flint neill	Client / Project	Job No.	Sub. Ref.	No. 1/2
SUBJECT: Frequency response from state-space eqns.		Date: 03/11/16	Prep. by: RIHY	Check. by

FREQUENCY RESPONSE (FROM STATE-SPACE EQNS)

State-space equations:

$$\begin{bmatrix} \underline{\dot{y}} \\ \underline{\ddot{y}} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \begin{bmatrix} \underline{y} \\ \underline{\dot{y}} \end{bmatrix} + \begin{bmatrix} 0 \\ -M^{-1} \end{bmatrix} \left[\underline{F}(t) \right]$$

ie
$$\underbrace{\underline{\dot{x}}(t)}_{(2N\times1)} = \underbrace{[A_c]}_{(2N\times2N)} \underbrace{\underline{x}(t)}_{(2N\times1)} + \underbrace{[B]}_{(2N\times N)} \underbrace{\underline{F}(t)}_{(N\times1)}$$

Also:
$$\underbrace{\underline{y_o}(t)}_{} = \underbrace{[C]}_{(N_o\times2N)} \underbrace{\underline{x}(t)}_{(2N\times1)} + \underbrace{[D]}_{(N_o\times N)} \underbrace{\underline{F}(t)}_{(N\times1)}$$

Note $\neq \underline{y}$. This is the output vector $(N_o\times1)$

These are "state-space equations" that define the problem

Frequency response is defined as follows:

$$\underline{y}_o(s) = \left[G(s)\right] \underline{F}(s)$$

Laplace $s \Leftrightarrow j\omega$ Fourier

$\underline{y}_o(s)$: Laplace (or Fourier) transform of $\underline{y_o}(t)$

$\underline{F}(s)$: Laplace (or Fourier) transform of $\underline{F}(t)$

Dimensions: $(N_o\times1) = (N_o\times N).(N\times1)$.

ie $[G(s)]$ is a $(N_o\times N)$ matrix, mapping inputs (force) to outputs.

Properties of Laplace transform:

$$\mathcal{L}(\dot{\underline{y}}(t)) = s\,\underline{y}(s)$$

$$\mathcal{L}(\ddot{\underline{y}}(t)) = s\,\dot{\underline{y}}(s) \qquad (= s^2\underline{y}(s))$$

Laplace transform of state equations:

$$\left.\begin{aligned}\mathcal{L}(\dot{\underline{x}}(t)) &= [A_c]\,\underline{x}(s) + [B]\,\underline{F}(s)\\ \mathcal{L}(\underline{y}_o(t)) &= [C]\,\underline{x}(s) + [D]\,\underline{F}(s)\end{aligned}\right\}$$

$$\left.\begin{aligned}s\,\underline{x}(s) &= [A_c]\,\underline{x}(s) + [B]\,\underline{F}(s)\\ \underline{y}_o(s) &= [C]\,\underline{x}(s) + [D]\,\underline{F}(s)\end{aligned}\right\}$$

Identity matrix

$$\Rightarrow \left(s[I] - [A_c]\right)\underline{x}(s) = [B]\,\underline{F}(s)$$

$$\Rightarrow \underline{x}(s) = \left(s[I]-[A_c]\right)^{-1}[B]\,\underline{F}(s)$$

$$\Rightarrow \underline{y}_o(s) = \left([C]\left(s[I]-[A_c]\right)^{-1}[B] + [D]\right)\underline{F}(s)$$

$$\Rightarrow \boxed{[G(s)] = [C]\left(s[I]-[A_c]\right)^{-1}[B] + [D]}$$

Defines $[G(s)]$ in terms of state-space matrices A, B, C, D.

Let $s = j\omega$; this gives frequency response $[G(\omega)]$ (which is complex)

$\sqrt{-1}$

$$\boxed{[G(\omega)] = [C]\left(j\omega[I]-[A_c]\right)^{-1}[B] + [D]}$$

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Frequency response – system with constraints

Let
$$\left.\begin{aligned} M' &= Z^T M Z \\ K' &= Z^T K Z \\ C' &= Z^T C Z \end{aligned}\right\} \quad \text{where } Z = \text{Null}(J).$$

$$\Rightarrow \quad A' = \begin{bmatrix} 0 & I \\ -M'^{-1}K' & -M'^{-1}C' \end{bmatrix}$$ is state matrix of unconstrained problem.

It can be shown that

$$\dot{\underline{x}}' = A'\underline{x}' + B'\underline{f}$$

$$\mathcal{L}: \quad s\underline{x}'(s) = A'\underline{x}'(s) + B'\underline{f}(s)$$

$$\Rightarrow \quad (sI - A')\underline{x}'(s) = B'\underline{f}(s)$$

$$\Rightarrow \quad \underline{x}'(s) = [sI - A']^{-1} B' \underline{f}(s)$$

Recall $\underline{x} = Z_2 \underline{x}'$ where $Z_2 = \begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix}$

$$\Rightarrow \quad \underline{x}(s) = Z_2 \underline{x}'(s)$$

$$\Rightarrow \quad \underline{x}(s) = \underbrace{\left[Z_2 [sI - A']^{-1} B' \right]}_{G(s)} \underline{f}(s).$$

Transfer matrix mapping applied loads to state variables