Q1.

a).

$$L(\lambda,\alpha)\cdot\pi(\lambda,\alpha)$$

$$=\lambda^{n\alpha}\left\{\prod_{i=1}^{n}x_i^{\alpha-1}\right\}\exp\left(-\lambda\sum_{i=1}^{n}x_i\right)\cdot\frac{1}{[\Gamma(\alpha)]^n}\cdot\frac{1}{10000}e^{-\lambda/100}\cdot e^{-\alpha/100}\quad \text{for } \lambda,\alpha>0$$

$$\prod_{i=1}^{n}x_i^{\alpha-1}=\exp\left(\log\left(\prod_{i=1}^{n}x_i\right)^{\alpha-1}\right)=\exp\left((\alpha-1)\sum_{i=1}^{n}\ln x_i\right)$$

$$=\exp\left(\alpha\sum_{i=1}^{n}\ln x_i\right)\cdot\exp\left(-\sum_{i=1}^{n}\ln x_i\right)$$

$$\pi(\alpha,\lambda\mid x_1\cdots x_n)=K(x_1\cdots x_n)\cdot L(\lambda,\alpha)\cdot\pi(\lambda,\alpha)$$

$$=K(x_1\cdots x_n)\cdot\frac{1}{[\Gamma(\alpha)]^n}\cdot\exp\left(\alpha\sum_{i=1}^{n}\ln x_i-\frac{\alpha}{100}\right)\cdot\exp\left(-\lambda\left(\sum_{i=1}^{n}x_i+\frac{1}{100}\right)\right)\cdot\lambda^{n\alpha}$$

$$\pi(\alpha\mid x_1\cdots x_n)=\int_0^\infty\pi(\alpha,\lambda\mid x_1\cdots x_n)\,d\lambda$$

$$=K(x_1\cdots x_n)\cdot\frac{1}{[\Gamma(\alpha)]^n}\cdot\exp\left(\alpha\sum_{i=1}^{n}\ln x_i-\frac{\alpha}{100}\right)\cdot\int_0^\infty\exp\left(-\lambda\left(\sum_{i=1}^{n}x_i+\frac{1}{100}\right)\right)\cdot\lambda^{n\alpha}\,d\lambda$$

When $\lambda\sim\text{Gamma}(n\alpha+1,\beta)$

$$\pi(\lambda)=\int_0^\infty\frac{1}{\Gamma(n\alpha+1)}\cdot\beta^{n\alpha+1}\cdot\lambda^{n\alpha}\cdot\exp(-\lambda\beta)\,d\lambda=1$$

Let $\beta=\sum_{i=1}^{n}x_i+\frac{1}{100}$

$$=\int_0^\infty\frac{1}{\Gamma(n\alpha+1)}\cdot\left(\sum_{i=1}^{n}x_i+\frac{1}{100}\right)^{n\alpha+1}\cdot\lambda^{n\alpha}\cdot\exp\left(-\lambda\left(\sum_{i=1}^{n}x_i+\frac{1}{100}\right)\right)d\lambda=1$$

Thus

$$\int_0^\infty\exp\left(-\lambda\left(\sum_{i=1}^{n}x_i+\frac{1}{100}\right)\right)\cdot\lambda^{n\alpha}\,d\lambda=\Gamma(n\alpha+1)\left(\sum_{i=1}^{n}x_i+\frac{1}{100}\right)^{-(n\alpha+1)}$$

$$\pi(\alpha \mid x_1 \cdots x_n) = \int_0^\infty \pi(\alpha, \lambda \mid x_1 \cdots x_n)\, d\lambda$$

$$= K(x_1 \cdots x_n) \cdot \frac{1}{[\Gamma(\alpha)]^n} \cdot \exp\left(\alpha \sum_{i=1}^n \ln x_i - \frac{\alpha}{100}\right) \cdot \int_0^\infty \exp\left(-\lambda\left(\sum_{i=1}^n x_i + \frac{1}{100}\right)\right) \cdot \lambda^{n\alpha}\, d\lambda$$

$$= K(x_1 \cdots x_n) \cdot \frac{\Gamma(n\alpha+1)}{[\Gamma(\alpha)]^n} \exp\left(\alpha \sum_{i=1}^n \ln x_i - \frac{\alpha}{100}\right) \left(\frac{1}{100} + \sum_{i=1}^n x_i\right)^{-(n\alpha+1)}$$

b) For pre-normalization, we noticed that

$$\pi(\alpha \mid x_1 \cdots x_n) = \frac{u(\theta)}{\int_\theta u(s)\, ds}$$

and $u(\theta)$ can computed as

$$u(\theta) = \exp\left[\ln \pi(\theta) + \ln \mathcal{L}(\theta) - \max_\theta \{\ln \pi(\theta) + \ln \mathcal{L}(\theta)\}\right]$$

