Lecture Content Summaries for MECH3750

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Week 02

With thanks to Alex Muirhead:

Tutors: Alex Muirhead, Bryce Hill, Kyle Mclaren, Luke Bartholomew, William Snell

Quizzes: Weeks 3, 6, 9

Content:

- Taylor Expansions;
- Newton's Method;
- Least Squares.

Taylor Series

$$f(x) = f(a) + \frac{f'(a)}{1!}\underbrace{(x-a)}_{\text{distance from the fixed point}} + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

a is the fixed point

function we want to approx

infinite series

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

But can't always have all infinte terms.

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \underbrace{O((x-a)^2)}_{\text{error term}}$$

aka Higher Order Terms in mult-variable

OR

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a)$$

Accurate to 2^{nd} order

Note : $f_x = \frac{\partial f}{\partial x}$ etc.

What about multivariable functions ?

$$f(x, y, \dots) \quad \text{OR} \quad f(x_0, x_1, \dots, x_n) = f(\vec{x})$$

$$f(x,y) = f(\underbrace{a,b}_{\text{two fixed points}}) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$\dots + \frac{1}{2}\left[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right]$$

$$\dots + \text{H.O.T} \leftarrow O\left(\sqrt{(x-a)^2 + (y-b)^2}\right)$$

In general:

$$f(\vec{x}) = f(\vec{x}_0) + [\nabla f(\vec{x}_0)]^T(\vec{x} - \vec{x}_0) + \frac{1}{2}(\vec{x} - \vec{x}_0)^T H (\vec{x} - \vec{x}_0) + \text{H.O.T}$$

where Hessian is $H = \nabla(\nabla f) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$

Newtons Method

Want to find roots of function:

$$f(x) = \emptyset$$

$$f(\underset{\substack{\uparrow \\ \text{initial guess}}}{x_0} + \underset{\substack{\uparrow \\ \text{step}}}{h}) \approx f(x_0) + f'(x_0)h$$

Let our next guess $x_1 = x_0 + h$ be a "root".

$$\therefore \quad f'(x_0)h = -f(x_0)$$

$$x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$\boxed{\text{Repeat until } f(x_n) \approx 0}$

For multi-variable functions ; and vector functions (aka multiple stacked equations).

$$\vec{f}(\vec{x}) = 0$$

$$\vec{f}'(\vec{x_0}) = \nabla \vec{f}(\vec{x_0}) = J \ \{\text{Jacobian}\}$$

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots \\ \frac{\partial f_2}{\partial x_1} & \ddots & \vdots \\ \vdots & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}, \quad \vec{f}(\vec{x}) = \begin{bmatrix} f_1(\vec{x}) \\ f_2(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{bmatrix}$$

$$\vec{f}(\vec{x_0} + \vec{h}) \approx \vec{f}(x_0) + J\vec{h} = \emptyset$$

$\therefore \quad J\vec{h} = -\vec{f}(\vec{x_0}) \leftarrow$ easier to solve numerically

OR $\quad \vec{h} = -J^{-1}\vec{f}(\vec{x_0}) \leftarrow$ easier to solve by hand IFF J is 2×2 matrix.

7mm A4

Least Squares

Want to approximate some vector $\vec{f}$ with vectors $\vec{p}^{(1)}, \vec{p}^{(2)}, \dots, \vec{p}^{(n)}$.

$$\vec{f} \approx \alpha_1 \vec{p}^{(1)} + \alpha_2 \vec{p}^{(2)} + \dots + \alpha_n \vec{p}^{(n)}$$

Define **positive** error :

square to ensure error is positive

$$E = \sum_i \left(\alpha_1 p_i^{(1)} + \alpha_2 p_i^{(2)} + \dots + \alpha_n p_i^{(n)} - f_i \right)^2$$

summing over components of the vectors

Minimise error by setting derivatives to zero.

