MECH3750 PBL Content Summary

Week 8

Content:

- Hyperbolic PDEs
  - Wave equation

Upcoming assessment:

- Problem Sheet 8 (due before Week 9 PBL session, after mid-sem break)

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1 Hyperbolic PDEs

This week in lectures we looked at the 2nd type of PDEs, hyperbolic. Hyperbolic PDEs can be first order (the convection equation),

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0,$$

or 2nd order (the wave equation),

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}.$$

The wave equation is the standard form of hyperbolic PDEs. In PBL *Question 1* you should obtain this equation. For oscillations (of a vibrating string), the constant c^2 is replaced by $F/\rho A$ (PBL *Question 2*).

1.1 Numerical Solution to Wave Equation

So how do we solve the wave equation using finite differences? 2nd order central differences are typically used for both the temporal and spatial derivatives,

$$\frac{u_j^{m-1} - 2u_j^m + u_j^{m+1}}{(\Delta t)^2} = c^2\left(\frac{u_{j-1}^m - 2u_j^m + u_{j+1}^m}{(\Delta x)^2}\right),$$

However, it can be seen that this approximation depends on the previous time step $(m-1)$, which is a problem at the first time step. To deal with this, we apply a three-point central difference approximation to the first derivative of the initial condition, $U_1(x)$, such that the update equation for the first time step uses $U_1(x)$,

$$u_j^1 = \frac{\sigma^2}{2}u_{j-1}^0 + (1-\sigma^2)u_j^0 + \frac{\sigma^2}{2}u_{j+1}^0 + \Delta t U_1(x),$$

recognising that the u^0 terms are simply the initial condition for displacement, $U_0(x)$. The general update equation for all subsequent steps is then,

$$u_j^{m+1} = \sigma^2 u_{j-1}^m + (2-2\sigma^2)u_j^m + \sigma^2 u_{j+1}^m - u_j^{m-1},$$

where $\sigma = c\frac{\Delta t}{\Delta x}$.

Similar to the diffusion equation last week, these update equations can be written in matrix form (refer to lecture notes).

1.2 Boundary conditions

To deal with boundaries in the wave equation, it is easiest to directly apply them at each time step. Common boundary conditions are fixed ends (Dirichlet),

$$u_0^{m+1} = u_0^m = 0; u_N^{m+1} = u_N^m = 0.$$

or specifying a zero-gradient (Neumann). By taking a 2nd order forward difference at the boundary nodes, the update for the zero-gradient condition can be obtained,

$$u_0^{m+1} = \frac{4u_1^{m+1} - u_2^{m+1}}{3}; u_{N-1}^{m+1} = \frac{4u_{N-2}^{m+1} - u_{N-2}^{N-3}}{3}.$$

Note, however, that both Dirichlet and Neumann boundaries can be non-zero and/or vary with time.