Since $\ln(a) + \ln(b) = \ln(ab)$

$$u(\theta) = \exp\left[\ln(\pi(\theta) \cdot \mathcal{L}(\theta)) - \max_\theta \{\ln(\pi(\theta) \cdot \mathcal{L}(\theta))\}\right]$$

In this case, $u(\theta) = u(\alpha)$

$$\pi(\theta)\mathcal{L}(\theta) = \pi(\alpha)\mathcal{L}(\alpha)$$

$$= \int_0^{\infty} \pi(\alpha, \lambda)\, \mathcal{L}(\alpha, \lambda)\, d\lambda$$

$$= \int_0^\infty \frac{1}{10000} \cdot \exp\left(-\frac{\lambda}{100}\right) \cdot \exp\left(-\frac{\alpha}{100}\right) \cdot \lambda^{n\alpha} \left\{\prod_{i=1}^n x_i^{\alpha-1}\right\} \exp\left(-\lambda \sum_{i=1}^n x_i\right) \cdot [\Gamma(\alpha)]^{-n}\, d\lambda \quad \text{for } \lambda, \alpha > 0$$

$$= \frac{1}{10000} \cdot \exp\left(-\frac{\alpha}{100}\right) \cdot \exp\left(\alpha \sum_{i=1}^n \ln x_i\right) \cdot \Gamma(\alpha)^{-n} \exp\left(-\sum_{i=1}^n \ln x_i\right) \int_0^\infty \exp\left(-\frac{\lambda}{100}\right) \cdot \lambda^{n\alpha} \cdot \exp\left(-\lambda \sum_{i=1}^n x_i\right) d\lambda$$

$$= \frac{1}{10000} \cdot \frac{\Gamma(n\alpha+1)}{\Gamma(\alpha)^n} \cdot \exp\left(\alpha \sum_{i=1}^n \ln x_i - \frac{\alpha}{100}\right) \cdot \left(\sum_{i=1}^n x_i + \frac{1}{100}\right)^{-(n\alpha+1)} \cdot \exp\left(-\sum_{i=1}^n \ln x_i\right)$$

Thus $\ln \pi(\alpha)\mathcal{L}(\alpha)$

$$= \ln\left[\frac{1}{10000} \cdot \frac{\Gamma(n\alpha+1)}{\Gamma(\alpha)^n} \cdot \exp\left(\alpha \sum_{i=1}^n \ln x_i - \frac{\alpha}{100}\right) \cdot \left(\sum_{i=1}^n x_i + \frac{1}{100}\right)^{-(n\alpha+1)} \cdot \exp\left(-\sum_{i=1}^n \ln x_i\right)\right]$$

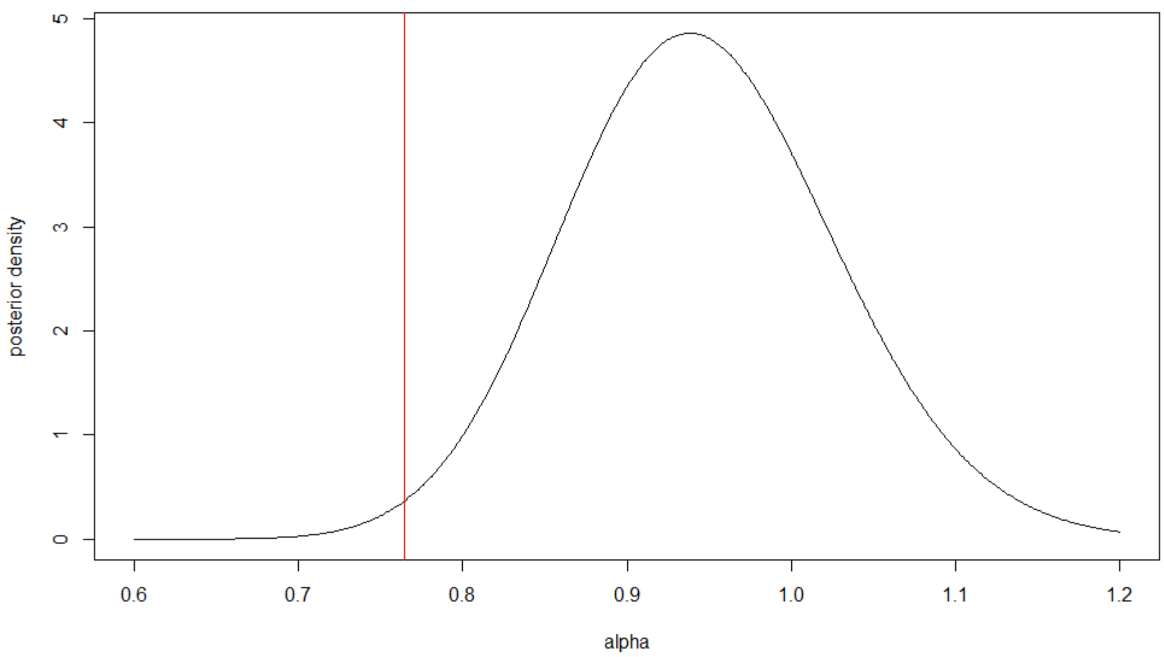
$$= \ln\left(\frac{1}{10000}\right) + \ln(\Gamma(n\alpha+1)) - n\ln(\Gamma(\alpha)) + \alpha \sum_{i=1}^n \ln x_i - \frac{\alpha}{100} - (n\alpha+1)\ln\left(\sum_{i=1}^n x_i + \frac{1}{100}\right) - \sum_{i=1}^n \ln x_i$$

