An Introduction to Tuned Mass Dampers

1 Introduction

In general, civil engineering structures are designed to avoid significant vibration at resonant frequencies. However, in some cases (e.g. tall buildings or long span bridges) it is more economic to control resonant response by the addition of damping, when compared to the cost of stiffening a structure to move its natural frequency away from that of the applied loading. For example, tall buildings may have natural frequencies corresponding to a high energy part of the wind gust velocity spectrum. Footbridges may have natural frequencies similar to walking frequencies. Both bridges and buildings may be subject to aero-elastic excitation (vortex shedding, for example).

Damping may be added to a structure in a number of ways; this paper presents the theory of one such method, namely tuned mass dampers (TMD). The theory is developed from the 'basic dynamics' module.

2 Definition of Damping Values

There are a number of ways in which damping can be quantified and it is important to understand the basis on which damping values are being presented (as several of the standard definitions are dimensionless). The following table gives the commonly used ways in which damping is presented together with conversion factors:

	To convert from:			
	ξ	α	c	δ
To:	**Multiply by:**			
Damping Ratio, ξ	-	1/100	$1/c_{crit}$	$1/2\pi$
% Critical Damping Ratio, α	100	-	$100/c_{crit}$	$100/2\pi$
Viscous Damping Coefficient, c	c_{crit}	$c_{crit}/100$	-	$c_{crit}/2\pi$
Logarithmic Decrement, $\delta^{(*)}$	2π	$2\pi/100$	$2\pi/c_{crit}$	-

Recall that c_{crit} is the critical damping ratio (= $2m\omega_n$), where m is the mass of the system and ω_n is its natural circular frequency (radians/second).

$^{(*)}$ The conversion for logarithmic decrement only applies for low values of damping, where $c \ll c_{crit}$.

For example, to convert from a logarithmic decrement of δ = 0.06 to the equivalent damping ratio (ξ), multiply by $1/2\pi$, so $\xi = 0.06/2\pi = 0.0095$.

3 Un-damped Two Degree of Freedom System

Consider the system shown in Figure 3-1:

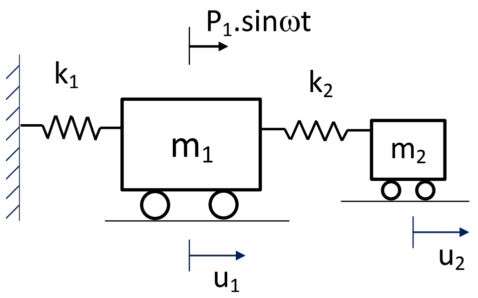
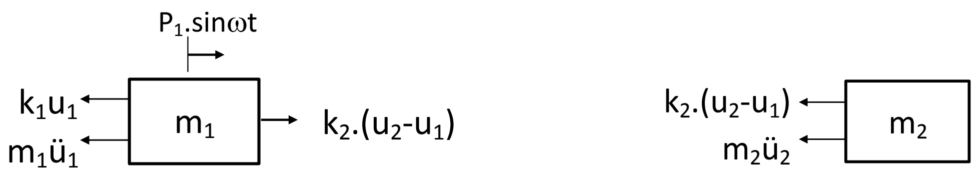


Figure 3-1: 2-DOF Un-damped System

Note that this is very similar to the worked example in Section 3.1 of the 'basic dynamics' module. However, individual values of spring stiffness and mass are defined and one spring has been removed so only m_1 is connected to the ground.

m_1 shall represent the main structure and m_2 shall represent and auxiliary mass added to the system with the purpose of reducing resonant response.

Consider the equilibrium of each mass in turn:



Hence the equations of equilibrium for this system can be written in matrix form as:

$$\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}.\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}.\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} = \begin{bmatrix} P_1(t) \\ 0 \end{bmatrix} \quad \text{(Eqn. 3-1)}$$

We wish to find the natural frequencies of the system, so first we shall consider the free vibration case (i.e. $P_1(t)=0$). We will also assume that harmonic vibration is taking place with natural circular frequency (i.e. in radians per second) ω. Because there are two degrees of freedom in our system, there will be two natural or normal modes. This means that, in each mode, both masses vibrate with the same frequency and that their amplitudes have the same relative values at all times.

Thus, $u_i = a_i.\sin(\omega t)$ where i = 1,2 for modes 1 and 2. By differentiating this expression twice, it follows that the corresponding accelerations can be expressed as: $\ddot{u}_i = -a_i.\omega^2.\sin(\omega t)$.

Substitution into the equations of equilibrium gives:

$$\begin{bmatrix} k_1+k_2-\omega^2.m_1 & -k_2 \\ -k_2 & k_2-\omega^2.m_2 \end{bmatrix}.\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{(Eqn.3-2)}$$

Note that, by dividing by sin(ωt), we have converted a differential equation to an algebraic equation. For a non-trivial solution, we must set the determinant of this matrix to zero and solve for ω:

$$(k_1+k_2-\omega^2.m_1)(k_2-\omega^2.m_2)-k_2^2=0 \quad \text{(Eqn. 3-3)}$$

Multiplying out, collecting terms and re-arranging, we obtain a quadratic equation in ω^2:

$$\omega^4-\omega^2.\left[\frac{k_1+k_2}{m_1}+\frac{k_2}{m_2}\right]+\frac{k_1.k_2}{m_1.m_2}=0 \quad \text{(Eqn. 3-4)}$$

It is convenient to define the ratio of the two masses as $\mu = \frac{m_2}{m_1}$

We shall also define the two limiting frequencies as:

$$\varpi_1^2 = \frac{k_1}{m_1+m_2} \text{ and } \varpi_2^2 = \frac{k_2}{m_2} \qquad \text{(Eqn. 3-5a \& b)}$$

If we substitute μ, ϖ_1^2 *and* ϖ_2^2 into the quadratic equation in ω^2 we obtain:

$$\omega^4 - \omega^2[(\varpi_1^2 + \varpi_2^2)(1+\mu)] + \varpi_1^2.\varpi_2^2(1+\mu) = 0 \qquad \text{(Eqn. 3-6)}$$

The roots of this equation are given by:

$$\omega_1^2, \omega_2^2 = \frac{[(\varpi_1^2 + \varpi_2^2)(1+\mu)] \pm \sqrt{[(\varpi_1^2 + \varpi_2^2)(1+\mu)]^2 - 4\varpi_1^2.\varpi_2^2(1+\mu)}}{2} \qquad \text{(Eqn.3-7)}$$

Example: Consider the system shown in Figure 3-1.

Set: $k_1/m_1 = k_2/m_2$ (so the frequency of the main system and the auxiliary mass are the same if treated in isolation).

Define the frequency of the main and auxiliary systems in isolation as Ω^2 = 27.5 radians/second

Mass ratio, μ = 0.05

Calculate the natural frequencies of the system and the relative displacements of the masses in the two modes of vibration.

So: $\varpi_2^2 = \frac{k_2}{m_2} = 27.5$; also $\varpi_1^2 = \frac{k_1}{m_1+m_2} = \frac{k_1}{m_1(1+\mu)} = \frac{27.5}{1.05} = 26.2$

$$\omega_1^2 = \frac{[(26.2+27.5)(1.05)]+\sqrt{[(26.2+27.5)(1.05)]^2-4x26.2x27.5x1.05}}{2} = 34.375 = 1.25\Omega^2$$

$$\omega_2^2 = \frac{[(26.2+27.5)(1.05)]-\sqrt{[(26.2+27.5)(1.05)]^2-4x26.2x27.5x1.05}}{2} = 22 = 0.80\Omega^2$$

Thus the two frequencies of the system are ω_1 = 5.86 rad/s and ω_2 = 4.69 rad/s. It can be seen that the effect of combining two separate spring-mass systems with the same natural frequency is to produce a combined system with natural frequencies split into higher and lower values.