$$\frac{\partial E}{\partial \alpha_j} = 0 \quad \forall j \in 1, 2, \dots, n$$

$$\frac{1}{2} \frac{\partial E}{\partial \alpha_j} = \sum_i p_i^{(j)} \left(\alpha_1 p_i^{(1)} + \alpha_2 p_i^{(2)} + \dots + \alpha_n p_i^{(n)} - f_i \right)$$

$$0 = \sum_i p_i^{(j)} \underbrace{\left(\alpha_1 p_i^{(1)} + \alpha_2 p_i^{(2)} + \dots + \alpha_n p_i^{(n)} \right)}_{\text{unknowns}} - \underbrace{\sum_i p_i^{(j)} f_i}_{\text{knowns}}$$

In matrix notation :

$$\underbrace{\begin{bmatrix} p^{(1)} \cdot f \\ p^{(2)} \cdot f \\ \vdots \\ p^{(n)} \cdot f \end{bmatrix}}_{P^T f} = \underbrace{\begin{bmatrix} p^{(1)} \cdot p^{(1)} & p^{(1)} \cdot p^{(2)} & \dots & p^{(1)} \cdot p^{(n)} \\ p^{(2)} \cdot p^{(1)} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ p^{(n)} \cdot p^{(1)} & \dots & \dots & p^{(n)} \cdot p^{(n)} \end{bmatrix}}_{P^T P} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}}_{\alpha}$$

$$\vec{\alpha} = (P^T P)^{-1} P^T \vec{f}$$

7mm A4

Week 03

With thanks to Alex Muirhead:

Tutors: Alex Muirhead, Bryce Hill, Kyle Mclaren, Luke Bartholomew, William Snell

Quiz: Weeks 3 - 1pm

Content:

- Least Squares continued;
- Inner product;
- Orthogonal polynomials;
- Fourier Series.

Least Squares (cont.)

Can represent discrete points on some function $f(x)$ by a vector.

$$f(\vec{x}) = \left(f(x_0), f(x_1), \dots, f(x_n) \right)^T = \vec{f}$$

Approximate vector with basis "vectors" made from functions.

$$\vec{p}^{(1)} = \left(p^{(1)}(x_0), p^{(1)}(x_1), \dots, p^{(1)}(x_n) \right)^T$$

etc.

Can be any set of **linearly independent** functions $\{p^{(1)}, p^{(2)}, \dots p^{(m)}\}$

$$\therefore \quad f(\vec{x}) = \vec{f} \approx \sum_m \alpha_m \vec{p}^{(m)}$$

$$\begin{bmatrix} \vec{p}^{(1)} \cdot \vec{p}^{(1)} & \vec{p}^{(1)} \cdot \vec{p}^{(2)} & \cdots & \vec{p}^{(1)} \cdot \vec{p}^{(m)} \\ \vec{p}^{(2)} \cdot \vec{p}^{(1)} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \vec{p}^{(m)} \cdot \vec{p}^{(1)} & \cdots & \cdots & \vec{p}^{(m)} \cdot \vec{p}^{(m)} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_m \end{bmatrix} = \begin{bmatrix} \vec{f} \cdot \vec{p}^{(1)} \\ \vdots \\ \vdots \\ \vec{f} \cdot \vec{p}^{(m)} \end{bmatrix}$$

Same form as before.

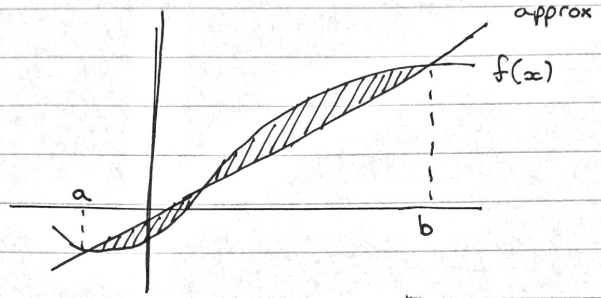
Written more concisely as:

$$P^T P \vec{\alpha} = P^T \vec{f}$$

$$\downarrow$$

$$\vec{\alpha} = (P^T P)^{-1} P^T \vec{f}$$

What if we want to approximate a continuous function over an interval ?



