GPS - HW3

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Problem I

This problem is to look at noise models.

Part A

Show mathematically that the error from the sum OR difference of two independent random measurements created from $y = 3a \pm 4b$, where a and b are both zero mean with variance σ_a^2 and σ_b^2 results in:

$$\mu_y = 0 \text{ and } \sigma_y = \sqrt{9\sigma_a^2 + 16\sigma_b^2}$$

Perform a 1000 run Monte-Carlo simulation to verify your results. Plot a histogram of the Monte-Carlo simulation to verify the output y is Gaussian.

Solution

The mean of y is the expectation of y. Following this algebraically, μ_y is

$$E[y] = E[3a + 4b]$$
$$E[y] = E[3a] + E[4b]$$
$$E[y] = 3E[a] + 4E[b]$$
$$\mu_y = 3\mu_a + 4\mu_b = 0$$

The standard deviation of y is the square root of the expectation of $(y - \overline{y})^2$. Following this algebraically, σ_y is

$$E\left[(y-\overline{y})^2\right] = E\left[y^2\right]$$
$$E\left[y^2\right] = E\left[(3a+4b)^2\right]$$
$$E\left[y^2\right] = E\left[3a^2+48ab+4b^2\right]$$
$$E\left[y^2\right] = 9E\left[a^2\right]+48E\left[ab\right]+16E\left[b^2\right]$$
$$E\left[y^2\right] = 9E\left[a^2\right]+48E\left[a\right]E\left[b\right]+16E\left[b^2\right]$$
$$\sigma_y^2 = 9\sigma_a^2+48\mu_a\mu_b+16\sigma_b^2$$
$$\sigma_y = \sqrt{9\sigma_a^2+16\sigma_b^2}$$

The expectation of ab can be separated as they are uncorrelated, and then the term reduces to zeros as the variables are zero mean. Also, the expectation of a^2 and b^2 is equal to their variance as they are zero mean.
The figure below shows the histogram of the Monte-Carlo Simulation.

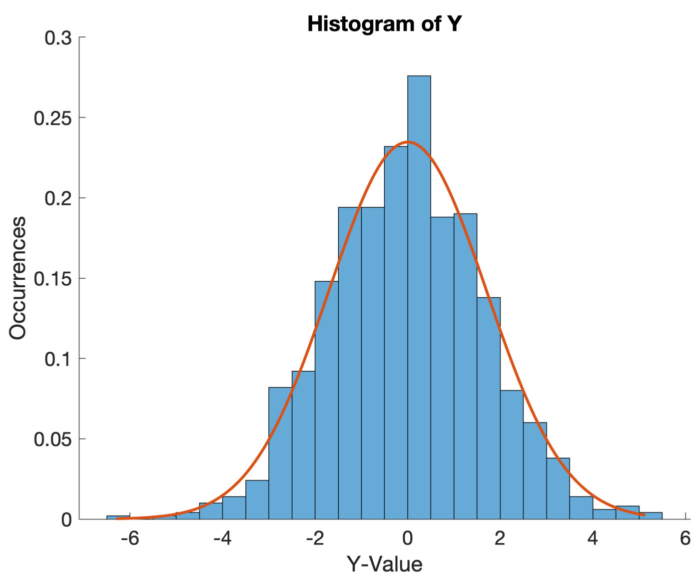

Figure 1: Histogram of the Monte Carlo Simulation

Problem II

Perform a 1000 run Monte-Carlo simulation (of 10 minutes) to look at the error growth of a random walk (integrated white noise). Use a white noise with 1-sigma value of 0.1 and 0.01 and compare the results. Plot the mean and standard deviation of the Monte-Carlo simulation along with one run of the simulation (show that the random walk is zero mean with a standard deviation is $\sigma_{\int w} = \sigma_w \Delta t \sqrt{k} = \sigma_w \sqrt{t \Delta t}$ (where k is the sample number).

