Lecture Content Summaries for MECH3750

Travis Mitchell

Updated: 15 August, 2019

Week 02

With thanks to Alex Muirhead:

Tutors: Alex Muirhead, Bryce Hill, Kyle Mclaren, Luke Bartholomew, William Snell

Quizzes: Weeks 3, 6, 9

Content:

- Taylor Expansions;
- Newton's Method;
- Least Squares.

Taylor Series

$$\underbrace{f(x)}_{\text{function we want to approx}} = f(\overset{\text{a is the fixed point}}{a}) + \frac{f'(a)}{1!}\underbrace{(x-a)}_{\text{distance from the fixed point}} + \frac{f''(a)}{2!}(x-a)^2 + \underbrace{\dots}_{\text{infinite series}}$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

But can't always have all infinte terms.

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \underbrace{O((x-a)^2)}_{\text{error term}}$$

aka Higher Order Terms in mult-variable

OR

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a)$$

Accurate to 2^{nd} order

Note : $f_x = \frac{\partial f}{\partial x}$ etc.

What about multivariable functions ?

$$f(x, y, \dots) \quad \text{OR} \quad f(x_0, x_1, \dots, x_n) = f(\vec{x})$$

$$f(x,y) = f(\underbrace{a,b}_{\text{two fixed points}}) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$\dots + \frac{1}{2}\left[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right]$$

$$\dots + \text{H.O.T} \longleftarrow O\left(\sqrt{(x-a)^2 + (y-b)^2}\right)$$

In general:

$$f(\vec{x}) = f(\vec{x}_0) + [\nabla f(\vec{x}_0)]^T(\vec{x} - \vec{x}_0) + \frac{1}{2}(\vec{x} - \vec{x}_0)^T H (\vec{x} - \vec{x}_0) + \text{H.O.T}$$

where Hessian is $H = \nabla(\nabla f) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$

7mm A4

Newtons Method

Want to find roots of function:

$$f(x) = \emptyset$$

$$f(\underset{\uparrow\text{ initial guess}}{x_0} + \underset{\uparrow\text{ step}}{h}) \approx f(x_0) + f'(x_0)h$$

Let our next guess $x_1 = x_0 + h$ be a "root".

$$\therefore \quad f'(x_0)h = -f(x_0)$$

$$x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)} \qquad \boxed{\begin{array}{l}\text{Repeat until}\\ f(x_n) \approx 0\end{array}}$$

For multi-variable functions; and vector functions (aka multiple stacked equations).

$$\vec{f}(\vec{x}) = 0$$

$$\vec{f}'(\vec{x}_0) = \nabla\vec{f}(\vec{x}_0) = J \quad \{\text{Jacobian}\}$$

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots \\ \frac{\partial f_2}{\partial x_1} & \ddots & \vdots \\ \vdots & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}, \quad \vec{f}(\vec{x}) = \begin{bmatrix} f_1(\vec{x}) \\ f_2(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{bmatrix}$$

$$\vec{f}(\vec{x}_0 + \vec{h}) \approx \vec{f}(x_0) + J\vec{h} = \emptyset$$

$\therefore \quad J\vec{h} = -\vec{f}(\vec{x}_0) \quad \leftarrow$ easier to solve numerically

OR $\quad \vec{h} = -J^{-1}\vec{f}(\vec{x}_0) \quad \leftarrow$ easier to solve by hand IFF J is 2×2 matrix.

7mm A4

Least Squares

Want to approximate some vector $\vec{f}$ with vectors $\vec{p}^{(1)}, \vec{p}^{(2)}, \dots, \vec{p}^{(n)}$.

$$\vec{f} \approx \alpha_1 \vec{p}^{(1)} + \alpha_2 \vec{p}^{(2)} + \dots + \alpha_n \vec{p}^{(n)}$$

Define **positive** error:

square to ensure error is positive

$$E = \underbrace{\sum_i}_{\text{summing over components of the vectors}} \left(\alpha_1 p_i^{(1)} + \alpha_2 p_i^{(2)} + \dots + \alpha_n p_i^{(n)} - f_i \right)^2$$

Minimise error by setting derivatives to zero.

$$\frac{\partial E}{\partial \alpha_j} = 0 \quad \forall j \in 1, 2, \dots, n$$

$$\frac{1}{2} \frac{\partial E}{\partial \alpha_j} = \sum_i p_i^{(j)} \left(\alpha_1 p_i^{(1)} + \alpha_2 p_i^{(2)} + \dots + \alpha_n p_i^{(n)} - f_i \right)$$

$$0 = \sum_i p_i^{(j)} \underbrace{\left(\alpha_1 p_i^{(1)} + \alpha_2 p_i^{(2)} + \dots + \alpha_n p_i^{(n)} \right)}_{\text{unknowns}} - \underbrace{\sum_i p_i^{(j)} f_i}_{\text{knowns}}$$