Define **positive** error based on the area between $f(x)$ and approximation.

$$E = \int_a^b \left(\sum_i^n \alpha_i p^{(i)}(x) - f(x) \right)^2 dx$$

Follow previous derivation, set $\partial_{\alpha_j} E = 0$

$$0 = \sum_i^n \int_a^b \alpha_i p^{(i)}(x)\, p^{(j)}(x) dx - \int_a^b p^{(j)}(x) f(x)\, dx$$

In matrix form:

$$\begin{bmatrix} \int_a^b p^{(1)} p^{(1)} dx & \cdots & \int_a^b p^{(1)} p^{(n)} dx \\ \vdots & \ddots & \vdots \\ \int_a^b p^{(n)} p^{(1)} dx & \cdots & \int_a^b p^{(n)} p^{(n)} dx \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \int_a^b p^{(1)} f\, dx \\ \vdots \\ \int_a^b p^{(n)} f\, dx \end{bmatrix}$$

Discrete vector form & continuous function forms of least squares very similar...

Generalise by introducing **inner product.**

$$\langle p, q \rangle$$

$$\vec{p}\cdot\vec{q} \;\xleftarrow{\text{vectors}}\; \langle p, q\rangle \;\xrightarrow{\text{functions}}\; \int p(x)q(x)\,dx$$

Some properties:

Norm (or length/siz) $\Rightarrow$ $\|p\| = \sqrt{\langle p, p \rangle}$

Distance $\Rightarrow$ $d(p,q) = \|p - q\|$

Orthogonality $\Rightarrow$ $\langle p, q \rangle \equiv 0$

Distance between two functions becomes:

$$d(p,q) = \sqrt{\int (p(x) - q(x))^2 \, dx}$$

$\therefore$ Least squares minimises distance squared!

There exist sets of linearly independent orthogonal functions. These simplify matrix form, as non-diagonal elements become $\emptyset$.

<u>Legendre Polynomials</u> $\displaystyle\int_0^1 p^{(i)} p^{(j)} \, dx \propto \delta_{ij} \equiv \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

Don't need to remember this!

Trigonometric Functions

Specifically sine and cosine, as they can be represented as:

$$e^{ix} = \cos x + i \sin x$$

Note that orthogonality holds:

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \begin{cases} \pi \text{ if } m=n \\ 0 \text{ otherwise} \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = \begin{cases} \pi \text{ if } m=n \ ^{*} \\ 0 \text{ otherwise} \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx = 0$$

* if $m, n > 0$

Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

exactly equal for an infinite sum

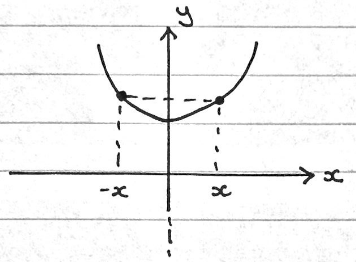
where $a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$

$b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$

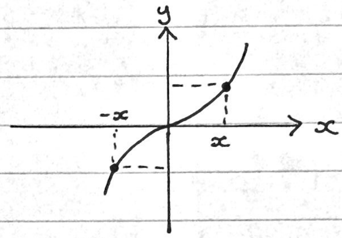
Shortcuts

Introduced **even** and **odd** functions.

Even



Odd



$$f(-x) = f(x)$$

$\left\{ \begin{array}{c} \text{Reflection Symmetry} \\ \text{about y-axis} \end{array} \right\}$

$$f(-x) = -f(x)$$

$\left\{ \begin{array}{c} \text{Rotation Symmetry} \\ \text{about origin} \end{array} \right\}$

Even x Even $\Rightarrow f(-x)g(-x) = f(x)g(x) \Rightarrow$ Even

Odd x Odd $\Rightarrow f(-x)g(-x) = (-1)^2 f(x)g(x) \Rightarrow$ Even

Even x Odd $\Rightarrow f(-x)g(-x) = -f(x)g(x) \Rightarrow$ Odd

$$\int_{-a}^{a} \text{Even } dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

$$\int_{-a}^{a} \text{Odd } dx = \int_{-a}^{0} g(x) + \int_{0}^{a} g(x)\,dx = \int_{0}^{a} -g(x) + g(x)\,dx = \emptyset$$

$\therefore$ As cosine is even & sine is odd

if $f(x)$ is even:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

if $f(x)$ is odd:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

Week 04

Tutors: Luke Bartholomew, Bryce Hill, Kyle Mclaren, Travis Mitchell, Alex Muirhead, William Snell

Assessment:

- Quiz 1 results are available for collection;
- Next quiz week 6;
- Assignment 1 due week 7.

