Time Series HomeWork (7)

钟瑜 222018314210044

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1. 对 AR(2) 模型 $X_t = -0.1X_{t-1} + 0.72X_{t-2} + \epsilon_t$, 计算自相关系数 $\rho_k, k = 1, 2, 3, 4, 5.$

解. 由

$$\rho_k = \sum_{i=1}^{p} a_i \rho_{k-i},\ \rho_0 = 1 \tag{1}$$

可得

$$\begin{aligned} \rho_1 &= a_1\rho_0 + a_2\rho_1 \\ \rho_2 &= a_1\rho_1 + a_2\rho_0 \\ \rho_3 &= a_1\rho_2 + a_2\rho_1 \\ \rho_4 &= a_1\rho_3 + a_2\rho_2 \\ \rho_5 &= a_1\rho_4 + a_2\rho_3 \end{aligned} \tag{2}$$

故

$$\begin{aligned} \rho_1 &= -0.3571 \\ \rho_2 &= 0.7557 \\ \rho_3 &= -0.3327 \\ \rho_4 &= 0.5774 \\ \rho_5 &= -0.2972 \end{aligned} \tag{3}$$

2. 对可逆 MA(2) 序列

$$X_t = \epsilon_t + b_1\epsilon_{t-1} + b_2\epsilon_{t-2}, \epsilon_t \in\sim WN(0, \sigma^2) \tag{4}$$

1. 计算它的偏相关系数 $a_{1,1}, a_{2,2}, a_{3,3}$.
2. 该 MA(2) 序列具有 p(某一整数值) 后截尾性吗?

解. 由定义,

$$\begin{aligned} a_{n,n} &= corr[X_1 - L(X_1|X_2, ..., X_n), X_{n+1} - L(X_{n+1}|X_2, ..., X_n)] \\ &= corr[X_1 - arg\min \mathbb{E}(X_1 - \hat{X_1})^2, X_{n+1} - arg\min \mathbb{E}(X_{n+1} - \hat{X_{n+1}})^2] \\ &= [(var(X_2, ..., X_n)^T)^{-1} \cdot cov((X_2, ..., X_n)^T, X_1)]^T (X_2, ..., X_n)^T \\ &= [(var(X_2, ..., X_n)^T)^{-1} \cdot (\gamma_1, ..., \gamma_{n-1})^T]^T (X_2, ..., X_n)^T \\ &= [\Gamma_{n-1}^{-1} \cdot (\gamma_1, ..., \gamma_{n-1})^T]^T (X_2, ..., X_n)^T \end{aligned} \tag{5}$$

而

$$\gamma_k = \sigma^2 \sum_{i=0}^{q-k} b_i b_{i+k} \tag{6}$$

得

$$\gamma_0 = \sigma^2 \sum_{i=0}^{q} b_i^2 = (1 + b_1^2 + b_2^2)\sigma^2,\ \gamma_1 = \sigma^2 \sum_{i=0}^{q-1} b_i b_{i+1} = (b_1 + b_1 b_2)\sigma^2 \tag{7}$$

$$\gamma_2 = \sigma^2 \sum_{i=0}^{q-2} b_i b_{i+2} = b_2 \sigma^2,\ \ \gamma_3 = \gamma_4 = ... = 0 \tag{8}$$

故

$$a_{1,1} = corr[X_1, X_2] = \rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{1 + b_1^2 + b_2^2}{b_1 + b_1 b_2} \tag{9}$$

同理

$$\begin{aligned}
a_{2,2} &= corr[X_1 - L(X_1|X_2), X_3 - L(X_2|X_2)] \\
&= corr[X_1 - (b_1 + b_1 b_2)X_2, X_3 - X_2] \\
&= corr(X_1 X_3) - (b_1 + b_1 b_2)corr(X_2 X_3) - corr(X_2 X_1) + (b_1 + b_1 b_2)corr(X_2^2) \\
&= \rho_2 - (b_1 + b_1 b_2 - 1)\rho_1 + b_1 + b_1 b_2 \\
&= \frac{b_2}{b_1 + b_1 b_2} - (b_1 + b_1 b_2 - 1)\frac{1 + b_1^2 + b_2^2}{b_1 + b_1 b_2} + (b_1 + b_1 b_2)
\end{aligned} \tag{10}$$

$$\begin{aligned}
a_{3,3} &= corr[X_1 - L(X_1|X_2, X_3), X_4 - L(X_3|X_2, X_3)] \\
&= corr[X_1 - \frac{(\gamma_0\gamma_1 - \gamma_2\gamma_1)X_2 - (\gamma_1^2 - \gamma_2\gamma_0)X_3}{\gamma_0^2 - \gamma_1^2}, X_4 - X_3] \\
&= \rho_3 - \rho_2 - \frac{\gamma_0\gamma_1 - \gamma_2\gamma_1}{\gamma_0^2 - \gamma_1^2}\rho_2 + \frac{\gamma_1^2 - \gamma_2\gamma_0}{\gamma_0^2 - \gamma_1^2}\rho_1 + \frac{\gamma_0\gamma_1 - \gamma_2\gamma_1}{\gamma_0^2 - \gamma_1^2}\rho_1 - \frac{\gamma_1^2 - \gamma_2\gamma_0}{\gamma_0^2 - \gamma_1^2}\rho_0 \\
&= -\frac{b_2}{b_1 + b_1 b_2} - \frac{b_1 + b_1^3 + b_1^3 b_2 + b_1 b_2^3}{(1 + b_1^2 + b_2^2)^2 - (b_1 + b_1 b_2)^2}\frac{1 + b_1^2 + b_2^2 - b_2}{b_1 + b_1 b_2} \\
&+ \frac{(b_1 + b_1 b_2)^2 - b_2(1 + b_1^2 + b_2^2)}{(1 + b_1^2 + b_2^2)^2 - (b_1 + b_1 b_2)^2}(\frac{1 + b_1^2 + b_2^2}{b_1 + b_1 b_2} - 1)
\end{aligned} \tag{11}$$

显然该 *MA(2)* 序列不具有 *p(某一整数值)* 后截尾性，因为这为可逆 *MA(2)*，那么 *MA(2)* 可以写成

$$\sum_{j=0}^{\infty} \varphi_j X_{t-j} = \epsilon_t \tag{12}$$

显然 p 不是有限实数.