MECH3750 - Tutorial 4

Question 1.
Show that

a. $\overline{\exp(iy)} = \exp(-iy)$

b. $\sin y = \frac{e^{iy}-e^{-iy}}{2i}$

c. $\cos y = \frac{e^{iy}+e^{-iy}}{2}$

Question 2.
Using the complex inner product $(u,v) = \sum_{i=0}^{4} \overline{u_i} v_i$ consider the vectors

$$q_n^{(k)} = \exp(ik\frac{2\pi n}{M}) \quad n = 0,1,2,3$$

a. Write $q^{(k)}$ explicitly for $k = 0,1,2,3$.

b. Use the inner product to find $\|q^{(k)}\|$ for $k = 0,1,2,3$.

c. Verify $q^{(0)}, q^{(1)} = 0$; $q^{(2)}, q^{(3)} = 0$; $q^{(0)}, q^{(2)} = 0$

Question 3.

In our interpretation of the DFT, the values a_k represent the coefficients of the vector:

$$p_n^{(k)} = \frac{1}{N} \exp\left(i\frac{2\pi nk}{N}\right)$$

in the signal f_n for $k = 0,1,\ldots,N-1$.

Verify that $p_n^{(N-1)} = p_n^{(-1)}$ and also $p_n^{(N-m)} = p_n^{(-m)}$. This is important for interpreting the values of the DFT for large k.

Question 4.

Find the DFT of:

(a) $\boldsymbol{f} = (1,\ 2,\ 0,\ 1)$

(b) $\boldsymbol{f} = (1,\ 1,\ \ldots,\ 1)$, for $N = 8$

Question 5.

Show that the DFT of: $\boldsymbol{f} = (f_0,\ f_1,\ \ldots,\ f_7)$ for:

$$f_n = \sin\frac{2\pi n}{8}$$

Is given by $(0,\ A, 0,\ 0,\ 0,\ 0, B)$, and determine $A,\ B$. You may use the orthogonality properties of $\boldsymbol{p}_n^{(k)}$.

Question 6.

The DFT of a signal $\boldsymbol{f}$ obtained at values $x_n = \dfrac{2\pi n}{8}$, $n = 0, \ldots, 7$ is: $\boldsymbol{a} = (8,\ 4-8i,\ 2,\ -i,\ 0,\ i,\ 2,\ 4+8i)$.

a. This means that the original signal can be expressed as

$$f_n = \sum_{k=0}^{N-1} a_k p_n^{(k)} \quad p_n^{(k)} = \frac{1}{N}\exp(ikx_n).$$

Rewrite the original signal in the form $\boldsymbol{f} = \alpha_1 + \sum_i \alpha_i \cos(\omega_i x) + \beta_i \sin(\omega_i x)$ where the coefficients and ω_i are to be determined. **Hint:** Use the property that $p_n^{N-m} = p_n^{-m}$

b. Use the inverse DFT to obtain the values of the $\boldsymbol{f}$ using Python.

Formula Sheet

Discrete Fourier Transform Given a vector $(f_0, f_1, \ldots f_{N-1})$ we define its DFT as

$$a_k = \sum_{n=0}^{N-1} f_n \exp(-ikx_n) \quad \text{with} \quad x_n = \frac{2\pi n}{N}, \quad n = 0, \ldots, N-1.$$

The original signal can be recovered using the inverse DFT $f_n = \dfrac{1}{N}\displaystyle\sum_{k=0}^{N-1} a_k \exp(ikx_n)$. The DFT relies on the orthogonality relationship:

$$\sum_{n=0}^{N-1} \exp(-ikx_n)\exp(ijx_n) = 0 \quad k \neq j \quad \& \quad = N \quad k = j$$