MECH3750 - Tutorial 4 Solutions

Question 1.
Show that:

$$\overline{\exp(iy)} = \exp(-iy)$$

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}$$

$$\cos y = \frac{e^{iy} + e^{-iy}}{2}$$

SOLUTIONS

$$\begin{aligned}
\overline{\exp(iy)} &= \overline{\cos y + i \sin y} && \\
&= \cos y - i \sin y && \text{since of the form } a + ib \text{ for real a,b} \\
&= \cos -y + i \sin -y && \text{since } \cos -y = \cos y \text{ and } \sin -y = -\sin y \\
&= \exp(-iy)
\end{aligned}$$

$$\begin{aligned}
\frac{e^{iy} - e^{-iy}}{2i} &= \frac{(\cos y + i \sin y) - (\cos y - i \sin y)}{2i} \\
&= \sin y
\end{aligned}$$

$$\begin{aligned}
\frac{e^{iy} + e^{-iy}}{2} &= \frac{(\cos y + i \sin y) + (\cos y - i \sin y)}{2} \\
&= \cos y
\end{aligned}$$

Question 2.

Using the complex inner product defined as:

$$(u, v) = \sum_{i=0}^{4} \overline{u_i} v_i$$

consider the vectors:

$$q_n^{(k)} = \exp\left(ik\frac{2\pi n}{M}\right) \qquad n = 0, 1, 2, 3$$

(a) Write $q^{(k)}$ explicitly for $k = 0, 1, 2, 3$.

(b) Use the inner product to find $\|q^{(k)}\|$ for $k = 0, 1, 2, 3$.

(c) Verify: $(q^{(0)}, q^{(1)}) = 0$; $(q^{(2)}, q^{(3)}) = 0$; $(q^{(0)}, q^{(2)}) = 0$

SOLUTIONS

(a) $M = 4$

$$q^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad q^{(1)} = \begin{bmatrix} 1 \\ \exp\left(i\frac{\pi}{2}\right) \\ \exp\left(i\pi\right) \\ \exp\left(i\frac{3\pi}{2}\right) \end{bmatrix} \quad q^{(2)} = \begin{bmatrix} 1 \\ \exp\left(i\pi\right) \\ \exp\left(i2\pi\right) \\ \exp\left(i3\pi\right) \end{bmatrix} \quad q^{(3)} = \begin{bmatrix} 1 \\ \exp\left(i\frac{3\pi}{2}\right) \\ \exp\left(i3\pi\right) \\ \exp\left(i\frac{9\pi}{2}\right) \end{bmatrix}$$

Now observe:

$$\begin{aligned} \exp(i\pi) = \exp(i3\pi) = \exp(in\pi) &= -1 && \text{for any odd } n \\ \exp(i0\pi) = \exp(i2\pi) = \exp(im\pi) &= 1 && \text{for any even } m \\ \exp\left(i\frac{3\pi}{2}\right) = \exp\left(i\frac{7\pi}{2}\right) = \exp\left(i\frac{(2n+1)\pi}{2}\right) &= -i && \text{for any odd } n \\ \exp\left(i\frac{\pi}{2}\right) = \exp\left(i\frac{5\pi}{2}\right) = \exp\left(i\frac{(2m+1)\pi}{2}\right) &= i && \text{for any even } m \end{aligned}$$

Therefore:

$$q^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad q^{(1)} = \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix} \quad q^{(2)} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \quad q^{(3)} = \begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix}$$

(b) $\|q^{(k)}\|^2 = \left(q^{(k)}, q^{(k)}\right)$, with conjugates:

$$\overline{q^{(0)}} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \overline{q^{(1)}} = \begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix} \quad \overline{q^{(2)}} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \quad \overline{q^{(3)}} = \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix}$$

Therefore, taking the inner product gives:

$$\|q^{(0)}\| = 2 \quad \|q^{(1)}\| = 2 \quad \|q^{(2)}\| = 2 \quad \|q^{(3)}\| = 2$$

(c)

$$\begin{aligned} \left(q^{(0)}, q^{(1)}\right) &= 1 - i - 1 + i = 0 \\ \left(q^{(0)}, q^{(2)}\right) &= 1 - 1 + 1 - 1 = 0 \\ \left(q^{(0)}, q^{(3)}\right) &= 1 + i - 1 - i = 0 \\ \left(q^{(1)}, q^{(2)}\right) &= 1 - i - 1 + i = 0 \\ \left(q^{(2)}, q^{(3)}\right) &= 1 - i - 1 + i = 0 \end{aligned}$$

As expected.