Now we shall calculate the relative displacements of the two masses. Returning to the equations of equilibrium:

$$(k_1 + k_2 - \omega^2.m_1).a_1 - k_2.a_2 = 0$$

Re-arranging and recognising that for this case $\mu = m_2/m_1 = k_2/k_1$ gives $\frac{a_2}{a_1} = \frac{1}{\mu}\left(1+\mu-\frac{\omega^2}{\Omega^2}\right)$

Therefore, for ω_1 we obtain $\frac{a_2}{a_1} = \frac{1}{0.05}(1+0.05-1.25) = -4$

And for ω_2 we obtain $\frac{a_2}{a_1} = \frac{1}{0.05}(1+0.05-0.80) = 5$

So we have demonstrated that the auxiliary mass moves in the opposite direction to the main mass for the mode associated with ω_1 and with the motion of the main mass for the mode associated with ω_2. It should also be noted that the motion of the smaller auxiliary mass is far greater than that of the main mass. This is intuitive as the smaller mass needs to travel further and generate greater accelerations to achieve dynamic equilibrium with the larger main mass.

We now wish to establish the response of the system shown in Figure 3-1 to the sinusoidal time varying load, $P_1.\sin\omega t$.

We shall convert the equations of equilibrium into a dimensionless form by defining the following terms:

$u_{st} = P_1/k_1$ The static deflection of the main mass under the action of P_1

$\omega_a^2 = k_2/m_2$ The natural frequency of the auxiliary system

$\Omega_n^2 = k_1/m_1$ The natural frequency of the main system

$\mu = m_2/m_1$ The mass ratio

Therefore, the equations of equilibrium become:

$$\begin{bmatrix} 1+\frac{k_2}{k_1}-\frac{\omega^2}{\Omega^2} & \frac{-k_2}{k_1} \\ \frac{-k_2}{k_1} & \frac{k_2}{k_1}\left(1-\frac{\omega^2}{\omega_a^2}\right) \end{bmatrix}.\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} u_{st} \\ 0 \end{bmatrix} \qquad \text{(Eqn. 3-8)}$$

Solving for a_1 and a_2 gives:

$$\frac{a_1}{u_{st}} = \frac{1-\frac{\omega^2}{\omega_a^2}}{\left(1-\frac{\omega^2}{\omega_a^2}\right).\left(1+\frac{k_2}{k_1}-\frac{\omega^2}{\Omega_n^2}\right)-\frac{k_2}{k_1}} \qquad \frac{a_2}{u_{st}} = \frac{1}{\left(1-\frac{\omega^2}{\omega_a^2}\right).\left(1+\frac{k_2}{k_1}-\frac{\omega^2}{\Omega_n^2}\right)-\frac{k_2}{k_1}} \qquad \text{(Eqn. 3-9a \& b)}$$

These two expressions give the response of the main and auxiliary masses as a multiplier of the static deflection of the main mass (u_{st}).

Consider the particular case where $\omega_a = \Omega_n$

This means that $k_1/m_1 = k_2/m_2$ or $k_2/k_1 = m_2/m_1 = \mu$. For this special case, we can write the expressions for the displacement of the main and auxiliary mass as:

$$\frac{u_1}{u_{st}} = \frac{1-\frac{\omega^2}{\omega_a^2}}{\left(1-\frac{\omega^2}{\omega_a^2}\right).\left(1+\mu-\frac{\omega^2}{\omega_a^2}\right)-\mu}.\sin(\omega t) \qquad \frac{u_2}{u_{st}} = \frac{1}{\left(1-\frac{\omega^2}{\omega_a^2}\right).\left(1+\mu-\frac{\omega^2}{\omega_a^2}\right)-\mu}.\sin(\omega t) \qquad \text{(Eqn. 3-10a \& b)}$$

It can be seen that the denominators for both of these equations are the same. Furthermore, they are quadratic equations in (ω^2/ω_a^2). If either root of the quadratic equation is substituted back into the expressions for displacement, the denominators become zero and the displacement of both masses becomes infinite. Thus, the roots of the quadratic equation give the resonant (or natural) frequencies of the system. The roots of the denominator are given by:

$$\frac{\omega}{\omega_a} = \sqrt{\left(1+\frac{\mu}{2}\right) \pm \sqrt{\mu+\frac{\mu^2}{4}}} \qquad \text{(Eqn. 3-11)}$$

This expression is plotted in Figure 3-2 and clearly illustrates the effect of adding a small auxiliary mass to a SDOF system. The resulting 2-DOF system has natural frequencies above and below that of the SDOF system.

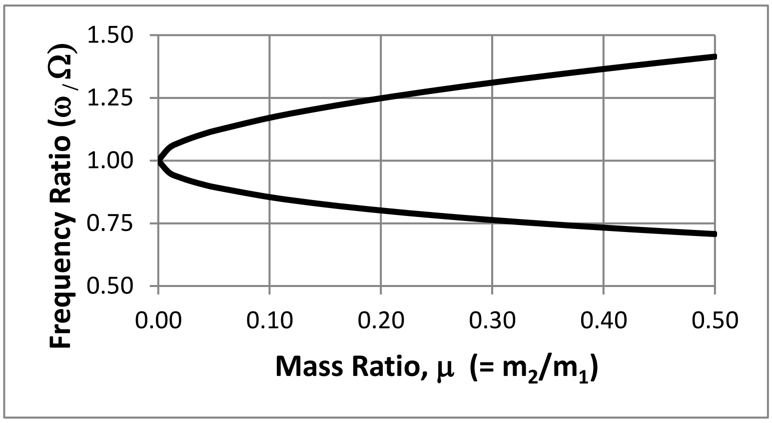


Figure 3-2: Resonant frequency as a function of mass ratio

Now let's consider the motion of the main and auxiliary masses. The following two figures plot their motion for the particular case of $\omega_a = \Omega_n$ and $\mu = 0.2$:

Looking first at the motion of the main mass:

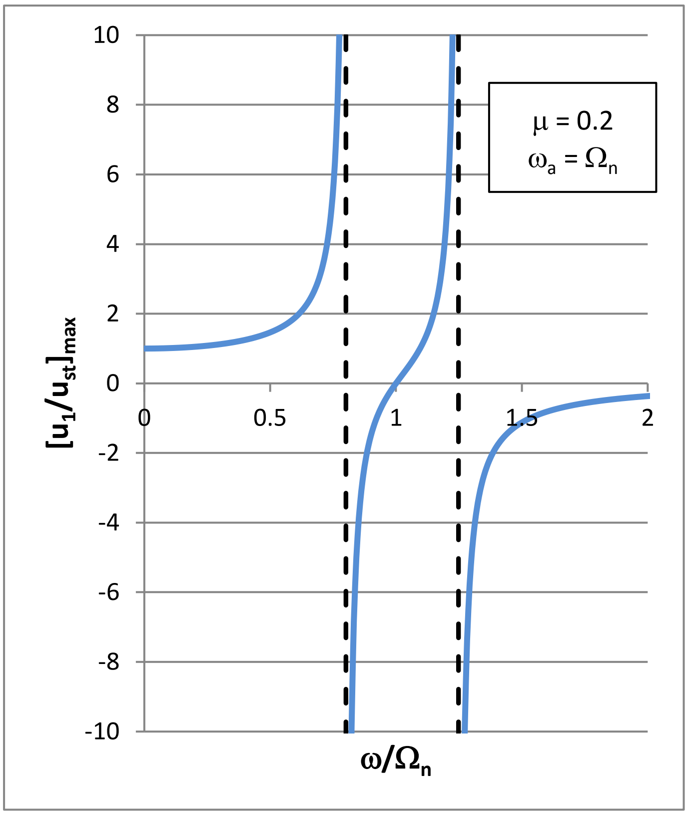


Figure 3-3: Response of the Main Mass

At a forcing frequency of $\omega = 0$, $[u_1/u_{st}]_{max} = 1$. In other words, the system responds statically to a static load.