Therefore, $\mathcal{U}(\alpha) = \exp\left[\ln(\pi(\alpha)\cdot\mathcal{L}(\alpha)) - \max_{\theta}\{\ln(\pi(\alpha)\cdot\mathcal{L}(\alpha))\}\right]$

where $\ln \pi(\alpha)\mathcal{L}(\alpha)$ equals to

$$= \ln\left(\frac{1}{10000}\right) + \ln(\Gamma(n\alpha+1)) - n\ln(\Gamma(\alpha)) + \alpha\sum_{i=1}^{n}\ln x_i - \frac{\alpha}{100} - (n\alpha+1)\ln\left(\sum_{i=1}^{n}x_i + \frac{1}{100}\right) - \sum_{i=1}^{n}\ln x_i$$

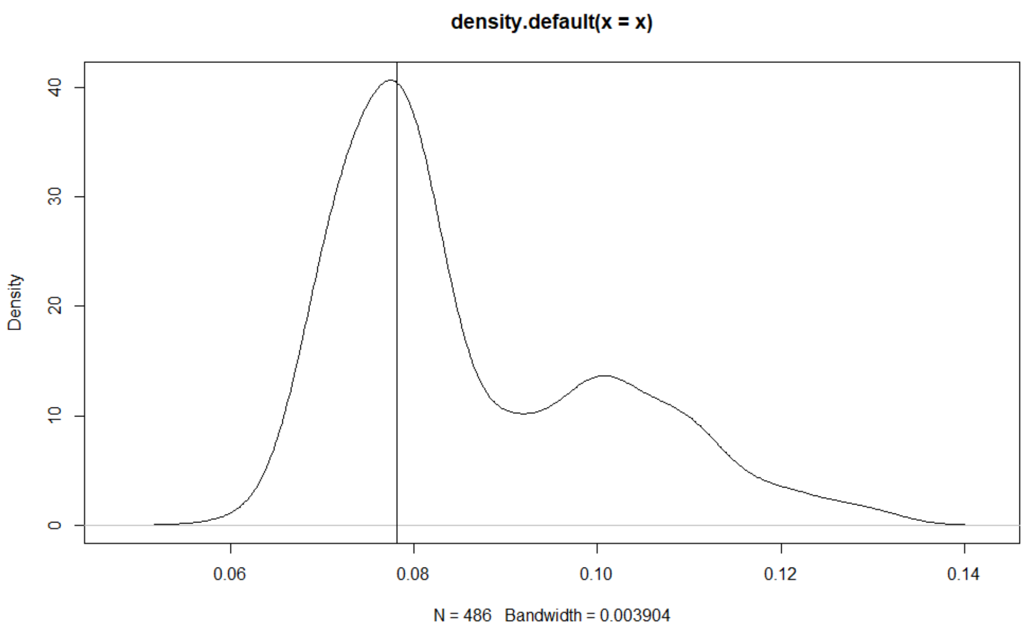
```
> prenorm <- function(x,alpha) {
+   n <- length(x)
+   r <-log(1/10000)+lgamma(n*alpha+1)-n*lgamma(alpha)+
+     alpha*sum(log(x))-alpha/100-(n*alpha+1)*log(sum(x)+1/100)-sum(log(x))
+   r <- r - max(r) # subtract maximum
+   pre <- exp(r) # pre-normalized
+   pre
+ }
>
> setwd("C:/Users/WLJY8/Desktop/资料/STA 355/HW3")
> air_data <- scan("aircon.txt")
Read 199 items
> alpha <- c(600:1200)/1000
> post <- prenorm(air_data,alpha)
> mult <- c(1/2,rep(1,599),1/2)
> norm <- sum(mult*post)/1000
> post <- post/norm # normalized posterior
> plot(alpha,post,type="l",ylab="posterior density")
> abline(v=mean(air_data)^2/var(air_data),col= "red")
```



