 \dot{\theta}_2 \\ \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \end{bmatrix} + \begin{bmatrix} -m_2 L_1 l_2 \cos(\theta_2) \\ \hat{J}_0 + \hat{J}_2 \sin^2(\theta_2) \\ -m_2 l_2 \sin(\theta_2) \left(\hat{J}_0 + \hat{J}_2 \sin^2(\theta_2) \right) \end{bmatrix}^T \begin{bmatrix} \tau_1 \\ \tau_2 \\ g \end{bmatrix} \right)}{\left(\hat{J}_0 \hat{J}_2 + \hat{J}_2^2 \sin^2(\theta_2) - m_2^2 L_1^2 l_2^2 \cos^2(\theta_2) \right)}$$

III. Controller Design and Simulation

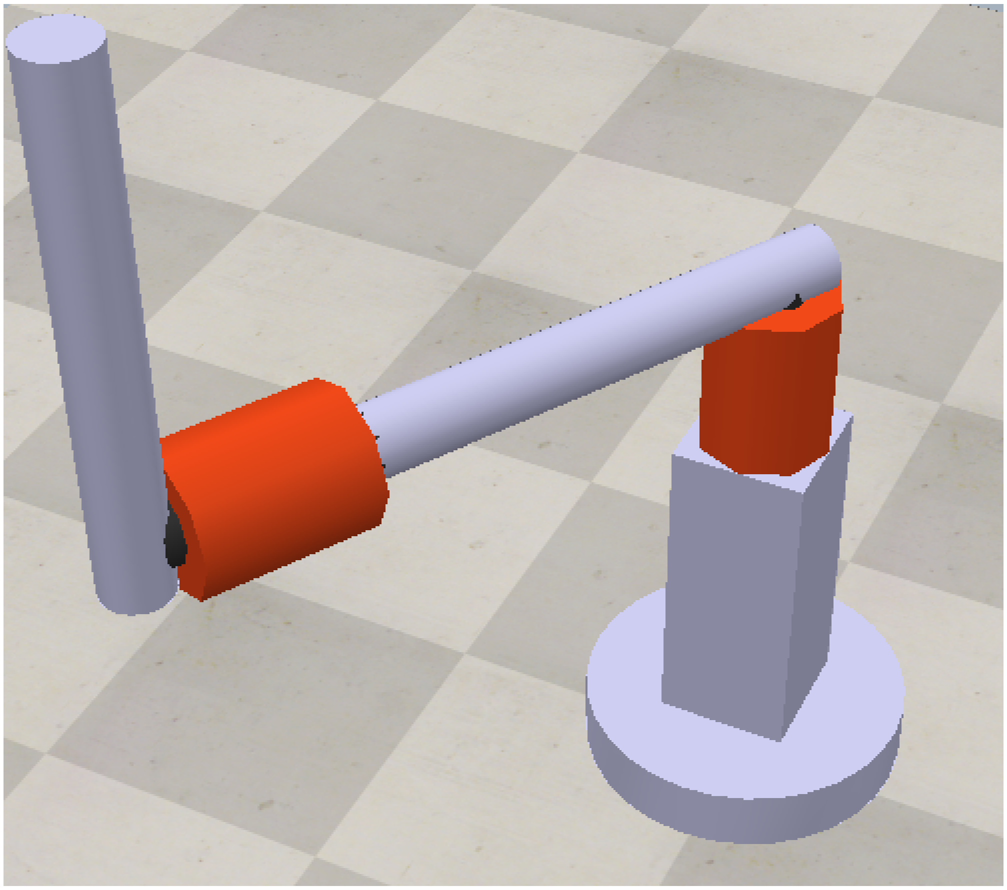

Figure 5. Coppeliasim Model of Furuta Pendulum System

IV. Appendix A: Simulation Code

```
% Group 1 member Daniel V, Isaac P, Phin, Emmett, Jacob


g = 9.81; % gravity m/s^2

% 1 - Arm ; 2 - Pendulum

m1 = 0.5; m2 = 0.5; % pendulum mass (kg)

L1 = 0.07; L2 = 0.16; % arms length (m)

M = 0.054; %

J = 3.5256e-4; % mass that is turning Pendulum inertia kg m^2

kb_p = 4.7940e-04 ;

kb_m = 6.75e-4 ;

ke = 0.5;

Re = 14.5; %Medido

alpha = J + (M+m1/3+m2)*L1^2;

beta = (M + m2/3)*L2^2;

gamma = (M + m2/2)*L2*L1;

sigma = (M + m2/2)*g*L2;


sims parameters

initial_state = pi;

Ts = 0.001;

dtDisc = 0.01;

Reference = [0 0 0 0];
```

```
Zn = 3; %Dead Zone

StepX = 10;

distrub = 12;

disturb = distrub*pi/180;


linearization

Ax matrix

A = zeros(4,4);

A(1,2) = 1;

A(2,3) = -(sigma*gamma)/(alpha*beta-gamma^2);

A(3,4) = 1;

A(4,3) = (alpha*sigma)/(alpha*beta-gamma^2);

% B matrix

B = zeros(4,2);

B(2,1) = beta/(alpha*beta-gamma^2);

B(2,2) = -gamma/(alpha*beta-gamma^2);

B(4,1) = -gamma/(alpha*beta-gamma^2);

B(4,2) = alpha/(alpha*beta-gamma^2);

% C matrix

C = [0 0 1 0;

 0 0 0 1];


pseudo_linear_system
```

```
pseudo Ax matrix

Ap = zeros(4,4);

Ap(1,2) = 1;

Ap(2,1) = 0; Ap(2,2) = -B(2,1)*(ke^2/Re + kb_m);

Ap(2,3) = A(2,3); Ap(2,4) = -B(2,2)*kb_p;

Ap(3,4) = 1;

Ap(4,1) = 0; Ap(4,2) = -B(4,1)*(ke^2/Re + kb_m);

Ap(4,3) = A(4,3); Ap(4,4) = -B(4,2)*kb_p;

% Pseudo B matrix

Bp = zeros(4,1);

Bp(2) = B(2,1)*ke/Re;

Bp(4) = B(4,1)*ke/Re;

% Controlability and Observability

Control = rank(ctrb(Ap,Bp));

Observ = rank(obsv(Ap,C));

Q = [0.1 0 0 0; 0 0.01 0 0; 0 0 100 0; 0 0 0 10];

R = 10;

[K, ~, ~] = lqr(Ap,Bp,Q,R);


while down (pi)

pseudo A matrix

Ap2 = zeros(4,4);

Ap2(1,2) = 1;
```

```
Ap2(2,1) = 0; Ap2(2,2) = -B(2,1)*(ke^2/Re + kb_m);

Ap2(2,3) = A(2,3); Ap2(2,4) = B(2,2)*kb_p;

Ap2(3,4) = 1;