As the forcing frequency increases, so does the response of the main mass. At the first natural frequency of the system, the response becomes infinite.

As the forcing frequency increases above the first natural frequency, the response of the main mass switches to a negative value. The physical significance of this is that the phase angle between the motion of the main mass and applied load has changed by 180°.

Response then reduces with increasing forcing frequency until at a forcing frequency of $\omega = \Omega_n$, $u_1/u_{st} = 0$. At this particular forcing frequency ratio, the motion of the main mass is completely damped.

As the forcing frequency increases beyond Ω_n, response once again becomes infinite at the second natural frequency and the phase angle between the response and the loading switches again by 180°. Note also that the motion of the main mass and the auxiliary mass is also now out of phase.

Looking at the motion of the auxiliary mass:

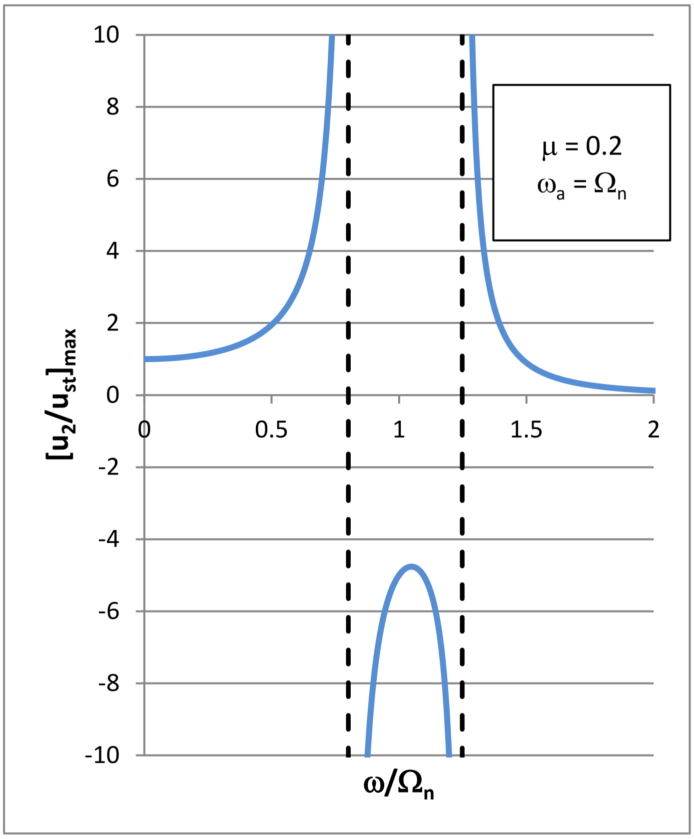


Figure 3-4: Response of the Auxiliary Mass

At a forcing frequency of $\omega = 0$, $[u_2/u_{st}]_{max} = 1$. In other words, the system responds statically to a static load.

As the forcing frequency increases, so does the response of the auxiliary mass. At the first natural frequency of the system, the response becomes infinite.

As the forcing frequency increases above the first natural frequency, the response of the auxiliary mass switches to a negative value. The physical significance of this is that the phase angle between the motion of the auxiliary mass and the main mass has changed by 180°.

Response then reduces with increasing forcing frequency to a minimum at a forcing frequency of $\omega = \Omega_n$. At this particular forcing frequency ratio, the motion of the auxiliary mass is $-1/\mu$ (= -5 for this example), whilst the motion of the main mass is completely damped.

As the forcing frequency increases beyond Ω_n, response once again becomes infinite at the second natural frequency and the phase angle between the response and the loading switches again by 180°.

So, in summary, it has been demonstrated that the addition of a small auxiliary mass to an un-damped SDOF system produces a 2-DOF system with natural frequencies "split" either side of the original SDOF natural frequency. Also, this 2-DOF system can provide effective reduction in resonant response but only at a particular constant forcing frequency (and a very narrow range either side).

In civil engineering applications, the un-damped vibration absorber is not a practical solution. Therefore, we must develop the concept to give a more robust solution.

4 Damped Two Degree of Freedom System – The "Classic" Vibration Absorber

Consider the system shown in Figure 4-1. Note that this is the same as the system shown in Figure 3-1 but with the addition of viscous damper between the main and auxiliary masses. Note that the main mass remains undamped.

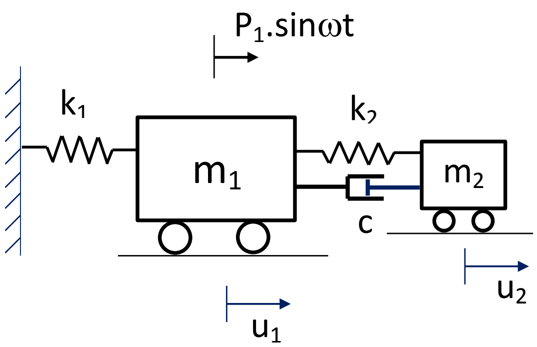
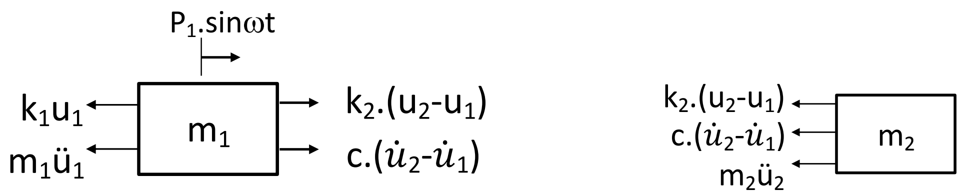


Figure 4-1: 2-DOF System – Damped Auxiliary Mass

Consider the equilibrium of each mass in turn:



Hence, the equations of equilibrium for the system can be written in matrix form as:

$$\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}.\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}+\begin{bmatrix} c & -c \\ -c & c \end{bmatrix}.\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix}+\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}.\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix}=\begin{bmatrix} P_1(t) \\ 0 \end{bmatrix} \quad \text{(Eqn. 4-1)}$$

It is reasonable to assume that vibration occurs at the same frequency as the forcing function. Thus a trial solution for the equations of equilibrium is:

$$u_1 = a_1 \sin \omega t + a_2 \cos \omega t$$
$$u_2 = a_3 \sin \omega t + a_4 \cos \omega t \quad \text{(Eqn. 4-2a \& b)}$$

Note that the introduction of viscous damping with terms proportional to $\dot{u}$ necessitate a solution with sine and cosine terms. The un-damped system described above only included terms in $\ddot{u}$ and u so the solution could be expressed in sine alone, as the sine function remained as such after double differentiation.

Alternatively (and in this case, more conveniently) the solution may be expressed in terms of complex numbers; so we can divide through by $e^{j\omega t}$ to convert from a differential to an algebraic equation:

$u_1 = x_1 . e^{j\omega t}$ and $u_2 = x_2 . e^{j\omega t}$ (Eqn. 4.3a & b)

The equations of equilibrium then become:

$$-m_1\omega^2 x_1 + k_1 x_1 + k_2(x_1 - x_2) + j\omega c(x_1 - x_2) = P_1$$
$$-m_2\omega^2 x_2 + k_2(x_2 - x_1) + j\omega c(x_2 - x_1) = 0 \quad \text{(Eqn. 4.4a \& b)}$$

Bringing together terms in x_1 and x_2 gives:

$$[-m_1\omega^2 + k_1 + k_2 + j\omega c]x_1 - [k_2 + j\omega c]x_2 = P_1$$
$$-[k_2 + j\omega c]x_1 + [-m_2\omega^2 + k_2 + j\omega c]x_2 = 0 \qquad \text{(Eqn. 4.5a \& b)}$$

This pair of simultaneous equations can be solved for x_1 and x_2. For example, by expressing the second equation in terms of x_1 and substituting for x_2 in the first equation gives:

$$x_1 = P_1 \frac{(k_2 - m_2\omega^2) + jc\omega}{(k_1 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2 m_2\omega^2 + j\omega c(k_1 - m_1\omega^2 - m_2\omega^2)} \qquad \text{(Eqn. 4-6)}$$

The physical meaning of expressing the motion in terms of complex numbers is that it consists of two components; one in phase with the applied loading (P_1) and another 90° ahead of it. The magnitude of the response is therefore the sum of these two vectors (i.e. the square root of the sum of their squares).