Q2

a)

```
> venter <- function(x, tau=0.39917695) {
+   x <- sort(x)
+   n <- length(x)
+   m <- ceiling(tau*n)
+   x1 <- x[1:(n-m+1)]
+   x2 <- x[m:n]
+   j <- c(1:(n-m+1))
+   len <- x2-x1
+   k <- min(j[len==min(len)])
+   (x[k]+x[k+m-1])/2
+ }
> setwd("C:/Users/WLJY8/Desktop/资料/STA 355/HW3")
> x <- scan("stamp.txt")
Read 486 items
> plot(density(x))
> abline(v=venter(x))
```



The τ need to be around 0.39917695 in order that the estimate "make sense".

The optimal τ is approximately between 0.39917694 and 0.39917696.

Q2 b)

```
set.seed(5789)
venter <- function(x, tau) {
  x <- sort(x)
  n <- length(x)
  m <- ceiling(tau*n)
  x1 <- x[1:(n-m+1)]
  x2 <- x[m:n]
  j <- c(1:(n-m+1))
  len <- x2-x1
  k <- min(j[len==min(len)])
  (x[k]+x[k+m-1])/2
}

##tau = 0.5, alpha=2,n=100
venter_est <- NULL
alpha <- 2
n <- 100

for (i in 1:10000){
  x <- rgamma(n = n, shape = alpha)
  venter_est[i] <- venter(x,0.5)
}
mean((venter_est - alpha)^2)
```

```
## [1] 0.6406268
```

```
##tau = 0.5, alpha=2,n=1000
venter_est <- NULL
alpha <- 2
n <- 1000

for (i in 1:10000){
  x <- rgamma(n = n, shape = alpha)
  venter_est[i] <- venter(x,0.5)
}
mean((venter_est - alpha)^2)
```

```
## [1] 0.6643079
```

```
##tau = 0.5, alpha=10,n=100
venter_est <- NULL
alpha <- 10
n <- 100

for (i in 1:10000){
  x <- rgamma(n = n, shape = alpha)
  venter_est[i] <- venter(x,0.5)
}
mean((venter_est - alpha) ^2)
```

```
## [1] 1.257029
```

```
##tau = 0.5, alpha=10,n=1000
venter_est <- NULL
alpha <- 10
n <- 1000

for (i in 1:10000){
  x <- rgamma(n = n, shape = alpha)
  venter_est[i] <- venter(x,0.5)
}
mean((venter_est - alpha) ^2)
```

```
## [1] 0.8317887
```

```
#####################

##tau = 0.1, alpha=2,n=100
venter_est <- NULL
alpha <- 2
n <- 100

for (i in 1:10000){
  x <- rgamma(n = n, shape = alpha)
  venter_est[i] <- venter(x,0.1)
}
mean((venter_est - alpha)^2)
```

```
## [1] 0.8473832
```

```
##tau = 0.1, alpha=2,n=1000
venter_est <- NULL
alpha <- 2
n <- 1000

for (i in 1:10000){
  x <- rgamma(n = n, shape = alpha)
  venter_est[i] <- venter(x,0.1)
}
mean((venter_est - alpha) ^2)
```

```
## [1] 0.9311603
```

```
##tau = 0.1, alpha=10,n=100
venter_est <- NULL
alpha <- 10
n <- 100

for (i in 1:10000){
  x <- rgamma(n = n, shape = alpha)
  venter_est[i] <- venter(x,0.1)
}
mean((venter_est - alpha) ^2)
```

```
## [1] 2.442192
```

```
##tau = 0.1, alpha=10,n=1000
venter_est <- NULL
alpha <- 10
n <- 1000

for (i in 1:10000){
  x <- rgamma(n = n, shape = alpha)
  venter_est[i] <- venter(x,0.1)
}
mean((venter_est - alpha) ^2)
```

```
## [1] 1.405943
```

The ventor estimator with $\alpha = 2$ seems better. With the same n and τ, $\alpha = 2$ will have smaller MSEs compare to $\alpha = 10$.

