ANNOUNCEMENT AND CALL FOR PAPERS

A symposium on 'Implementation of Computer Procedures and Stress–Strain Laws in Geotechnical Engineering' will be held during 3–6 August 1981 at Blacksburg, Virginia, U.S.A. The meeting is co-sponsored by the Computer and Numerical Methods Committee of the Geotechnical Engineering Division, American Society of Civil Engineers. The main purpose of the symposium will be the aspect of *theory to practice* of available computer procedures and the important topic of development and determination of stress–strain models for geological media.

Emphasis will be given to the methodology and guidelines for using available procedures based on techniques such as the finite element, finite difference, boundary integral and computer-oriented analytical techniques. Various topics in geotechnical engineering such as static and dynamic analysis, soil–structure interaction, flow and consolidation with mass transport, and stress–strain laws under static and repeated loads will be included. Papers describing applications of computer methods and stress–strain laws, comparison of prediction with actual behaviour, laboratory determination of stress–strain parameters and relation of field data with constitutive characteristics for incorporation in numerical models, difficulties encountered in implementation and identification of needed improvements and development of numerical schemes for handling such improvements will be relevant.

Short abstracts, not longer than one page, are invited by 30 June 1980. Based on these abstracts, authors will be advised to proceed with preparation of the final manuscript by 30 July 1980, and the final approval will be based on the completed papers due by 30 October 1980. In addition, a number of state-of-the-art papers in various topics emphasizing the methodology for application of available procedures will be presented.

It is planned to organize an informal workshop session in which details of derivation of stress–strain parameters from comprehensive series of laboratory tests on geological media leading to their incorporation in numerical procedures will be presented and discussed.

Send titles and abstracts to Prof. C. S. Desai or Prof. T. Kuppusamy, Department of Civil Engineering, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061, U.S.A.

INTERNATIONAL ASSOCIATION OF SEISMOLOGY AND PHYSICS OF THE EARTH'S INTERIOR

Preliminary announcement:
A General Assembly of IASPEI will be held at London, Ontario, Canada, 21–30 July 1981.

Major Topics will be:
Earthquake Ground Motions and their Effects on Critical Structures (in association with the International Association for Earthquake Engineering).
Earthquake Prediction and Risk.

Among the themes to be included in other scientific symposia will be:
Heterogeneity within the Earth.
Studies of the Earthquake Source.
Thermal Aspects of Plate Interactions.
Digital Seismology.
Properties of Materials at High Pressures and High Temperatures.
Structure of the Arctic.

In addition, meetings and workshops will be arranged by the Association's technical commissions and working groups.

Further information from:
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OPTIMUM ABSORBER PARAMETERS FOR SIMPLE SYSTEMS

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SUMMARY

In the classical problem a damped one degree-of-freedom absorber system is attached to a main system, which has one degree of freedom and is undamped. The optimum values of absorber stiffness and damping, which will minimize the resonant response of the main mass, are well known. In this paper the effect on these optimum conditions of light damping in the main system is studied. The authors show that optimum parameters for absorbers, which are attached to beams and plates, can be obtained simply and accurately from those for an equivalent one degree-of-freedom main system. This depends upon the concept of an effective mass for the elastic body and the representation of its response by the single relevant mode. It will be shown in a later paper that for more complex elastic bodies such as cylindrical shells, for which the natural frequencies are more closely spaced, these simple concepts do not predict accurately optimum absorber parameters.

INTRODUCTION

For many years there has been considerable interest in the design of dynamic vibration absorbers that, when added to an engineering system, will reduce significantly the vibration response. The classical problem is that of an absorber, which consists of a mass, spring and viscous damper, attached to an undamped single degree-of-freedom system of which the mass is subjected to harmonic vibrations.[1–3] If response curves for the main mass are plotted for different values of the absorber damping (with other parameters unchanged), all curves pass through two invariant or fixed points. The existence of these invariant points has lead to analysis for optimizing the parameters. The ratio of the natural frequency of the absorber to that of the main system is altered until the responses associated with these two invariant points are equal. This gives the optimum tuning for the absorber. The two values of absorber damping which cause the response curve to have zero gradient at the lower and upper invariant points, respectively, are determined. The difference between these values increases as the ratio of the absorber mass to the main system mass increases. It is usual to take a mean, which is based on these two values, as the optimum damping and to assume that the maximum response with optimum absorber parameters is equal to the response at the invariant points.

Real systems contain some damping. Although absorbers are likely to be added only to lightly damped systems, the effect of including damping in the main system on optimum absorber design is of importance and has been studied by several authors.[4–8] Important contributions have been made recently by Ioi and Ikeda,[6] who determined correction factors for the absorber parameters in terms of the main system damping with the proviso that the latter damping is light, and by Randall *et al*,[4] who present design charts for the optimum absorber parameters.

There has been considerable interest in the application of damped vibration absorbers to simple elastic systems, such as rods in extension,[9] beams in flexure,[10] plates[11, 12] and cylindrical shells,[13] etc. In most of this

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work the concept of the invariant points of the classical problem has been used to determine optimum parameters for absorbers which are attached to elastic bodies. The senior author investigated the effect of vibration absorbers upon the response of cylindrical shells and showed the variation of the optimum tuning ratio, i.e. the ratio of the absorber natural frequency to the fundamental natural frequency of the main system, against the effective mass ratio.[13] The latter is defined as the ratio of the absorber mass to the effective mass of the main system. The effective mass of the main system is determined by equating the kinetic energy of the main system in the mode of vibration under consideration to that of a lumped mass which is located at the point of attachment of the absorber and has the same direction constraints as the point of attachment. Expressions for effective masses of such bodies as beams in flexure are quite simple. With the introduction of the concept of the effective mass ratio it was shown that the curves of optimum tuning ratio versus effective mass ratio for some elastic bodies were very similar to that for the classical problem. However, there were major departures from the latter curve when absorbers were applied to cylindrical shells, and he gave some reasons for this behaviour.

A major step in our understanding of the relations between the classical problem and that of the absorber attached to an elastic body was made in a recent paper by Jacquot.[14] He considered viscous, and also hysteretically, damped absorbers attached to an undamped beam, which is vibrating in flexure and subjected to a harmonic force. He showed that if the beam response was approximated by the fundamental mode, invariant points exist as in the classical problem and expressions for optimum tuning ratio, optimum damping ratio and the corresponding maximum response of the beam were similar to those in the classical problem. The mass ratio, which occurs in the classical problem, has to be replaced by a factor equal to the absorber mass × (normalized beam mode evaluated at the point of attachment of the absorber)^2/mass of the beam. This effective mass ratio is compatible with the senior author's earlier definition of the concept.[13] One purpose of the present paper is to demonstrate that Jacquot's work can be generalized to other elastic bodies with attached vibration absorbers, provided that it is assumed that the vibration of the elastic body can be represented by a single normal mode. With this assumption we obtain standard curves for optimum tuning and damping ratios and optimum maximum response when plotted against the effective mass ratio. Two major questions now arise, and it is the purpose of this paper to consider these. Firstly, what is the effect of damping in the main system on the resulting optimizing process; this is of importance because real systems are bound to contain some damping. Second, in what circumstances is it impossible, or at least inaccurate, to represent the vibration of an elastic body by a single mode shape.

An immediate result of the introduction of damping into the main system, whether the classical single degree-of-freedom system or an elastic body, is the disappearance of the invariant points. It should be noted that the absorbers to be considered here consist of a mass, which is connected to the main system through a parallel spring and viscous damper. The Lanchester absorber, which consists of a mass and a viscous damper, cannot be tuned and is less efficient, in general. Recent analysis shows that the single invariant point, which exists on the response curves for an undamped main system with attached absorber, is not suppressed when damping in the main system is included.[8] However, this type of absorber will not be considered in this paper. Returning to the tuned and damped absorber, when damping in the main system is included, the optimum values are changed, but the main difference is a reduction in the optimum maximum response. The importance of these results is that for specified damping in the main system they provide standard curves, which can be used with elastic bodies, provided that Jacquot's assumption of a single mode shape and the senior author's concept of effective mass ratio are used.

Specifically, standard curves are generated for the optimized response of a main system which has light hysteretic damping, as this facilitates comparison with hysteretically damped elastic bodies. Additionally, numerical results for main systems with viscous damping show very close agreement with the recently developed expressions for optimum parameters of Ioi and Ikeda.[6]

The response of an elastic body at frequencies in the vicinity of a specified resonance can be represented adequately by assuming the relevant single mode, if (a) the natural frequencies of other modes which are stimulated by the excitation are not close to the specified resonance, and (b) the presence of the absorber system does not affect significantly the mode shape. Uniform beams have well-spaced natural frequencies, and consequently satisfy condition (a). Numerical results for simply supported beams show that the single mode assumption is justified even for very large absorber systems (considerably larger than are likely to be used in practice). In a subsequent paper the authors will present extensive results for the optimized parameters of absorbers, which are attached to cylindrical shells.[15] The spacing of the natural frequencies of the latter can be altered by appropriate changes in geometrical ratios. In this manner bodies with closely spaced natural frequencies, which are typical of many real engineering structures, are investigated.