In matrix notation:

$$\underbrace{\begin{bmatrix} p^{(1)} \cdot f \\ p^{(2)} \cdot f \\ \vdots \\ p^{(n)} \cdot f \end{bmatrix}}_{P^T f} = \underbrace{\begin{bmatrix} p^{(1)} \cdot p^{(1)} & p^{(1)} \cdot p^{(2)} & \dots & p^{(1)} \cdot p^{(n)} \\ p^{(2)} \cdot p^{(1)} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ p^{(n)} \cdot p^{(1)} & \dots & \dots & p^{(n)} \cdot p^{(n)} \end{bmatrix}}_{P^T P} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}}_{\alpha}$$

$$\vec{\alpha} = (P^T P)^{-1} P^T \vec{f}$$

Week 03

With thanks to Alex Muirhead:

Tutors: Alex Muirhead, Bryce Hill, Kyle Mclaren, Luke Bartholomew, William Snell

Quiz: Weeks 3 - 1pm

Content:

- Least Squares continued;
- Inner product;
- Orthogonal polynomials;
- Fourier Series.

Least Squares (cont.)

Can represent discrete points on some function $f(x)$ by a vector.

$$f(\vec{x}) = \left(f(x_0), f(x_1), \dots, f(x_n) \right)^T = \vec{f}$$

Approximate vector with basis "vectors" made from functions.

$$\vec{p}^{(1)} = \left(p^{(1)}(x_0), p^{(1)}(x_1), \dots, p^{(1)}(x_n) \right)^T$$

etc.

Can be any set of **linearly independent** functions $\{p^{(1)}, p^{(2)}, \dots p^{(m)}\}$

$$\therefore \quad f(\vec{x}) = \vec{f} \approx \sum_m \alpha_m \vec{p}^{(m)}$$

$$\begin{bmatrix} \vec{p}^{(1)} \cdot \vec{p}^{(1)} & \vec{p}^{(1)} \cdot \vec{p}^{(2)} & \cdots & \vec{p}^{(1)} \cdot \vec{p}^{(m)} \\ \vec{p}^{(2)} \cdot \vec{p}^{(1)} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \vec{p}^{(m)} \cdot \vec{p}^{(1)} & \cdots & \cdots & \vec{p}^{(m)} \cdot \vec{p}^{(m)} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_m \end{bmatrix} = \begin{bmatrix} \vec{f} \cdot \vec{p}^{(1)} \\ \vdots \\ \vdots \\ \vec{f} \cdot \vec{p}^{(m)} \end{bmatrix}$$

Same form as before.

Written more concisely as:

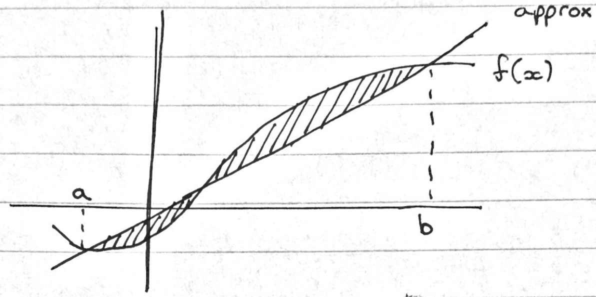
$$P^T P \vec{\alpha} = P^T \vec{f}$$

$$\downarrow$$

$$\vec{\alpha} = (P^T P)^{-1} P^T \vec{f}$$

7mm A4

What if we want to approximate a continuous function over an interval?



Define **positive** error based on the area between $f(x)$ and approximation.

$$E = \int_a^b \left(\sum_i^n \alpha_i p^{(i)}(x) - f(x) \right)^2 dx$$

Follow previous derivation, set $\partial_{\alpha_j} E = 0$

$$0 = \sum_i^n \int_a^b \alpha_i p^{(i)}(x)\, p^{(j)}(x) dx - \int_a^b p^{(j)}(x) f(x)\, dx$$

In matrix form:

$$\begin{bmatrix} \int_a^b p^{(1)} p^{(1)} dx & \cdots & \int_a^b p^{(1)} p^{(n)} dx \\ \vdots & \ddots & \vdots \\ \int_a^b p^{(n)} p^{(1)} dx & \cdots & \int_a^b p^{(n)} p^{(n)} dx \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \int_a^b p^{(1)} f\, dx \\ \vdots \\ \int_a^b p^{(n)} f\, dx \end{bmatrix}$$

7mm A4

Discrete vector form & continuous function forms of least squares very similar...

Generalise by introducing **inner product**.

$$\langle p, q \rangle$$

vectors ↙ ↘ functions

$$\vec{p} \cdot \vec{q} \qquad \int p(x) q(x)\, dx$$

Some properties:

Norm (or length/siz) $\Rightarrow \|p\| = \sqrt{\langle p, p \rangle}$
Distance $\Rightarrow d(p,q) = \|p - q\|$
Orthogonality $\Rightarrow \langle p, q \rangle \equiv 0$

Distance between two functions becomes:

$$d(p,q) = \sqrt{\int (p(x) - q(x))^2 \, dx}$$

$\therefore$ Least squares minimises distance squared!

