MECH3750 - Tutorial 3

Question 1.

Suppose $p_1,\ p_2,\ p_3$ are three vectors (or functions). Suppose we wish to find another vector (or function) $y = a_1 p_1 + a_2 p_2 + a_3 p_3$ which minimizes the distance squared:

$$d^2\left(y,\ f\right) = \|y - f\|^2 = (y - f, y - f)$$

Where d is the "distance" as introduced in lectures, and u, v is a well defined inner product.

Obtain $a_1,\ a_2,\ a_3$ (to find the best approximation) for the following cases:

(a) $p_1 = (1,\ 0,\ -1,\ 0)$, $p_2 = (1,\ 1,\ 1,\ 0)$, $p_3 = (1,\ -2,\ 1,\ 0)$. With inner product: $(u, v) = \sum_i u_i v_i$ and:

$$\text{(i) } f = (4,\ 0,\ 2,\ 0) \qquad \text{(ii) } f = (0,\ 0, 0,\ 1)$$

SOLUTIONS

In all following questions, the coefficients may be solved from the normal equations as:

$$\begin{bmatrix} (p_1,p_1) & (p_1,p_2) & (p_1,p_3) \\ (p_2,p_1) & (p_2,p_2) & (p_2,p_3) \\ (p_3,p_1) & (p_3,p_2) & (p_3,p_3) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} (p_1, f) \\ (p_2, f) \\ (p_3, f) \end{bmatrix} \tag{1}$$

(i) The inner product for vectors defines the *dot* product. Using Equation 1, and evaluating of each inner product, we obtain the diagonal matrix:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 x \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 6 \end{bmatrix}$$

So $a_1 = 1$, $a_2 = 2$ and $a_3 = 1$ and $y = p_1 + 2p_2 + p_3 = (4,\ 0,\ 2,\ 0) = f$ with $d^2 = 0$.

(ii) As the p vectors are unchanged, the matrix takes the same value as in (i), evaluation of each inner product on the RHS of 1, results in:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So $a_1 = 0$, $a_2 = 0$ and $a_3 = 0$ and $y = (0,\ 0,\ 0,\ 0) = f$ and $d^2 = 1$.

(b) $p_1 = \sin x$, $p_2 = \sin 2x$, $p_3 = \sin 3x$. With inner product: $(u, v) = \int_0^\pi u(x)v(x)\ \mathrm{d}x$ and:

$$\text{(i) } f(x) = \sin 2x + \sin 3x \qquad \text{(ii) } f(x) = \sin 4x$$

Hint: Use the result established in lectures that:

$$(\sin nx,\ \sin mx) = \begin{cases} \frac{\pi}{2} & m = n \\ 0 & m \neq n \end{cases}$$

SOLUTIONS

(i) Note that $f(x) = p_2 + p_3$.
Therefore: $a_1 = 0,\ a_2 = 1,\ a_3 = 1$ and $y = \sin 2x + \sin 3x = f$ with $d^2 = 0$.

(ii) The normal equations are:

$$\begin{bmatrix} \frac{\pi}{2} & 0 & 0 \\ 0 & \frac{\pi}{2} & 0 \\ 0 & 0 & \frac{\pi}{2} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} (\sin x,\ \sin 4x) \\ (\sin 2x,\ \sin 4x) \\ (\sin 3x,\ \sin 4x) \end{bmatrix}$$

However, using the identity for the inner product of sin terms, every element of the RHS vector is equal to zero, hence we obtain $a_1 = a_2 = a_3 = 0$ and therefore $y = 0$ with

$$d^2 = \int_0^{\pi} \sin^2 4x \ \mathrm{d}x = \frac{\pi}{2}$$

.

(c) $p_1 = 1$, $p_2 = x$, $p_3 = \frac{1}{2}\left(3x^2 - 1\right)$. With inner product: $(u, v) = \int_{-1}^{1} u(x)v(x) \ \mathrm{d}x$ and:

$$\text{(i) } f(x) = x^2 \qquad \text{(ii) } f(x) = 5x^3 - 3x$$

SOLUTIONS

From (1) we have the normal equations:

$$\begin{bmatrix} (p_1, p_1) & (p_1, p_2) & (p_1, p_3) \\ (p_2, p_1) & (p_2, p_2) & (p_2, p_3) \\ (p_3, p_1) & (p_3, p_2) & (p_3, p_3) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} (p_1, f) \\ (p_2, f) \\ (p_3, f) \end{bmatrix}$$

Now observe however, that $p_1,\ p_2,\ p_3$ are the *legendre polynomials* from lectures and Tutorial 2 (with subscript shifted up by 1). Recall from Tutorial 2, question 4c, we derived the following results:

$$\int_{-1}^{1} p_1 p_1 = 2$$
$$\int_{-1}^{1} p_1 p_2 = 0$$
$$\int_{-1}^{1} p_1 p_3 = 0$$
$$\int_{-1}^{1} p_2 p_2 = \frac{2}{3}$$
$$\int_{-1}^{1} p_3 p_3 = \frac{2}{5}$$

And also note $\int_{-1}^{1} p_2 p_3 \ \mathrm{d}x = 0$ since $p_2 p_3$ is odd. Therefore the normal equations simplify to:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{2}{5} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} (p_1, f) \\ (p_2, f) \\ (p_3, f) \end{bmatrix}$$

Where the property of inner products such that $(u, v) = (v, u)$ was used to express the remaining inner products in terms of known results.