Solution

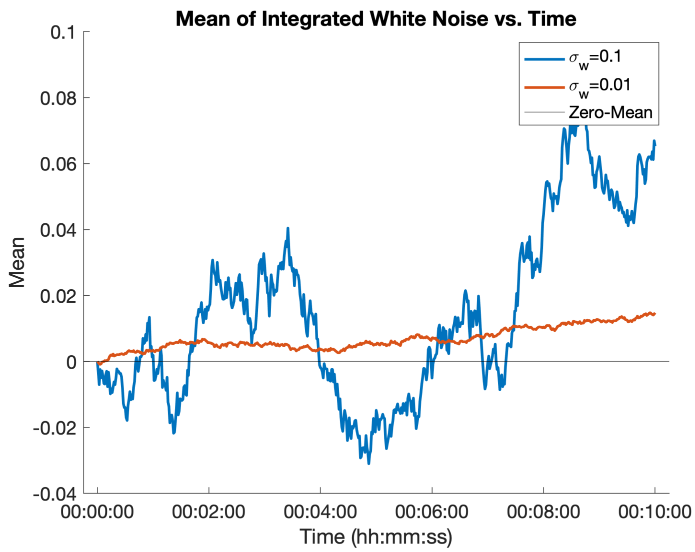

Figure 2: Mean of the Monte Carlo Simulation of Random Walk

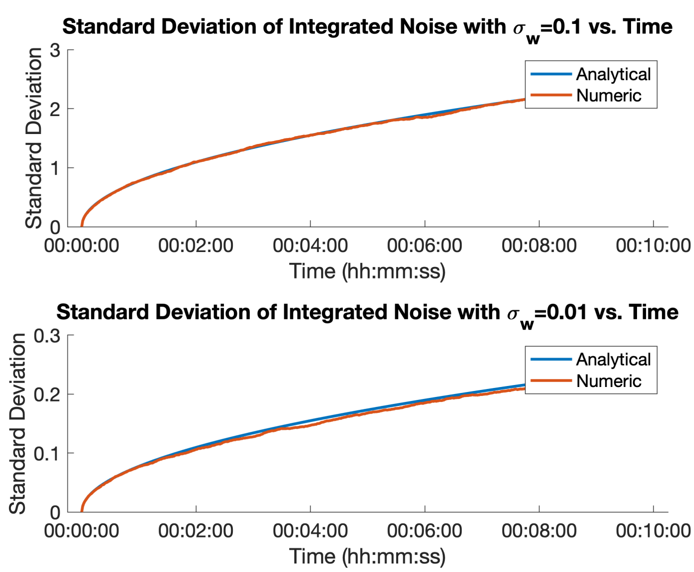

Figure 3: Standard Deviation of the Monte Carlo Simulation of Random Walk

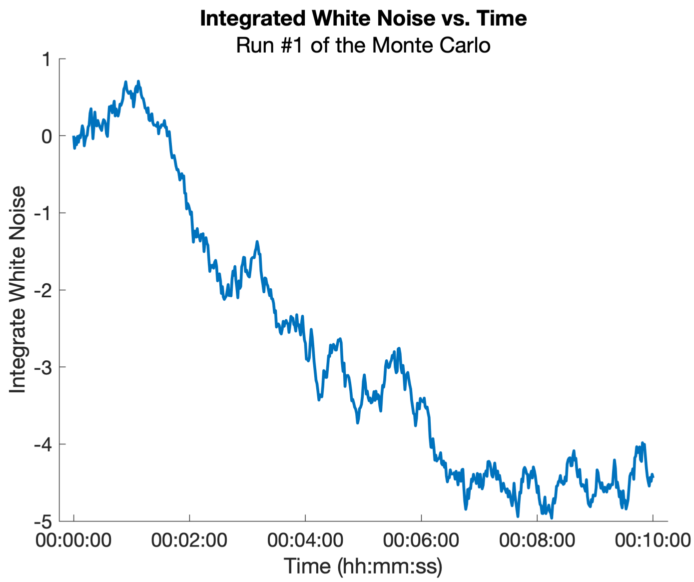

Figure 4: Integrated White Noise vs. Time (Run 1 of Monte Carlo)

Bonus

Provide a Histogram of the Monte-Carlo simulation at a few select time slots.

Solution

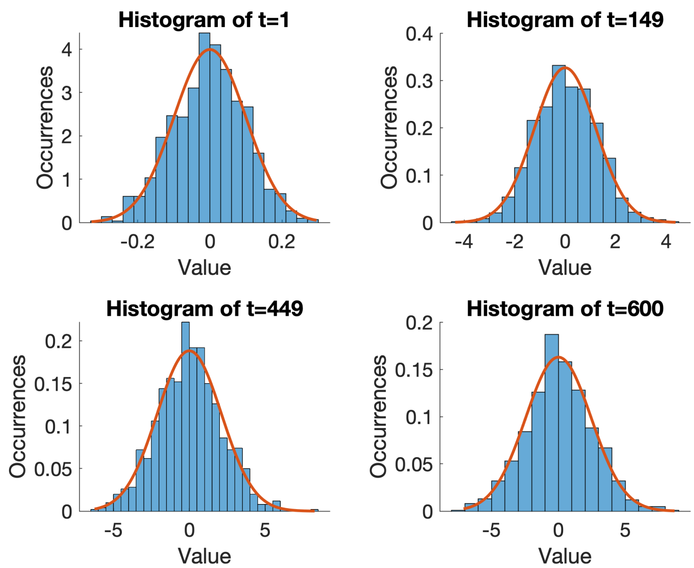

Figure 5: Histograms of Individual Time Points

Part B

Perform a 1000 Monte-Carlo simulation to look at the error growth of a 1st order Markov process (integrated filtered noise) of the form $\dot{x} = -\frac{1}{\tau}x + w$. Use the same noise characteristics as above and compare the results with a 1 second and 100 second time constant (this results in 4 combinations). Comment on how changing the time constant and changing the standard deviation of the noise effects the error. Show that the 1st order Markov process is zero mean with a standard deviation of is $\sigma_x = \sigma_w \Delta t \sqrt{\frac{A^{2t}-1}{A^2-1}}$ where $A = (1 - \frac{\Delta t}{\tau})$. Note that for a positive time constant (i.e. stable system) the standard deviation has a steady-state value.

