Optimal - HW2

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Question I

Two random variables x_1 and x_2 have a joint PDF that is uniform inside the circle (in the $x_1 - x_2$ plane) with radius 2, and zero outside the circle.

Part A

Find the math expression of the joint PDF function.

Solution

To begin solving for the mathematical expression for the joint PDF of a uniform circle with radius 2, the equation of a circle is needed.

$$(x - c_x)^2 + (y - c_y)^2 = r^2 \tag{1}$$

Given the above equation and the fact this circle is centered at the origin, the symbolic joint PDF is:

$$P_{x_1 x_2}(x_1, x_2) = \left\{ \begin{array}{ll} \frac{1}{\pi R^2}, & x_1^2 + x_2^2 \leq r^2 \\ 0, & x_1^2 + x_2^2 > r^2 \end{array} \right\} \tag{2}$$

Taking this symbolic equation and plugging in 2 for the radius, the joint pdf is:

$$P_{x_1 x_2}(x_1, x_2) = \left\{ \begin{array}{ll} \frac{1}{4\pi}, & x_1^2 + x_2^2 \leq 4 \\ 0, & x_1^2 + x_2^2 > 4 \end{array} \right\} \tag{3}$$

Part B

Find the conditional PDF $P_{x_2|x_1}(x_2|x_1 = 0.5)$?

Solution

Before performing the calculations to obtain the conditional PDF, the constraints on x_1 given x_2 need to be found and vice versa. These are as follows:

$$x_1 \leq \sqrt{4 - x_2^2} \tag{4}$$

$$x_2 \leq \sqrt{4 - x_1^2} \tag{5}$$

Given these, the bounds of x_2 given that $x_1 = 0.5$ can be found.

$$x_2 = \sqrt{4 - x_1^2}$$
$$x_2 = \sqrt{4 - 0.5^2}$$
$$x_2 = \sqrt{3.75} \quad x_2 = \pm 1.9365$$

With these bounds calculated, the PDF of x_1 can be calculated.

$$P_{x_1}(x_1) = \int_{-\infty}^{\infty} P_{x_1 x_2}(x_1, x_2) dx_2$$

$$P_{x_1}(x_1) = \int_{-1.9365}^{1.9365} \frac{1}{4\pi}(x_1, x_2) dx_2$$

$$P_{x_1}(x_1) = \frac{x_2}{4\pi}\Big|_{x_2 = 1.9365} - \frac{x_2}{4\pi}\Big|_{x_2 = -1.9365}$$

$$P_{x_1}(x_1) = 0.3082$$

Now with the PDF of x_1 solved for, the conditional PDF of $P_{x_2|x_1}(x_2|x_1 = 0.5)$ can be solved for.

$$P_{x_2|x_1}(x_2|x_1 = 0.5) = \frac{P_{x_1, x_2}(x_1, x_2)}{P_{x_1}(x_1 = 0.5)}$$

$$P_{x_2|x_1}(x_2|x_1 = 0.5) = \frac{\frac{1}{4\pi}}{0.3082}$$

$$P_{x_2|x_1}(x_2|x_1 = 0.5) = 0.2582$$

Part C

Are the two random variables uncorrelated?

Solution

To prove whether two random variables are uncorrelated, it needs to be proven that their covariance is 0 as shown below

$$E\left[(x_1 - \bar{x_1})(x_2 - \bar{x_2})^T\right] = 0 \tag{6}$$

First, the mean of the variables needs to be solved for.

$$E[x_1] = \int_{-\infty}^{\infty} x_1 P_{x_1}(x_1) dx_1$$

$$E[x_1] = \int_{-2}^{2} x_1 0.3082 dx_1$$

$$E[x_1] = \frac{x_1^2}{4\pi}\Big|_{x_1 = 2} - \frac{x_1^2}{4\pi}\Big|_{x_1 = -2}$$

$$E[x_1] = 0$$

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Given the symmetry of the joint PDF, it is clear that the solution above also applies to x_2. Now the covariances of x_1 and x_2 can be solved for.

$$E\left[(x_1)(x_2)^T\right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 P_{x_1,x_2}(x_1,x_2) dx_1 dx_2$$

$$E\left[(x_1)(x_2)^T\right] = \int_{-2}^{2} \int_{-2}^{2} x_1 x_2 P_{x_1,x_2}(x_1,x_2) dx_1 dx_2$$

$$E\left[(x_1)(x_2)^T\right] = \int_{-2}^{2} \int_{-2}^{2} x_1 x_2 \frac{1}{4\pi} dx_1 dx_2$$

$$E\left[(x_1)(x_2)^T\right] = \frac{x_1^2 x_2^2}{4\pi}\bigg|_{x_1=2,x_2=2} - \frac{x_1^2 x_2^2}{4\pi}\bigg|_{x_1=-2,x_2=-2}$$

$$E\left[(x_1)(x_2)^T\right] = 0$$

Because the covariance of x_1 and x_2 is equal to 0, the two variables are uncorrelated.

Part D

Are the two random variables statistically independent?