INCLUSION OF DAMPING IN THE MAIN SYSTEM FOR THE CLASSICAL PROBLEM

The main system (Figure 1) consists of the mass M_M, spring of stiffness k_M and viscous damper c_M; the mass is subjected to a harmonic force represented by $Pe^{i\omega t}$. The absorber system has mass M_A, a spring stiffness k_A and viscous damping c_A. It is easily shown that the steady-state response of the main mass is [16].

$$x_1 = \frac{(k_A - M_A\omega^2 + i\omega c_A)\,Pe^{i\omega t}}{[k_M + k_A - M_M\omega^2 + i\omega(c_M + c_A)](k_A - M_A\omega^2 + i\omega c_A) - (k_A + i\omega c_A)^2} \tag{1}$$

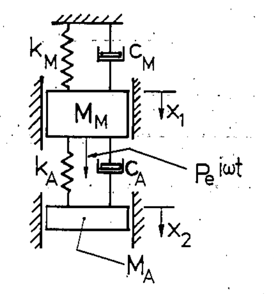

Figure 1. Single degree-of-freedom main system with attached absorber

We introduce the parameters:

Mass ratio	$\mu = M_A/M_M$	
	$\omega_A^2 = k_A/M_A$	
	$\omega_M^2 = k_M/M_M$	
Tuning ratio	$f = \omega_A/\omega_M$	(2)
Forced frequency ratio	$r = \omega/\omega_M$	
Absorber damping	$\gamma_A = c_A/2M_A\omega_A$	
Main system damping	$\gamma_M = c_M/2M_M\omega_M$	

The dynamic magnification factor for the response of the main mass is $k_M X_1/P$, where X_1 is the amplitude of vibration of M_M. It can be expressed as

$$R = \frac{k_M X_1}{P} = \left[\frac{A^2 + B^2}{C^2 + D^2}\right]^{\frac{1}{2}} \tag{3}$$

where

$$\begin{aligned} A &= f^2 - r^2, \quad B = 2\gamma_A rf, \quad C = f^2(1-r^2) - \mu f^2 r^2 - r^2(1-r^2) - 4\gamma_A\gamma_M f r^2, \\ D &= 2\gamma_A rf(1 - r^2 - \mu r^2) + 2\gamma_M r(f^2 - r^2) \end{aligned} \tag{4}$$

Putting the main system damping equal to zero ($\gamma_M = 0$), we obtain Den Hartog's expression.[3] In this case invariant points exist because A and C are independent of γ_A and B and D are proportional to γ_A, so that the response is independent of γ_A if $A/C = B/D$. From this relation we obtain a quadratic equation in r^2 for the frequencies of the two invariant points. When the optimizing condition, i.e. the responses at the two invariant points should be equal, is applied, the frequencies at the invariant points are given by

$$r_{1,2}^2 = \left(\frac{1}{1+\mu}\right)\left[1 \pm \left(\frac{\mu}{2+\mu}\right)^{\frac{1}{2}}\right] \tag{5}$$

the optimum tuning ratio

$$f_{opt} = 1/(1+\mu) \tag{6}$$

and the response at the invariant points

$$R_{opt} = (1+2/\mu)^{\frac{1}{2}} \tag{7}$$

If the slope of the response curve is equated to zero at each of the invariant points, the absorber damping is given by

$$\gamma_A^2 = \frac{\mu[3 \pm \{\mu/(\mu+2)\}^{\frac{1}{2}}]}{8(1+\mu)} \tag{8}$$

The positive signs in equations (5) and (8) correspond. Den Hartog recommends the useful average value from equation (8)[1, 3]

$$\gamma_{A,opt}^2 = 3\mu/8(1+\mu) \tag{9}$$

The above results are well known but have been quoted for purposes of comparison.

Returning to the general system, $\gamma_M > 0$, the form of expressions (4) shows that invariant points will not exist. Depending upon the values chosen for the parameters, the response curve from equation (3) has one or two peaks, but in the vicinity of the optimum conditions there are always two peaks. Accordingly, the authors undertook a computer study of the dependence of these maxima on the system parameters in order to find the optimum values of the tuning ratio f and absorber damping ratio γ_A for specified values of the mass ratio μ and damping ratio γ_M. For the optimum condition the maximum value of the DMF of equation (3) over the full frequency range is minimized.

Table I gives optimum values of the maximum dynamic magnification factor for the main mass, R_{opt}, the tuning ratio f_{opt} and the absorber damping ratio $\gamma_{A,opt}$ for specified values of the mass ratio and the main system

Table 1. Optimum values of absorber parameters for main system with viscous damping

Mass ratio (μ)	Main system damping (γ_M)	Optimum values R_{opt}	f_{opt}	$\gamma_{A,opt}$	Values of r for equal peaks r_1	r_2
0·01	0	14·18	0·9901	0·061	0·960	1·030
	0·01	11·37	0·9886	0·062	0·956	1·032
	0·02	9·465	0·9869	0·064	0·953	1·033
	0·05	6·251	0·9807	0·068	0·942	1·034
	0·1	3·967	0·9663	0·073	0·923	1·030
0·1	0	4·589	0·9091	0·185	0·848	1·059
	0·01	4·270	0·9051	0·187	0·843	1·058
	0·02	3·991	0·9009	0·188	0·838	1·058
	0·05	3·337	0·8875	0·193	0·823	1·054
	0·1	2·622	0·8619	0·199	0·795	1·043
1·0	0	1·746	0·499	0·448	0·487	0·928
	0·01	1·714	0·494	0·448	0·481	0·924
	0·02	1·683	0·489	0·449	0·476	0·921
	0·05	1·600	0·473	0·454	0·462	0·904
	0·1	1·482	0·446	0·455	0·434	0·882
'Standard' optimum values from equations (5), (6), (7) and (9)						
0·01	0	14·18	0·9901	0·0609	0·959	1·030
0·1	0	4·583	0·9091	0·1846	0·843	1·052
1·0	0	1·732	0·500	0·433	0·460	0·888

damping ratio, namely $\mu = 0{\cdot}01, 0{\cdot}1$ and $1{\cdot}0$ and $\gamma_M = 0, 0{\cdot}01, 0{\cdot}02, 0{\cdot}05$ and $0{\cdot}1$. As the main system damping increases, there is a small decrease in the optimum value of the tuning ratio, and this change in f_{opt} is more marked when the mass ratio is large. The absorber damping ratio increases slightly as the main system damping ratio increases. Thus allowance for damping in the main system, which has been assumed to be relatively small, has only a small effect on the values of f_{opt} and $\gamma_{A,opt}$. For a specified value of the mass ratio the optimum value of the dynamic magnification factor decreases as the main system damping ratio increases, and this decrease is considerably more marked when the mass ratio is small. In the extreme case when the two masses are equal, allowance for main system damping has a relatively small effect upon the optimum dynamic magnification factor. Some of these effects are illustrated graphically by Randall *et al.*[4] who present useful design curves for R_{opt}, f_{opt} and $\gamma_{A,opt}$ for the range of values $0{\cdot}01 \leqslant \mu \leqslant 0{\cdot}4$ and $0 \leqslant \gamma_M \leqslant 0{\cdot}5$. For the main system without an absorber present the maximum dynamic magnification factor is $1/(2\gamma_M)$. In order to assess the effectiveness of adding an absorber to the main system the optimum values of the table should be compared with the latter ratio. For a small absorber, $\mu = 0{\cdot}01$, and a moderately damped main system, $\gamma_M = 0{\cdot}1$, the addition of the absorber has a relatively small effect on diminishing the dynamic magnification factor. Higher values of main system damping than 0·1 have not been considered, as for such systems the maximum dynamic magnification factor without an absorber present would hardly be sufficiently large to warrant the addition of an absorber.

In a recent paper Ioi and Ikeda investigated the same problem.[6] If R_1 and R_2 are the two maximum values of the dynamic magnification factor, their optimum conditions are

$$R_1 = R_2$$

and

$$\frac{\partial R_1}{\partial \gamma_A} = -\frac{\partial R_2}{\partial \gamma_A} \tag{10}$$

Using an iterative procedure, which starts from the optimum conditions for an undamped main system [equations (6), (7) and (9)], they develop expressions for the optimum parameters (f_{opt} and $\gamma_{A,opt}$) and response (R_{opt}) in terms of ascending powers of μ and γ_M. They state that the accuracy of these expressions is better than 1 per cent for $0{\cdot}03 \leqslant \mu \leqslant 0{\cdot}4$ and $\gamma_M \leqslant 0{\cdot}15$. Values of R_{opt} and f_{opt} in Table I are believed to be of high accuracy, but those for $\gamma_{A,opt}$ are not so accurate, as the optimum conditions are less sensitive to small changes in γ_A. Comparison of the tabulated results with values from Ioi and Ikeda's expressions shows very close agreement. For $\mu = 0{\cdot}1$ agreement is within 0·2, 0·1 and 0·5 per cent for R_{opt}, f_{opt} and $\gamma_{A,opt}$ respectively. For $\mu = 0{\cdot}4$ and $\gamma_M = 0{\cdot}1$ the corresponding percentage differences are 0·1, 0·8 and 0·3. For $\mu = 0{\cdot}01$, which is outside Ioi and Ikeda's recommended range, their expressions overestimate R_{opt} by up to 2·3 per cent and underestimate f_{opt} by up to 1·2 per cent.