Solution

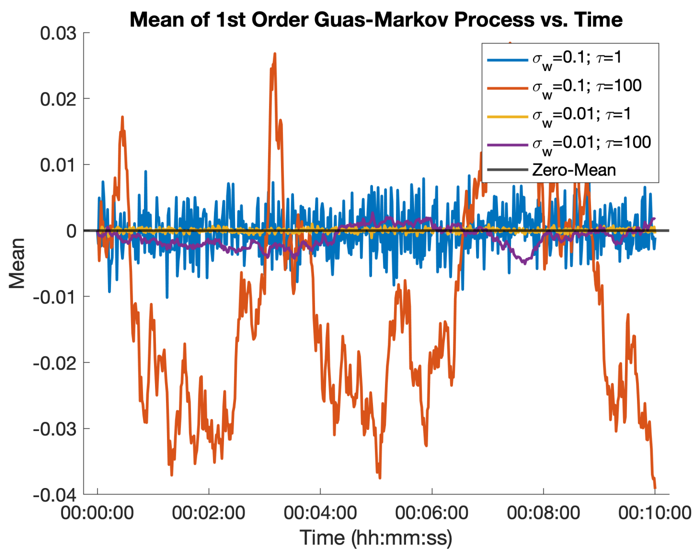

Figure 6: Mean of the Monte Carlo Simulation of Gauss Markov

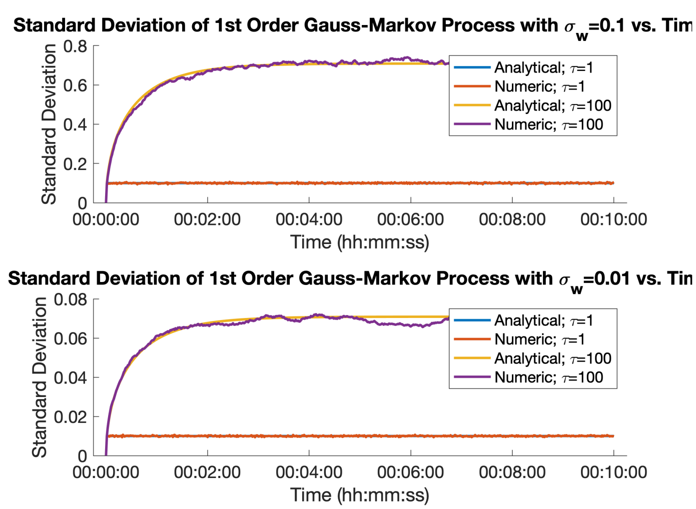

Figure 7: Standard Deviation of the Monte Carlo Simulation of Gauss Markov

As seen from the above figures, increasing τ increases both the rise time and the steady state of σ.

Problem III

Determine the expected uncertainty for an L1-L5 ionosphere free pseudorange measurement and L2-L5 ionosphere free pseudorange. Assuming all measurements have the same accuracy (L1, L2, L5) which will provide the best ionosphere estimate?

Solution

The equation for a pseudorange can be found in Equation (1)

$$\rho = r + c\delta t + I + M + T + \nu_{rcvr} \tag{1}$$

The only non-deterministic part of Equation (1) is the receiver noise, ν_{rcvr}. The equation for an L1+L2 Ionospheric-Free pseudorange is shown in Equation (2)

$$\rho_{IF} = \frac{f_{L1}^2}{f_{L1}^2 - f_{L2}^2}\rho_{L1} + \frac{f_{L2}^2}{f_{L1}^2 - f_{L2}^2}\rho_{L2} \tag{2}$$

If we subtract off the pseudorange itself, and consider the fact that the receiver noise is the only non-determinant part of the pseudorange equation, we can turn Equation (2) into the following:

$$\delta\rho_{IF} = \frac{f_{L1}^2}{f_{L1}^2 - f_{L2}^2}\nu_{rcvr} + \frac{f_{L2}^2}{f_{L1}^2 - f_{L2}^2}\nu_{rcvr} \tag{3}$$

The receiver noises are not frequency-dependent, so given the signals are read with the same receiver, the noises would be equal. Following Equation (3) through algebraically, the σ on $\delta\rho$ is:

$$\begin{aligned}
E\left[(\delta\rho_{IF})^2\right] &= E\left[\left(\frac{f_{L1}^2}{f_{L1}^2 - f_{L2}^2}\nu_{rcvr} + \frac{f_{L2}^2}{f_{L1}^2 - f_{L2}^2}\nu_{rcvr}\right)\right] \\
E\left[(\delta\rho_{IF})^2\right] &= \frac{f_{L1}^2}{f_{L1}^2 - f_{L2}^2}E\left[\nu_{rcvr}\right] + \frac{f_{L2}^2}{f_{L1}^2 - f_{L2}^2}E\left[\nu_{rcvr}\right] \\
\sigma_{\rho_{IF}}^2 &= \frac{f_{L1}^2}{f_{L1}^2 - f_{L2}^2}\sigma_{rcvr}^2 + \frac{f_{L2}^2}{f_{L1}^2 - f_{L2}^2}\sigma_{rcvr}^2 \\
\sigma_{\rho_{IF}}^2 &= \sigma_{rcvr}^2\left(\frac{f_{L1}^2}{f_{L1}^2 - f_{L2}^2} + \frac{f_{L2}^2}{f_{L1}^2 - f_{L2}^2}\right) \\
\sigma_{\rho_{IF}} &= \sqrt{\sigma_{rcvr}^2\left(\frac{f_{L1}^2}{f_{L1}^2 - f_{L2}^2} + \frac{f_{L2}^2}{f_{L1}^2 - f_{L2}^2}\right)}
\end{aligned}$$

Using the above equation for the standard deviation on the Ionospheric-Free pseudorange, an estimate of that value can be calculated for any combination of frequencies, just replacing f_{L1} and f_{L2} with the utilized frequencies. Doing this shows that L1+L5 is the best combination for obtaining Ionospheric-Free psuedoranges as it produces the smallest scale factor to the standard deviation. A table of the scale values on σ is shown below for the different frequency combinations.

Frequencies	Scale for σ
L1+L2	2.9793
L1+L5	2.5840
L2+L5	16.4918

Problem IV

Show that the differential GPS problem is linear. In other words derive the following expression:

$$\Delta\rho = \begin{bmatrix} -uv_x & -uv_y & -uv_z & 1 \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \\ c\delta t_{ab} \end{bmatrix}$$

Solution

To show the problem of differential GPS is linear, it begins with the nonlinear, matrix equation for the user's pseudorange as shown in Equation (4)

$$\delta\rho_u = \begin{bmatrix} -uv_x & -uv_y & -uv_z & 1 \end{bmatrix} \begin{bmatrix} \delta x_u \\ \delta y_u \\ \delta z_u \\ c\delta t_u \end{bmatrix} \tag{4}$$

Given that the base and the user are close enough, the unit vectors for the base and the user can be assumed to be the same. The user psuedorange, as shown above, can be differenced with a similar looking equation for the base resulting in the below equation.

$$\Delta\rho = \begin{bmatrix} -uv_x & -uv_y & -uv_z & 1 \end{bmatrix} \begin{bmatrix} \delta x_u - \delta x_b \\ \delta y_u - \delta y_b \\ \delta z_u - \delta z_b \\ c\delta t_u - c\delta t_b \end{bmatrix} \tag{5}$$

Because the unit vectors for the base and the user are assumed to be the same, they are factored out. Now, given that $\delta x = x - \hat{x}$ and the assumption that the base and user pseudorange equations are linearized around the same point($\hat{x_u} = \hat{x_b}$), the above equation reduces to:

$$\Delta\rho = \begin{bmatrix} -uv_x & -uv_y & -uv_z & 1 \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \\ c\delta t_{ub} \end{bmatrix}$$

where $r_x = x_u - x_b$.