Content:

- Fourier series continued;
- Discrete Fourier Series.

1 Fourier Series continued

Example 1.1.
Find the Fourier Series of $f(x) = x^2$ on $[-\pi, \pi]$.

This example shows a key point to note (particularly for quizzes/exams), and that is to check if the function is *odd* or *even*! The first step here is to determine our Fourier coefficients:

$$\begin{aligned}
b_j &= \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x^2}_{even}\underbrace{\sin(jx)}_{odd}\,dx \\
&= 0 \\
a_j &= \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x^2}_{even}\underbrace{\cos(jx)}_{even}\,dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} x^2\cos(jx)dx
\end{aligned}$$

The coefficients, a_j, can then be determined through integration by parts to be,

$$a_j = \left(\frac{4(-1)^j}{j^2}\right)$$

We now evaluate, a_0, and then we can determine the Fourier series,

$$\begin{aligned}
a_0 &= \frac{2}{\pi}\int_0^{\pi} x^2 dx \\
&= 2\pi^2/3
\end{aligned}$$

And therefore, as $f(x)$ is even,

$$\begin{aligned}
f(x) &= a_0/2 + \sum_{n=1}^{\infty} a_n cos(nx) \\
&= \pi^2/3 - \frac{4\cos(x)}{1^2} + \frac{4\cos(2x)}{2^2} - \frac{4\cos(3x)}{3^2} + \cdots
\end{aligned}$$

Remarks from Fourier Theorem, namely is we can restate the formulation as,

$$\begin{aligned}
f(x) &= \sum_{j=-n}^{n} c_j e^{ijx} \\
c_j &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ijx}dx, \quad j = -n, \ldots, -1, 0, 1, \ldots, n
\end{aligned}$$

This makes use of Euler's formula (or we can also show with Taylor series) that, $e^{ijx} = \cos jx + i\sin(jx)$.

1.1 Extension to arbitrary domain

Here we first state the result for the interval $[-L, L]$,

$$\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j\cos\left(\frac{j\pi x}{L}\right) + b_j\sin\left(\frac{j\pi x}{L}\right), \\
a_j &= \frac{1}{L}\int_{-L}^{L}\cos\left(\frac{j\pi x}{L}\right)f(x)dx, \quad j = 0, 1, 2, \ldots \\
b_j &= \frac{1}{L}\int_{-L}^{L}\sin\left(\frac{j\pi x}{L}\right)f(x)dx
\end{aligned}$$

To come to this result, we simply make a transformation in which we search for the Fourier series of $F(z)$ on the domain $[-\pi, \pi]$, but set $f(x) = F(z)$ with $z = \pi x/L$.

2 Discrete Fourier Transform (D.F.T.)

Namely, we have looked at how to fit a Fourier series to a function, $f(x)$, but what if we don't know the function? E.g. we have experimental data, so we want to fit a Fourier series to a *discrete* set of data.

2.1 *Aside:* Complex inner product and complex vectors

Here we take two complex numbers,

$$\mathbf{u} = (u_1, u_2) = (a + bi, c + di)$$
$$\mathbf{v} = (v_1, v_2) = (e + fi, g + hi)$$

Therefore, the inner product is defined using the conjugate as,

$$\langle \mathbf{u}, \mathbf{v} \rangle = \bar{\mathbf{u}} \cdot \mathbf{v} = \sum_{i=1}^{n} \bar{u}_i v_i$$

Note that $\langle \mathbf{u}, \mathbf{v} \rangle \neq \langle \mathbf{v}, \mathbf{u} \rangle$

2.2 So how does the method work?

So we have a vector of complex or real data that we will state as,

$$\mathbf{f} = (f_0, f_1, \ldots, f_{N-1}).$$

To do this, we decide to approximate the discrete points using a basis, $\mathbf{p}^{(k)}$,

$$\mathbf{y} = a_0 \mathbf{p}^{(0)} + a_1 \mathbf{p}^{(1)} + \cdots + a_{N-1} \mathbf{p}^{(N-1)}.$$

From here, least squares is used such that we minimise $||\mathbf{y} - \mathbf{f}||^2$. This gives a normal set of equations (as seen in the least squares methods), and as we have an orthogonal basis set, all off diagonal terms of our design matrix are 0. As shown in the lectures, the diagonal terms are given by, $1/N$.