Equation 4-6 is currently of the form $x_1 = P_1 \frac{A+jB}{C+jD}$ but we want it in the form $x_1 = P_1(A_1 + jB_1)$ so we may calculate the magnitude of the response as $x_1 = P_1\sqrt{A_1^2 + B_1^2}$. The following transformation can be used:

$$x_1 = P_1 \frac{A + jB}{C + jD} = P_1 \frac{A + jB}{C + jD} . \frac{C - jD}{C - jD} = P_1 \frac{(AC + BD) + j(BC - AD)}{C^2 + D^2} \qquad \text{(Eqn. 4-7)}$$

Hence the length of the vector is:

$$\frac{x_1}{P_1} = \sqrt{\left(\frac{AC + BD}{C^2 + D^2}\right)^2 + \left(\frac{BC - AD}{C^2 + D^2}\right)^2} = \sqrt{\frac{A^2C^2 + B^2D^2 + B^2C^2 + A^2D^2}{(C^2 + D^2)^2}}$$
$$= \sqrt{\frac{(A^2 + B^2)(C^2 + D^2)}{(C^2 + D^2)^2}} = \sqrt{\frac{A^2 + B^2}{C^2 + D^2}} \qquad \text{(Eqn. 4-8)}$$

Applying this to equation 4-6 we obtain the following expression for the motion of the main mass:

$$\frac{u_1}{P_1} = \frac{x_1}{P_1} e^{j\omega t}$$
$$= \left\{\frac{(k_2 - m_2\omega^2)^2 + c^2\omega^2}{[(k_1 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2 m_2\omega^2]^2 + c^2\omega^2(k_1 - m_1\omega^2 - m_2\omega^2)^2}\right\}^{\frac{1}{2}} . e^{j\omega t} \qquad \text{(Eqn. 4-9)}$$

The method may also be applied to obtain the following expression for the motion of the auxiliary mass:

$$\frac{u_2}{P_1} = \frac{x_2}{P_1} e^{j\omega t}$$
$$= \left\{\frac{{k_2}^2 + c^2\omega^2}{[(k_1 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2 m_2\omega^2]^2 + c^2\omega^2(k_1 - m_1\omega^2 - m_2\omega^2)^2}\right\}^{\frac{1}{2}} . e^{j\omega t} \qquad \text{(Eqn. 4-10)}$$

We are primarily interested in the motion of the main mass. Therefore, we shall convert the equation for x_1/P_1 into a dimensionless form by defining the following terms:

$u_{st} = P_1/k_1$ The static deflection of the main mass under the action of P_1

$\omega_a^2 = k_2/m_2$ The natural frequency of the auxiliary system

$\Omega_n^2 = k_1/m_1$ The natural frequency of the main system

$\mu = m_2/m_1$ The mass ratio

$f = \omega_a/\Omega_n$ The ratio of the natural frequency of the auxiliary system to the main system

$g = \omega/\Omega_n$ The ratio of the forcing frequency to the natural frequency of the main system

$c_c = 2m_2\omega_a$ Critical damping for the auxiliary system

$\xi = c/c_c$ Critical damping ratio

Substitution into the equation for u_1/P_1 gives:

$$\frac{x_1}{u_{st}} = \sqrt{\frac{(2\xi g f)^2 + (g^2 - f^2)^2}{(2\xi g f)^2 (g^2 - 1 + \mu g^2)^2 + [\mu f^2 g^2 - (g^2 - 1)(g^2 - f^2)]^2}} \qquad \text{(Eqn. 4-11)}$$

Thus it can be seen that the motion of the main mass is a function of four essential variables: (i) the mass ratio, **μ**, (ii) the damping ratio, **ξ**, (iii) the ratio of the natural frequency of the auxiliary system to the main system, **f** and (iv) the ratio of the forcing frequency to the natural frequency of the main system, **g**.

Consider the case where $k_1/m_1 = k_2/m_2$ (i.e. **f**=1) and $\mu = 0.05$. x_1/u_{st} is plotted for a range of damping values (i.e. the limiting values of $\xi = 0$ and $\xi = \infty$, together with intermediate values of $\xi = 0.1$ and $\xi = 0.3$.

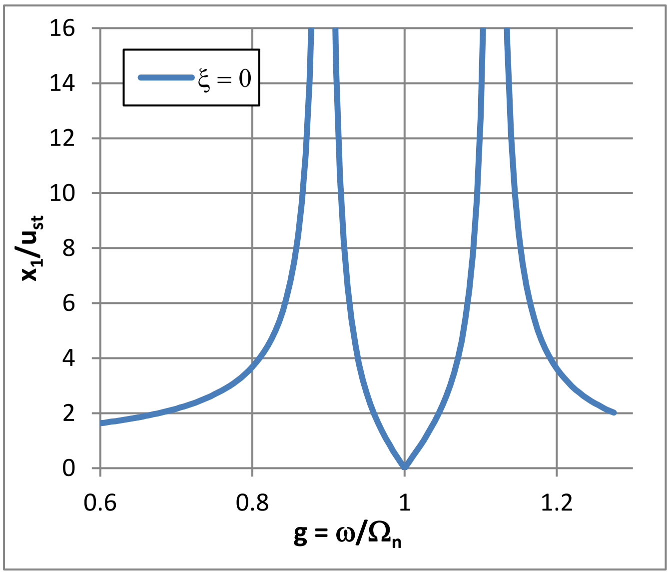


Figure 4-2: Response of Main Mass – zero damping

Figure 4-2 shows the limiting case where the system is un-damped. It can be seen that the solution collapses to that described in Section 2 of these lecture notes.

If the main mass is excited at its natural frequency (Ω_n), the system is perfectly damped. In other words, the main mass does not move and all the energy from the exciting force is dissipated in motion of the auxiliary mass.

If the frequency of excitation (ω) is different to Ω_n, the damping effect of the auxiliary mass is quickly lost. If excited at either of the natural frequencies of the two degree of freedom system, the response of the main mass becomes infinite.

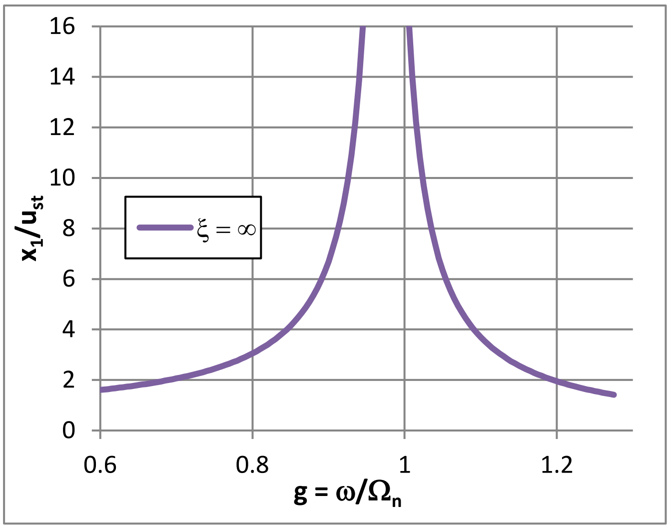


Figure 4-3: Response of Main Mass – infinite damping

Figure 4-3 shows the limiting case where the damping is infinite. In physical terms, this means that the main and auxiliary masses are now connected by a rigid link and so act as one.