Problem V

Set up your own 2D planar trilateration problem. Place the SVs at (0,300) (100,400), (700,400), and (800,300). Generate a range measurement for a base station at (400,0) and a user at (401,0).

Part A

Solve for the position of the user using 2 SVs and then 4 SVs assuming no clock errors. How does the PDOP change for the two cases?

Part A Comments

The PDOP was ~5x larger with only 2 SVs as compared to 4.

Part B

Solve for the position of the user assuming you need to solve for the user clock bias. What is the PDOP with all 4 satellites.

Part C

Calculate a differential solution between the base and user using a single difference model and assuming you must solve for a clock bias between the base station and user. What is the PDOP with all 4 satellites?

Part D

Calculate a differential solution between the base and user using a double difference model to remove the clock bias between the base station and user. What is the PDOP with all 4 satellites?

Part E

Assuming the range error is zero mean with unit variance, what is the order of accuracy in the above 4 solution methods?

Solution

Scenario	x	y	b
A-2SVs	401	∼0	-
A-4SVs	401	∼0	-
B-4SVs + Bias	401	∼0	∼0
C-DGPS-Single	401	-0.0014	-0.0005
D-DGPS-Double	401	-0.0014	-

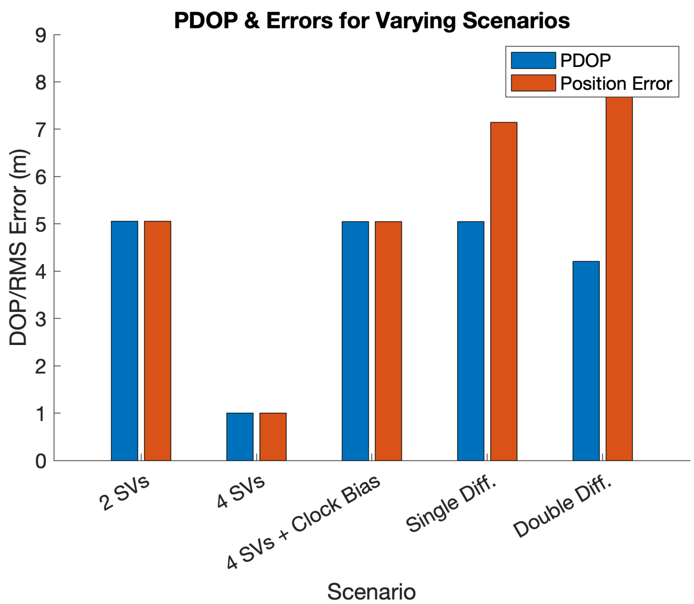

Figure 8: PDOP & Error for Varying Circumstances

Problem VI

(Bonus for Undergrads/Required for Grads). Repeat problem #4 using 4 and 8 SV positions from Lab #2 (4,7,8,9,16,21,27,30). Comment on any difference or similarities with the planar problem in #3.

Solution

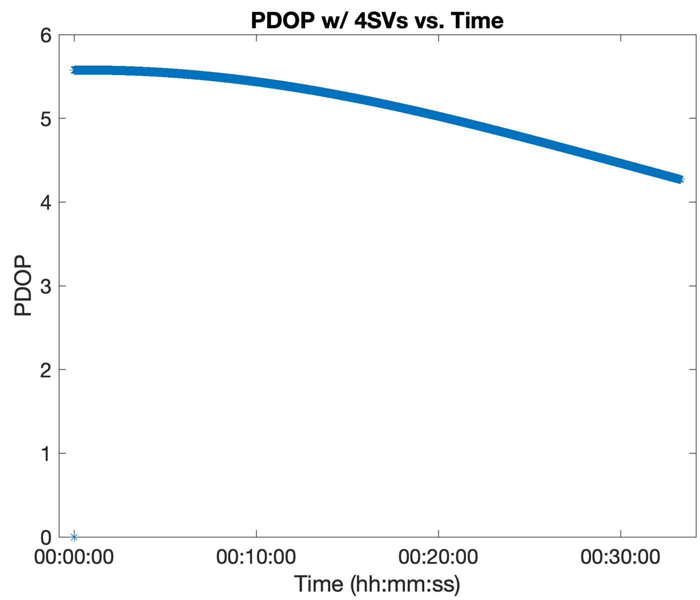

Figure 9: PDOP for Real Data w/ 4SVs

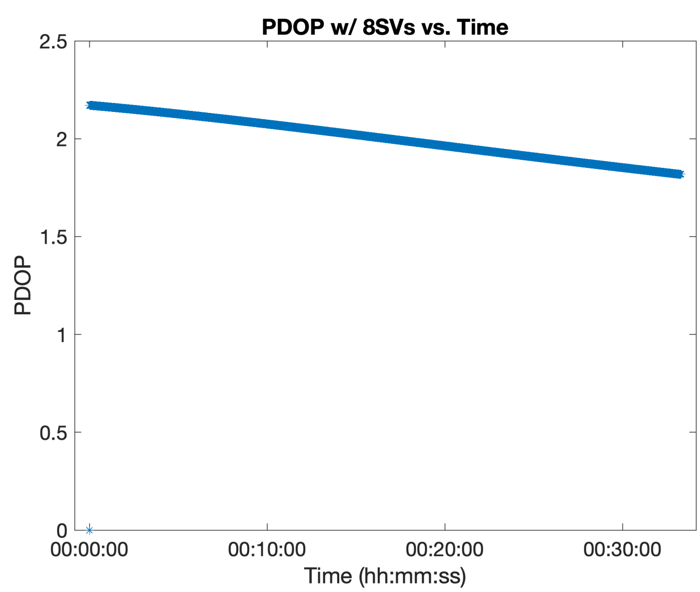

Figure 10: PDOP for Real Data w/ 8SVs

Though not exact, the 4SV scenario with the real satellite data has a PDOP 3x larger than the 8SV scenario. This is a similar difference to what was shown between the 4SV and 2SV scenarios in Part V.