It will be noted that optimum conditions are determined for a specified value of the mass ratio μ. The optimum value of R_{opt} decreases as μ increases—this is true for $\mu \leqslant 1{\cdot}0$, which is a wider range of values than is likely to be used in practice. Youseff and Popplewell, who considered a constant tuned absorber, $f = 1$, could not determine a unique optimum solution when μ and γ_A were varied simultaneously.[7]

It is instructive to compare the results in Table I for an undamped main system ($\gamma_M = 0$) with the classical values. The optimum tuning ratio, the frequency factors for the invariant points, the response at the invariant points when tuning is optimized and the recommended absorber damping value are given by equations (6), (5), (7) and (9) respectively, and numerical values are included in Table I. The optimum value of absorber damping is slightly higher than that given by equation (9). In Figure 2 response curves are plotted for frequencies in the vicinity of the invariant points for $\mu = 0{\cdot}1$ and three values of γ_A. For $\gamma_A = 0{\cdot}1778$ and $0{\cdot}1912$ the response curve has a maximum at the lower and upper invariant point respectively, but the other peak is slightly higher, being $R = 4{\cdot}614$ and $4{\cdot}605$ respectively. The third curve is for the optimum damping value $\gamma_{A,opt} = 0{\cdot}1851$, for which the two peaks are equal and of magnitude 4·5892. Equation (9) gives $\gamma_A = 0{\cdot}1846$; the corresponding response curve is practically coincident with the optimum curve of Figure 2, but has maxima which are 0·02 per cent larger and smaller respectively than the equal peaks of the optimum curve and a minimum between the invariant points which is 0·25 per cent lower than that for the optimum curve. Similar behaviour for $\mu = \frac{1}{3}$ has been reported by Umbach.[5]

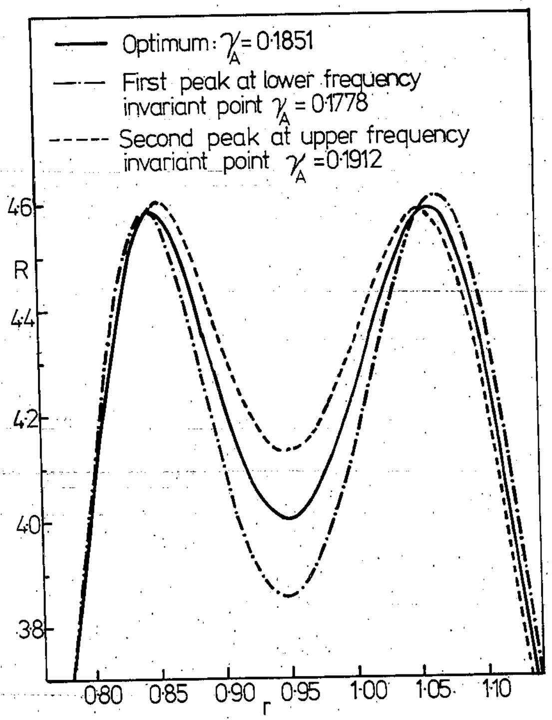

Figure 2. Response curves illustrating the dependence of the maximum amplitude of the main mass on the absorber damping γ_A; $\mu = 0{\cdot}1$, $f = 0{\cdot}9091$, $\gamma_M = 0$.

As will be shown later, optimum conditions which have been determined for absorbers attached to damped single degree-of-freedom systems can be used in certain circumstances for absorbers to be attached to elastic bodies. For the latter, internal damping is hysteretic and for most results in the literature a non-dimensional hysteretic damping $\eta = 0{\cdot}01$ has been used. Consequently, it is useful to determine parameters for an absorber which is attached to a single degree-of-freedom system with hysteretic damping of $\eta = 0{\cdot}01$. Changing from viscous to hysteretic damping in the main system of Figure 1, the complex stiffness $(k_M + i\omega c_M)$ is replaced by $k_M(1+i\eta)$. In expressions (4) $2\gamma_M r$ is replaced by η. Table II gives optimum values for the absorber parameters when the main system damping $\eta = 0{\cdot}01$.†

Table II. Optimum values of absorber parameters for main system with hysteretic damping, $\eta = 0{\cdot}01$

Mass ratio (μ)	Optimum values			Values of r for equal peaks	
	R_{opt}	f_{opt}	$\gamma_{A,opt}$	r_1	r_2
0·01	12·62	0·9897	0·0617	0·958	1·031
0·03	7·684	0·9702	0·1053	0·923	1·046
0·1	4·418	0·9079	0·186	0·846	1·059
0·3	2·718	0·7673	0·299	0·712	1·039
1·0	1·725	0·497	0·449	0·484	0·927

APPLICATION TO ELASTIC STRUCTURES

The basic assumption of this method is that the response of the elastic body with attached absorber can be represented by the single representative mode of the body without the absorber. This has been investigated by

† An alternative graphical method of determining optimum parameters for this case has been described by Falcon *et al.*[21]

Jacquot for the flexural vibrations of an undamped beam.[14] The method can be applied to any uniform elastic body for which exact mode shapes can be specified; this includes longitudinal and torsional vibrations of bars, flexural vibrations of beams, transverse vibrations of circular plates and of rectangular plates, which have two parallel edges simply supported, and simply supported cylindrical shells. However, as the Rayleigh method with single term approximations for mode shapes yields accurate results for natural frequencies and resonant response of rectangular plates and cylindrical shells with any combination of standard boundary conditions, the limitations on applicable boundary conditions could be removed.[17–19] Here the vibrations of rectangular plates and cylindrical shells will be considered; hysteretic damping in the plate or shell is assumed to exist. Comparative results for other elastic bodies can be inferred.

Rectangular plate

We consider a rectangular plate of uniform thickness h and sides a and b with the middle surface, which is assumed to be unstretched as in conventional small vibration theory, lying in the XY plane. The free undamped transverse vibrations for mode j/n of the plate without an absorber have a natural frequency ω_{jn} and can be expressed as

$$w(x, y, t) = \phi_j(x)\,\psi_n(y)\sin\omega_{jn} t \tag{11}$$

The absorber (Figure 3) is attached to the plate at point (x_A, y_A); the mass of the absorber M_A is constrained to move in the Z-direction and has a displacement $q_0(t)$. It is assumed that the response of the system to a

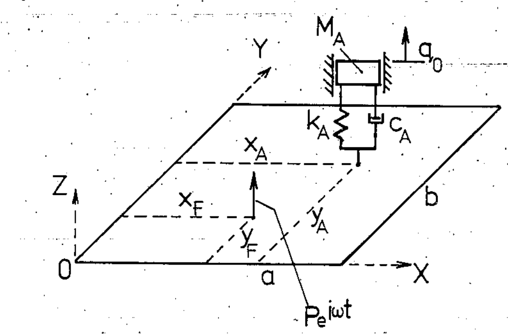

Figure 3. Rectangular plate with attached absorber

transverse harmonic force $Pe^{i\omega t}$, which is applied to the plate at point (x_F, y_F), can be represented in terms of the single mode j/n, where the function of the absorber is to reduce the resonant response associated with mode j/n. Thus for the plate

$$w(x, y, t) = \phi_j(x)\,\psi_n(y)\,q_{jn}(t) \tag{12}$$

where the functions $\phi_j(x)$ and $\psi_n(y)$ are normalized so that

$$\rho h \int_0^a \int_0^b \phi_j^2(x)\,\psi_n^2(y)\,dy\,dx = M_P \tag{13}$$

where M_P is the mass of the plate. At the point of attachment of the absorber

$$w_A = \phi_j(x_A)\,\psi_n(y_A)\,q_{jn} \tag{14}$$

The kinetic energy of the system

$$T = \tfrac{1}{2}M_P\,\dot{q}_{jn}^2 + \tfrac{1}{2}M_A\,\dot{q}_0^2 \tag{15}$$

The strain energy of the system can be written

$$S = \tfrac{1}{2}M_P\,\omega_{jn}^2\,q_{jn}^2 + \tfrac{1}{2}k_A(q_0 - w_A)^2 \tag{16}$$

The absorber damping contributes a term $\tfrac{1}{2}c_A(\dot{q}_0 - \dot{w}_A)^2$ to the dissipation function. The hysteretic damping in the plate, η, is introduced through the conventional complex modulus approach, which leads effectively to ω_{jn}^2

in S being replaced by $\omega_{jn}^2(1+i\eta)$. Applying the Lagrange equation yields the two equations

$$\begin{aligned} &M_P[\ddot{q}_{jn}+\omega_{jn}^2(1+i\eta)\,q_{jn}]+c_A[\phi_j^2(x_A)\,\psi_n^2(y_A)\dot{q}_{jn}-\phi_j(x_A)\,\psi_n(y_A)\,\dot{q}_0] \\ &\quad +k_A[\phi_j^2(x_A)\,\psi_n^2(y_A)\,q_{jn}-\phi_j(x_A)\,\psi_n(y_A)\,q_0]=Pe^{i\omega t}\,\phi_j(x_f)\,\psi_n(y_F) \\ &M_A\,\ddot{q}_0+c_A[\dot{q}_0-\phi_j(x_A)\,\psi_n(y_A)\,\dot{q}_{jn}] \\ &\quad +k_A[q_0-\phi_j(x_A)\,\psi_n(y_A)\,q_{jn}]=0 \end{aligned} \tag{17}$$

We define the effective mass of the plate, M_{eff}, as the lumped mass, which when placed at the point of attachment of the absorber gives the same kinetic energy as that for the plate vibrating in the relevant mode, i.e.