We now have a single degree of freedom (SDOF) system with a mass of $m_1(1+\mu)$ and a stiffness of k_1. The natural frequency of this system is $\sqrt{(k_1/m_1(1+\mu))}$. For the chosen mass ratio of μ=0.05, this gives a natural frequency of 0.976Ω_n.

Thus the plot of x_1/u_{st} is the magnification factor for an un-damped SDOF system subject to forced harmonic motion presented in the Basic Dynamic lecture notes, but with the x-axis offset by the ratio of $1/(1+\mu)$.

Neither limiting case represents a practical solution; in both cases there is a possibility that the amplitude of the main mass can become infinite. Damping limits vibration by dissipation of energy – in other words, when the damping force does work (i.e. work = damping force x displacement). When ξ = 0, the damping force is zero and so no work in done. When $\xi = \infty$, there is no relative displacement between the two masses and again no work is done. In between these two limiting cases is a value of damping for which the product of the damping force and the displacement is a maximum, and the amplitude at resonance will be minimised.

The following two plots investigate the effect of introducing a finite damping value (c).

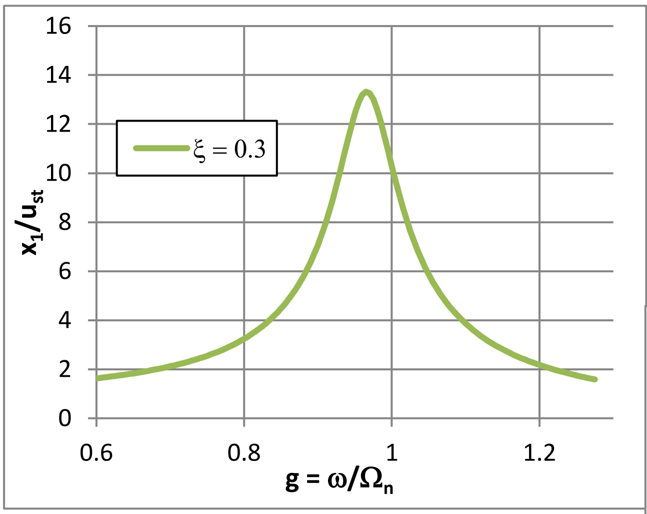


Figure 4-4: Damped Response of Main Mass

By setting ξ = 0.3 Figure 4-4 shows the performance of the system has been improved. There is no longer a forcing frequency at which the amplitude of the main mass becomes infinite. However, the system performs better for forcing frequencies above the natural frequency of the main system.

By setting a lower value of ξ = 0.1, Figure 4-5 shows the performance of the system is improved further, with significant reductions in response between $0.92 < g < 1.075$ and marginal penalty outside this range.

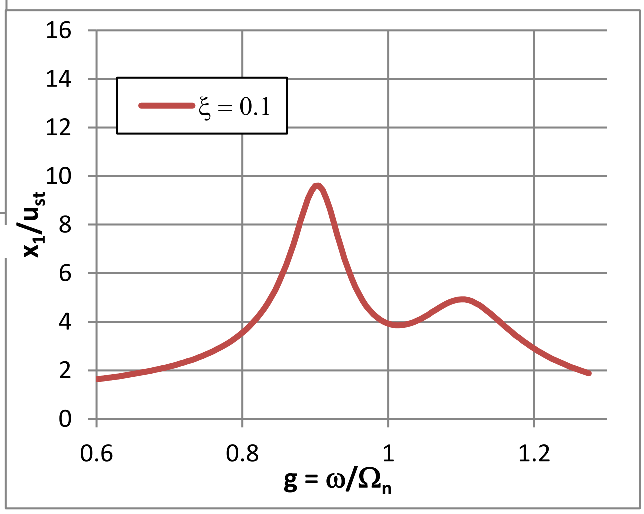


Figure 4-5: Damped Response of Main Mass

It is instructive to overlay Figure 4-2, Figure 4-3, Figure 4-5 and Figure 4-4:

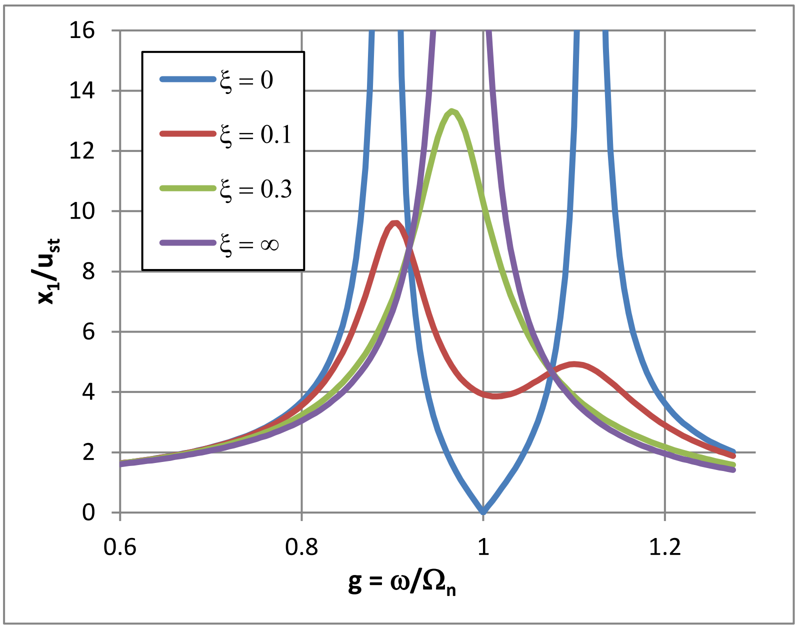


Figure 4-6: Response of main mass for a range of damping

It can be seen that there are two points on the combined plot through which all four curves pass. For the given mass ratio (μ) and the given frequency ratio (f), the response curve for the system for any value of damping (ξ) would pass through these two points.

We can postulate that the optimal solution is one where the intersection points are at the same height and the response curve has a horizontal tangent at one of the two intersection points.

Returning to Equation 4-11, we wish to establish whether there are any values of g for which the response u_1/u_{st} is independent of damping. We can see that Equation 4-11 is of the form:

$$\frac{x_1}{u_{st}} = \sqrt{\frac{A\xi^2 + B}{C\xi^2 + D}}$$ This is independent of damping if A/C = B/D (Eqn 4-12)

Or, if written out fully:

$$\left(\frac{1}{g^2 - 1 + \mu g^2}\right)^2 = \left(\frac{g^2 - f^2}{\mu f^2 g^2 - (g^2-1)(g^2-f^2)}\right)^2 \qquad \text{(Eqn. 4-13)}$$

We can cancel the square sign on both sides of the equation but then have to put a $\pm$ in front of one side of the equation. Putting a minus sign in front of the right hand side leads to the trivial results that at **g**=0 the amplitude is x_{st}.

