MECH3750 PBL Content Summary
Week 6

Content:

- Separation of Variables (Fourier's Method)

Upcoming assessment:

- Problem Sheet 6 (due before Week 7 PBL session)
- Assignment 1 (Due Fri. Week 7)

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1 Separation of Variables (Fourier's Method)

The method of separation of variables assumes a PDE is *separable*, i.e., it can be written in the form,

$$u(x,t) = F(x)g(t).$$

1.1 Wave Equation

Suppose we have a system (e.g. string occupying $0 \le x \le L$) which is described by the wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$
$$u_{tt} = c^2 u_{xx}$$

with initial conditions,

$$u(x,0) = f(x),$$
$$u_t(x,0) = g(x),$$

and boundary conditions,

$$u(0,t) = 0,$$
$$u(L,t) = 0.$$

To solve the PDE, firstly substitute the separated equation, $u(x,t) = F(x)g(t)$, into the boundary conditions,

$$u(0,t) = 0 = F(0)g(t),$$
$$u(L,t) = 0 = F(L)g(t),$$

to obtain the useful results $F(0) = 0$ and $F(L) = 0$.

Next, substitute the separated equation into the original PDE,

$$u_{tt} = c^2 u_{xx},$$
$$F(x)g''(t) = c^2 F''(x)g(t),$$

to obtain the result,

$$\frac{g''(t)}{g(t)c^2} = \frac{F''(t)}{F(t)} = k.$$

In the lectures, it was shown that the solution to this equation is,

$$\begin{aligned}
&F(x) = ax + b && \text{for } k = 0,\\
&F(x) = a\,cosh(\mu x) + b\,sinh(\mu x) && \text{for } k = \mu^2,\\
&F(x) = a\,cos(px) + b\,sin(px) && \text{for } k = -p^2.
\end{aligned}$$

These results provide the starting point for PBL *Question 1.*

By applying the boundary conditions, it was shown that the case $k = -p^2$ provides the only useful solution, $p = n\pi/L$, such that

$$F(x) = b\,sin\left(\frac{n\pi}{L}x\right),$$

and,

$$g(t) = a\,cos\left(\frac{n\pi}{L}ct\right) + b\,sin\left(\frac{n\pi}{L}ct\right),$$

where $c = x/t$. Substituting these results into the separated equation, $u(x,t) = F(x)g(t)$, we obtain,

$$u(x,t) = \left[A_n\,cos\left(\frac{n\pi}{L}ct\right) + B_n\,sin\left(\frac{n\pi}{L}ct\right)\right]\,sin\left(\frac{n\pi}{L}x\right).$$

Seeing as any value of n gives a solution to the original PDE, and any linear combination of the equations for any n is also a solution, the general solution is,

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n\,cos\left(\frac{n\pi}{L}ct\right) + B_n\,sin\left(\frac{n\pi}{L}ct\right)\right]\,sin\left(\frac{n\pi}{L}x\right).$$

Like our previous work with Fourier series, the summation can now be thought of as breaking the solution to the PDE into a combination of trigonometric functions of different frequencies.

Imposing the initial conditions ($f(x) = u(x,0)$ and $g(x) = u_t(x,0)$) expressions for the coefficients were found in lectures,

$$A_n = \frac{2}{L}\int_0^L sin\left(\frac{n\pi x}{L}\right) f(x)dx,$$

$$B_n = \frac{2}{n\pi c}\int_0^L sin\left(\frac{n\pi x}{L}\right) g(x)dx.$$

Separation of variables may also be applied to the 1D heat equation and steady state heat conduction in 2D, as derived in lectures.