$$\tfrac{1}{2}M_{eff}\,\dot{w}_A^2=\tfrac{1}{2}M_P\,\dot{q}_{jn}^2 \tag{18}$$

Thus

$$M_{eff}\,\phi_j^2(x_A)\,\psi_n^2(y_A)=M_P \tag{19}$$

We define also an effective stiffness for the plate k_{eff} so that the strain energy of the plate in mode j/n is $\frac{1}{2}k_{eff}\,q_{jn}^2$. Thus

$$k_{eff}=M_P\,\omega_{jn}^2 \tag{20}$$

Using equation (19), the steady-state solution of equations (17) is

$$q_{jn}=\frac{Pe^{i\omega t}\,\phi_j(x_F)\,\psi_n(y_F)\,(k_A-M_A\,\omega^2+i\omega c_A)}{\phi_j^2(x_A)\,\psi_n^2(y_A)\,[\{M_{eff}\,\omega_{jn}^2(1+i\eta)-M_{eff}\,\omega^2+k_A+i\omega c_A\}(k_A-M_A\,\omega^2+i\omega c_A)-(k_A+i\omega c_A)^2]} \tag{21}$$

Introducing the non-dimensional parameters, mass ratio $\mu_{eff}=M_A/M_{eff}$, frequency ratio $r=\omega/\omega_{jn}$, tuning ratio $f=\omega_A/\omega_{jn}$ and ω_A and γ_A defined in equation (2)

$$\frac{M_{eff}\,\omega_{jn}^2\,q_{jn}}{P}=\frac{e^{i\omega t}\,\phi_j(x_F)\,\psi_n(y_F)}{\phi_j^2(x_A)\,\psi_n^2(y_A)}\left(\frac{A+iB}{C+iD}\right) \tag{22}$$

where A, B, C and D are functions of r, μ_{eff}, f, γ_A and η defined in equation (4) with μ and $2\gamma_M r$ in C and D replaced by μ_{eff} and η respectively. Finally, the amplitude of vibration at point (x, y) is given by

$$k_{eff}\frac{W(x,y)}{P}=\phi_j(x_F)\,\psi\hat{n}(y_F)\,\phi_j(x)\,\psi_n(y)\left(\frac{A^2+B^2}{C^2+D^2}\right)^{\frac{1}{2}} \tag{23}$$

Comparing equations (3) and (23), optimizing conditions for an absorber attached to a single degree-of-freedom system and to a rectangular plate will be identical, provided that the two systems have the same values of μ_{eff} and of η. Thus the results of Table II can be used to predict optimum conditions for an absorber attached to a hysteretically damped rectangular plate. The optimum response of the plate $W(x, y)$ can be found from the optimum response of the equivalent one degree-of-freedom system R_{opt} from the relation

$$\frac{k_{eff}\,W(x,y)}{P\phi_j(x_F)\,\psi_n(y_F)\,\phi_j(x)\,\psi_n(y)}=R_{opt} \tag{24}$$

Cylindrical shell

We consider a thin cylindrical shell of uniform thickness h, mean radius a and length L. If the boundary conditions at $x=0$ and $x=L$ are simply supported without axial constraint, the axial, tangential and radial

components of displacement, u, v and w respectively, for mode j/n of the shell without an absorber are

$$\begin{aligned} u(x,\theta,t)&=\alpha_{jn}\cos\frac{j\pi x}{L}\cos n\theta\sin\omega_{jn}t \\ v(x,\theta,t)&=\beta_{jn}\sin\frac{j\pi x}{L}\sin n\theta\sin\omega_{jn}t \\ w(x,\theta,t)&=\sin\frac{j\pi x}{L}\cos n\theta\sin\omega_{jn}t \end{aligned} \tag{25}$$

The natural frequency ω_{jn} and the amplitude ratios α_{jn} and β_{jn} are determined conventionally, using a consistent shell theory, for example that of Novozhilov.[20] The absorber (Figure 4) is attached to the shell at point (x_A, θ_A); its mass M_A is constrained to move in the radial direction and has a displacment $q_0(t)$. When a

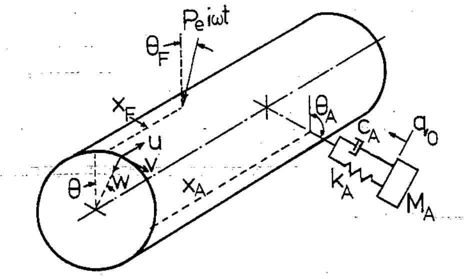

Figure 4. Cylindrical shell with attached absorber

radial harmonic force $Pe^{i\omega t}$ is applied to the shell at point (x_F, θ_F), it is assumed that the response of the complete system can be represented in terms of the single mode j/n, if the function of the absorber is to reduce the resonant response associated with mode j/n. Thus

$$\begin{aligned} u(x,\theta,t)&=2\alpha_{jn}\cos\frac{j\pi x}{L}\cos n\theta\,q_{jn}(t) \\ v(x,\theta,t)&=2\beta_{jn}\sin\frac{j\pi x}{L}\sin n\theta\,q_{jn}(t) \\ w(x,\theta,t)&=2\sin\frac{j\pi x}{L}\cos n\theta\,q_{jn}(t) \end{aligned} \tag{26}$$

The factors 2 have been introduced so that

$$\rho h\int_0^L\int_0^{2\pi}\left(2\sin\frac{j\pi x}{L}\cos n\theta\right)^2 a\,d\theta\,dx=M_S \tag{27}$$

where M_S is the mass of the shell.

The kinetic energy of the system

$$T=\tfrac{1}{2}M_S(1+\alpha_{jn}^2+\beta_{jn}^2)\dot{q}_{jn}^2+\tfrac{1}{2}M_A\dot{q}_0^2 \tag{28}$$

At the point of attachment of the absorber

$$w_A=2\sin\frac{j\pi x_A}{L}\cos n\theta_A\,q_{jn} \tag{29}$$

The strain energy of the system

$$S = \tfrac{1}{2}M_S(1+\alpha_{jn}^2+\beta_{jn}^2)\,\omega_{jn}^2\, q_{jn}^2 + \tfrac{1}{2}k_A(q_0 - w_A)^2 \tag{30}$$

Absorber damping and hysteretic damping in the shell are treated in the same manner as in the plate analysis. Applying the Lagrange equation leads to two equations, which are similar to equations (17). On the basis of equality of kinetic energy the effective mass of the shell M_{eff} is given by

$$\tfrac{1}{2}M_{eff}\,\dot{w}_A^2 = \tfrac{1}{2}M_S(1+\alpha_{jn}^2+\beta_{jn}^2)\,\dot{q}_{jn}^2 \tag{31}$$

Thus

$$4M_{eff}\sin^2\frac{j\pi x_A}{L}\cos^2 n\theta_A = M_S(1+\alpha_{jn}^2+\beta_{jn}^2) \tag{32}$$

One of the authors introduced the concept of effective mass in an earlier attempt to correlate optimum tuning ratios for absorbers attached to different elastic bodies.[13] However, in that work he approximated the expression for M_{eff} by replacing the factor $(1+\alpha_{jn}^2+\beta_{jn}^2)$ in equation (31) by 1. This is not an essential simplification and could cause an error as this factor may be significantly greater than unity, as its order of magnitude is $(1+1/n^2)$. If the strain energy of the shell is written as $\frac{1}{2}k_{eff}\,q_{jn}^2$,

$$k_{eff} = M_S(1+\alpha_{jn}^2+\beta_{jn}^2)\,\omega_{jn}^2 \tag{33}$$

Using the non-dimensional parameters previously defined, q_{jn} is given by equation (21) with $2\sin(j\pi x/L)\cos n\theta$ replacing $\theta_j(x)\,\psi_n(y)$. Finally, the amplitude of vibration of the shell at point (x, θ) is given by

$$k_{eff}\frac{W(x,\theta)}{P} = 4\sin\frac{j\pi x_F}{L}\cos n\theta_F \sin\frac{j\pi x}{L}\cos n\theta\left(\frac{A^2+B^2}{C^2+D^2}\right)^{\frac{1}{2}} \tag{34}$$

Comparison of equations (3), (23) and (34) shows that provided appropriate definitions of mode shape, M_{eff} and k_{eff} are used optimum absorber parameters can be obtained for any elastic body, for which the response can be represented in terms of a single mode, from known results for the one degree-of-freedom main system.

For example, for a uniform beam with an absorber attached at x_A and subjected to a transverse force $Pe^{i\omega t}$ at x_F, the amplitude of vibration $V(x)$ is given by

$$k_{eff}\frac{V(x)}{P} = \phi_j(x_F)\,\phi_j(x)\left(\frac{A^2+B^2}{C^2+D^2}\right)^{\frac{1}{2}} \tag{35}$$

where $\phi_j(x)$ is the relevant mode of the beam. This is normalized so that

$$\rho A\int_0^L \phi_j^2(x)\,\mathrm{d}x = M_B \tag{36}$$

where ρA is the mass per unit length and $M_B\,(=\rho AL)$ is the mass of the beam.