(i). Substituting $p_1 = 1,\ p_2 = x,\ p_3 = \frac{1}{2}\left(3x^2 - 1\right), f = x^2$ into the above gives:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{2}{5} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} (1, x^2) \\ (x, x^2) \\ (\frac{1}{2}\left(3x^2 - 1\right), x^2) \end{bmatrix}$$

Evaluating the LHS inner products and solving the diagonal system results in:

$$a_1 = \frac{1}{3},\ a_2 = 0,\ a_3 = \frac{2}{3}$$

Therefore $y = x^2$ and $d^2 = 0$

(ii). With $f = 5x^3 - 3x$ we have:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{2}{5} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} (1, 5x^3 - 3x) \\ (x, 5x^3 - 3x) \\ (\frac{1}{2}\left(3x^2 - 1\right), 5x^3 - 3x) \end{bmatrix}$$

The first and third inner products can be observed to be equal to 0, since the resulting integrals are odd. Evaluating the second inner product as an integral using wolfram alpha reveals that $(x, 5x^3 - 3x) = 0$ also, and therefore:

$$a_1 = 0,\ a_2 = 0,\ a_3 = 0$$

Therefore $y = 0$ and $d^2 = \int_{-1}^{1} \left(5x^3 - 3x\right)\ \mathrm{d}x = 0$

(d) Comment on why $d^2(y,\ f) = (y - f, y - f) = 0$ in all (i) cases. Comment on (ii) results and why all these cases are non zero.

SOLUTIONS

In all (i) cases, f is in the set covered by $\{p_1,\ p_2,\ p_3\}$, thus it can be approximated exactly by y.

In all (ii) cases, we find $y = 0$. Problem a(ii) shows what is going on. The vector f is orthogonal to $\{p_1,\ p_2,\ p_3\}$. The best approximation is in fact zero. Think of the closest point in the x-y plane to $(0, 0, 1)$. It is the point $0, 0, 0$ directly "below" $(0, 0, 1)$. It is not a very good approximation. Since it has no components in the set $\{p_1,\ p_2,\ p_3\}$ it cannot be approximated by anything in the set.
(e) There is an alternative way of solving all (i) cases. Why?

SOLUTIONS

If f can be represented exactly by the set, then we can just let:

$$f = a_1 p_1 + a_2 p_2 + a_3 p_3$$

and solve the equation for three unknowns.

Question 2.

(a) Find the best approximation to the function $f(x) = 1$ on the interval $[0,\ \pi]$ using the set $\sin nx,\ \ n = 1, 2, ..., N$. On the worksheet, you are asked to plot your answer for different values of N.

SOLUTIONS

We seek an approximation of the form of Equation 2 of $f(x) = 1$ on $[0, \pi]$:

$$y = \sum_{n=1}^{N} a_n \sin nx \tag{2}$$

This is the Fourier sine series of $[0, \pi]$

$$a_n = \frac{2}{\pi} \int_0^\pi 1 \times \sin nx dx \tag{3}$$

Hence, for $f(x) = 1$, equation (3) becomes:

$$a_n = \frac{2}{\pi} \int_0^\pi \sin nx \ \mathrm{d}x = \frac{2}{\pi n} \left(-\cos(\pi n) + 1\right) = \begin{cases} 0 & n \text{ even} \\ 4/\pi n & n \text{ odd} \end{cases}$$

Finally, substituting each a_n into equation 2, we obtain:

$$y = \frac{4}{\pi} \left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \cdots + \frac{\sin Nx}{N} \right) \qquad N \text{ odd}$$

$$y = \frac{4}{\pi} \left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \cdots + \frac{\sin((N-1)x)}{N-1} \right) \qquad N \text{ even}$$

on $[0, \pi]$

(b) Find the best approximation to the function $f(x) = x$ on the interval $[0,\ \pi]$ using the set $\sin nx,\ \ n = 1, 2, ..., N$. On the worksheet, you are asked to plot your answer for different values of N.

SOLUTION

The formula for sine series coefficient is:

$$a_n = \frac{2}{\pi} \int_0^\pi x \sin nx dx$$

Evaluating the integral results in:

$$a_n = \frac{2}{\pi} \left(\frac{\sin(\pi n) - \pi n \cos(\pi n)}{n^2} \right)$$

$$a_n = \frac{2}{n} \begin{cases} -1 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$$

Hence substituting a_n into Equation 2, we obtain:

$$y = 2 \left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \cdots + (-1)^{N+1} \frac{\sin Nx}{N} \right) \qquad \text{on } [0, \pi]$$

(c) Find the best approximation to the function $f(x) = x$ on the interval $[0,\ \pi]$ using the set $1,\ \cos nx,\ n = 1, 2, ..., N$.

SOLUTION

The Fourier cosine series coefficients are:

$$a_0 = \frac{1}{\pi}\int_0^\pi x\ \mathrm{d}x = \frac{\pi}{2}$$

and for the n'th coefficient $(n > 0)$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\ \mathrm{d}x = \frac{2}{\pi n^2}\left(\pi n \sin(\pi n) + \cos(\pi n) - 1\right)$$

$$a_n = \begin{cases} 0 & n \text{ even} \\ -4/\pi n^2 & n \text{ odd} \end{cases}$$

This gives the solution:

$$y = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1} + \frac{\cos 3x}{9} + \cdots + \frac{\cos Nx}{N^2}\right) \qquad N \text{ odd}$$

$$y = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1} + \frac{\cos 3x}{9} + \cdots + \frac{\cos (N-1)x}{(N-1)^2}\right) \qquad N \text{ even}$$

on $[0, \pi]$.