Question 3.

In our interpretation of the DFT, the values a_{p_k} represent the coefficients of the vector:

$$p_n^{(k)} = \frac{1}{N} \exp\left(i\frac{2\pi nk}{N}\right)$$

in the signal f_n for $k = 0, 1, \ldots, N-1$.

Verify that $p_n^{(N-1)} = p_n^{(-1)}$ and also $p_n^{(N-m)} = p_n^{(-m)}$. This is important for interpreting the values of the DFT for large k.

SOLUTIONS

$$p_n^{(N-1)} = \frac{1}{N} \exp\left(i\frac{2\pi n(N-1)}{N}\right) = \frac{1}{N} \exp\left(i\frac{2\pi nN}{N} - i\frac{2\pi n}{N}\right) = \frac{1}{N} \exp\left(i2\pi n\right) \exp\left(-i\frac{2\pi n}{N}\right)$$

But $\exp\left(i2\pi n\right) = 1$, since it is a rotation around the unit circle n times, hence:

$$p_n^{(N-1)} = \frac{1}{N} \exp\left(-i\frac{2\pi n}{N}\right) = p_n^{(-1)}$$

Under a similar argument

$$p_n^{(N-m)} = \frac{1}{N} \exp\left(i\frac{2\pi n(N-m)}{N}\right) = \frac{1}{N} \exp\left(i\frac{2\pi nN}{N} - i\frac{2\pi nm}{N}\right) = \frac{1}{N} \exp\left(i2\pi n\right) \exp\left(-i\frac{2\pi nm}{N}\right)$$

So

$$p_n^{(N-m)} = \frac{1}{N} \exp\left(-i\frac{2\pi nm}{N}\right) = p_n^{(-m)}$$

Question 4.

Find the DFT of:

(a) $\boldsymbol{f} = (1,\ 2,\ 0,\ 1)$

(b) $\boldsymbol{f} = (1,\ 1,\ \ldots,\ 1)$, for $N = 8$

SOLUTIONS

Recall:

$$a_k = \sum_{n=0}^{N-1} \exp(-ikx_n) f_n \quad \text{for } x_n = \frac{2\pi n}{N}$$

(a) For $\boldsymbol{f} = (1,\ 2,\ 0,\ 1)$, coefficients are given by:

$$a_k = \sum_{n=0}^{3} \exp\left(-ik\frac{\pi n}{2}\right) f_n$$

$$\begin{aligned}
a_0 &= 1 \cdot 1 + 1 \cdot 2 + 1 \cdot 0 + 1 \cdot 1 && = 4 \\
a_1 &= 1 \cdot 1 + \exp\left(-i\frac{\pi}{2}\right) \cdot 2 + \exp\left(-i\pi\right) \cdot 0 + \exp\left(-i\frac{3\pi}{2}\right) \cdot 1 && = 1 - 2i + i \\
a_2 &= 1 \cdot 1 + \exp\left(-i2\frac{\pi}{2}\right) \cdot 2 + \exp\left(-i2\pi\right) \cdot 0 + \exp\left(-i2\frac{3\pi}{2}\right) \cdot 1 && = 1 - 2 - 1 \\
a_3 &= 1 \cdot 1 + \exp\left(-i3\frac{\pi}{2}\right) \cdot 2 + \exp\left(-i3\pi\right) \cdot 0 + \exp\left(-i3\frac{3\pi}{2}\right) \cdot 1 && = 1 + 2i - i
\end{aligned}$$

So $\boldsymbol{a} = (4, 1 - i. - 2, 1 + i)$

(b) For $\boldsymbol{f} = (1,\ 1,\ \ldots,\ 1)$, $N = 8$ we use orthogonality. Recall that $f_n = 1 = \exp(i.0.x_n)$, which is orthogonal to $f_n = \exp(i.k.x_n)$.
Thus

$$a_k = \sum_{n=0}^{7} \exp(i.0.x_n) \exp\left(-ikx_n\right) = 0$$

unless $k = 0$, in which case

$$a_0 = \sum_{n=0}^{7} \exp(i.0.x_n) \exp\left(-i.0.x_n\right) = \sum_{n=0}^{7} 1 \times 1 = 8$$