Putting a plus sign before the right hand side, multiplying out and collecting terms gives:

$$\boldsymbol{g}^4 - 2\boldsymbol{g}^2\frac{1 + f^2 + \mu f^2}{2+\mu} + \frac{2f^2}{2+\mu} = 0 \qquad \text{(Eqn. 4-14)}$$

This is a quadratic equation in $\mathbf{g}^2$, the roots of which will give the locations of the two intersection points. If these two roots ($\mathbf{g}_1$ & $\mathbf{g}_2$) are substituted into Eqn. 4-11, the amplitudes of vibration at the intersection points can also be found. We have postulated that the most efficient vibration absorber is one where the amplitudes of vibration at the two intersection points are equal. It can be shown (by substituting $\mathbf{g}_1$ & $\mathbf{g}_2$ into Eqn. 4-11 and equating) that this occurs when:

$$f = \frac{1}{1+\mu} = \frac{natural\ frequency\ of\ auxiliary\ system}{natural\ frequency\ of\ main\ system} \qquad \text{(Eqn. 4-15)}$$

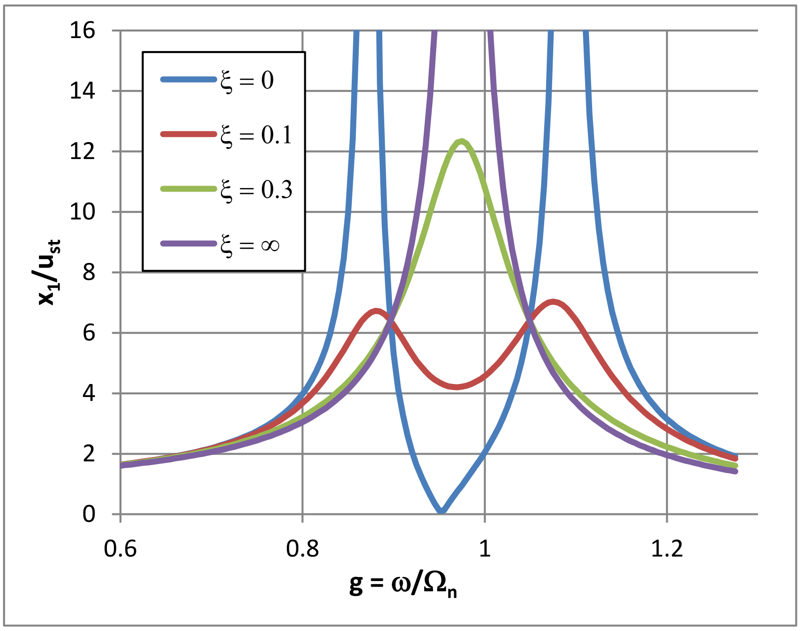


Figure 4-7: Response of Main Mass with 'tuned' auxiliary mass

Returning to the example presented above in Figure 4-6 where the mass ratio was set at μ = 0.05, the optimal ratio of auxiliary to main system frequencies is f = 1/(1+0.05) = 20/21 ≈ 0.952.

Re-creating Figure 4-6 and changing the frequency ratio of the auxiliary to main mass from **f** = 1 to **f** = 20/21 gives the desired effect of equating the amplitude of the response at two intersection points.

However, it can be seen that the performance of the system could be improved further by choosing an optimal value of the damping ratio (ξ).

This will occur when the response curve has a horizontal tangent at one of the two intersection points. In order to do this, we substitute the optimum value of frequency ratio given by Eqn. 4-15 into Eqn. 4-11 and differentiate with respect to **g**. We can set this equation equal to zero at either the left hand or the right hand intersection point, obtaining:

$$\xi = \sqrt{\frac{\mu\left(3-\sqrt{\mu/(\mu+2)}\right)}{8(1+\mu)^3}} \quad \text{and} \quad \xi = \sqrt{\frac{\mu\left(3+\sqrt{\mu/(\mu+2)}\right)}{8(1+\mu)^3}} \qquad \text{(Eqn. 4-16a \& b)}$$

For design purposes, it is convenient to use the average value:

$$\xi = \sqrt{\frac{3\mu}{8(1+\mu)^3}} \qquad \text{(Eqn. 4-17)}$$

So for the example considered above, the optimum value of damping is given by $\xi = \sqrt{3x0.05/(8(1+0.05)^3} = 0.127$

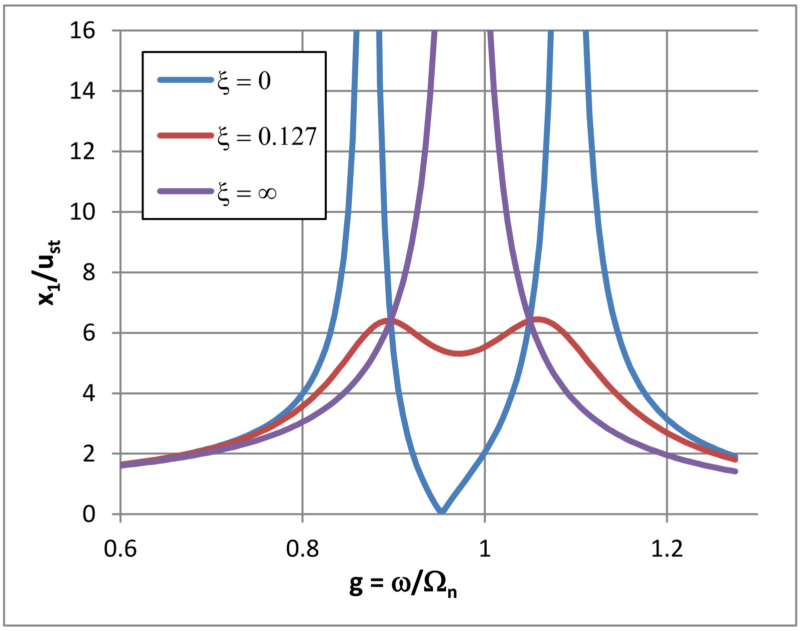


Figure 4-8: Response of Main Mass with 'tuned' auxiliary mass and optimised damping

The maximum response at either of the peaks can be calculated by substituting the appropriate root of Eqn. 4-14 into Eqn. 4-11. It can be shown that the maximum response is given by:

$$\frac{x_1}{u_{st}} = \sqrt{1+\frac{2}{\mu}}$$

It can also be seen that the damper is relatively "broad-band", reducing response amplitudes significantly between frequency ratios of 0.895 < **g** < 1.05 (i.e. the intersection points).

5 Relative motion of main and absorber masses

Now we shall establish the relative motion between the two masses. This is important for two primary reasons: (i) to calculate the force in spring k_2, and (ii) to determine how much space is required to allow mass m_2 to vibrate freely.

An exact solution is possible by returning to the original differential equations. However, a good approximate calculation can be made which should suffice for most practical applications. The approximation is based on the premise that at (or near) resonant response, the phase angle between the force and the motion is 90°.

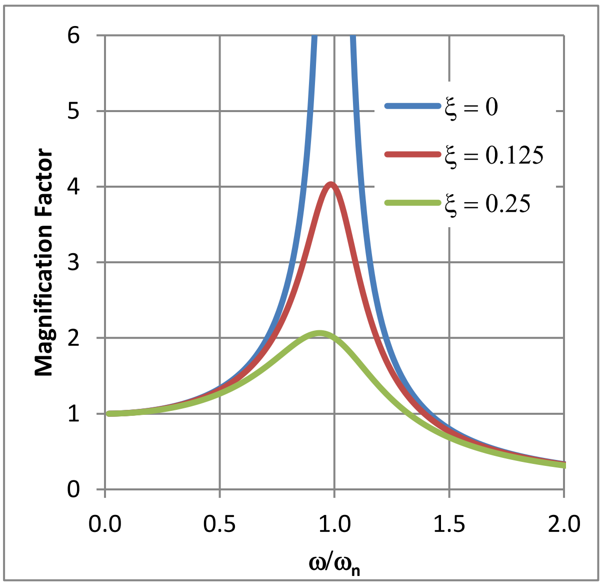
In the basic dynamics module, the harmonic excitation of a damped single degree of freedom system was introduced. The steady state dynamic magnification was derived as:

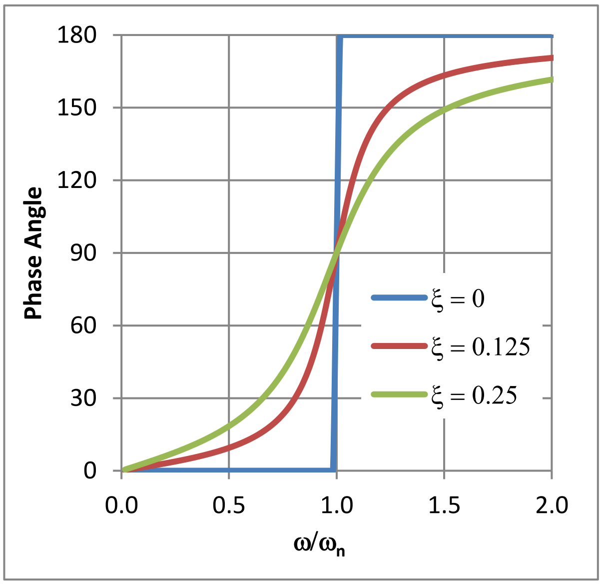
$$Magnification\ factor = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left(2\frac{c}{c_c}\frac{\omega}{\omega_n}\right)^2}} \qquad \text{(Eqn. 5-1)}$$

The phase angle between the load and the response is defined as:

$$\tan(\varphi) = \frac{2\frac{c}{c_c}\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2} \qquad \text{(Eqn. 5-2)}$$

By plotting these two expressions, it can be seen that the premise is true at or about resonance:

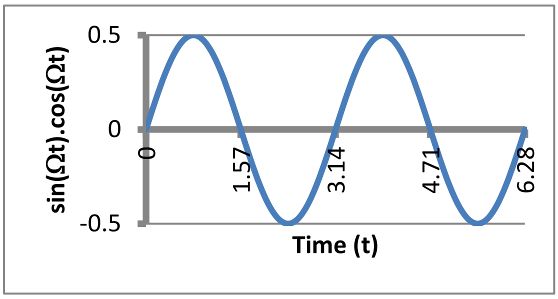




Now we shall calculate the work done by the load in one cycle of harmonic motion. Work done = force x distance moved, so the work done by a force *P* during a small displacement *dx* equals *P.dx*, which can be written as: $P.\frac{dx}{dt}.dt$. Let us define the force acting on the body as $P = P_1.sin(\omega t + \varphi)$ and the motion of the body as $x = x_1.sin\omega t$. During one cycle ωt varies between 0 and 2π; therefore, *t* varies between 0 and $2\pi/\omega$.

Therefore, the work done in one cycle is:

$$\int_0^{\frac{2\pi}{\omega}} P\frac{dx}{dt}dt = \frac{1}{\omega}\int_0^{2\pi} P\frac{dx}{dt}d(\omega t) = P_1x_1\int_0^{2\pi}\sin(\omega t+\varphi).\cos\omega t.d(\omega t)$$
$$= P_1x_1\int_0^{2\pi}\cos\omega t\,[\sin\omega t\,.\cos\varphi + \cos\omega t\,.\sin\varphi]\,d(\omega t) \qquad \text{(Eqn. 5-3)}$$
$$= P_1x_1 cos\,\varphi\int_0^{2\pi}\sin(\omega t).\cos(\omega t).d(\omega t) + P_1x_1 sin\,\varphi\int_0^{2\pi}\cos^2(\omega t).d(\omega t)$$



As the thumbnail plot shows, the first integral is zero over a full cycle (positive and negative areas cancel out). The second integral equals π.

Therefore, the work done per cycle is:

$$W = \pi P_1 x_1 sin\,\varphi \qquad \text{(Eqn. 5-4)}$$

However, we have already established that, at or close to resonance the phase angle (φ) between the force and the motion is 90°. Therefore, the work done per cycle at or close to resonance is: $W = \pi P_1 x_1$.

The work dissipated per cycle by the viscous damping is equal to the damping force times the relative amplitude. Using the same argument as developed for the work done by the load, the work dissipated equals π x damping force x relative amplitude (x_{rel}). Therefore:

$$W_{dissipated} = \pi.(c.x_{rel}.\omega).x_{rel} = \pi.c.\omega.x_{rel}^2 \qquad \text{(Eqn. 5-5)}$$

Equating the work done by the load to the work dissipated by the damper gives: $\pi P_1 x_1 = \pi.c.\omega.x_{rel}^2$ giving: $x_{rel}^2 = P_1x_1/c.\omega$ (Eqn. 5-6)

Written in non-dimensionless form, this becomes:

$$\left(\frac{x_{rel}}{u_{st}}\right)^2 = \frac{x_1}{u_{st}}.\frac{1}{2\mu g\xi} \qquad \text{(Eqn. 5-6)}$$

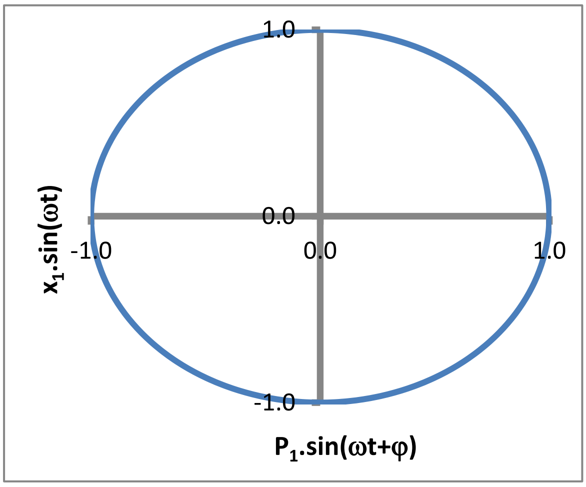


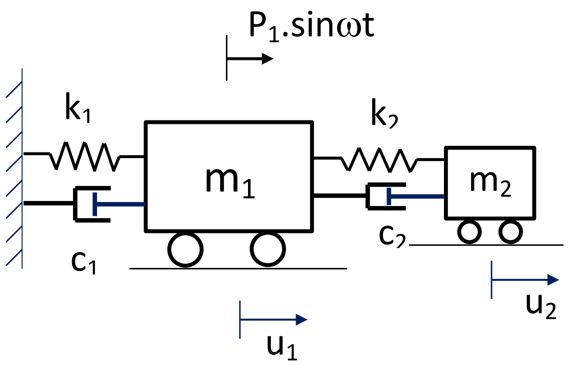
Figure 5-1

Alternatively, to calculate the work done, we can plot the force acting on the body against the motion of the body. This is illustrated in Figure 5-1 for φ = 90°. It can be seen that the plot is an ellipse. The area inside the plot equals the work done and the area of an ellipse is π times the length of the semi-major and semi minor axes. In other words, the area of the ellipse = $\pi.P_1.x_1$.

At other values of phase angle, φ, the axes of the ellipse are rotated relative to the Cartesian axes. It can be shown by geometry that, for the general case, the area of the ellipse = $\pi P_1 x_1 sin\,\varphi$.

6 Damped Main System and Damped Auxiliary System

In practice, real structures contain some damping. Although if a tuned mass damper is to be fitted, it is likely that the real structure will only be lightly damped.



We extend the 2-DOF system used previously by adding a dashpot (c_1) between the main mass (m_1) and the ground.

