1 Separation of Variables Conti.

Last week we managed to conclude:

$$\begin{aligned} u(x,t) = F(x)G(t) &= b\sin(n\pi x/L)[A\cos(n\pi ct/L) + B\sin(n\pi ct/L)] \\ &= [A_n\cos(n\pi ct/L) + B_n\sin(n\pi ct/L)]\sin(n\pi x/L) \end{aligned}$$

Aside: A note on linearity
As each value of n gives us a new solution to the equation, as they are linear equations, the superposition of these is also a solution. This means we can write,

$$u(x,t) = \sum_1^\infty \left(A_n\cos\left(\frac{n\pi ct}{L}\right) + B_n\sin\left(\frac{n\pi ct}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right).$$

Here, we now apply initial conditions to get our coefficients. This aligns quite closely with our past work on Fourier series. We obtain:

$$A_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)f(x)dx$$

where, $f(x) = u(x,0)$. The other initial condition, $g(x) = u_t(x,0)$ gives,

$$B_n = \frac{2}{n\pi c}\int_0^L \sin\left(\frac{n\pi x}{L}\right)g(x)dx$$

2 Fourier's method for 1-D heat equation

Looking to solve:

$$\partial_t u = k\partial_{xx}u$$

and we are considering thermally insulated ends (boundary conditions),

$$\begin{aligned} \partial_x u(0,t) = 0, \\ \partial_x u(L,t) = 0. \end{aligned}$$

We also have the initial condition,

$$u(x,0) = f(x).$$

As per the previous case, we are assuming we can find a solution in a separable form:

$$u(x,t) = F(x)G(t).$$

Now apply the boundary conditions again and eliminate useless solutions to obtain,

$$F'(0) = F'(L) = 0.$$

Now formulate the partials and sub them into the heat equation. Separating our variables onto the left and right hand side again gives us,

$$\frac{G'}{c^2G} = \frac{F''}{F}.$$

This is again a function of t equal to a function of x, so it is only possible if they equal a constant. Therefore, we now have two partial differential equations that we know how to solve,

$$\begin{aligned} &G' = kc^2G \\ &F'' = kF, \quad F'(0) = F'(L) = 0. \end{aligned}$$

If we go through the process, we again find that the constant needs to be negative so for convenience set, $k = -p^2$. Thus we need to solve,

$$\begin{aligned} F'' &= -p^2 F \\ \therefore F &= a\cos(px) + b\sin(px). \end{aligned}$$

Applying our boundary conditions for F gives,

$$\begin{aligned} b = 0, \quad p &= \frac{n\pi}{L}, \\ \implies F(x) &= a\cos\left(\frac{n\pi x}{L}\right). \end{aligned}$$

Therefore, our temporal function becomes

$$\begin{aligned} G' &= kc^2 G \\ \implies G(t) &= D\exp\left[-\left(\frac{n\pi c}{L}\right)^2 t\right] \end{aligned}$$

Combining these equations we get,

$$\begin{aligned} u(x,t) &= A_0 \\ u_n(x,t) &= A_n \exp\left[-\left(\frac{n\pi c}{L}\right)^2 t\right]\cos\left(\frac{n\pi x}{L}\right). \\ \therefore u(x,t) &= A_0 + \sum_n u_n. \end{aligned}$$

This currently satisfies the PDE and the BCs, so we then use the initial condition to define the coefficients. From this,

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

2.1 Steady-state, 2D heat equation

We can apply the same ideas to,

$$u_{xx} + u_{yy} = 0.$$

Doing the normal procedure, $u(x,y) = F(x)G(y)$ we achieve the two ODEs,

$$\begin{aligned} F'' &= kF \\ G'' &= -kG. \end{aligned}$$

Again, taking k as negative, we find,

$$\begin{aligned} F &= A\cos(px) + B\sin(px), \quad F(0) = F(a) = 0 \\ \implies F(x) &= B_n \sin\left(\frac{n\pi}{a}x\right) \\ G &= C\cosh\left(\frac{n\pi y}{a}\right) + D\sinh\left(\frac{n\pi y}{a}\right), \quad G(0) = 0 \\ G(y) &= D\sinh\left(\frac{n\pi y}{a}\right). \end{aligned}$$

So combining these,

$$u(x,y) = \sum_n B_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

After this process there is conveniently one BC left to fix B_n at $u(x,b) = f(x)$. Using a Fourier sine series we find,

$$B_n = \frac{2}{a\sinh(2\pi b/a)}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx$$