Q1

i)

$$\frac{1}{mn} H_m \hat{Z} H_n = \frac{1}{mn} H_m (H_m Z H_n) H_n \qquad \text{since } \hat{Z} = H_m Z H_n$$
$$= \frac{1}{m} H_m H_m \; Z \; \frac{1}{n} H_n H_n$$

since m & n are power of 2, $H_m^{-1} = H_m/m$ & $H_n^{-1} = H_n/n$

Thus, $\frac{1}{mn} H_m \hat{Z} H_n = \left\{\frac{H_m}{m} \cdot \frac{m}{H_m}\right\} Z \left\{\frac{H_n}{n} \cdot \frac{n}{H_n}\right\} = Z$

$$Z = \frac{1}{mn} H_m \hat{Z} H_n$$

STA410

Q1-b

```
fwht2d <- function(x) {
  h <- 1
  len <- ncol(x)
  while (h < len) {
    for (i in seq(1,len,by=h*2)) {
      for (j in seq(i,i+h-1)) {
        a <- x[,j]
        b <- x[,j+h]
        x[,j] <- a + b
        x[,j+h] <- a - b
      }
    }
    h <- 2*h
  }
  h <- 1
  len <- nrow(x)
  while (h < len) {
    for (i in seq(1,len,by=h*2)) {
      for (j in seq(i,i+h-1)) {
        a <- x[j,]
        b <- x[j+h,]
        x[j,] <- a + b
        x[j+h,] <- a - b
      }
    }
    h <- 2*h
  }
  x
}
```

```
hard.thresholding<-function(x,lambda){
  xhat<-fwht2d(x) # W-H transform
  xhat<-ifelse(abs(xhat)<=lambda,0,xhat)
  xx<-fwht2d(xhat)/((ncol(xhat))^2) # inverse transform
  xx
}

soft.thresholding<-function(x,lambda){
  xhat<-fwht2d(x) # W-H transform
  xhat<-sign(xhat)*pmax(abs(xhat)-lambda,0)
  xx<-fwht2d(xhat)/((ncol(xhat))^2) # inverse transform
  xx
}
```

Q1-c

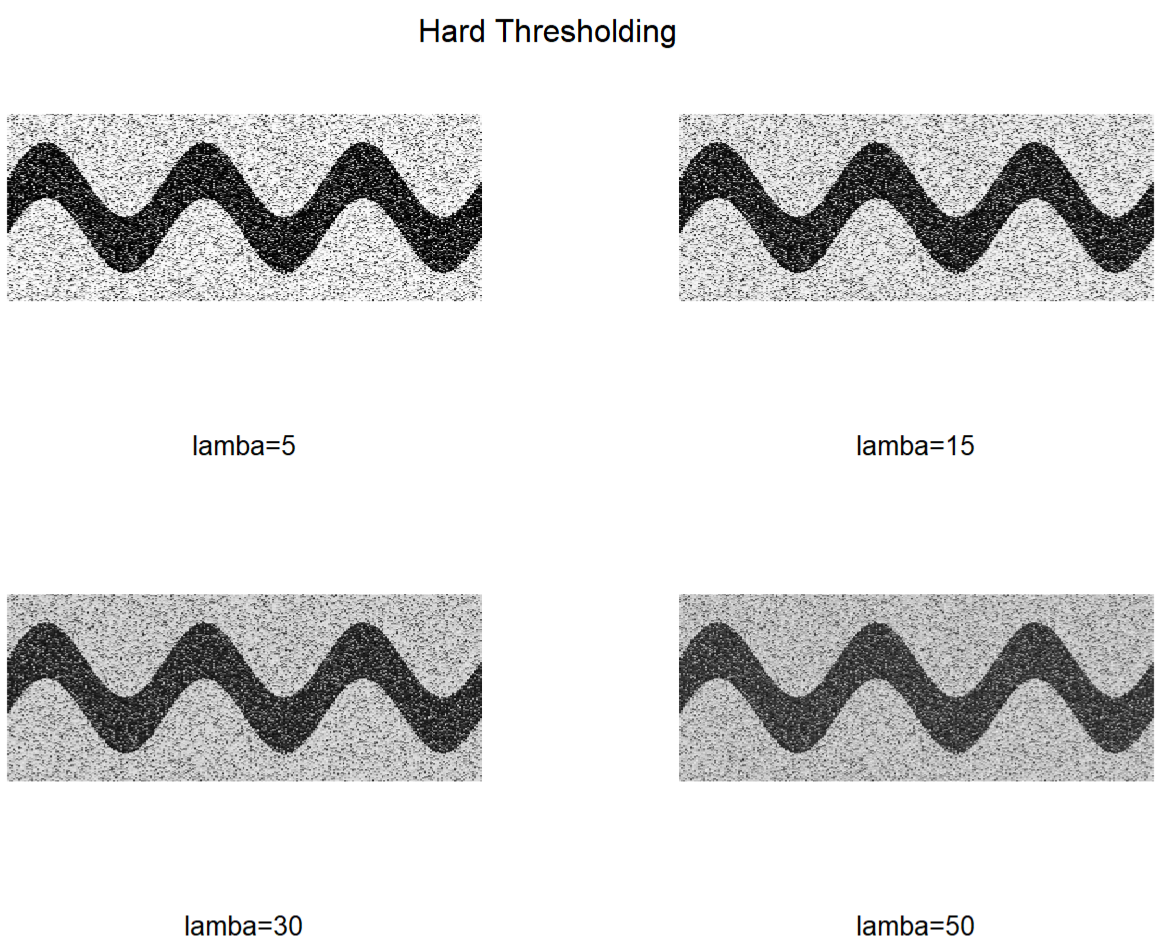
```
design<-matrix(scan("C:/Users/WLJY8/Desktop/Courses/YEAR 4/STA410/A1/design.txt"),ncol = 256,byrow
= T)
colours <-grey(seq(0,1,length=256))
#image(design, axes=F, col=colours)
```