Problem VII

Chapter 2, Problem 1a and 1b for PRN #4. Repeat 1a for PRN #7

Solution

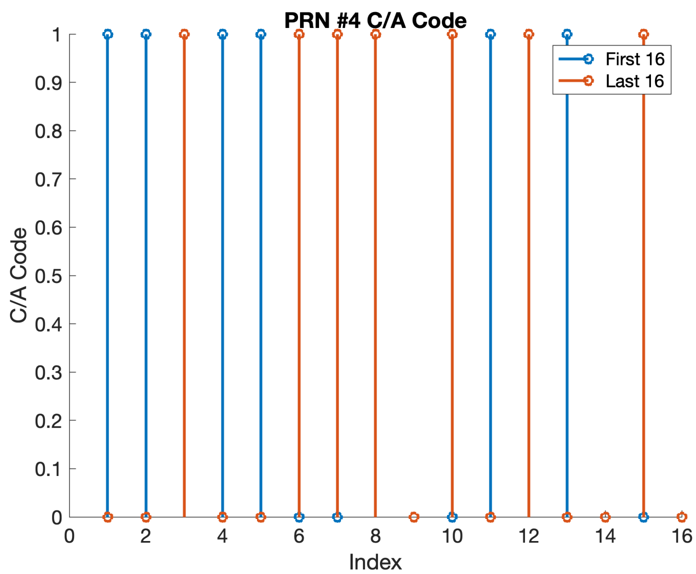

Figure 11: First 16 bits of C/A Code for PRN 4

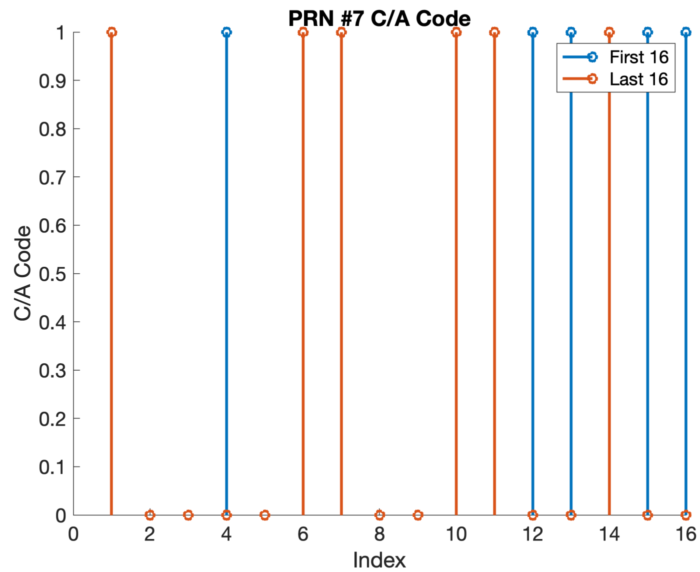

Figure 12: First 16 bits of C/A Code for PRN 7

Problem VIII

Using your PRN sequence for PRN 4 and 7, repeat problem #2 from HW#1. Compare the results to the results for your made up sequence.

Solution

The results from HW1 - Problem II greatly resemble the results of the same analysis done on the PRN signals. This is indicative of the fact that the PRNs contain a lot of the same properties as a random sequence which makes sense given they are named, "Pseudo Random Number."

