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Study on dynamic characteristics of the cubic nonlinear economic system with time-varying parameters

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In this work, the homotopy-perturbation method (HPM) proposed by J.-H. He, one of the most effective methods, is implemented to find approximate solutions of a cubic nonlinear economic system with time-varying parameters. Comparisons are made between the results of the proposed method and the numerical solutions. It illustrates the validity and the great potential of the HPM in solving nonlinear partial differential equations. HPM is also suitable for large parameter problems.

KEYWORDS
cubic nonlinear economic system, homotopy-perturbation method, numerical solution

MSC CLASSIFICATION
35J25; 35 q35

1 | INTRODUCTION

The early models of economic fluctuation are mostly linear econometrics models, and the methods used are mainly statistical methods. However, with the in-depth study of economic fluctuation, it is found that there is a complex nonlinear relationship between the variables of the economic model. The interference of various random factors inside and outside the model further aggravates the complexity of the model, which makes the previous linear model research methods unable to explain the nonlinear phenomenon in the economic model. In the 1930s, The French physicist Le Corbeiller[1] put forward the vibration theory of nonlinear mechanics to study the economic cycle problem. In the following 20 years, the research on nonlinear economic cycle models was flourishing, mainly including Kaldor[2] and Kalecki models.[3] The mechanical structure of the model proposed by Kaldor is similar to that of the Van der Pol equation. Its main feature is that the investment function and savings function in the model are in nonlinear form, which results in limit cycles in the evolution analysis results of the model. And the Kalecki model further sets the time-delayed investment function on the basis of the Kaldor model, which leads to the fluctuation of the economic cycle model. In the 1950s, Goodwin[4] combined a time-delay induced investment and equilibrium national income determination model to improve Hicks consumption function and established a nonlinear 'multiplication-acceleration number' economic cycle model for the first time. Then, he studied the limit cycle of the economic cycle model with the kinetic method when the spontaneous investment and consumption are zero. In recent years, based on the study of Kaldor and Kalecki models, scholars such as Agliari Brock and Hommes, Lorenz, Bischi and Kopel, Puu, and Anna Aglian have studied and improved the economic cycle model from different angles.[5–9] Szydłowski et al[10] discussed the stability of the Kaldor–Kalecki economic cycle model with time delay and analyzed the conditions for the occurrence of Hopf biforking. Agliari[6] studied the global bifurcation and the attractor of the nonlinear discrete business cycle model. Bäurle and Burren[11] studied the boundary value problem of economic cycle models. Yoshida and Asada[12] used the kinetic method to study the Keynesian Goodwin economic model under policy lag. Hallegatte et al[13] studied the chaotic phenomenon of the economic cycle model with distributed tax lag.

Economists have found that the economic system is composed of complex and uncertain nonlinear elements interacting with each other and the operation of economic system is often accompanied by the time-varying characteristics of system parameters, which means a smoothly run economy is a rare ideal. The dynamic equation of a continuous nonlinear economic system is a nonlinear differential equation, and most of the nonlinear differential equations have no exact analytic solutions, so we can only use some approximate methods to solve them. Classical perturbation methods such as Lindstedt-Poincare (L-P)[14], multiscale,[15] average, [16] and Krylov–Bogoliubov–Mitropolsky methods[17] are effective for small-parameter nonlinear dynamic equations but difficult to apply in large-parameter nonlinear dynamic equations. For differential equations with large parameters, the approximate analytic methods include the modified Linstedt–Poincare method,[18] harmonic balance method,[19] and so on. According to homotopy in topology, He Jihuan proposed the homotopy-perturbation method (HPM)[20–25] to solve nonlinear problems. The validity of the homotopy perturbation method is independent of whether small parameters exist in the differential equation, and the homotopy perturbation method has been successfully applied to solve many nonlinear problems.

In this paper, based on the Samuelson–Hicks economic model established in the literature,[26] the parameters of the model are changed into time-varying parameters, and HPM is used to solve the approximate analytic solution of the newly built model.