Hard-thresholding images with lambda = 5,10,30,50

```
par(mfrow=c(2,2))
pic5<-hard.thresholding(design,5)
pic15<-hard.thresholding(design,15)
pic30<-hard.thresholding(design,30)
pic50<-hard.thresholding(design,50)

image(pic5,axes=F,col = colours,sub="lamba=5")
image(pic15,axes=F,col = colours,sub="lamba=15")
image(pic30,axes=F,col = colours,sub="lamba=30")
image(pic50,axes=F,col = colours,sub="lamba=50")

mtext("Hard Thresholding", side = 3, line = -2, outer = TRUE,cex=1)
```

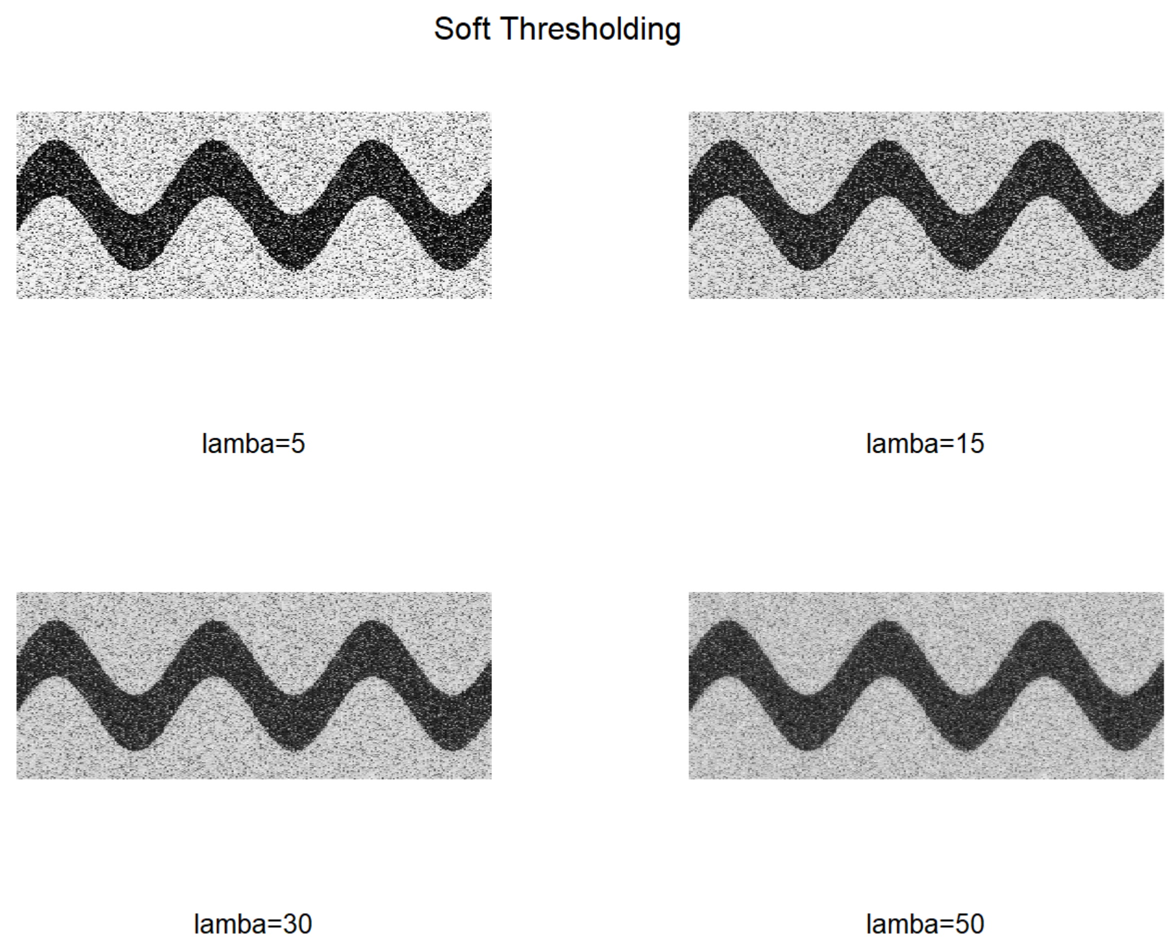


Soft-thresholding images with lambda = 5,10,15,20

```
par(mfrow=c(2,2))
pic_5<-soft.thresholding(design,5)
pic_15<-soft.thresholding(design,15)
pic_30<-soft.thresholding(design,30)
pic_50<-soft.thresholding(design,50)

image(pic_5,axes=F,col = colours,sub="lamba=5")
image(pic_15,axes=F,col = colours,sub="lamba=15")
image(pic_30,axes=F,col = colours,sub="lamba=30")
image(pic_50,axes=F,col = colours,sub="lamba=50")

mtext("Soft Thresholding", side = 3, line = -2, outer = TRUE,cex=1)
```



For both methods, With larger \(\lambda\), the clarity of graph decreases. With the same lambda, Hard-thresholding method is better compare to Soft-thresholding method.

Q2

1)

$$E(t^S \mid N=n) = E(t^{X_1+\cdots+X_n}) \qquad \text{since } S = X_1 + \cdots + X_n$$

$$= E(t^{X_1})E(t^{X_2})\cdots E(t^{X_n})$$

$$= [\phi(t)]^n \qquad \text{since } \phi(t) = E(t^{X_i})$$

$$E(t^S) = \sum_{n=0}^{\infty} E(t^S \mid N=n)\, P(N=n)$$