HW1 - Problem II Results

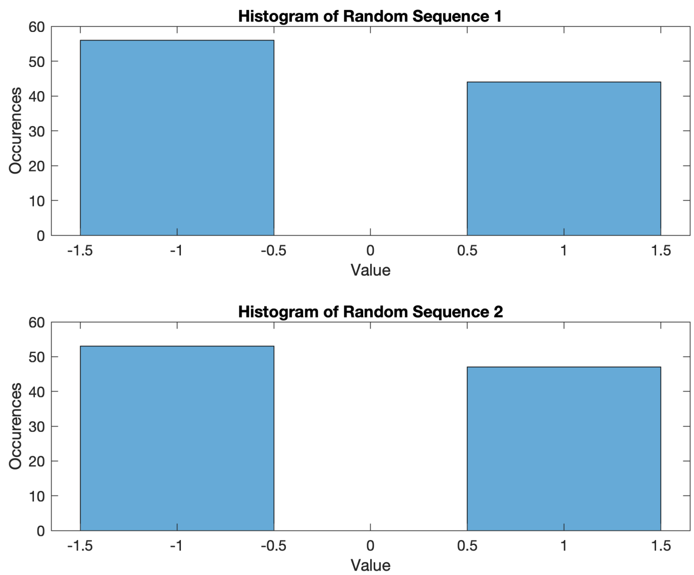

Figure 13: HW1 Histogram of Random Sequences

Figure 14: HW1 PSD of Random Sequences

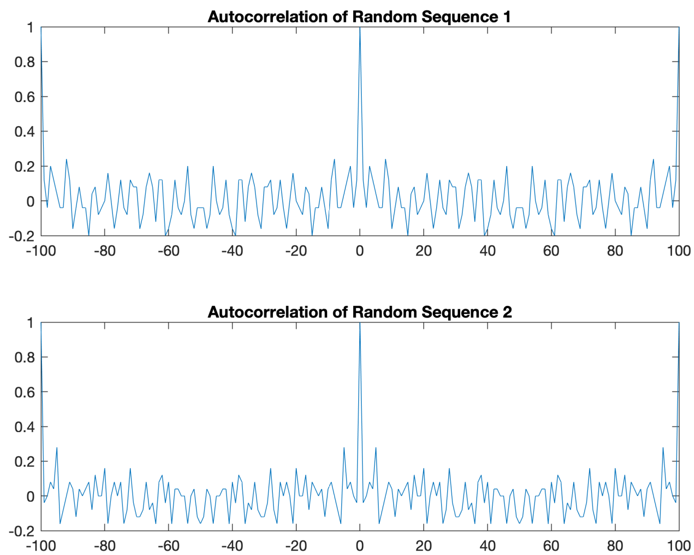

Figure 15: HW1 Autocorrelation of Random Sequences

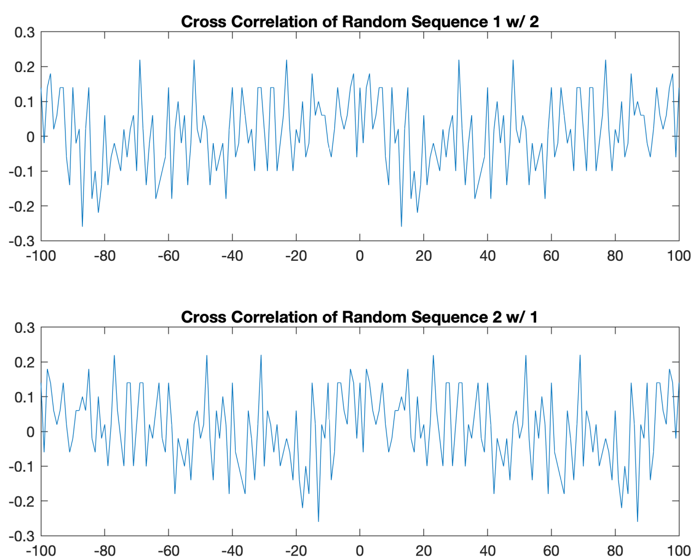

Figure 16: HW1 Cross Correlation of Random Sequences

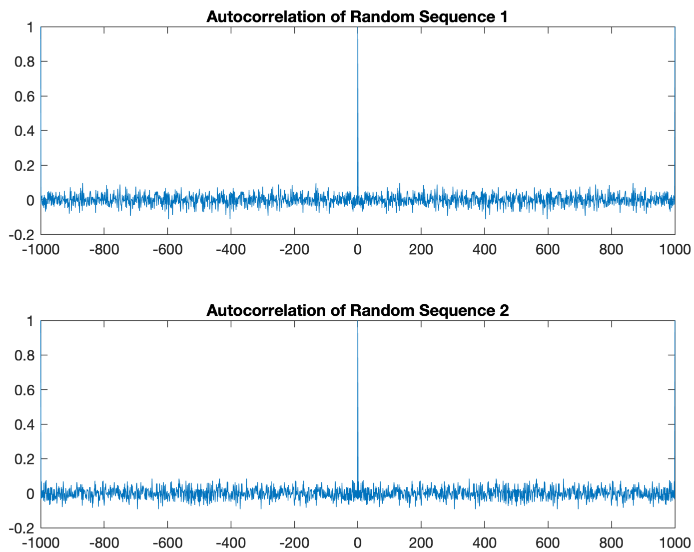

Figure 17: HW1 Cross Correlation of Long Random Sequences

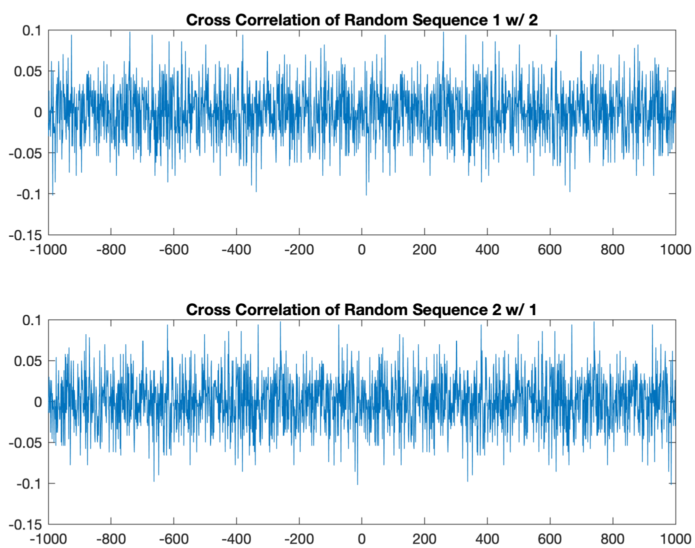

Figure 18: HW1 Cross Correlation of Long Random Sequences