The effective stiffness

$$\begin{aligned} k_{eff} &= M_B\,\omega_j^2 \\ &= EI(\lambda_j L)^4/L^3 \end{aligned} \tag{37}$$

where $\lambda_j L$ is the jth non-dimensional frequency factor for the beam (without the absorber system) and EI is the flexural rigidity. As usual, the functions A, B, C and D contain the mass ratio $\mu_{eff} = M_A/M_{eff}$; the effective mass for the beam is given by

$$M_{eff}\,\phi_j^2(x_A) = M_B \tag{38}$$

Substituting equations (4), (37) and (38) in equation (35) and neglecting the damping in the beam, we obtain Jacquot's expression.[14]

RESULTS

The analogy between the determination of optimum absorber parameters for an elastic body and for a one degree-of-freedom main system depends upon the assumption that the response of the body with the absorber attached can be represented with adequate accuracy by a single mode of the body without the absorber. In order to justify this assumption comparison must be made with results that have been determined from a closed-form solution for a one-dimensional body or from a conventional multi-mode approach. In this section results for a simply supported beam will be presented and other existing results discussed. Extensive results for cylindrical shells will be given in a later paper.[15] In general, there are two factors which affect the accuracy of the single mode approach. Adjacent resonant frequencies should be well removed from the natural frequency to which the absorber is being tuned. If the latter is the fundamental frequency, then ω_2/ω_1 should be large, where ω_1 and ω_2 are the fundamental and second natural frequencies of the elastic body. This criterion is known to be important when determining resonant response by the single mode approach for an elastic body without an attached absorber. Second, the accuracy will decrease as the size of the absorber increases, i.e. as the mass ratio μ_{eff} increases.

We consider a hysteretically damped, uniform simply supported beam. A transverse harmonic force is applied at mid-span. To preserve symmetry either an absorber with properties k_A, M_A and c_A is attached at mid-span or two identical absorbers, which each have properties $\frac{1}{2}k_A$, $\frac{1}{2}M_A$ and $\frac{1}{2}c_A$, are attached at positions x_A and $L-x_A$ (Figure 5). This allows the previous analysis to be used, but in a full modal analysis only symmetric

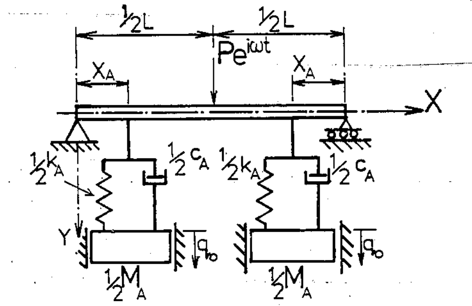

Figure 5. Simply supported beam with attached absorber

modes of the beam have to be included. For specified values of the absorber mass ratio M_A/M_B (and hence of μ_{eff}) and hysteretic damping $\eta = 0{\cdot}01$, optimization of the displacement response at the four points, $x/L = \frac{1}{8}, \frac{1}{4}, \frac{3}{8}$ and $\frac{1}{2}$, was investigated for excitation frequencies in the vicinity of the fundamental resonance of the beam for four absorber locations, $x_A/L = \frac{1}{8}, \frac{1}{4}, \frac{3}{8}$ and $\frac{1}{2}$. The optimum tuning ratio f_{opt} and the optimum absorber damping ratio $\gamma_{A,\,opt}$ are plotted against the mass ratio $\mu_{eff}\,(=M_A/M_{eff})$ in Figures 6 and 7 respectively. The non-dimensional amplitude ratio at position x is $k_{eff}\,V(x)/P$, where

$$\begin{aligned} k_{eff} &= M_B\,\omega_1^2 \\ &= EI\pi^4/L^3 \end{aligned} \tag{39}$$

From equation (35) the single mode approach gives

$$[k_{eff}\,V(x)]/P = \phi_1(x_F)\,\phi_1(x)\,R \tag{40}$$

where $R = [(A^2+B^2)/(C^2+D^2)]^{\frac{1}{2}}$ is the dynamic magnification factor for the equivalent one degree-of-freedom system and $\phi_1(x) = 2^{\frac{1}{2}}\sin(\pi x/L)$. Optimized values of R_{opt} are plotted against μ_{eff} in Figure 8. In each of Figures 6, 7 and 8 the full line represents the result from the single mode approach. (Although they are drawn for $\eta = 0{\cdot}01$, separate lines for an undamped main system would occur only for $\mu_{eff} > 0{\cdot}5$, $\mu_{eff} > 0{\cdot}1$ and $\mu_{eff} < 0{\cdot}5$ in Figures 6, 7 and 8 respectively.) Results from the full modal analysis have been added to Figures 6–8. At a specified value of μ_{eff} there could be four values of the optimized quantity, which correspond to optimization with respect to the displacement at the four stations, $x/L = \frac{1}{8}, \frac{1}{4}, \frac{3}{8}$ and $\frac{1}{2}$. Where possible the

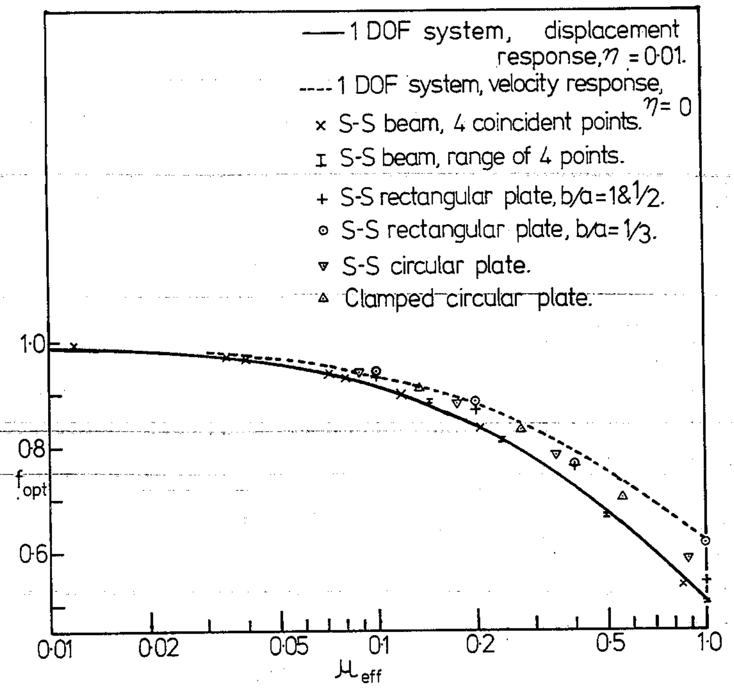

Figure 6. Optimum tuning ratio, f_{opt}, against effective mass ratio, μ_{eff}, for single degree-of-freedom systems, simply supported beams and circular and rectangular plates

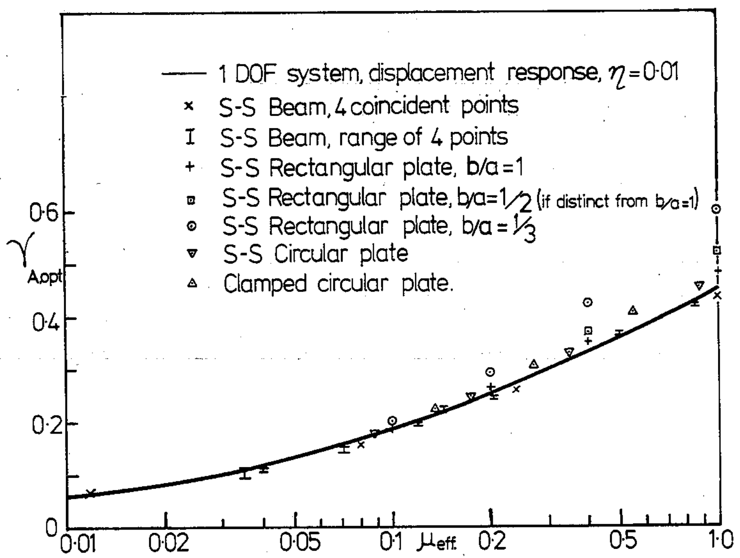

Figure 7. Optimum absorber damping, $\gamma_{A,opt}$, against effective mass ratio, μ_{eff}, for single degree-of-freedom systems, simply supported beams and circular and rectangular plates

extremes of the four values are indicated, but in the majority of cases the values are so close that they can only be represented by a single point. The figures show that the single mode analysis can be used to predict optimum absorber parameters for this beam. As expected, there is more scatter and the differences between values from the single-mode and multi-mode analyses are larger for the optimized absorber damping, but the agreement is still reassuringly good. For symmetric vibrations the frequency of the second resonant mode (neglected in the single mode analysis) is 9 × the fundamental natural frequency. For this large frequency ratio the single mode

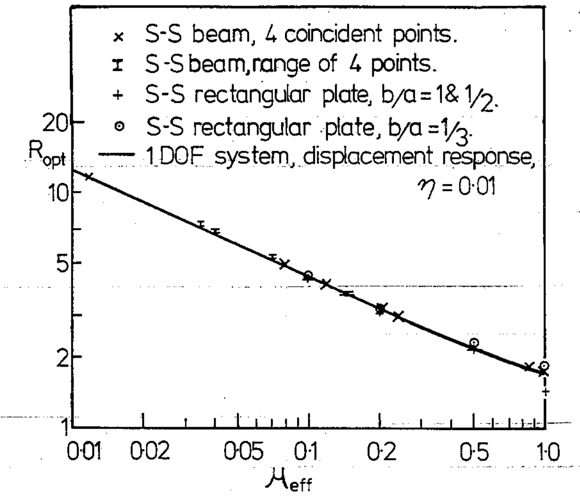

Figure 8. Optimum maximum response, R_{opt}, against effective mass ratio, μ_{eff}, for single degree-of-freedom systems, simply supported beams and rectangular plates

approach is accurate even for large values of the mass ratio μ_{eff}. (The graphs include higher values of μ_{eff} than are likely to occur for practical absorbers.)