For the D.F.T. the basis set is written as,

$$\mathbf{p}_n^{(k)} = \frac{e^{ikx_n}}{N}, \quad \text{where } x_n = \frac{2\pi n}{N}, \quad n = 0, 1, \ldots, N-1, \quad k = 0, 1, \ldots, N-1.$$

If we work through the normal equations, we end up finding:

$$a_k = \sum_{n=0}^{N-1} e^{-ikx_n} f_n$$

This then gives us the coefficients for the approximation, $\mathbf{y}$, above. One particular point here is that we have used N data points and N vectors to fit our data, so we can achieve an exact representation of our vector $\mathbf{f}$,

$$f_n = \frac{1}{N} \sum_{k=0}^{N-1} a_k e^{ikx_n} = \sum_{n=0}^{N-1} a_n \mathbf{p}^{(n)}$$

The inverse DFT, which allows us to recover values of, $\mathbf{f}$, from our coefficients gives,

$$f_n = \frac{1}{N} \sum_{k=0}^{N-1} a_k e^{ikx_n} = \sum_{n=0}^{N-1} a_n \mathbf{p}^{(n)}$$

However, we note here that we do not always use all N points, in which case the equalities above become approximations.

Week 05

Tutors: Luke Bartholomew, Nathan di Vaira, Bryce Hill, Kyle Mclaren, Travis Mitchell, Alex Muirhead, William Snell

Assessment:

- Next quiz week 6;
- Assignment 1 due week 7.

Content:

- Introduction to PDEs;
- Introduction to separation of variables.

1 Introduction to partial differential equations (PDEs)

There are three core PDEs that we will look at:

- Wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

- Heat or diffusion equation:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

- Laplace equation:

$$0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

These equations here are presented in lower dimensions, but not that when we have 2D or 3D analysis the operator, $\nabla^2 = \nabla(\nabla \cdot u)$ is used:

$$\nabla^2 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

1.1 Initial Conditions

In order to solve these equations, we typically need to know what they originally look like in order to resolve them forward through time. For example, to solve the wave equation we typically need,

$$\begin{aligned} u(x, 0) &= \text{Initial position of e.g. a string at } t = 0, \\ \frac{\partial u}{\partial t}(x, 0) &= \text{velocity of e.g. a string at } t = 0. \end{aligned}$$

1.2 Boundary Conditions

For solving these equations, it is also important to know what the boundaries are - i.e. what is limiting the motion of our system. Again, an example of this for the wave equation could be,

$$\begin{aligned} u(0, t) = u_0 &= \text{the start of the string is fixed for all } t, \\ u(L, t) = u_L &= \text{the end of the string may also be fixed for all , } t. \end{aligned}$$

2 Introduction to the heat (or diffusion) equation

Used to model how heat would diffuse for example through a plate or how a concentration of particles may diffuse in space. In the lectures, we used a population of walkers on a 2D grid in order to derive the equation where we found a discrete model,

$$\begin{aligned} \frac{u(x, y, t + \Delta t) - u(x, y, t)}{\Delta t} &= \frac{p(\Delta x)^2}{\Delta t} \times \\ &\left(\frac{u(x + \Delta x, y, t) - 2u(x, y, t) + u(x - \Delta x, y, t)}{\Delta x^2} + \frac{u(x, y + \Delta y, t) - 2u(x, y, t) + u(x, y - \Delta y, t)}{\Delta x^2} \right) \\ \frac{\partial u}{\partial t} &= k \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \end{aligned}$$

2.1 Initial Conditions

For this system, as was seen by our lattice example, only an initial concentration is required,

$$u(x, y, 0) = f(x, y).$$

2.2 Boundary Conditions

Here, we can define the boundaries to be sinks like was done in the lecture, where $u = 0$ on all boundaries. This type of 'fixed-value' boundary is typically termed a **Dirichlet boundary condition**, and we will touch on these throughout the course. We note here, that the boundaries don't need to be fixed at 0. For example, if we consider the diffusion equation in terms of the diffusion of heat... We could have a boundary fixed at a particular temperature.