Part A

Plot the histogram on each sequence

Solution

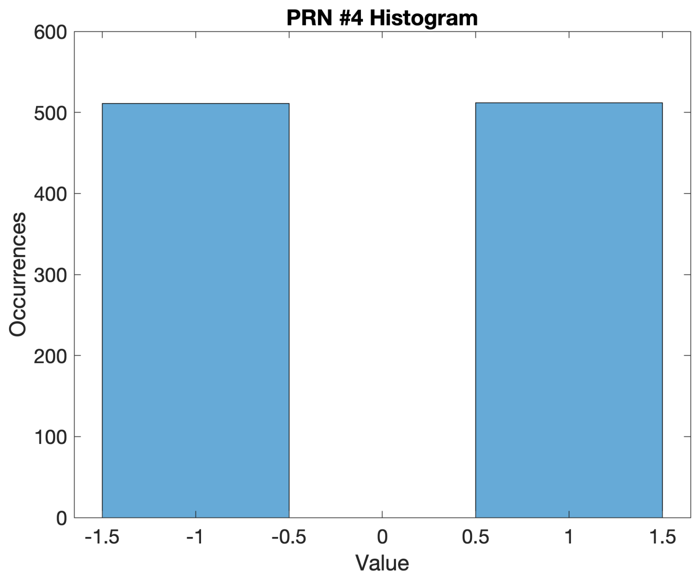

Figure 19: PRN 4 Histogram

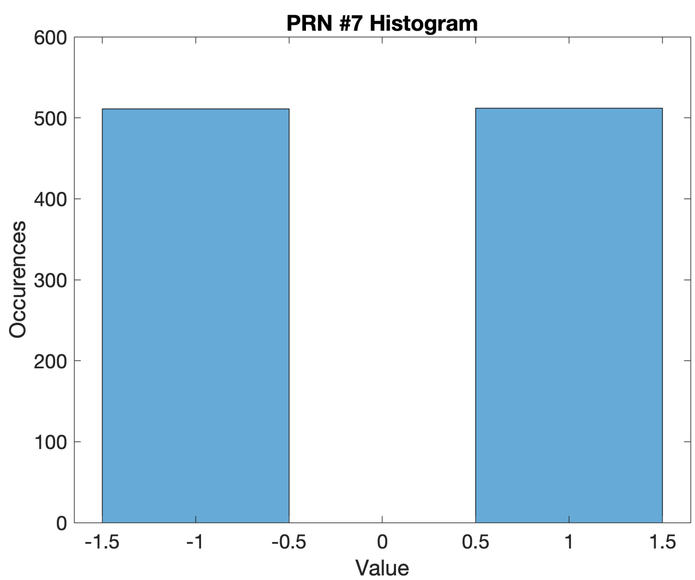

Figure 20: PRN 7 Histogram

Part B

Plot the spectral analysis on each sequence

Solution

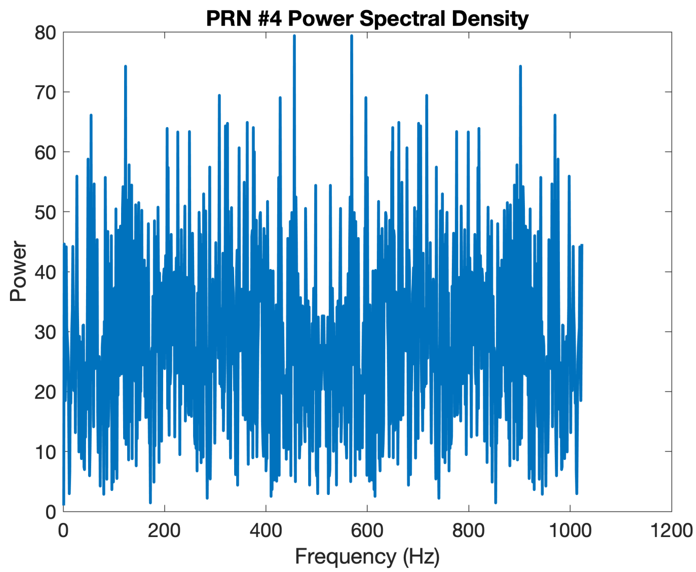

Figure 21: PRN 4 Power Spectral Density

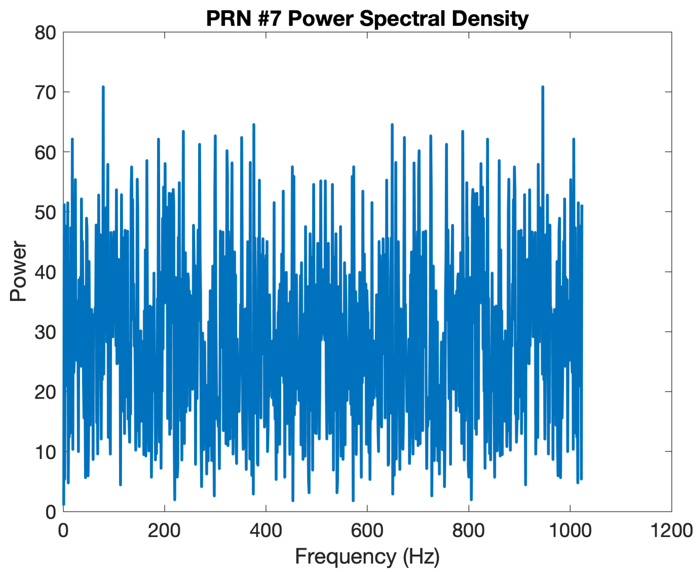

Figure 22: PRN 7 Power Spectral Density

Part C

Plot the auto correlation each sequence with itself (i.e. a sequence delay cross correlation)

