MECH3750

Tutors : Alex Muirhead
Bryce Hill
Kyle Mclaren
Luke Bartholomew
William Snell

Quiz : 1pm $\begin{cases} \text{24 S402} & \text{Surnames A-K} \\ \text{24 S304} & \text{" L-Z} \end{cases}$

Content :

- Least Squares continued
- Inner product
- Orthogonal polynomials
- Fourier Series

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Least Squares (cont.)

Can represent discrete points on some function $f(x)$ by a vector.

$$f(\vec{x}) = \left(f(x_0), f(x_1), \ldots, f(x_n) \right)^T = \vec{f}$$

Approximate vector with basis "vectors" made from functions.

$$\vec{p}^{(1)} = \left(p^{(1)}(x_0), p^{(1)}(x_1), \ldots, p^{(1)}(x_n) \right)^T$$

etc.

Can be any set of **linearly independent** functions $\{p^{(1)}, p^{(2)}, \ldots p^{(m)}\}$

$$\therefore \quad f(\vec{x}) = \vec{f} \approx \sum_m \alpha_m \vec{p}^{(m)}$$

$$\begin{bmatrix} \vec{p}^{(1)} \cdot \vec{p}^{(1)} & \vec{p}^{(1)} \cdot p^{(2)} & \cdots & \vec{p}^{(1)} \cdot \vec{p}^{(m)} \\ \vec{p}^{(2)} \cdot \vec{p}^{(1)} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \vec{p}^{(m)} \cdot \vec{p}^{(1)} & \cdots & \cdots & \vec{p}^{(m)} \cdot \vec{p}^{(m)} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_m \end{bmatrix} = \begin{bmatrix} \vec{f} \cdot \vec{p}^{(1)} \\ \vdots \\ \vdots \\ \vec{f} \cdot \vec{p}^{(m)} \end{bmatrix}$$

Same form as before.

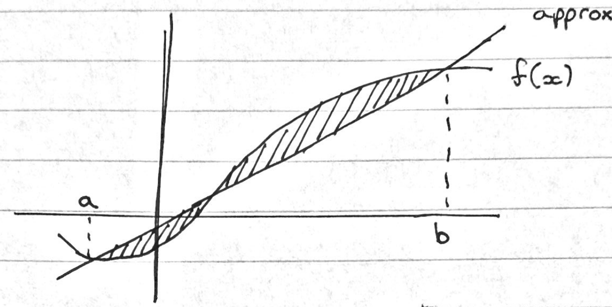
Written more concisely as:

$$P^T P \vec{\alpha} = P^T \vec{f}$$

$$\downarrow$$

$$\vec{\alpha} = (P^T P)^{-1} P^T \vec{f}$$

What if we want to approximate a continuous function over an interval ?



Define **positive** error based on the area between $f(x)$ and approximation.

$$E = \int_a^b \left(\sum_i^n \alpha_i p^{(i)}(x) - f(x) \right)^2 dx$$

Follow previous derivation, set $\partial_{\alpha_j} E = 0$

$$0 = \sum_i^n \int_a^b \alpha_i p^{(i)}(x)\, p^{(j)}(x)\, dx - \int_a^b p^{(j)}(x)\, f(x)\, dx$$

In matrix form:

$$\begin{bmatrix} \int_a^b p^{(1)} p^{(1)}\, dx & \cdots & \int_a^b p^{(1)} p^{(n)}\, dx \\ \vdots & \ddots & \vdots \\ \int_a^b p^{(n)} p^{(1)}\, dx & \cdots & \int_a^b p^{(n)} p^{(n)}\, dx \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \int_a^b p^{(1)} f\, dx \\ \vdots \\ \int_a^b p^{(n)} f\, dx \end{bmatrix}$$

Discrete vector form & continuous function forms of least squares very similar...

Generalise by introducing **inner product**.

$$\langle p, q \rangle$$

vectors ↙ ↘ functions

$$\vec{p} \cdot \vec{q} \qquad \qquad \int p(x) q(x)\, dx$$

Some properties:

Norm (or length/siz) $\Rightarrow$ $\|p\| = \sqrt{\langle p, p \rangle}$

Distance $\Rightarrow$ $d(p,q) = \|p-q\|$

Orthogonality $\Rightarrow$ $\langle p, q \rangle \equiv 0$

Distance between two functions becomes:

$$d(p,q) = \sqrt{\int (p(x) - q(x))^2\, dx}$$

$\therefore$ Least squares minimises distance squared!

There exist sets of linearly independent orthogonal functions. These simplify matrix form, as non-diagonal elements become $\emptyset$.

<u>Legendre Polynomials</u> $\quad \int_0^1 p^{(i)} p^{(j)}\, dx \propto \delta_{ij} \equiv \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

Don't need to remember this!

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Trigonometric Functions

Specifically sine and cosine, as they can be represented as:

$$e^{ix} = \cos x + i \sin x$$

Note that orthogonality holds:

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \begin{cases} \pi & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = \begin{cases} \pi & \text{if } m = n^{*} \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx = 0$$

$*$ if $m, n > 0$

Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

↑ exactly equal for an infinite sum

where $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

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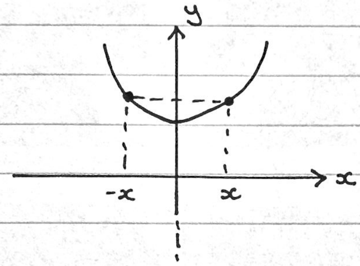
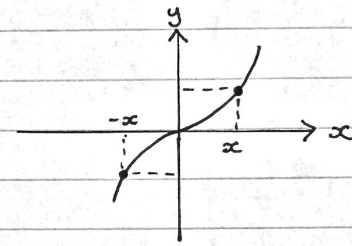
Shortcuts

Introduced **even** and **odd** functions.

Even



Odd



$$f(-x) = f(x)$$

$$\left\{ \begin{array}{c} \text{Reflection Symmetry} \\ \text{about y-axis} \end{array} \right\}$$

$$f(-x) = -f(x)$$

$$\left\{ \begin{array}{c} \text{Rotation Symmetry} \\ \text{about origin} \end{array} \right\}$$

Even x Even $\Rightarrow$ $f(-x)g(-x) = f(x)g(x) \Rightarrow$ Even

Odd x Odd $\Rightarrow$ $f(-x)g(-x) = (-1)^2 f(x)g(x) \Rightarrow$ Even

Even x Odd $\Rightarrow$ $f(-x)g(-x) = -f(x)g(x) \Rightarrow$ Odd

$$\int_{-a}^{a} \text{Even}\, dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

$$\int_{-a}^{a} \text{Odd}\, dx = \int_{-a}^{0} g(x) + \int_{0}^{a} g(x)\,dx = \int_{0}^{a} -g(x) + g(x)\,dx = \varnothing$$

$\therefore$ As cosine is even & sine is odd

if $f(x)$ is even:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

if $f(x)$ is odd:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$