$$= \sum_{n=0}^{\infty} [\phi(t)]^n \, P(N=n)$$

$$= \sum_{n=0}^{\infty} [\phi(t)]^n (1-\theta)\theta^n \qquad \text{Since } P(N=n) = (1-\theta)\theta^n \text{ for } n=1,2\cdots$$

$$= (1-\theta) \sum_{n=0}^{\infty} [\phi(t)]^n \theta^n$$

$$= (1-\theta) \sum_{n=0}^{\infty} [\theta\phi(t)]^n$$

$$= (1-\theta) \frac{1}{1-\theta\phi(t)}$$

$$= \frac{1-\theta}{1-\theta\phi(t)}$$

Since $g(t) = \dfrac{1-\theta}{1-\theta t}$, $\quad g(\theta(t)) = \dfrac{1-\theta}{1-\theta\phi(t)}$

Thus, $\quad g(\theta(t)) = E(t^S)$

2) $x_i \leq l$, $S = \sum_{i=1}^{N} x_i \leq \sum_{i=1}^{N} l = Nl$

$$P(S \geq M) \leq P(Nl \geq M) = P(N \geq \frac{M}{l})$$

Let $\frac{M}{l} = m$

$$P(S \geq M) = P(S \geq ml) \leq P(N \geq m) \leq \varepsilon$$

Since $P(N \geq m) \leq \varepsilon$ then $P(S \geq ml) \leq \varepsilon$ and so we can take $M \geq ml$.

c)

We have $P(S \geq ml) \leq \varepsilon$, $M \geq ml$

then $P(S \geq M) \leq \varepsilon$

$$P(S \geq M) = P(t^S \geq t^M) \leq \frac{E(t^S)}{t^M} = \frac{1}{t^M}\left(\frac{1-\theta}{1-\theta\phi(t)}\right)$$

Let $\dfrac{E(t^S)}{t^M} = \varepsilon$

$$\frac{1}{t^M}\left(\frac{1-\theta}{1-\theta\phi(t)}\right) = \varepsilon$$

$$\frac{1-\theta}{(1-\theta\phi(t))\varepsilon} = t^M$$

$$\ln\left(\frac{1-\theta}{(1-\theta\phi(t))\varepsilon}\right) = \ln(t^M)$$

$$\ln(1-\theta) - \ln(1-\theta\phi(t)) - \ln\varepsilon = M\ln(t)$$

$$M = \inf_{1<\phi(t)<\theta^{-1}} \frac{\ln(1-\theta) - \ln(1-\theta\phi(t)) - \ln(\varepsilon)}{\ln(t)}$$