Figure 1 shows the improved Samuelson–Hicks economic model with time-varying parameters, and its dynamic equation can be established according to the nonlinear vibration theory, with specific equation as follows:

$$m(t)x''(t)+\left[\frac{\mathrm{d}m(t)}{\mathrm{d}t}+c\right]x'(t)+\left(k+3k_1\delta_{st}^2\right)x(t)+k_1x^3(t)-3k_1\delta_{st}x^2(t)=\frac{\mathrm{d}m(t)}{\mathrm{d}t}u_0+m(t)g-m_0g, \tag{1}$$

where $x(t)$ is the income, "′" is the differential with respect to time t, k is the marginal rate of consumption, k_1 is the investment multiplier, c is the marginal savings rate, δ_{st} is the initial consumption value, and m_0 is the total amount of the system at the initial state. The initial consumption value can be solved by the following formula:

$$m_0g=k\delta_{st}+k_1\delta_{st}^3. \tag{2}$$

The initial conditions of the system are

$$x(0)=s_0,\ x'(0)=v_0. \tag{3}$$

For the convenience of discussion, it is assumed that the total amount of the system in Equation 1 changes linearly with time t, namely,

$$m(t)=m_0(1+\alpha t), \tag{4}$$

where m_0 and α are real numbers.

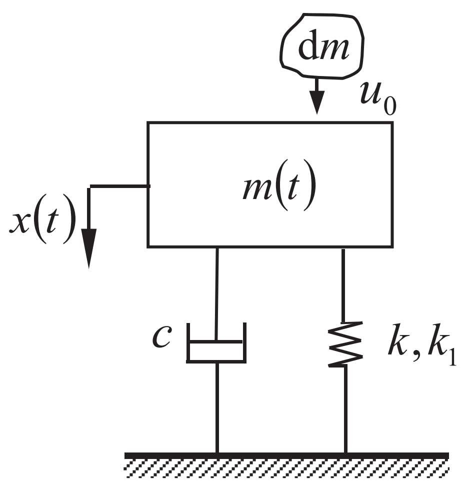


FIGURE 1 The improved Samuelson–Hicks economic model with time-varying parameters

2 | SOLVING PROCEDURE

The solving process of the HPM method can be carried out in accordance with the steps in He[20]. By embedding a parameter p that changes in [0,1], the variable $x(t)$ is transformed into $X(t,p)$, so that Equation 1 becomes

$$\begin{aligned} H(X,p) = (1-p)\left\{m(t)X''(t,p) + \left[\frac{\mathrm{d}m(t)}{\mathrm{d}t} + c\right]X'(t,p) + kX(t,p) - m(t)x_0''(t) - \left[\frac{\mathrm{d}m(t)}{\mathrm{d}t} + c\right]x_0'(t) - kx_0(t)\right\} \\ + p\left\{m(t)X''(t,p) + \left[\frac{\mathrm{d}m(t)}{\mathrm{d}t} + c\right]X'(t,p) + k_1X^3(t,p) + \left(k + 3k_1\delta_{st}^2\right)X(t,p) - 3k_1\delta_{st}X^2(t,p) - \frac{\mathrm{d}m(t)}{\mathrm{d}t}u_0 - m(t)g + m_0g\right\} = 0. \end{aligned} \tag{5}$$

The initial condition of the system is rewritten as

$$X(0,P) = s_0,\ X'(0,P) = v_0, \tag{6}$$

where p is called the homotopy parameter and $x_0(t)$ is the solution to the linear homogeneous equation of Equation 1, and it satisfies the initial conditions:

$$m(t)x_0''(t) + [\mathrm{d}m(t)/\mathrm{d}t + c]x_0'(t) + kx_0(t) = 0, \tag{7}$$

$$x_0(0) = s_0,\ x_0'(0) = v_0. \tag{8}$$

According to the HPM method, Equation 5 shows that when p changes from 0 to 1, the system maps from the initial approximate solution $x_0(t)$ to the exact solution $X(t,p)$. Transform the solution of Equation 5 into the following series form:

$$X(t,p) = x_0(t) + px_1(t) + p^2x_2(t) + \cdots. \tag{9}$$

When $p = 1$, the approximate solution of Equation 1 can be expressed as the extreme value of $X(t,p)$, namely,

$$x(t) = \lim_{p\to 1} X(t,p) = x_0(t) + x_1(t) + x_2(t) + \cdots. \tag{10}$$

Substituting Equation 9 into Equation 5 and comparing the coefficients of the same power of p, we can get the equations of the coefficients of each power of p:

P^1:

$$\begin{aligned} m(t)x_1''(t) + [\mathrm{d}m(t)/\mathrm{d}t + c]x_1'(t) + kx_1(t) + m(t)x_0''(t) + [\mathrm{d}m(t)/\mathrm{d}t + c]x_0'(t) \\ + k_1x_0^3(t) + \left(k + 3k_1\delta_{st}^2\right)x_0(t) - 3k_1\delta_{st}x_0^2(t) - \mathrm{d}m(t)/\mathrm{d}tu_0 - m(t)g + m_0g = 0, \end{aligned} \tag{11}$$

P^2:

$$m(t)x_2''(t) + [\mathrm{d}m(t)/\mathrm{d}t + c]x_2'(t) + kx_2(t) + 3k_1\delta_{st}^2x_1(t) + 3k_1x_0^2(t)x_1(t) - 6k_1\delta_{st}x_0(t)x_1(t) = 0. \tag{12}$$

P^3:

After solving the initial approximate solution $x_0(t)$ in Equation 7, substitute it into Equation 11 to solve the first-order approximate solution $x_1(t)$. In the same way, we can get higher order approximations $x_2(t)$, $x_3(t)$, and so on.

3 | VARIOUS ORDER APPROXIMATE SOLUTIONS

Equation 7 represents a linear Samuelson–Hicks economic model with time-varying parameters. The literature[27] has studied the approximate solution process of such equations in detail. According to the literature,[27] the solution $x_0(t)$ of Equation 7 can be set as

$$x_0(t) = A(t)\cos[q(t)], \tag{13}$$

where

$$A(t) = A_0 m(t)^{-1/4} e^{-\frac{c}{2}\int \frac{dt}{m(t)}}. \tag{14}$$

A_0 is a real number determined by the initial conditions.

Equation 4 is substituted into Equation 14, and it can be obtained as

$$A(t) = A_0 {m_0}^{-1/4}(1+\alpha t)^{-\frac{1}{4}-\frac{c}{2m_0\alpha}}. \tag{15}$$

Substituting Equations 13 and 15 into Equation 7, we can obtain

$$[\mathrm{d}q(t)/\mathrm{d}t]^2 = a/(1+\alpha t) + b/(1+\alpha t)^2, \tag{16}$$

where $a = k/m_0$, $b = \alpha^2/16 - c^2/\left(4m_0^2\right)$, and the solution of Equation 16 is

$$q(t) = \begin{cases} \dfrac{2}{\alpha}\sqrt{a(1+\alpha t)+b} + \dfrac{1}{\alpha\sqrt{b}}\ln\left[\dfrac{\sqrt{a(1+\alpha t)+b}-\sqrt{b}}{\sqrt{a(1+\alpha t)+b}+\sqrt{b}}\right] + \theta, & \text{when } \mathrm{b} \geq 0 \\ \dfrac{2}{\alpha}\sqrt{a(1+\alpha t)+b} + \dfrac{2}{\alpha\sqrt{-b}}\arctan\sqrt{\dfrac{a(1+\alpha t)+b}{-b}} + \theta, & \text{when } \mathrm{b} \geq 0 \end{cases}, \tag{17}$$

where θ is a real number determined by the initial conditions and A_0 and θ can be solved according to the initial conditions:

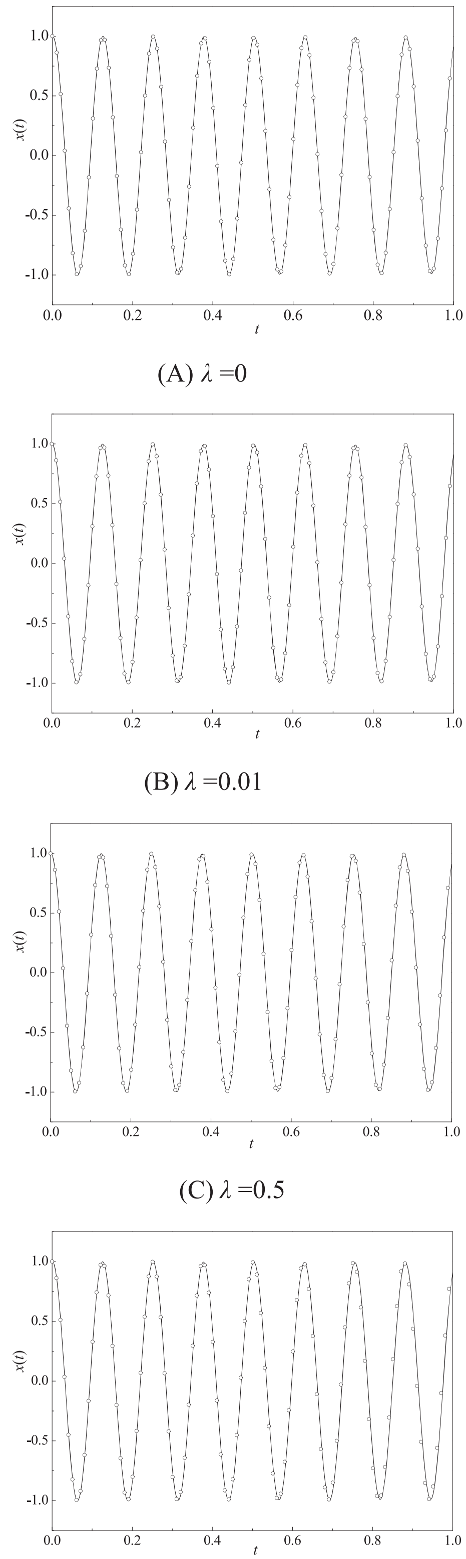
$$\begin{cases} A_0^2 = \dfrac{{m_0}^{1/2}}{a+b}\left[\dfrac{m_0\alpha + 2c}{4m_0}s_0 + v_0\right]^2 + {m_0}^{1/2}s_0^2 \\ \theta = \arccos\left(\dfrac{s_0 m_0^{1/4}}{A_0}\right) - q_0 \end{cases}, \tag{18}$$

where

$$q_0 = \begin{cases} \dfrac{2}{\alpha}\sqrt{a+b} + \dfrac{1}{\alpha\sqrt{b}}\ln\left[\dfrac{\sqrt{a+b}-\sqrt{b}}{\sqrt{a+b}+\sqrt{b}}\right], & \text{when } b \geq 0 \\ \dfrac{2}{\alpha}\sqrt{a+b} + \dfrac{2}{\alpha\sqrt{-b}}\arctan\sqrt{\dfrac{a+b}{-b}}, & \text{when } b \geq 0 \end{cases}. \tag{19}$$

The initial approximate solution $x_0(t)$ of the linear Samuelson–Hicks system with time-varying parameters is

$$x_0(t) = A_0 {m_0}^{-1/4}(1+\alpha t)^{-\frac{1}{4}-\frac{c}{2m_0\alpha}}\cos[q(t)]. \tag{20}$$



(A) λ =0

(B) λ =0.01

(C) λ =0.5

(D) λ =1.0

FIGURE 2 Comparison of the first-order approximate solution and numerical solution of the homotopy-perturbation method (HPM) method under different lambda values (solid line: HPM result; filled circles: numerical solution)

Substituting Equation 20 into Equation 11, we can obtain the first-order approximate analytic solution $x_1(t)$ of the system:

$$
\begin{aligned}
x_1(t) \approx & \frac{-k_1 A^3(t)}{4\sqrt{[8k+9m_0 b/(1+\alpha t)]^2+9(m_0\alpha+c)^2\left[a/(1+\alpha t)+b/(1+\alpha t)^2\right]}}\cos[3q(t)+\theta_1] \\
& + \frac{3k_1\delta_{st}^2 A^2(t)}{2\sqrt{[3k+4m_0 b/(1+\alpha t)]^2+4(m_0\alpha+c)^2\left[a/(1+\alpha t)+b/(1+\alpha t)^2\right]}}\cos[2q(t)+\theta_2] \\
& - \frac{3k_1 A(t)\left[\delta_{st}^2+A^2(t)/4\right]}{\sqrt{[m_0 b/(1+\alpha t)]^2+(m_0\alpha+c)^2\left[a/(1+\alpha t)+b/(1+\alpha t)^2\right]}}\cos[q(t)+\theta_3] \\
& + \frac{3k_1\delta_{st}A^2(t)}{2k}+\frac{m_0\alpha u_0}{k}+\frac{m_0\alpha g}{k}\left[t-\frac{m_0\alpha+c}{k}\right],
\end{aligned}
\tag{21}
$$

