1 Fourier Series continued

Example 1.1.
Find the Fourier Series of $f(x) = x^2$ on $[-\pi, \pi]$.

This example shows a key point to note (particularly for quizzes/exams), and that is to check if the function is *odd* or *even*! The first step here is to determine our Fourier coefficients:

$$\begin{aligned} b_j &= \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x^2}_{even} \underbrace{\sin(jx)_{odd} dx} \\ &= 0 \\ a_j &= \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x^2}_{even} \underbrace{\cos(jx)_{even} dx} \\ &= \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos(jx) dx \end{aligned}$$

The coefficients, a_j, can then be determined through integration by parts to be,

$$a_j = \left(\frac{4(-1)^j}{j^2}\right)$$

We now evaluate, a_0, and then we can determine the Fourier series,

$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_0^{\pi} x^2 dx \\ &= 2\pi^2/3 \end{aligned}$$

And therefore, as $f(x)$ is even,

$$\begin{aligned} f(x) &= a_0/2 + \sum_{n=1}^{\infty} a_n cos(nx) \\ &= \pi^2/3 - \frac{4\cos(x)}{1^2} + \frac{4\cos(2x)}{2^2} - \frac{4\cos(3x)}{3^2} + \cdots \end{aligned}$$

Remarks from Fourier Theorem, namely is we can restate the formulation as,

$$\begin{aligned} f(x) &= \sum_{j=-n}^{n} c_j e^{ijx} \\ c_j &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-ijx} dx, \quad j = -n, \ldots, -1, 0, 1, \ldots, n \end{aligned}$$

This makes use of Euler's formula (or we can also show with Taylor series) that, $e^{ijx} = \cos jx + i\sin(jx)$.

1.1 Extension to arbitrary domain

Here we first state the result for the interval $[-L, L]$,

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos\left(\frac{j\pi x}{L}\right) + b_j \sin\left(\frac{j\pi x}{L}\right), \\ a_j &= \frac{1}{L}\int_{-L}^{L} \cos\left(\frac{j\pi x}{L}\right) f(x) dx, \quad j = 0, 1, 2, \ldots \\ b_j &= \frac{1}{L}\int_{-L}^{L} \sin\left(\frac{j\pi x}{L}\right) f(x) dx \end{aligned}$$

To come to this result, we simply make a transformation in which we search for the Fourier series of $F(z)$ on the domain $[-\pi, \pi]$, but set $f(x) = F(z)$ with $z = \pi x/L$.

2 Discrete Fourier Transform