Q2 C)

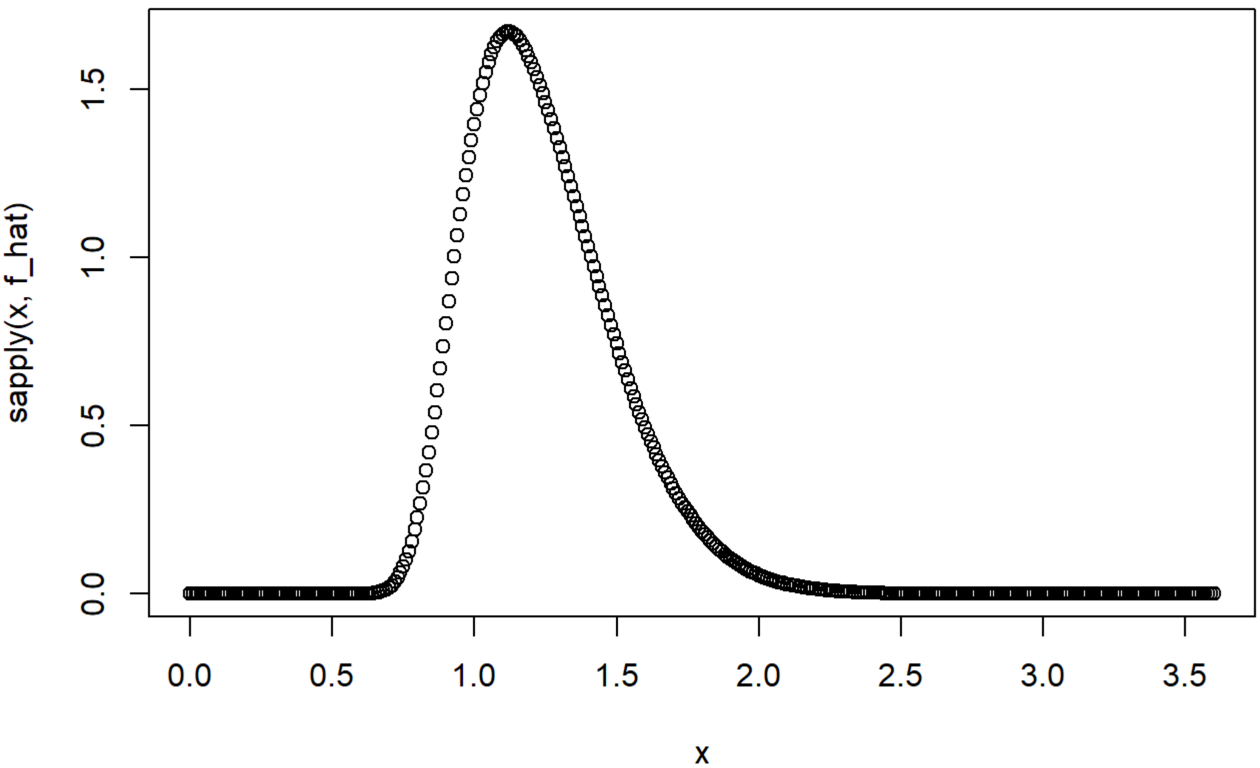
```
set.seed(5789)
venter <- function(x, tau) {
  x <- sort(x)
  n <- length(x)
  m <- ceiling(tau*n)
  x1 <- x[1:(n-m+1)]
  x2 <- x[m:n]
  j <- c(1:(n-m+1))
  len <- x2-x1
  k <- min(j[len==min(len)])
  (x[k]+x[k+m-1])/2
}

venter_est <- NULL
ti<-NULL
alpha <- 2
n <- 100

for (i in 1:10000){
  x <- rgamma(n = n, shape = alpha)
  ti[i] <- sum(x)
  venter_est[i] <- venter(x,0.5)
}

f_hat<- function(x){
  fx<-mean(dgamma(x,shape=n*alpha,rate =ti/venter_est))
}

x<- seq(0,3.6,0.01)
plot(x,sapply(x,f_hat))
```


sapply(x, f_hat)
0.0
0.5
1.0
1.5
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
x