COMPARISON WITH OTHER WORK

Neubert considered the extensional vibrations of a uniform bar with free ends; an absorber was attached to, and an excitation applied at, one end.[9] Optimization was with respect to the velocity response of the excited end. The standard curve for optimization of the velocity response of an undamped single degree-of-freedom system lies slightly above the corresponding curve for displacement response, equation (6), and is given by [9]

$$f_{opt} = (1+\tfrac{1}{2}\mu)^{\frac{1}{2}}/(1+\mu) \tag{41}$$

Neubert's values of f_{opt} for the optimization of the fundamental resonance of a bar in extension are shown in Figure 9 and lie close to the standard curve of equation (41). For a bar in extension M_{eff} = 0·5 (mass of bar); this has been used to generate values of μ_{eff}. Figure 10 shows that optimum values of the absorber damping for the bar lie close to the standard curve; equation (9) has been used for the latter, as it is believed that the change from optimization of displacement response to velocity response will have a small effect. For this type of vibration the ratio of the frequency of the lowest neglected mode to the fundamental frequency is 2.

Snowdon has published extensive work on the reduction of the vibrations of elastic bodies by attached absorbers; his results for cantilever beams will be considered initially.[10] The excitation force is at the tip of the beam; the absorber is attached at the tip or at mid-span. Optimization of the force transmitted to the frame (i.e. the shear force at the root of the cantilever) is investigated with respect to the fundamental, second or third resonance. In order to compare Snowdon's results, which are based on a closed-form solution for the problem (a type of solution that is available only for one-dimensional bodies), we require a modified form of equaion (35), which expresses the transmissibility ratio. The force transmitted to the frame

$$F_0 = -EI\left(\frac{\partial^3 v}{\partial x^3}\right)_{x=0} \tag{42}$$

Using this expression, equations (35)–(38) and standard expressions for the flexural vibrations of uniform beams, the transmissibility ratio

$$\frac{F_0}{P} = \frac{\phi_j'''(0)\,\phi_j(L)}{\lambda_j L}\left(\frac{A^2+B^2}{C^2+D^2}\right)^{\frac{1}{2}} \tag{43}$$

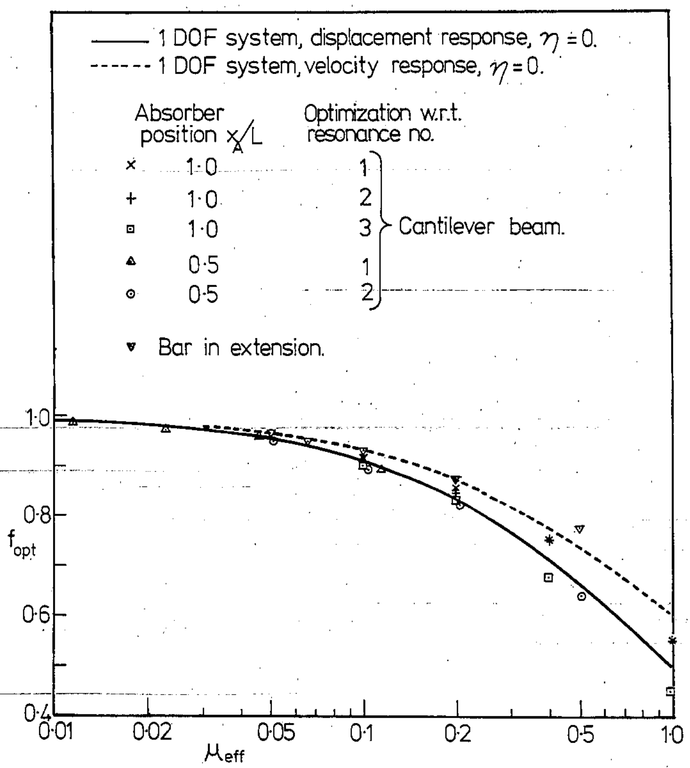

Figure 9. Optimum tuning ratio, f_{opt}, against effective mass ratio, μ_{eff}, for single degree-of-freedom systems, cantilever beams and bars in extension

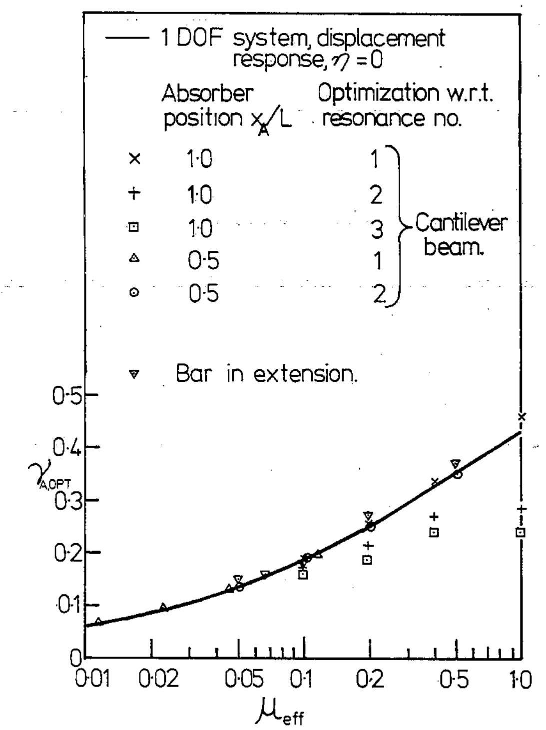

Figure 10. Optimum absorber damping, $\gamma_{A,\,opt}$, against effective mass ratio, μ_{eff}, for single degree-of-freedom systems, cantilever beams and bars in extension

where $\phi_j'''(x) = (1/\lambda_j^3)(d^3\phi_j/dx^3)$; it has been assumed that the excitation force acts at the tip, $x = L$ and the standard definitions apply for A, B, C and D.

Alternatively the frame force F_0 can be determined from the equilibrium equation for the beam, which includes F_0, P and the total transverse inertia force. Using Snowdon's approach, the two methods lead eventually to identical expressions for the transmissibility ratio. If a normal mode series solution is used, the two series will give identical numerical results, although the rate of convergence of the two series will differ. With the approximate one mode method the two methods yield different results. It would be preferable to use the second method, which does not depend upon the third differential of the approximate mode shape, but only the former method yields an expression for the ratio that is directly comparable with that developed for a single degree-of-freedom main system.

In Figures 9–11 the standard curves for optimized displacement response are plotted. As Snowdon's results relate to an undamped beam, equations (6), (7) and (9) for an undamped main system are used. Points, which correspond to five sets of Snowdon's results, are added to the figures. For optimization with respect to the first or second resonances they are scaled from Figures 10.12–10.17 of Reference 10 and for optimization with respect to the third resonance they are scaled from Table 10.2. Snowdon plots, or tabulates, transmissibility ratio; from this value and equation (43) an equivalent value of R_{opt} is obtained and plotted in Figure 11. The

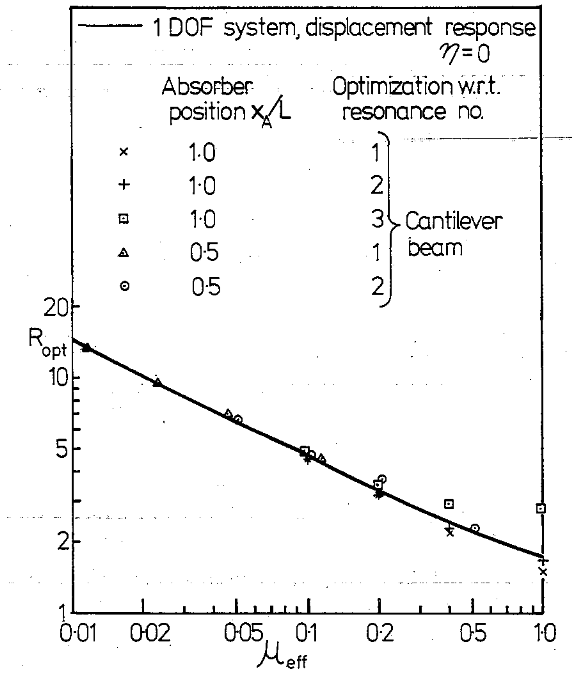

Figure 11. Optimum maximum response, R_{opt}, against effective mass ratio, μ_{eff}, for single degree-of-freedom systems and cantilever beams

standard curves predict with reasonable accuracy the optimum dynamic magnification factor, although at high values of μ_{eff} there is difficulty in representing all Snowdon's results by a single curve. For optimum damping Snowdon's curves for optimization of the second or third resonances with an absorber at the tip (but not his curve for optimization of the second resonance with an absorber at mid-span) lie significantly below the standard curve. The tuning ratio for optimization of velocity response, equation (41), is shown as a broken line in Figure 9. Its possible relevance to this problem, if the second method of determining the transmissibility ratio is used, is discussed later for rectangular plates and in detail in Appendix I. Here it should be noted that tuning ratios for optimization of the first and second resonances with an absorber at the tip lie between the two standard curves. For optimization with respect to resonance j the frequency ratio ω_{j+1}/ω_j indicates the significance of neglected modes and is 6·27, 2·80 and 1·96 for $j = 1$, 2 and 3 respectively.