It is interesting to see how complicated the direct calculation is
(DO NOT USE IN AN ASSIGNMENT, QUIZ or EXAM):

$$\begin{aligned}
a_0 &= 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 && = 8 \\
a_1 &= 1 + \frac{1-i}{\sqrt{2}} - i + \frac{-1-i}{\sqrt{2}} - 1 + \frac{-1+i}{\sqrt{2}} + i + \frac{1+i}{\sqrt{2}} && = 0 \\
a_2 &= 1 - i - 1 + i + 1 - i - 1 + i && = 0 \\
a_3 &= 1\frac{-1-i}{\sqrt{2}} + i + \frac{1-i}{\sqrt{2}} - 1 + \frac{1+i}{\sqrt{2}} - i + \frac{-1+i}{\sqrt{2}} && = 0 \\
a_4 &= 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 && = 0 \\
a_5 &= 1 + \frac{-1+i}{\sqrt{2}} - i + \frac{1+i}{\sqrt{2}} - 1 + \frac{1-i}{\sqrt{2}} + i + \frac{-1-i}{\sqrt{2}} && = 0 \\
a_6 &= 1 + i - 1 - i + 1 + i - 1 - i && = 0 \\
a_7 &= 1 + \frac{1+i}{\sqrt{2}} + i + \frac{-1+i}{\sqrt{2}} - 1 + \frac{-1-i}{\sqrt{2}} - i + \frac{1-i}{\sqrt{2}} && = 0
\end{aligned}$$

$\boldsymbol{a} = (8, 0, 0, 0, 0, 0, 0, 0)$

Question 5.

Show that the DFT of: $\boldsymbol{f} = (f_0,\ f_1,\ \ldots,\ f_7)$ for:

$$f_n = \sin\frac{2\pi n}{8}$$

Is given by $(0,\ A, 0,\ 0,\ 0,\ 0,\ 0, B)$, and determine $A,\ B$. You may use the orthogonality properties of $\boldsymbol{p}_n^{(k)}$.

SOLUTIONS

First note that:

$$\sin\frac{2\pi n}{8} = \frac{1}{2i}\left(\exp\left(\frac{i2\pi n}{8}\right) - \exp\left(\frac{-i2\pi n}{8}\right)\right)$$

And that:

$$\exp\left(\frac{-i2\pi n}{8}\right) = \exp\left(\frac{i(8-1)2\pi n}{8}\right) = \frac{1}{2i}\left(\exp\left(i.1.x_n\right) - \exp\left(i.7.x_n\right)\right)$$

$$a_k = \sum_{n=0}^{7}\exp\left(-ikx_n\right)\frac{1}{2i}\left(\exp\left(i.1.x_n\right) - \exp\left(i.7.x_n\right)\right)$$

We now use the orthogonality relationship

$$\sum_{n=0}^{7}\exp(-ikx_n)\exp\left(ijx_n\right) = 0$$

unless $k = 1$ or $k = 7$ in which case it equals 8. Therefore $\boldsymbol{a} = (0, 8/2i, 0, 0, 0, 0, 0, -8/2i) = (0, -4i, 0, 0, 0, 0, 0, 4i)$

Question 6.

The DFT of a signal $\boldsymbol{f}$ is: $(8,\ 4-8i,\ 2,\ -i,\ 0,\ i,\ 2,\ 4+8i)$

(a) This means that the original signal can be expressed as

$$f_n = \sum_{k=0}^{N-1} a_k p_n^{(k)}, \quad p_n^{(k)} = \frac{1}{N}\exp\left(ikx_n\right)$$

Rewrite the original signal in the form:

$$f = \alpha_0 + \sum_{k=0}^{N-1}\left[\alpha_i\cos\left(\omega_i x\right) + \beta_i \sin\left(\omega_i x\right)\right]$$

Where the coefficients and ω_i are yet to be determined. **Hint:** Use the property that $p_n^{N-m} = p_n^{-m}$.