Solution

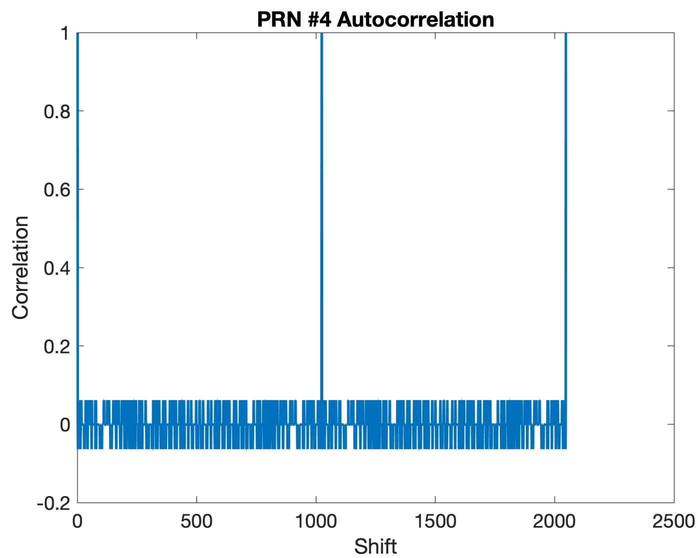

Figure 23: PRN 4 Autocorrelation

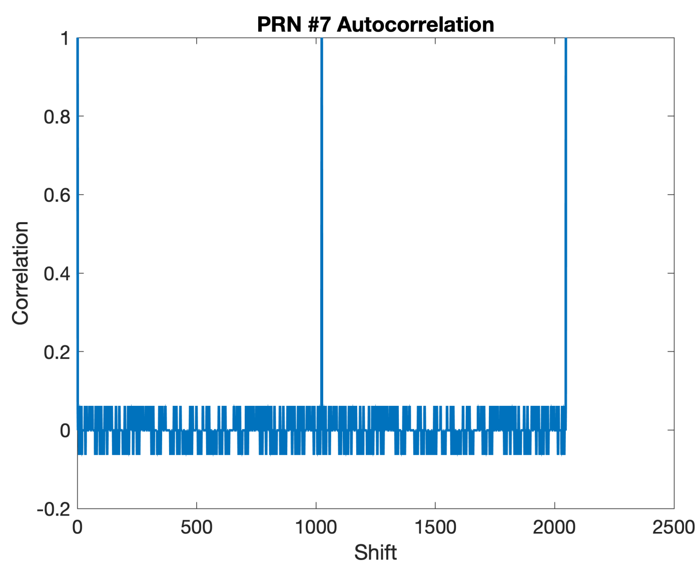

Figure 24: PRN 7 Autocorrelation

Part D

Plot the cross auto correlation between the two sequences

Solution

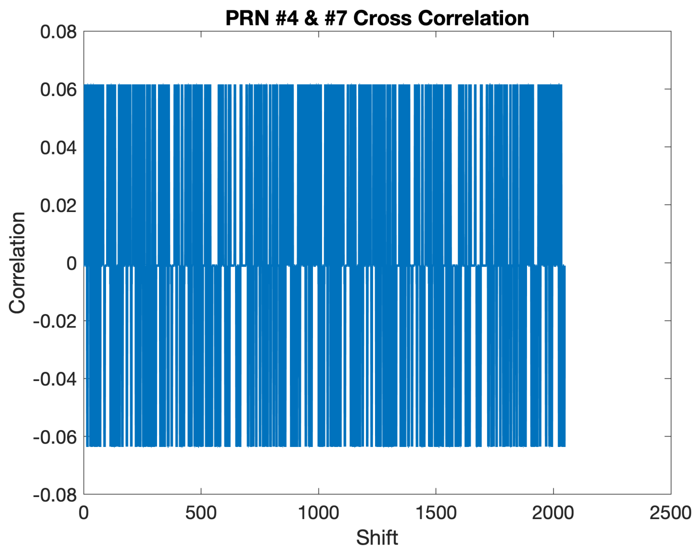

Figure 25: PRN 4 & 7 Cross Correlation

Problem IX

Take your C/A code from problem above (i.e. PRN #4 and #7) and multiply it times the L1 Carrier (your C/A code must be in the form -1 and +1). Perform a spectral analysis (magnitude) on the resultant signal. You will need to make sure to "hold" your C/A code bits for the correct length of time (I suggest using a sample rate 10x the L1 carrier frequency – meaning each chip of the C/A code will be used for 10 samples of the sine wave).

Solution

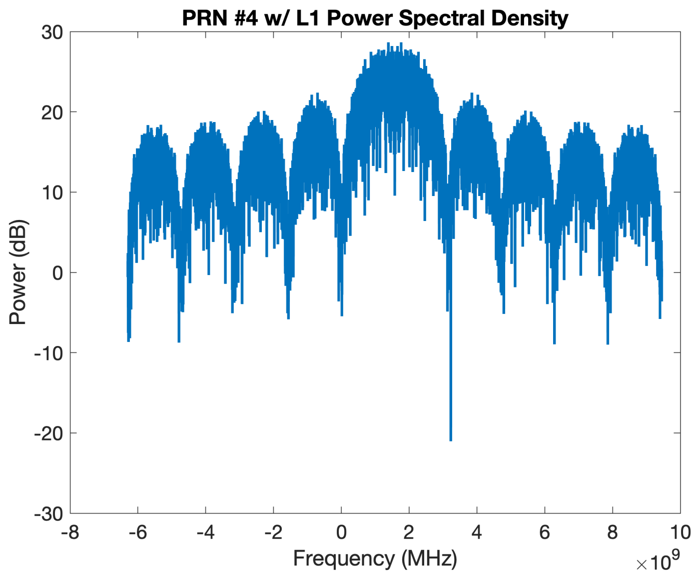

Figure 26: C/A Code for PRN 4 + L1 Carrier PSD

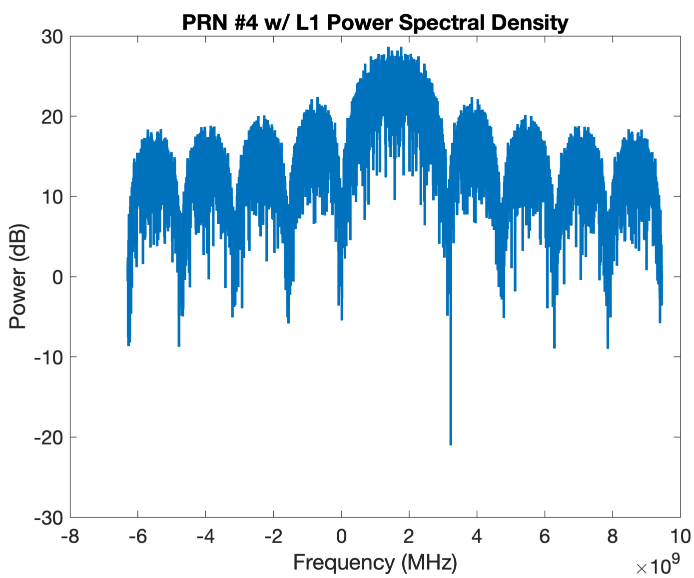

Figure 27: C/A Code for PRN 4 + L1 Carrier PSD