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	Subject: Eigenproperties of system with constraints	Prepared: RIKY	Date: 20/07/18
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Eigenproperties of system with constraints

Problem: $$M\ddot{\underline{y}} = -K\underline{y} - C\dot{\underline{y}} + \underline{f} \qquad (1)$$

subject to $$J\ddot{\underline{y}} = 0. \qquad (2)$$

where (M, K, C, J) are all time-invariant.

Let $Z = \mathrm{Null}(J)$ (ie rectangular matrix whose columns span the null space of J).

By definition $JZ = 0$

Let $Z\underline{u} = \underline{y}$

$\Rightarrow Z\dot{\underline{u}} = \dot{\underline{y}}$

$Z\ddot{\underline{u}} = \ddot{\underline{y}}$

} noting that J is not a function of time (ie is time-invariant).

Note →
$$\underbrace{\begin{bmatrix} y \\ \dot{y} \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix}}_{Z_2} \underbrace{\begin{bmatrix} u \\ \dot{u} \end{bmatrix}}_{\underline{x}'}$$

State-space representation of (1):

$$\begin{bmatrix} \dot{\underline{y}} \\ \ddot{\underline{y}} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \begin{bmatrix} \underline{y} \\ \dot{\underline{y}} \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} \begin{bmatrix} \underline{f} \end{bmatrix}$$

ie $$\dot{\underline{x}} = A\,\underline{x} + B\,\underline{f}$$

Substitute for $(\ddot{y}, \dot{y}, y)$ in eqn (1); then pre-multiply by Z^T:

$$(Z^T M Z)\ddot{\underline{u}} = -(Z^T K Z)\underline{u} - (Z^T C Z)\dot{\underline{u}} + Z^T \underline{f}.$$

Let
$$\left.\begin{aligned} M' &= Z^T M Z \\ K' &= Z^T K Z \\ C' &= Z^T C Z \end{aligned}\right\} \Rightarrow \ddot{\underline{u}} = -M'^{-1}K'\underline{u} - M'^{-1}C'\dot{\underline{u}} + (M'^{-1}Z^T)\underline{f}$$

Thus:

$$\begin{bmatrix} \dot{\underline{u}} \\ \ddot{\underline{u}} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M'^{-1}K' & -M'^{-1}K' \end{bmatrix} \begin{bmatrix} \underline{u} \\ \dot{\underline{u}} \end{bmatrix} + \begin{bmatrix} 0 \\ M'^{-1}Z^T \end{bmatrix} \begin{bmatrix} \underline{f} \end{bmatrix}$$

ie $\dot{\underline{x}}' = A' \underline{x}' + B' \underline{f}$

where. $\underline{x}'$ is state vector for unconstrained problem.

A' is state matrix of unconstrained problem.

For eigensolution we seek solutions of the form

$\underline{x}'(t) = \underline{x}_0' e^{\lambda t}$ with $\underline{f} = 0$ (free vibration)

$\Rightarrow \dot{\underline{x}}'(t) = \lambda \underline{x}_0' e^{\lambda t}$

$\Rightarrow \lambda \underline{x}_0' \cancel{e^{\lambda t}} = A' \underline{x}_0' \cancel{e^{\lambda t}}$

$\Rightarrow \lambda \underline{x}_0' = A' \underline{x}_0'$ ie classic eigenproblem

$\left.\begin{array}{l} \underline{x}_0' \text{ are eigenvectors of } A' \\ \lambda \text{ are eigenvalues of } A' \end{array}\right\}.$

Hence by solving for eigenproperties of A' (state matrix of unconstrained problem) we can obtain the information we need.

Once determined, eigenvectors can be converted back to eigenvectors of constrained problem:

$$\underline{x}_0 = Z_2 \underline{x}_0' \quad \text{where } Z_2 = \begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix}.$$