Figure 6-1: 2-DOF System – Damped main and auxiliary masses

The equations of equilibrium become:

$$\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} . \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} . \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} + \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} . \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} = \begin{bmatrix} P_1(t) \\ 0 \end{bmatrix} \quad \text{(Eqn. 6-1)}$$

Applying the same method as used for the un-damped main system and damped auxiliary system gives:

$$u_1 = \frac{(k_2 - m_2\omega^2 + j\omega c_2)}{[k_2 + k_1 - m_1\omega^2 + j\omega(c_1+c_2)].(k_2 - m_2\omega^2 + j\omega c_2) - (k_2 + j\omega c_2)^2} . P_1 . e^{j\omega t} \quad \text{(Eqn. 6-2)}$$

Introducing the following non-dimensional parameters:

$u_{st} = P_1/k_1$	The static deflection of the main mass under the action of P_1
$\omega_a^2 = k_2/m_2$	The natural frequency of the auxiliary system
$\Omega_n^2 = k_1/m_1$	The natural frequency of the main system
$\mu = m_2/m_1$	The mass ratio
$f = \omega_a/\Omega_n$	The ratio of the natural frequency of the auxiliary system to the main system
$g = \omega/\Omega_n$	The ratio of the forcing frequency to the natural frequency of the main system
$c_{c1} = 2m_1\Omega_n$	Critical damping for the main system
$c_{c2} = 2m_2\omega_a$	Critical damping for the auxiliary system
$\xi_1 = c_1/c_{c1}$	Critical damping ratio for the main system
$\xi_2 = c_2/c_{c2}$	Critical damping ratio for the auxiliary system

So we may define:

$$\frac{x_1}{u_{st}} = \sqrt{\frac{A^2 + B^2}{C^2 + D^2}} \qquad \begin{aligned} &A = f^2 - g^2 \qquad\qquad B = 2\xi_2 gf \\ &C = f^2(1-g^2) - \mu f^2 g^2 - g^2(1-g^2) - 4\xi_1\xi_2 f g^2 \\ &D = 2\xi_2 gf(1 - g^2 - \mu g^2) + 2\xi_1 g(f^2 - g^2) \end{aligned} \qquad \text{(Eqn. 6-3)}$$

If we set $\xi_1 = 0$ we obtain Equation 4-11 (the "Classic" solution). As previously noted, in this case we obtain two points on the plot of forcing frequency ratio against response that are invariant (i.e. independent of damping). This is because A and C are independent of ξ_2 and B and D are proportional to ξ_2. Therefore, we obtain invariant points when A/C = B/D and can use these to obtain a closed form optimum solution. However, this is not the case if ξ_1 has a finite value; in such circumstances, the optimum solution needs to be determined iteratively.

Consider an example where the mass ratio, μ = 0.10, the tuning ratio is optimised for an the "Classic" solution so f = 1/(1+μ) = 0.909 and the critical damping ratio for the auxiliary system is set to ξ_2 = 0.185. The response of the main mass is presented in the following figure for different values of structural damping:

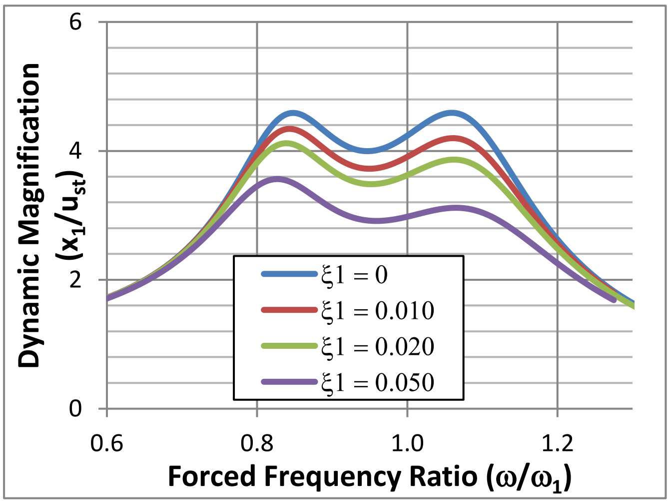


Figure 6-2: The effect of structural damping on response of the main mass

It can be seen in Figure 6-2 that the addition of main system damping (ξ_1) leads to an imbalance in the peaks of the response curve, implying the chosen values of tuning ratio (f) and auxiliary system damping (ξ_2) are no longer optimal. However, at low levels of main system damping, the effect is marginal.

Figure 6-3 presents the same example, but with the values of tuning ratio (f) and auxiliary system damping (ξ_2) optimised to give balanced peak responses and a horizontal tangent at the peaks.

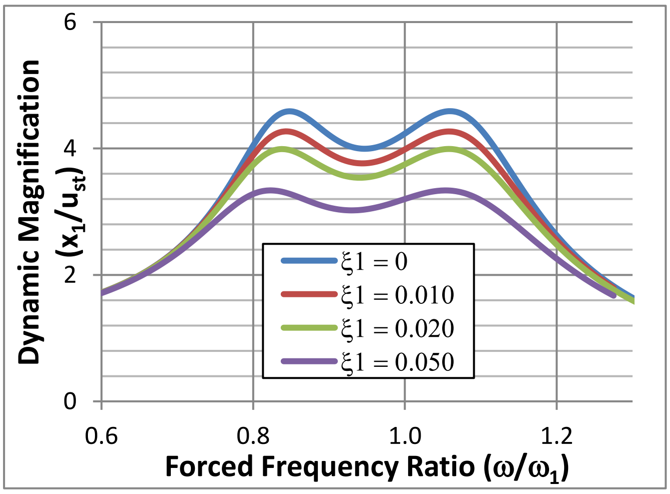


Figure 6-3: Response of the main mass for an optimised system

For comparison, the values of f and ξ_2 used in generating Figure 6-2 and Figure 6-3 are tabulated below, together with the dynamic magnification factor (R) at the two peaks.

Note that when an absorber is not present ($m_2 = \xi_2 = 0$), the maximum dynamic magnification factor (at g = 1) is $1/2\xi_1$. Therefore, in order to assess the effect of adding an absorber to a damped system, the values of dynamic magnification obtained should be compared to this ratio.

		Optimum Values			Value of frequency ratio (g) at the two peaks	
Mass Ratio (μ)	**Main System Damping (ξ_1)**	**R**	**f**	**ξ_2**	**LHS**	**RHS**
0.10	0	4.583	0.9091	0.185	0.848	1.059
	0.01	4.270	0.9051	0.187	0.843	1.058
	0.02	3.991	0.9009	0.188	0.838	1.058
	0.05	3.337	0.8875	0.193	0.823	1.054

The following conclusions can be drawn from the figures in the table:

- The optimum value of the tuning ratio (**f**) reduces with increasing damping of the main system, but the change is small.
- The optimum value of the absorber damping ratio (ξ_2) increases slightly with increasing damping of the main system.
- The effect of variation in main system damping on the value of the frequency ratio at the two peaks in the response curve is very small.

Thus the effect of a low level of structural damping (ξ_1) on the design of a TMD tuned on the assumption that ξ_1 = 0 is small. In practice, the level of damping in a structure (i.e. main system) is often unknown and so it is necessary to ensure that the design of the TMD gives the required level of performance over the credible range of values of ξ_1.

7 Generalised force, mass and damping

The 2-DOF models used in Sections 3, 4 and 6 may be used to represent real structures through the concept of generalised force, mass and damping. Provided modes of vibration are uncoupled and their natural frequencies are well separated:

$$P_{1,j} = \int \phi_j(s).P_1(s).ds \quad \text{(Eqn. 7-1)}$$

$$m_{i,j} = \int \phi_j^2(s).m_i(s).ds \quad \text{(Eqn. 7-2)}$$

$$c_{i,j} = \int \phi_j^2(s).c_i(s).ds \quad \text{(Eqn. 7-3)}$$

Where:

i = 1 for the main (structure) mass and i = 2 for the auxiliary (damper) mass

j represents the number of mode under consideration

$\phi_j(s)$ represents the normalised mode shape of mode j

… and the integrals are calculated over the whole structure (ds).

8 References and Further Reading

[1] J.P. Den Hartog – Mechanical Vibrations – McGraw-Hill

[2] Warburton, G.B. and Ayorinde, E.O., "Optimum absorber parameters for simple systems". Earthquake Engineering and Structural Dynamics, 8, pp.197-217, 1980,

[3] Warburton GB. Optimum absorber parameters for various combinations of response and excitation parameters. Earthquake Engineering and Structural Dynamics 1982; 10:381–401,

[4] T.T. Soong and G.F. Dargush, 'Passive Energy Dissipation Systems in Structural Engineering', John Wiley & Sons (ISBN 0-471-96821-8)