where

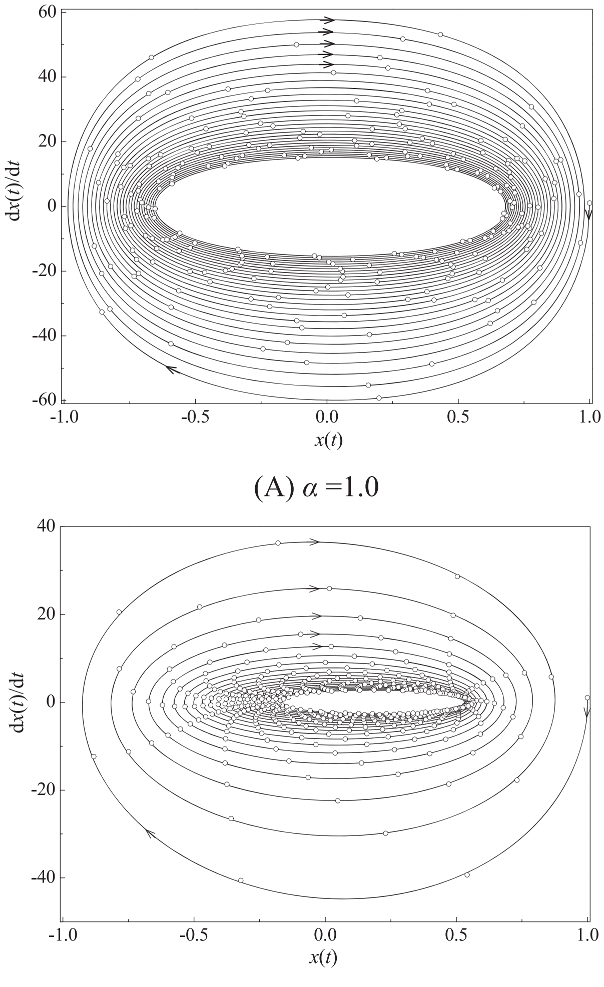


FIGURE 3 Comparison of the first-order homotopy-perturbation method (HPM) approximation of phase trajectory with the numerical solution by various α. (solid line: HPM result; circles: numerical solution)

$$\tan\theta_1 = \frac{3(m_0\alpha + c)\sqrt{a/(1+\alpha t) + b/(1+\alpha t)^2}}{8k + 9m_0 b/(1+\alpha t)}$$

$$\tan\theta_2 = \frac{2(m_0\alpha + c)\sqrt{a/(1+\alpha t) + b/(1+\alpha t)^2}}{3k + 4m_0 b/(1+\alpha t)}$$

$$\tan\theta_3 = \frac{(m_0\alpha + c)\sqrt{a/(1+\alpha t) + b/(1+\alpha t)^2}}{m_0 b/(1+\alpha t)}.$$

Substituting Equations 20 and 21 into Equation 12, we can obtain the second-order approximate solution of the system. In this paper, we only solve the first approximation solution.

4 | EXAMPLES

To illustrate the effectiveness of the above method, we consider the situation of $m_0 = 1$, $\alpha = 0.01$, $c = 0.01$, $k = 2{,}500$, $u_0 = 1.0$, $s_0 = 1.0$, $v_0 = 1.0$, and $g = 9.8$, assuming $\lambda = k_1/k$. The first-order approximate solution obtained by the above method is in good agreement with the numerical solution obtained by the Runge–Kutta method, as shown in Figure 2.

Figure 3 shows the phase diagram of the corresponding first-order approximate solution under different α, and the solution with loops is the numerical solution.

5 | CONCLUSIONS

In this paper, we apply the HPM method to obtain approximate analytic solutions of cubic nonlinear economic systems with time-varying parameters and compare them with the numerical solutions. We clearly see the effectiveness of the HPM method for analytic solutions of broader partial differential equations.

CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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