Snowdon has determined optimum parameters for absorbers which are attached to the centre point of simply supported rectangular plates.[11] The excitation is applied also at the centre point. He uses a modal solution, which contains contributions only from those modes which are symmetric in both co-ordinate directions. He optimizes the transmissibility ratio (i.e. the ratio of the total force transmitted to the supports to the applied force) for the fundamental resonance of the plate. As the plates have hysteretic damping $\eta = 0{\cdot}01$, points which correspond to his results have been added to Figures 6–8, so that comparison is made with the standard curves for a main system with appropriate hysteretic damping. For simply supported boundary conditions the mode shape of equation (12) becomes

$$\phi_j(x)\,\psi_n(y) = 2\sin\frac{j\pi x}{a}\sin\frac{n\pi y}{b} \tag{44}$$

in order to satisfy normalizing condition (13). From the relation (19) between the effective mass M_{eff} and the mass of the plate M_{P} for a centrally attached absorber ($x_{\mathrm{A}} = \frac{1}{2}a$, $y_{\mathrm{A}} = \frac{1}{2}b$), $4M_{\mathrm{eff}} = M_{\mathrm{P}}$. The total force transmitted to the supports is obtained by integrating the transverse shear resultant along the complete boundary as

$$F_0 = -\frac{8\pi^2\,abD^*}{jn}\left(\frac{j^2}{a^2}+\frac{n^2}{b^2}\right)^2 q_{jn} \tag{45}$$

where the generalized co-ordinate q_{jn} is given by equation (22), and the flexural rigidity of the plate $D^* = Eh^3/12(1-\nu^2)$. Hence the transmissibility ratio

$$\frac{F_0}{P} = -\frac{16}{\pi^2 jn}\sin\frac{j\pi x_{\mathrm{A}}}{a}\sin\frac{n\pi y_{\mathrm{A}}}{b}\,\mathrm{e}^{\mathrm{i}\omega t}\left(\frac{A+\mathrm{i}B}{C+\mathrm{i}D}\right)$$

and

$$\left|\frac{F_0}{P}\right| = \frac{16R}{\pi^2} \tag{46}$$

for optimization at the fundamental resonance. Using Snowdon's values for optimized transmissibility, values of R_{opt} are obtained from equation (46) and plotted in Figure 8.

In Figure 6 and 8 results from Snowdon's analysis for aspect ratios b/a equal to 1 and $\frac{1}{2}$ are insufficiently separated to be represented by separate points. The standard curve for R_{opt} represents well all Snowdon's results. The optimum damping ratios diverge from the standard curve as the effective mass ratio increases to large values and this divergence is more significant as the aspect ratio decreases. For this symmetric problem and optimization with respect to the fundamental resonance the lowest neglected mode, which would contribute to the response, is $j/n = 3/1$; the frequency ratio ω_{31}/ω_{11} is 5·0, 2·6 and 1·8 for aspect ratios of 1, $\frac{1}{2}$ and $\frac{1}{3}$ respectively. Snowdon's results for optimum tuning ratio lie above the standard curve. As discussed above for beams, the transmissibility ratio could be determined from the equilibrium equation in the Z-direction for the plate and absorber system. This approach would be more accurate, as it avoids the use of differentials of the approximate mode shape, but cannot be correlated directly with known standard curves. It is noted that Snowdon's results lie close to the optimum tuning curve for velocity response, which is given by equation (41) (for an undamped main system) and is shown by a broken line on Figure 6. Ioi and Ikeda determine optimized parameters for an absorber which is attached to a main system, for which the support is subjected to a harmonic displacement.[6] For an undamped main system and optimization of the relative displacement of the main mass the optimum tuning ratio is given by equation (41), while the optimum damping and response are again given by equations (9) and (7) respectively. As shown in Appendix I, there is a similarity of form between the equations for the total inertia force in the plate and absorber system and the relative displacement of the main mass for support excitation of the system of Figure 1. However, a numerical factor, which depends upon modal properties, should be unity for direct correlation, but is 1·62 for all the rectangular plates considered here. For cantilever beams which were considered above there is a similar numerical factor that should be unity. For optimization of the first and second resonances with the absorber at the tip the factor is 1·57 and 0·87 respectively and Snowdon's results lie between the two standard curves. For the other beams the factor is of the order of $\frac{1}{2}$ and his results lie closer to the standard curve for displacement response (Figure 9). Thus the optimum tuning curve for velocity is relevant to some of Snowdon's results and there are reasons for the apparent scatter shown in Figures 6 and 9.

Snowdon has obtained absorber parameters for the optimization of force transmissibility for uniform circular plates, which are subjected to a central transverse harmonic force.[12] The absorber is attached to the centre of the plate, optimization is with respect to the fundamental resonance and the outer boundary is either simply supported or clamped. For the fundamental mode of vibration ($j = 1$, $n = 0$) the transverse displacement at point (r^*, θ) can be written

$$\begin{aligned} w(r^*,\theta,t) &= W(r^*,\theta)\,q_{10}(t) \\ &= \zeta[J_0(\kappa_{10}r^*)+\xi I_0(\kappa_{10}r^*)]\,q_{10}(t) \end{aligned} \tag{47}$$

where J_0 and I_0 are the Bessel function and modified Bessel function respectively of the first kind and zero order; $\kappa_{jn}^4 = \rho h\omega_{jn}^2/D^*$. Imposing the conditions at the outer boundary, $r^* = a^*$, yields the constant ξ and the non-dimensional natural frequency factor $\kappa_{10}a^*$. The constant ζ is chosen to satisfy the normalizing condition.

$$\rho h\int_0^{a^*}\int_0^{2\pi} 2\pi r W^2(r^*,\theta)\,\mathrm{d}\theta\,\mathrm{d}r = M_{\mathrm{P}} \tag{48}$$

where M_{P} is the mass of the plate. Following equation (19) the effective mass is given by

$$\tfrac{1}{2}M_{\mathrm{eff}}\,W^2(0,0)\,\dot{q}_{10}^2 = \tfrac{1}{2}M_{\mathrm{P}}\,\dot{q}_{10}^2 \tag{49}$$

From equations (47)–(49), $M_{\mathrm{eff}} = 0{\cdot}183\,M_{\mathrm{P}}$ and $0{\cdot}284\,M_{\mathrm{P}}$ for clamped and simply supported boundaries respectively. These ratios have been used to re-plot Snowdon's results in Figures 6 and 7. As his plates have hysteretic damping $\eta = 0{\cdot}01$, it is appropriate to compare his values with the standard curves of those figures. For low values of μ_{eff} his values of the tuning ratio f_{opt} lie close to the velocity response curve. As in his work on beams and rectangular plates, Snowdon has optimized the force transmitted to the frame; for circular plates there may be some approximate relation between values of f_{opt} for the velocity response of a one-degree-of-freedom system and force transmissibility for a circular plate, but this has not been investigated. The damping ratios, $\gamma_{\mathrm{A,\,opt}}$, for circular plates are slightly higher than those predicted by the standard curve, particularly for large values of μ_{eff} (Figure 7). For this symmetric problem the relevant frequency ratio ω_{20}/ω_{10} is 3·9 and 6·0 for clamped and simply supported boundaries respectively.

CONCLUSION

Optimum conditions for absorbers which are attached to undamped single degree-of-freedom systems are well known. As real systems must contain some damping, it is useful to extend this to allow for viscous and hysteretic damping in the main system; the latter type of damping is more relevant to practical problems. Numerical results are given for both types of damping; they supplement and corroborate the recently published expressions for optimum conditions.[6] If the damping in the main system is very small, the effect on the optimum tuning ratio and absorber damping is also very small but, as expected, there is a decrease in the optimum dynamic magnification factor.

Standard curves for optimum tuning ratio, absorber damping and maximum response of the main system can be plotted against the mass ratio. This can be done for either damped or undamped main systems, and for optimization with respect to various response parameters, using results in the literature or in some cases by generation of new results. The main purpose of this paper is to show that for any elastic body, which can be represented by a single mode, the standard curves can be used to determine optimum absorber parameters with the minimum of additional work. This is an extension of the work of Jacquot, who showed that the expression for the optimum response of an undamped beam with an attached absorber showed small modifications from that of a single degree-of-freedom system. The analogy depends upon the concept of an effective mass for the structure; this is defined as the mass which must be placed at the point of attachment of the absorber in order to give the same kinetic energy as that of the whole structure, when it vibrates in the mode whose resonance is minimized. Provided that the ratio of the absorber mass to the effective structural mass is used, the standard curve can be used directly to predict the optimum tuning and damping ratios. For the determination of the maximum response of the structure the maximum dynamic magnification factor from the standard curve together with the appropriate mode shape of the structure are required. Thus the only additional work that is required to determine the complete solution is an estimate of the relevant natural frequency and mode shape of the particular structure.

