1 Fourier Series continued

Example 1.1.
Find the Fourier Series of $f(x) = x^2$ on $[-\pi, \pi]$.

This example shows a key point to note (particularly for quizzes/exams), and that is to check if the function is *odd* or *even*! The first step here is to determine our Fourier coefficients:

$$\begin{aligned}
b_j &= \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x^2}_{even}\underbrace{\sin(jx)}_{odd}\,dx \\
&= 0 \\
a_j &= \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x^2}_{even}\underbrace{\cos(jx)}_{even}\,dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} x^2\cos(jx)dx
\end{aligned}$$

The coefficients, a_j, can then be determined through integration by parts to be,

$$a_j = \left(\frac{4(-1)^j}{j^2}\right)$$

We now evaluate, a_0, and then we can determine the Fourier series,

$$\begin{aligned}
a_0 &= \frac{2}{\pi}\int_0^{\pi} x^2 dx \\
&= 2\pi^2/3
\end{aligned}$$

And therefore, as $f(x)$ is even,

$$\begin{aligned}
f(x) &= a_0/2 + \sum_{n=1}^{\infty} a_n cos(nx) \\
&= \pi^2/3 - \frac{4\cos(x)}{1^2} + \frac{4\cos(2x)}{2^2} - \frac{4\cos(3x)}{3^2} + \cdots
\end{aligned}$$

Remarks from Fourier Theorem, namely is we can restate the formulation as,

$$\begin{aligned}
f(x) &= \sum_{j=-n}^{n} c_j e^{ijx} \\
c_j &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ijx}dx, \quad j = -n, \ldots, -1, 0, 1, \ldots, n
\end{aligned}$$

This makes use of Euler's formula (or we can also show with Taylor series) that, $e^{ijx} = \cos jx + i\sin(jx)$.

1.1 Extension to arbitrary domain

Here we first state the result for the interval $[-L, L]$,

$$\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j\cos\left(\frac{j\pi x}{L}\right) + b_j\sin\left(\frac{j\pi x}{L}\right), \\
a_j &= \frac{1}{L}\int_{-L}^{L}\cos\left(\frac{j\pi x}{L}\right)f(x)dx, \quad j = 0, 1, 2, \ldots \\
b_j &= \frac{1}{L}\int_{-L}^{L}\sin\left(\frac{j\pi x}{L}\right)f(x)dx
\end{aligned}$$

To come to this result, we simply make a transformation in which we search for the Fourier series of $F(z)$ on the domain $[-\pi, \pi]$, but set $f(x) = F(z)$ with $z = \pi x/L$.

2 Discrete Fourier Transform (D.F.T.)

Namely, we have looked at how to fit a Fourier series to a function, $f(x)$, but what if we don't know the function? E.g. we have experimental data, so we want to fit a Fourier series to a *discrete* set of data.

2.1 *Aside:* Complex inner product and complex vectors

Here we take two complex numbers,

$$\mathbf{u} = (u_1, u_2) = (a + bi, c + di)$$
$$\mathbf{v} = (v_1, v_2) = (e + fi, g + hi)$$

Therefore, the inner product is defined using the conjugate as,

$$\langle \mathbf{u}, \mathbf{v} \rangle = \bar{\mathbf{u}} \cdot \mathbf{v} = \sum_{i=1}^{n} \bar{u}_i v_i$$

Note that $\langle \mathbf{u}, \mathbf{v} \rangle \neq \langle \mathbf{v}, \mathbf{u} \rangle$

2.2 So how does the method work?

So we have a vector of complex or real data that we will state as,

$$\mathbf{f} = (f_0, f_1, \ldots, f_{N-1}).$$

To do this, we decide to approximate the discrete points using a basis, $\mathbf{p}^{(k)}$,

$$\mathbf{y} = a_0 \mathbf{p}^{(0)} + a_1 \mathbf{p}^{(1)} + \cdots + a_{N-1} \mathbf{p}^{(N-1)}.$$

From here, least squares is used such that we minimise $||\mathbf{y} - \mathbf{f}||^2$. This gives a normal set of equations (as seen in the least squares methods), and as we have an orthogonal basis set, all off diagonal terms of our design matrix are 0. As shown in the lectures, the diagonal terms are given by, $1/N$.

For the D.F.T. the basis set is written as,

$$\mathbf{p}_n^{(k)} = \frac{e^{ikx_n}}{N}, \quad \text{where } x_n = \frac{2\pi n}{N}, \quad n = 0, 1, \ldots, N-1, \quad k = 0, 1, \ldots, N-1.$$

If we work through the normal equations, we end up finding:

$$a_k = \sum_{n=0}^{N-1} e^{-ikx_n} f_n$$

This then gives us the coefficients for the approximation, $\mathbf{y}$, above. One particular point here is that we have used N data points and N vectors to fit our data, so we can achieve an exact representation of our vector $\mathbf{f}$,

$$f_n = \frac{1}{N} \sum_{k=0}^{N-1} a_k e^{ikx_n} = \sum_{n=0}^{N-1} a_n \mathbf{p}^{(n)}$$

The inverse DFT, which allows us to recover values of, $\mathbf{f}$, from our coefficients gives,

$$f_n = \frac{1}{N} \sum_{k=0}^{N-1} a_k e^{ikx_n} = \sum_{n=0}^{N-1} a_n \mathbf{p}^{(n)}$$

However, we note here that we do not always use all N points, in which case the equalities above become approximations.