There exist sets of linearly independent orthogonal functions. These simplify matrix form, as non-diagonal elements become $\emptyset$.

<u>Legendre Polynomials</u> $\displaystyle\int_0^1 p^{(i)} p^{(j)}\, dx \propto \delta_{ij} \equiv \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

$P_0(x) = 1$
$P_1(x) = x$
$P_2(x) = \frac{1}{2}(3x^2 - 1)$
$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

Don't need to remember this!

Trigonometric Functions

Specifically sine and cosine, as they can be represented as:

$$e^{ix} = \cos x + i \sin x$$

Note that orthogonality holds:

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \begin{cases} \pi & \text{if } m=n \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = \begin{cases} \pi & \text{if } m=n^{*} \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx = 0$$

* if $m, n > 0$

Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

↑ exactly equal for an infinite sum

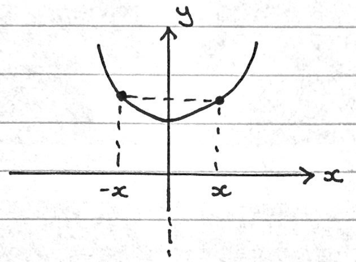
where $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\,dx$

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\,dx$

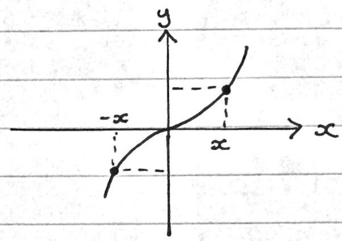
Shortcuts

Introduced **even** and **odd** functions.

Even



Odd



$$f(-x) = f(x)$$

$\left\{ \begin{array}{c} \text{Reflection Symmetry} \\ \text{about y-axis} \end{array} \right\}$

$$f(-x) = -f(x)$$

$\left\{ \begin{array}{c} \text{Rotation Symmetry} \\ \text{about origin} \end{array} \right\}$

Even x Even $\Rightarrow$ $f(-x)g(-x) = f(x)g(x) \Rightarrow$ Even

Odd x Odd $\Rightarrow$ $f(-x)g(-x) = (-1)^2 f(x)g(x) \Rightarrow$ Even

Even x Odd $\Rightarrow$ $f(-x)g(-x) = -f(x)g(x) \Rightarrow$ Odd

$$\int_{-a}^{a} \text{Even } dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

$$\int_{-a}^{a} \text{Odd } dx = \int_{-a}^{0} g(x) + \int_{0}^{a} g(x)\,dx = \int_{0}^{a} -g(x) + g(x)\,dx = \emptyset$$

$\therefore$ As cosine is even & sine is odd

if $f(x)$ is even:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

if $f(x)$ is odd:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

Week 04

Tutors: Luke Bartholomew, Bryce Hill, Kyle Mclaren, Travis Mitchell, Alex Muirhead, William Snell

Assessment:

- Quiz 1 results are available for collection;
- Next quiz week 6;
- Assignment 1 due week 7.

Content:

- Least Squares continued;
- Inner product;
- Orthogonal polynomials;
- Fourier Series.

1 Fourier Series continued

Example 1.1.
Find the Fourier Series of $f(x) = x^2$ on $[-\pi, \pi]$.

This example shows a key point to note (particularly for quizzes/exams), and that is to check if the function is *odd* or *even*! The first step here is to determine our Fourier coefficients:

$$\begin{aligned}
b_j &= \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x^2}_{even} \underbrace{\sin(jx)_{odd} dx} \\
&= 0 \\
a_j &= \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x^2}_{even} \underbrace{\cos(jx)_{even} dx} \\
&= \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos(jx) dx
\end{aligned}$$

The coefficients, a_j, can then be determined through integration by parts to be,

$$a_j = \left(\frac{4(-1)^j}{j^2}\right)$$

We now evaluate, a_0, and then we can determine the Fourier series,

$$\begin{aligned}
a_0 &= \frac{2}{\pi}\int_0^{\pi} x^2 dx \\
&= 2\pi^2/3
\end{aligned}$$

And therefore, as $f(x)$ is even,

$$\begin{aligned}
f(x) &= a_0/2 + \sum_{n=1}^{\infty} a_n cos(nx) \\
&= \pi^2/3 - \frac{4\cos(x)}{1^2} + \frac{4\cos(2x)}{2^2} - \frac{4\cos(3x)}{3^2} + \cdots
\end{aligned}$$

Remarks from Fourier Theorem, namely is we can restate the formulation as,

$$\begin{aligned}
f(x) &= \sum_{j=-n}^{n} c_j e^{ijx} \\
c_j &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-ijx} dx, \quad j = -n, \ldots, -1, 0, 1, \ldots, n
\end{aligned}$$

This makes use of Euler's formula (or we can also show with Taylor series) that, $e^{ijx} = \cos jx + i\sin(jx)$.

2 Discrete Fourier Transform