MECH3750 PBL Content Summary
Week 5

Content:

- Introduction to Partial Differential Equations
- Initial Conditions & Boundary Conditions

Upcoming assessment:

- Problem Sheet 5 (due before Week 6 PBL session)
- Quiz 2 (Week 6)
- Assignment 1 (Due Fri Week 7)

Tutors: Nathan Di Vaira, Alex Muirhead, William Snell, Tristan Samson, Robert Watt

1 Introduction to Partial Differential Equations (PDEs)

In lectures we were introduced to the three core PDEs.

Wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$u_{tt} = c^2 u_{xx}$$

Heat (diffusion) equation:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

$$u_t = k^2 u_{xx}$$

Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$u_{xx} + u_{yy} = 0$$

u may represent any quantity, such as a displacement (wave equation), a concentration or a temperature (heat/diffusion equation).

These PDEs are written here in their lowest dimensional forms, but may be extended to higher spatial dimensions (y, z), in which case the Laplace operator is commonly used,

$$\bigtriangledown^2 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

1.1 Initial Conditions

In order to solve a PDE as we progress forward through time, we need to know what it originally looked like, i.e., at $t = 0$.

For the wave equation, we must know the original position and velocity,

$$u(x, 0),$$

$$\frac{\partial u}{\partial t}(x, 0).$$

For the heat equation, only initial concentration is required,

$$u(x, 0) = f(x).$$

Sometimes we may be required to solve for initial conditions given the solution to $u(x, t)$ (PBL *Question 3*).

1.2 Boundary Conditions

Like initial conditions, we must also know what the boundaries limiting the motion of our system are in order to solve a PDE.

For the wave equation, we typically know (or solve) the value of u at the beginning and end of the spatial interval $[0, L]$ (PBL *Question 3*),

$$u(0, t),$$
$$u(L, t).$$

These are an example of fixed, or *Dirichlet*, boundary conditions. The other common boundary condition is the *Neumann* boundary condition,

$$\frac{\partial u}{\partial x}(L, x) = f(x),$$

which effectively specifies a flow, or flux, over a boundary. This is common to diffusion problems. We'll come back to *Dirichlet* and *Neumann* boundary conditions as we progress through solving PDEs throughout the rest of the course.

Note that for ODEs where time is the only variable, $y(t)$, a boundary value problem requires solving $y(t_1)$ and $y(t_2)$ (PBL *Questions 1 & 2*).