MECH3750

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Quizzes : Weeks 3, 6, 9

Content :

- Taylor Expansions
- Newtons Method
- Least Squares

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Taylor Series

a is the fixed point

$$f(x) = f(a) + \frac{f'(a)}{1!}\underbrace{(x-a)}_{\text{distance from the fixed point}} + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

function we want to approx

infinite series

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

But can't always have all infinte terms.

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \underbrace{O((x-a)^2)}_{\text{error term}}$$

error term

aka Higher Order Terms in mult-variable

OR

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a)$$

Accurate to 2^{nd} order

Note : $f_x = \frac{\partial f}{\partial x}$ etc.

What about multivariable functions ?

$f(x, y, \dots)$ OR $f(x_0, x_1, \dots, x_n) = f(\vec{x})$

two fixed points

$$f(x,y) = f(\underbrace{a, b}_{\text{two fixed points}}) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$\dots + \frac{1}{2}\left[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2 \right]$$

$$\dots + \text{H.O.T} \leftarrow O\left(\sqrt{(x-a)^2 + (y-b)^2}\right)$$

In general:

$$f(\vec{x}) = f(\vec{x}_0) + [\nabla f(\vec{x}_0)]^T(\vec{x} - \vec{x}_0) + \frac{1}{2}(\vec{x} - \vec{x}_0)^T H (\vec{x} - \vec{x}_0) + \text{H.O.T}$$

where Hessian is $H = \nabla(\nabla f) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$

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Newtons Method

Want to find roots of function:

$$f(x) = \emptyset$$

$$f(x_0 + h) \approx f(x_0) + f'(x_0)h$$

x_0: initial guess, h: step

Let our next guess $x_1 = x_0 + h$ be a "root".

$$\therefore \quad f'(x_0)h = -f(x_0)$$

$$x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$\boxed{\text{Repeat until } f(x_n) \approx 0}$

For multi-variable functions; and vector functions (aka multiple stacked equations).

$$\vec{f}(\vec{x}) = 0$$

$$\vec{f}'(\vec{x}_0) = \nabla \vec{f}(\vec{x}_0) = J \quad \{\text{Jacobian}\}$$

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots \\ \frac{\partial f_2}{\partial x_1} & \ddots & \vdots \\ \vdots & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}, \quad \vec{f}(\vec{x}) = \begin{bmatrix} f_1(\vec{x}) \\ f_2(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{bmatrix}$$

$$\vec{f}(\vec{x}_0 + \vec{h}) \approx \vec{f}(x_0) + J\vec{h} = \emptyset$$

$\therefore \quad J\vec{h} = -\vec{f}(\vec{x}_0)$ ← easier to solve numerically

OR $\quad \vec{h} = -J^{-1}\vec{f}(\vec{x}_0)$ ← easier to solve by hand IFF J is 2x2 matrix.

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Least Squares

Want to approximate some vector $\vec{f}$ with vectors $\vec{p}^{(1)}, \vec{p}^{(2)}, \dots, \vec{p}^{(n)}$

$$\vec{f} \approx \alpha_1 \vec{p}^{(1)} + \alpha_2 \vec{p}^{(2)} + \dots + \alpha_n \vec{p}^{(n)}$$

Define **positive** error :

square to ensure error is positive

$$E = \underbrace{\sum_i}_{\text{summing over components of the vectors}} \left(\alpha_1 p_i^{(1)} + \alpha_2 p_i^{(2)} + \dots + \alpha_n p_i^{(n)} - f_i \right)^2$$

Minimise error by setting derivatives to zero.

$$\frac{\partial E}{\partial \alpha_j} = 0 \quad \forall j \in 1, 2, \dots, n$$

$$\frac{1}{2} \frac{\partial E}{\partial \alpha_j} = \sum_i p_i^{(j)} \left(\alpha_1 p_i^{(1)} + \alpha_2 p_i^{(2)} + \dots + \alpha_n p_i^{(n)} - f_i \right)$$

$$0 = \underbrace{\sum_i p_i^{(j)} \left(\alpha_1 p_i^{(1)} + \alpha_2 p_i^{(2)} + \dots + \alpha_n p_i^{(n)} \right)}_{\text{unknowns}} - \underbrace{\sum_i p_i^{(j)} f_i}_{\text{knowns}}$$

In matrix notation :

$$\underbrace{\begin{bmatrix} p^{(1)} \cdot f \\ p^{(2)} \cdot f \\ \vdots \\ p^{(n)} \cdot f \end{bmatrix}}_{P^T f} = \underbrace{\begin{bmatrix} p^{(1)} \cdot p^{(1)} & p^{(1)} \cdot p^{(2)} & \dots & p^{(1)} \cdot p^{(n)} \\ p^{(2)} \cdot p^{(1)} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ p^{(n)} \cdot p^{(1)} & \dots & \dots & p^{(n)} \cdot p^{(n)} \end{bmatrix}}_{P^T P} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}}_{\alpha}$$

$$\vec{\alpha} = (P^T P)^{-1} P^T \vec{f}$$

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