Ap2(4,1) = 0; Ap2(4,2) = B(4,1)*(ke^2/Re + kb_m);

Ap2(4,3) = -A(4,3); Ap2(4,4) = -B(4,2)*kb_p;

% pseudo B matrix

Bp2 = zeros(4,1);

Bp2(2) = B(2,1)*ke/Re;

Bp2(4) = -B(4,1)*ke/Re;


K2 = place(Ap2,Bp2,[-5 -4 -2+2j -2-2j]);

R2 = 1;

Q2=[1 0 0 0; 0 10 0 0; 0 0 1000 0;  0 0 0 10];

[K2, ~, E] = lqr(Ap2,Bp2,Q2,R2);


break in code

%The following  is for the Furuta Pendulums broken donw even more

sim=remApi('remoteApi'); % using the prototype file (remoteApiProto.m)

sim.simxFinish(-1); % just in case, close all opened connections

   clientID=sim.simxStart('127.0.0.1',19999,true,true,5000,5);

if (clientID>-1)

else

   disp('Failed connecting to remote API server');
```

```
end
    sim.delete(); % call the destructor!

    disp('Program ended');
view(135,20) %staring point
AL = 5; %linits of graphs
axis([-AL AL -AL AL -AL AL]);
grid on
L1=3; % arm length
L2=2; %pendulum length
Xh=[0 ; L1]';
Yh=[0 ; 0]';
Zh=[0 ; 0]';
Xv=[Xh(2) ; L1]';
Yv=[Yh(2) ; 0]';
Zv=[Zh(2) ; -L2]';
hold on
Harm = fill3(Xh,Yh,Zh,'b');
Varm = fill3(Xv,Yv,Zv,'g');
s=8;
M=scatter3(Xv(2),Yv(2),Zv(2),s,'filled','MarkerFaceColor','b','MarkerEdgeColor','k');
theta=0;
phi=0;
```

```
c = [0 0 0];
% video
v = VideoWriter('simulation');
open(v);
TXT=title('Time: ');
for t=1:10:size(sind(theta),1)
 TXT2=sprintf('Time:%.2f',sind(t));
 set(TXT,'String',TXT2);
 %
 phi=sim(t);
 theta =-1*sim(t);
 %
 Xh(2)= L1*cos(phi);
 Yh(2)=L1*sin(phi);
 %
 Xva = 0;
 Yva = L2*sin(theta);
 Zva = -L2*cos(theta);
 %
 Xvb = Xva*cos(phi)-Yva*sin(phi)+L1*cos(phi);
 Yvb = Xva*sin(phi)+Yva*cos(phi) + L1*sin(phi);
 Zvb = Zva;
 %
```

```
Xv=[Xh(2);Xvb]';

Yv=[Yh(2);Yvb]';

Zv=[ 0 ;Zvb]';

%

set(Harm,'XData',Xh);

set(Harm,'YData',Yh);

set(Harm,'ZData',Zh);

%

set(Varm,'XData',Xv);

set(Varm,'YData',Yv);

set(Varm,'ZData',Zv);

%

rem(t,30)

%

%tracer lines of the pendulum

%scatter3(Xv(2),Yv(2),Zv(2),s,'filled','MarkerFaceColor',c,'MarkerEdgeColor','k');

%

set(M,'XData',Xv(2));

set(M,'YData',Yv(2));

set(M,'ZData',Zv(2));

%

%save video

frame = getframe(gcf);
```

```
 writeVideo(v, frame);
 if(simt(t) > 5) %break after given time
 break;
 end
end
close(v);
```

V. References

Apkarian, J., Lacheray, H., & Martin, P. (2012). *Student Workbook Inverted Pendulum Experiment for Matlab®/Simulink® Users*. Quanser.

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