2-d

```
dist<-function(theta,px,M){
  px<-c(px,rep(0,M-(length(px)-1)-1))
  pj_hat<-fft(px)
  ps_hat<-(1-theta)/(1-theta*pj_hat)
  ps <- Re(fft(ps_hat,inv=T))/M
  df <- list(y=c(0:(M-1)),probs=ps)
  df
}

x<-c(0:10)
px<-choose(10,x)*(1/2)^10
t <- c(1010:12000)/1000
pgf <- ((1+t)^10)/(2^10)

a <- log(1-0.9*pgf)
```

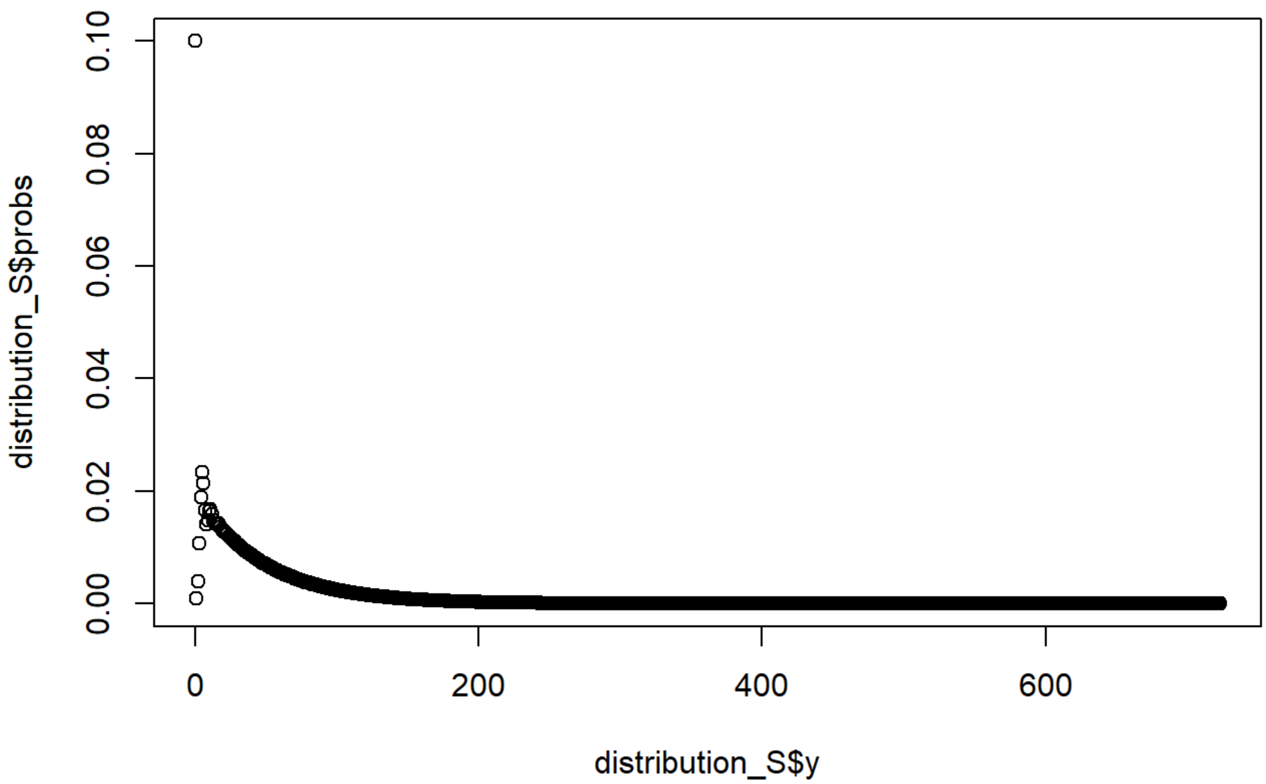
```
## Warning in log(1 - 0.9 * pgf): NaNs produced
```

```
b <- cbind(t,a)
col <- na.omit(b)

M <- min((log(0.1)-log(10^(-5))-col[,2])/(log(col[,1])))
M
```

```
## [1] 724.8205
```

```
distribution_S<-dist(0.9,px,M)
plot(distribution_S$y,distribution_S$probs)
```



distribution_S$probs
0.00
0.02
0.04
0.06
0.08
0.10
0
200
400
600
distribution_S$y