The other boundary condition that we commonly see is a **Neumann boundary condition**, which we would write for example as

$$\frac{\partial u}{\partial x}(L, y) = g(y),$$

on the right hand side boundary of our 2D domain. This effectively specifies a flow, or flux, of a property. Again considering heat, a flow of heat in or out of the system could be seen. A particular type of interest is an insulating boundary, in which no heat can flow, $g(y) = 0$.

3 Laplace Equation

If we consider the heat on a plate as $t \to \infty$, such that the heat has reached a steady-state. We can then determine that,

$$\frac{\partial u}{\partial t} = 0.$$

And then the diffusion equation can be seen to satisfy the Laplace equation as $u(x, y, t) \to u(x, y)$. This equation is also commonly seen in incompressible fluid mechanics,

$$0 = \nabla u.$$

4 Fourier's method for the wave equation

We will move to a short-hand notation in which the wave equation can be written as,

$$u_{tt} = c^2 u_{xx}$$

with

$$\begin{aligned} \text{Initial cond's} \quad & u(x, 0) = f(x) \\ & u_t(x, 0) = g(x) \\ \text{Boundary cond's} \quad & u(0, t) = u(L, t) = 0. \end{aligned}$$

To solve this, we will seek a solution of the form,

$$u(x, t) = F(x)G(t),$$

namely, we are assuming the function is *separable.*

4.1 Introduction to separation of variables

Let us try to apply this to the initial conditions of our previous system,

$$u(0,t) = 0 = F(0)G(t), u(L,t) \qquad = 0 = F(L)G(t)$$

So either $G(t) = 0$ (bad) or $F(0) = 0$, and similarly we have $F(L) = 0$. Let us now look if we can apply this to the actual wave equation, so the left-hand and right-hand side of the equation will require,

$$u_{tt} = F(x)G''(t),$$
$$u_{xx} = F''(x)G(t).$$

Substituting this in and separating our variables we find,

$$\frac{G''(t)}{c^2G(t)} = \frac{F''(x)}{F(x)}.$$

As one of these is a function of time and the other of space, we conclude that these ratios must be a constant. Which if this is true, we have two ODE's to solve,

$$F''(x) - kF(x) = 0$$
$$G''(t) - c^2kG(t) = 0.$$

There are solutions to these, and for which we consider the constant k to be 0, μ^2 (positive) and

From this, we find $k = 0$ to be useless. From k positive we find,

$$\begin{aligned} F(x) &= Ae^{\mu x} + Be^{-\mu x} \\ &= a\cosh(\mu x) + b\sinh(\mu x). \\ \therefore F(0) = 0 &\implies a = 0 \\ F(L) = 0 &\implies b\sinh(\mu L) = 0 \end{aligned}$$

This again is not useful, so we conclude that $k < 0$, so set $k = -p^2$ which gives,

$$\begin{aligned} F(x) &= Ae^{ipx} + Be^{-ipx} \\ &= a\cos(px) + b\sin(px) \end{aligned}$$

We can then apply the boundary conditions $(F(0) = F(L) = 0)$, which for us leave,

$$p = n\pi/L, \quad n = 1, 2, 3, \ldots$$

Now apply this value of k into the ODE for $G(t)$,

$$\begin{aligned} &G'' + p^2c^2G = 0 \\ &\implies G(t) = A\cos(n\pi ct/L) + B\sin(n\pi ct/L) \end{aligned}$$

with this, we can now conclude that the solutions for u are,

$$\begin{aligned} u(x,t) = F(x)G(t) &= b\sin(n\pi x/L)[A\cos(n\pi ct/L) + B\sin(n\pi ct/L)] \\ &= [A_n\cos(n\pi ct/L) + B_n\sin(n\pi ct/L)]\sin(n\pi x/L) \end{aligned}$$