MECH3750 PBL Content Summary

Week 10

Content:

- Advanced solution techniques for convection
  - Higher-order schemes
  - Non-linear convection
- Boundary value problems

Upcoming assessment:

- Problem Sheet 10 (due before Week 11 PBL session)
- Assignment II (Due Fri. Week 11)

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1 Advanced solution techniques for convection

1.1 Higher-order schemes

Modified finite difference schemes are required to universally achieve second order spatial and temporal convergence, $O((\Delta x)^2)$ and $O((\Delta t)^2)$, for convection problems, while maintaining stability. The Lax-Wendroff scheme uses a second order Taylor series as its basis,

$$
\begin{aligned}
u_j^{m+1} &= u_j^m + \Delta t \frac{\partial u_j^m}{\partial t} + \frac{(\Delta t)^2}{2}\frac{\partial^2 u_j^m}{\partial t^2} + H.O.T. \\
&\approx u_j^m - v\Delta t \frac{\partial u_j^m}{\partial x} + v^2 \frac{(\Delta t)^2}{2}\frac{\partial^2 u_j^m}{\partial x^2} \\
&\approx u_j^m - v\Delta t \frac{u_{j+1}^m - u_{j-1}^m}{2\Delta x} + v^2 \frac{(\Delta t)^2}{2}\frac{u_{j+1}^m - 2u_j^m + u_{j-1}^m}{(\Delta x)^2},
\end{aligned}
$$

where, rearranging and substituting $\sigma = v\Delta t \Delta x$, we obtain the update equation,

$$
u_j^{m+1} \approx \frac{\sigma}{2}(1+\sigma)u_{j-1}^m + (1-\sigma^2)u_j^m - \frac{\sigma}{2}(1-\sigma)u_{j+1}^m.
$$

The MacCormack Method, which is based on an explicit predictor-correct finite difference scheme, presents a more general method for solving hyperbolic systems, both linear and non-linear.

1.2 Non-linear convection

Non-linear convection arises in the inviscid euler equations, which are described by the inviscid Burgers Equation,

$$
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0,
$$

where the convective velocity is now dependent on the convected quantity, u. These can be written in its non-conservative (above) and conservative form,

$$
\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0.
$$

This conservative form is useful when shocks in solution can form, at which point PDEs are no longer applicable.

2 Boundary value problems

The most simple boundary value problems (BVP) are based on one-dimensional, linear ODEs,

$$\frac{d^2u}{dx^2} + a\frac{du}{dx} + bu = 0.$$

To solve the BVP, we need the values of u at each end of our solution domain $[0, L]$,

$$u(0) = U_0,$$

$$u(L) = U_L.$$

These types of BVPs are typically for steady-state problems, such as trajectories or temperature distributions (today's PBL sheet),

$$\frac{d^2T}{dx^2} + q(T) = 0.$$

Note that this is the 1D Laplace (elliptical) equation, and that adding a temporal term would result in a diffusion equation,

$$\frac{\partial T}{\partial t} + \frac{d^2T}{dx^2} + q(T) = 0.$$