Solution

To prove whether two random variables are independent, it needs to be proven that:
$$P_{x_1}(x_1)P_{x_2}(x_2) = P_{x_1,x_2}(x_1,x_2) \tag{7}$$
Plugging in the values calculated previously into the above equation yields:

$$(0.3082)(0.3082) \neq \frac{1}{4\pi}$$

Because these two are not equal, the two random variables are not independent.

Question II

The station process $x(t)$ has an autocorrelation function of the form:
$$R_x(\tau) = \sigma^2 e^{\beta|\tau|}$$
Another process $y(t)$ is related to $x(t)$ by the deterministic equation:
$$y(t) = ax(t) + b$$
where the constants a and b are known.

Part A

What is the autocorrelation function for $y(t)$?

Solution

The autocorrelation function of $y(t)$ is calculated as follows:

$$R_y(\tau) = E\left[y(t)y(t+\tau)\right] = E\left[(ax(t)+b)(ax(t+\tau)+b)\right]$$
$$R_y(\tau) = E\left[a^2 x(t)x(t+\tau) + abx(t) + abx(t+\tau) + b^2\right]$$
$$R_y(\tau) = a^2 E\left[x(t)x(t+\tau)\right] + abE\left[x(t)\right] + abE\left[x(t+\tau)\right] + b^2$$
$$R_y(\tau) = a^2 R_x + abx\bar{(t)} + abx(\bar{t} + \tau) + b^2$$
$$R_y(\tau) = a^2 \sigma^2 e^{\beta|\tau|} + 2ab\mu_x + b^2$$

One note about this calculation is that because $x(t)$ is a stationary process, $E\left[x(t)\right] = E\left[x(t+\tau)\right] = \mu_x$.

Part B

What is the crosscorrelation function $R_{xy} = E\left[x(t)y(t+\tau)\right]$

Solution

The crosscorrelation function of $x(t)$ and $y(t)$ is calculated as follows:

$$R_{xy}(\tau) = E\left[x(t)y(t+\tau)\right] = E\left[x(t)ax(t+\tau) + bx(t)\right]$$
$$R_{xy}(\tau) = aR_x + b\mu_x$$
$$R_{xy}(\tau) = a\sigma^2 e^{-\beta|\tau|} + b\mu_x$$

Question III

Use least squares to identify a gyroscope scale factor (a) and bias (b). Simulate the gyroscope using:

$$g(k) = ar(k) + b + n(k)$$
$$n \sim N(0, \sigma = 0.3deg/s)$$
$$r(k) = 100sin(\omega t)$$

Part A

Perform the least squares with 10 samples (make sure to pick ω so that you get one full cycle in 10 samples.)

Part B

Repeat part (a) 1000 times and calculate the mean and standard deviation of the estimate errors (this is known as a Monte Carlo Simulation). Compare the results to the theoretically expected mean and standard deviation.

Part C

Repeat part (a) and (b) using 1000 samples. What does the theoretical and Monte Carlo standard deviation of the estimated errors approach?

Part D

Set up the problem to run as a recursive least squares and plot the coefficients and theoretical standard deviation of the estimate error and the actual estimate error as a function of time.

Question IV

Least Squares for System I.D. Simulate the following discrete system with a normal random input and output noise:

$$G(z) = \frac{0.25(z-0.8)}{z^2 - 1.90z + 0.95}$$

Part A

Develop the H matrix for the least squares solution.

Part B

use least squares to estimate the coefficients of the above Transfer Function. How good is the fit? Plot the bode response of the I.D. TF and the simulated TF on the same plot. How much relative noise has been added (SNR - signal to noise ratio), plot y and Y on the same plot.

Part C

Repeat the estimation orocess about 10 times using new values for the noise vector each time. Compute the mean and standard deviation of your parameter estimates. Compare the computed values of the parameter statistics with those predicted by the theory based on the known value of the noise statistics.

Part D

Now use sigma between 0.1 and 1.0 and repeat parts b and c.

Part E

What can you conclude about using least squares for sys id with large amounts of noise?

Question V

Justification of white noise for certain problems. Consider two problems: (i) Simple first order low-pass filter with bandlimited white noise as the input: $y = G(s)\omega$, so that $S_y(j\omega) = |G(j\omega)|^2 S_w(j\omega)$, and the noise has the PSD:

$$S_1(\omega) = \left\{ \begin{array}{ll} A, & |\omega| \leq \omega_c \\ 0, & |\omega| > \omega_c \end{array} \right\}$$

$$G(s) = \frac{1}{T_w s + 1}$$

(ii) The same low pass system, but with pure white noise as the input.

$$S_2(\omega) = A, \forall \omega$$

$$G(s) = \frac{1}{T_w s + 1}$$

The first case seems quite plausible, but the second case has an input with infinite variance and so is not physically realizable. However, the white noise assumption simplifies the system analysis significantly, so it is important to see if the assumption is justified. We test this with our two examples above.

Part A

Sketch the noise PSD and $|G(j\omega)|$ for a reasonable value of T_w and ω_c to compare the two cases.

Part B

Determine the $S_y(j\omega)$ for the two cases. Sketch these too.

Part C

Determine $\left[y^2\right]$ for the two cases.

Part D

Use these results to justify the following statement: $\widehat{\text{If}}$ the input spectrum is flat considerably beyond the system bandwidth, there is little error introduced by assuming that the input spectrum is flat out to infinity.