(b) Use the inverse DFT to obtain the values of f using python, and using direct calculation. (Note, part (b) is not suitable for an exam question because of the number of calculations involved.)

SOLUTIONS

(a) Expanding out the inverse DFT formulae:

$$
\begin{aligned}
f_n &= 8p_n^{(0)} + (4-8i)p_n^{(1)} + 2p_n^{(2)} - ip_n^{(3)} + ip_n^{(5)} + 2p_n^{(6)} + (4+8i)p_n^{(7)} \\
&= 8p_n^{(0)} + 4\left(p_n^{(1)} + p_n^{(7)}\right) - 8i\left(p_n^{(1)} - p_n^{(7)}\right) + 2\left(p_n^{(2)} + p_n^{(6)}\right) - i\left(p_n^{(3)} - p_n^{(5)}\right)
\end{aligned}
$$

But since $p_n^{N-m} = p_n^{-m}$:

$$
f_n = 8p_n^{(0)} + 4\left(p_n^{(1)} + p_n^{(-1)}\right) - 8i\left(p_n^{(1)} - p_n^{(-1)}\right) + 2\left(p_n^{(2)} + p_n^{(-2)}\right) - i\left(p_n^{(3)} - p_n^{(-3)}\right)
$$

And note that:

$$
\begin{aligned}
p_n^{(k)} + p_n^{(-k)} &= \frac{1}{N}\left[\exp\left(ikx_n\right) + \exp\left(-ikx_n\right)\right] \\
&= \frac{2}{N}\left[\cos\left(kx_n\right)\right] \\
p_n^{(k)} - p_n^{(-k)} &= \frac{1}{N}\left[\exp\left(ikx_n\right) - \exp\left(-ikx_n\right)\right] \\
&= \frac{2}{N}\left[i\sin\left(kx_n\right)\right]
\end{aligned}
$$

Therefore, f_n may be expressed in terms of the trig functions, also note, from the definition of $p_n^{(k)}$, we have $p_n^{(0)} = 1/8$

$$
\begin{aligned}
f_n &= \frac{1}{8}\left[8 + 8\cos\left(x_n\right) + 16\sin\left(x_n\right) + 4\cos\left(2x_n\right) + 2\sin\left(3x_n\right)\right] \\
&= 1 + \cos\left(\frac{\pi n}{4}\right) + 2\sin\left(\frac{\pi n}{4}\right) + \frac{1}{2}\cos\left(\frac{\pi n}{2}\right) + \frac{1}{4}\sin\left(\frac{3\pi n}{4}\right)
\end{aligned}
$$

(b) An implementation in python comparing the numerical inverse fast fourier transform to our analytical inverse dsicrete fourier transform is given below:

```python
import numpy, math

# Define the complex DFT vector
W = numpy.array([8 , 4.0-8.j , 2, -1.j , 0, 1.j, 2  ,4+8.j], dtype=complex)

print("W = ")
[print("{0:.3f}".format(ak)) for ak in W]

# calculate the inverse DFT to retrieve the original signal f
f = numpy.fft.ifft(W)

print("\n numerical f = ")
[print("{0:.3f}".format(fn)) for fn in f]

# Verify that the numerical solution matches our analytical one
def fn(n):
        "Analytical inverse fourier transform, value for the n'th element of the signal"
        return 1 + math.cos(math.pi * n /4) + 2 * math.sin(math.pi * n / 4) +\
        1/2 * math.cos(math.pi * n /2) + 1/4 *math.sin(3* math.pi * n /4) + 0.j

print("\n analytical f = ")
[print("{0:.3f}".format(fn(n))) for n in range(8)]
```

Which outputs:

W = 8.000+0.000j
4.000-8.000j
2.000+0.000j
-0.000-1.000j
0.000+0.000j
0.000+1.000j
2.000+0.000j
4.000+8.000j

numerical f =
2.500+0.000j
3.298+0.000j
2.250+0.000j
1.884+0.000j
0.500+0.000j
-1.298-0.000j
-1.250+0.000j
0.116-0.000j

analytical f =
2.500+0.000j
3.298+0.000j
2.250+0.000j
1.884+0.000j
0.500+0.000j
-1.298+0.000j
-1.250+0.000j
0.116+0.000j

And it can be seen that our analytical solution agrees with the numerical one.