The concepts in this paper depend upon the ability to represent the response of the structure with attached absorber by a single mode shape, which is one of the modes of the unmodified structure. Thus it is necessary to give results that verify this assumption. Extensive results have been given for a simply supported beam with an attached absorber and show good agreement. The accurate results which are used to check this theory are based on a multi-mode solution. Further results have been taken from the literature and replotted in the format of the paper. These relate to bars in extension, cantilever beams, rectangular and circular plates. Again there is good agreement between the exact results and those of the simple theory, but there are some discrepancies which require comment. For some of the results optimization has been undertaken for higher modes than the fundamental, and not surprisingly the simple theory is less accurate in such cases. Also the optimized quantity in Snowdon's work on cantilever beams and plates is the force transmissibility ratio, rather than the maximum response of the structure. This force ratio can be determined for the idealized system from the displacement response, but it would be more accurate to obtain it from another, as yet unknown, optimized standard curve. It seems likely that this standard curve bears some resemblance to the standard curve for optimum tuning ratio when velocity response is considered. With this proviso the correlation between optimum tuning ratios from Snowdon's work and from simple theory appears to be reasonable. Correlation of absorber damping is less good as might be expected. The results suggest that, provided that the natural frequency of the lowest mode, which contributes to the response but is neglected in the single mode analysis, is greater than about twice the fundamental frequency, the standard curves can be used to predict absorber parameters with adequate accuracy. This frequency ratio would allow the theory to be used for many practical structures, particularly tall structures. The results for cylindrical shells, where the frequency spacing is closer and thus the ratio is less than 2, will be given in a subsequent paper, where it will be demonstrated that there is divergence from the standard curves.[15]

The emphasis in the paper is on the response to harmonic excitation, and effectively this leads to the minimization of the standard transfer function or receptance. In a recent paper Youssef and Popplewell state that minimization of this receptance is the best condition for general excitations.[7]

APPENDIX I

Support excitation of two degree-of-freedom system

If a harmonic displacement, $x_0 = X_0\,e^{i\omega t}$, is applied to the support of the system of Figure 1 (with the force $Pe^{i\omega t}$ removed), the relative displacement of the main mass, $y_1 = x_1 - x_0$, is given by

$$y_1 = [M_M\omega^2(k_A - M_A\omega^2 + i\omega c_A) + M_A\omega^2(k_A + i\omega c_A)]\,X_0\,e^{i\omega t}/F(\omega) \tag{50}$$

where $F(\omega)$ is the denominator of equation (1).

Transmissibility through equilibrium equation—cantilever beam

For an excitation force $Pe^{i\omega t}$ at the tip, $x = L$, and an absorber attached at x_A the force transmitted to the support

$$F_0 = Pe^{i\omega t} - F_{in} \tag{51}$$

where the total inertia force

$$F_{in} = \rho A^*\int_0^L \frac{\partial^2 v}{\partial t^2}\,dx + M_A\frac{d^2 q_0}{dt^2} \tag{52}$$

Now $v(x,t) = \phi_j(x)\,q_j(t)$. At the point of attachement of the absorber to the beam

$$v_A = \phi_j(x_A)\,q_j \tag{53}$$

From the analysis of the absorber system

$$(k_A - M_A\omega^2 + i\omega c_A)\,q_0 = (k_A + i\omega c_A)\,v_A \tag{54}$$

Thus

$$F_{in} = -\rho A^*\omega^2 q_j\int_0^L \phi_j\,dx - M_A\omega^2 q_0$$

$$= \frac{M_B\,\omega^2 q_j}{\lambda_j L}\,\phi_j'''(0) - M_A\omega^2 q_0 \tag{55}$$

where

$$\frac{1}{\lambda_j^3}\frac{d^3\phi_j}{dx^3} = \phi_j'''(x)$$

By analogy with equation (21)

$$q_j = \frac{Pe^{i\omega t}\,\phi_j(L)\,(k_A - M_A\omega^2 + i\omega c_A)}{\phi_j^2(x_A)\,F(\omega)} \tag{56}$$

where M_{eff} and $M_{eff}\,\omega_j^2$ replace M_M and k_M respectively in the definition of $F(\omega)$. Using equations (53), (54) and (56) in equation (55),

$$F_{in} = \frac{Pe^{i\omega t}\,\phi_j(L)}{\phi_j(x_A)\,F(\omega)}\left[\frac{M_B\,\omega^2\,\phi_j'''(0)}{\lambda_j L\phi_j(x_A)}(k_A - M_A\omega^2 + i\omega c_A) - M_A\omega^2(k_A + i\omega c_A)\right]$$

$$= \frac{Pe^{i\omega t}\,\phi_j(L)}{\phi_j(x_A)\,F(\omega)}\left[\frac{M_{eff}\,\omega^2\,\phi_j(x_A)\,\phi_j'''(0)}{\lambda_j L}(k_A - M_A\omega^2 + i\omega c_A) - M_A\omega^2(k_A + i\omega c_A)\right] \tag{57}$$

Equations (50) and (57) are of similar form; they would be equivalent if $-\phi_j(x_A)\,\phi_j'''(0)/\lambda_j L \to 1{\cdot}0$. If $x_A = L$, $-\phi_j(x_A)\,\phi_j'''(0)/\lambda_j L = 1{\cdot}566, 0{\cdot}868$ and $0{\cdot}509$ for $j = 1, 2$ and 3 respectively; for $x_A = \frac{1}{2}L$, the factor equals $0{\cdot}532$ and $0{\cdot}619$ for $j = 1$ and 2 respectively.

Rectangular plate with simply supported edges

In this case the total inertia force

$$F_{in} = \rho h\int_0^a\int_0^b \frac{\partial^2 w}{\partial t^2}\,dy\,dx + M_A\frac{d^2 q_0}{dt^2}$$

$$= -\frac{8M_P\,\omega^2 q_{jn}}{\pi^2 jn} - M_A\omega^2 q_0 \tag{58}$$

using equations (12) and (44). From equation (21), as $x_F = x_A = \frac{1}{2}a$, $y_F = y_A = \frac{1}{2}b$,

$$q_{jn} = Pe^{i\omega t}(k_A - M_A\omega^2 + i\omega c_A)/2F(\omega) \tag{59}$$

Also

$$q_0 = 2q_{jn}(k_A + i\omega c_A)/(k_A - M_A\omega^2 + i\omega c_A) \tag{60}$$

Substituting in equation (58) and using $M_P = 4M_{eff}$,

$$F_{in} = -\frac{Pe^{i\omega t}}{F(\omega)}\left[\frac{16M_{eff}\,\omega^2}{\pi^2 jn}(k_A - M_A\omega^2 + i\omega c_A) + M_A\omega^2(k_A + i\omega c_A)\right] \tag{61}$$

For the fundamental resonance $j = n = 1$; then equations (50) and (61) are equivalent apart from the factor $1{\cdot}621$ $(=16/\pi^2)$.

The similarity of equations (57) and (61) to (50) and the fact that optimization of equation (50) leads to the tuning ratio of equation (41) suggest that the latter curve may be relevant to some of Snowdon's results; accordingly it has been shown by broken lines in Figures 6 and 9.

APPENDIX II

Nomenclature

a, b	lengths of sides of rectangular plate
a^*	mean radius of cylindrical shell or radius of circular plate
A, B, C, D	functions of r, μ (or μ_{eff}), γ_A and γ_M (or η)
A^*	cross-sectional area of beam
c_A, c_M	viscous damping coefficients for absorber and main systems
D^*	flexural rigidity of plate
E	Young's modulus
$f = \omega_A/\omega_M$ (or ω_A/ω_{jn})	tuning ratio
F_{in}	total inertia force
F_0	force transmitted to frame
h	thickness of plate or shell
I	second moment of area for beam
j, n	modal integers
k_A, k_M	stiffness for absorber and main systems
k_{eff}	effective stiffness of elastic body, corresponding to mode j/n (strain energy of elastic body $= \frac{1}{2}k_{eff}\,q_{jn}^2$)
L	length of beam or shell
M_A, M_M	masses of absorber and main systems
M_B, M_P, M_S	masses of beam, plate and shell
M_{eff}	effective mass of elastic body (kinetic energy of elastic body $= \frac{1}{2}M_{eff}\,\dot{w}_A^2$)
P	amplitude of applied force
q_{jn}	principal co-ordinate in mode j/n for elastic body
q_0	principal co-ordinate for absorber
r^*	radial co-ordinate
$r = \omega/\omega_M$ (or ω/ω_{jn})	frequency ratio
$R = [(A^2+B^2)/(C^2+D^2)]^{\frac{1}{2}}$	dynamic magnification factor
S	strain energy
t	time
T	kinetic energy
u, v and w	components of displacement in X, Y, (or θ) and Z directions
v_A, w_A	displacement of beam and plate or shell at point of attachment of absorber
V, W	amplitude in Y and Z directions
x, y, z	cartesian co-ordinates
x_A, y_A	co-ordinates of elastic body at point of attachment of absorber
x_F, y_F	co-ordinates of elastic body at point of application of force P
x_1, x_2	displacements of main and absorber masses
X_1	amplitude of main mass
x_0, X_0	displacement and amplitude of frame
y_1, y_2	displacement of main and absorber masses relative to frame
α_{jn}, β_{jn}	amplitude ratios for mode j/n of cylindrical shell
$\gamma_A = c_A/2M_A\,\omega_A$	absorber damping ratio
$\gamma_M = c_M/2M_M\,\omega_M$	main system damping ratio
η	hysteretic damping constant
θ	angular co-ordinate
$\kappa_{jn} = (\rho h\omega_{jn}^2/D^*)^{\frac{1}{4}}$	frequency parameter for circular plate
$\lambda_j = (\rho A^*\,\omega_j^2/EI)^{\frac{1}{4}}$	frequency parameter for beam
$\mu = M_A/M_A$	mass ratio
$\mu_{eff} = M_A/M_{eff}$	effective mass ratio
ν	Poisson's ratio
ρ	density
ϕ_j, ψ_n	mode functions in the X and Y directions
ω	excitation frequency
$\omega_A = (k_A/M_A)^{\frac{1}{2}}$	natural frequency of absorber system
$\omega_M = (k_M/M_M)^{\frac{1}{2}}$	natural frequency of main system
ω_j, ω_{jn}	natural frequency of mode j for beams and mode j/n for plates and shells

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