MECH3750 PBL Content Summary
Week 3

Content:

- Inner Products, Norms, Distances and Orthogonality
- Least Squares using Inner Products

Upcoming assessment:

- Problem Sheet 3 (due before Week 4 PBL session)

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1 Inner Products

The inner product, $\langle p, q\rangle$, is an operation between two quantities (e.g., functions or vectors) which produces a scalar quantity.

It can be thought of as a generalisation of the dot product for vectors,

$$\langle \mathbf{p}, \mathbf{q}\rangle = \mathbf{p} \cdot \mathbf{q},$$

and can be defined for functions as,

$$\langle p, q\rangle = \int_a^b p(x)q(x)dx.$$

1.1 Norms, Distances and Orthogonality

These are important properties of the inner product.

Norm (magnitude)	$\Rightarrow$	$\|\mathbf{p}\| = \sqrt{\langle \mathbf{p}, \mathbf{p}\rangle}$
Distance	$\Rightarrow$	$\|\mathbf{p} - \mathbf{q}\| = \sqrt{\langle \mathbf{p} - \mathbf{q}, \mathbf{p} - \mathbf{q}\rangle}$
Orthogonality	$\Rightarrow$	$\langle \mathbf{p}, \mathbf{q}\rangle = 0$

For the inner product for functions shown above, these properties become,

Norm	$\Rightarrow$	$\|p\| = \sqrt{\int_a^b p(x)^2 dx}$
Distance	$\Rightarrow$	$\|p - q\| = \sqrt{\int_a^b \left(p(x) - q(x)\right)^2 dx}$
Orthogonality	$\Rightarrow$	$\langle p, q\rangle = \int_a^b p(x)q(x)dx = 0$

2 Least Squares using Inner Products

The least squares matrix equations from last week can be written in a generalised form using the inner product (known as the normal equations),

$$\begin{bmatrix} \langle f, p_1 \rangle \\ \langle f, p_2 \rangle \\ \vdots \\ \langle f, p_n \rangle \end{bmatrix} = \begin{bmatrix} \langle p_1, p_1 \rangle & \langle p_1, p_2 \rangle & \dots & \langle p_1, p_n \rangle \\ \langle p_2, p_1 \rangle & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \langle p_n, p_1 \rangle & \dots & \dots & \langle p_n, p_n \rangle \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

The normal equations can now be used to approximate any function or vector, f, with a set of other functions or vectors, p_i, where the approximation is $y = \sum_{i=1}^{n} \alpha_i p_i$ (PBL *Questions 1 & 2*).

In other words, for the least squared problem, we are minimising $\|\mathbf{y} - \mathbf{f}\|^2$.

If the approximating functions or vectors are mutually orthogonal, the above matrix is diagonal.

2.1 Orthogonal Polynomials

There exist sets of mutually orthogonal polynomials for which the normal equation matrix is diagonal. One such set is the shifted Legendre Polynomials,

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= 2x - 1 \\ P_2(x) &= 6x^2 - 6x + 1 \\ P_3(x) &= 20x^3 - 30x^2 + 12x - 1 \end{aligned}$$