Time Series HomeWork (7)

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1. 对 AR(2) 模型 $X_t = -0.1X_{t-1} + 0.72X_{t-2} + \epsilon_t$, 计算自相关系数 $\rho_k, k = 1, 2, 3, 4, 5$.

解.

$$A(z) = 1 + 0.1z - 0.72z^2 \tag{1}$$

A(z) 的零点 $z_1 = -\frac{10}{9}, z_2 = \frac{5}{4}$ *都在单位圆外, 满足最小相位条件.*

Wold 系数的递推公式:

$$\varphi_m = \begin{cases} 1 & m = 0, \\ a_1\varphi_{m-1} + a_2\varphi_{m-2} & m = 1, 2, ... \end{cases}$$

即解差分方程

$$\varphi_{m+2} + 0.1\varphi_{m+1} - 0.72\varphi_m = 0 \tag{2}$$

其中初始条件为 $\varphi_0 = 1, \varphi_1 = a_1\varphi_0 = -0.1$ 解差分得

$$\varphi_m = \frac{8}{17}(\frac{4}{5})^m + \frac{9}{17}(-\frac{9}{10})^m, m = 1, 2, ...$$

故由

$$\gamma_k = \sum_{i=1}^{p} a_i\gamma_{k-i} = -0.1\gamma_{k-1} + 0.72\gamma_{k-2}, k = 1, 2, ... \tag{3}$$

$$\gamma_k = \sigma^2 \sum_{i=0}^{\infty} \varphi_i\varphi_{i+k}, k = 0, 1, 2, ... \tag{4}$$

$$\gamma_0 = \sigma^2 + \sum_{i=0}^{p} a_i\varphi_i = \sigma^2 - 0.1\varphi_1 + 0.72\varphi_2 \tag{5}$$

有

$$\begin{aligned} \rho_k &= \sum_{i=1}^{p} a_i\rho_{k-i} \\ &= -0.1\rho_k + 0.72\rho_{k-1} \\ &= \frac{1}{\gamma_0}[-0.1\gamma_k + 0.72\gamma_{k-1}] \\ &= \frac{1}{\sigma^2 - 0.1\varphi_1 + 0.72\varphi_2}[-0.1\sigma^2 \sum_{i=0}^{\infty} \varphi_i\varphi_{i+k} + 0.72\sigma^2 \sum_{i=0}^{\infty} \varphi_i\varphi_{i+k-1}] \end{aligned} \tag{6}$$

2. 对可逆 MA(2) 序列

$$X_t = \epsilon_t + b_1\epsilon_{t-1} + b_2\epsilon_{t-2}, \epsilon_t \in\sim WN(0,\sigma^2) \tag{7}$$

1. 计算它的偏相关系数 $a_{1,1}, a_{2,2}, a_{3,3}$.
2. 该 MA(2) 序列具有 p(某一整数值) 后截尾性吗?

解. 由定义,

$$\begin{aligned} a_{n,n} &= corr[X_1 - L(X_1|X_2,...,X_n), X_{n+1} - L(X_{n+1}|X_2,...,X_n)] \\ &= corr[X_1 - arg\min \mathbb{E}(X_1 - \hat{X_1})^2, X_{n+1} - arg\min \mathbb{E}(X_{n+1} - \hat{X_{n+1}})^2] \\ &= [(var(X_2,...,X_n)^T)^{-1} \cdot cov((X_2,...,X_n)^T, X_1)]^T (X_2,...,X_n)^T \\ &= [(var(X_2,...,X_n)^T)^{-1} \cdot (\gamma_1,...,\gamma_{n-1})^T]^T (X_2,...,X_n)^T \\ &= [\Gamma_{n-1}^{-1} \cdot (\gamma_1,...,\gamma_{n-1})^T]^T (X_2,...,X_n)^T \end{aligned} \tag{8}$$

而

$$\gamma_k = \sigma^2 \sum_{i=0}^{q-k} b_i b_{i+k} \tag{9}$$

得

$$\gamma_0 = \sigma^2 \sum_{i=0}^{q} b_i^2 = (1 + b_1^2 + b_2^2)\sigma^2,\ \gamma_1 = \sigma^2 \sum_{i=0}^{q-1} b_i b_{i+1} = (b_1 + b_1 b_2)\sigma^2 \tag{10}$$

$$\gamma_2 = \sigma^2 \sum_{i=0}^{q-2} b_i b_{i+2} = b_2\sigma^2,\ \ \gamma_3 = \gamma_4 = ... = 0 \tag{11}$$

故

$$a_{1,1} = corr[X_1, X_2] = \rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{1 + b_1^2 + b_2^2}{b_1 + b_1 b_2} \tag{12}$$

同理

$$\begin{aligned} a_{2,2} &= corr[X_1 - L(X_1|X_2), X_3 - L(X_2|X_2)] \\ &= corr[X_1 - (b_1 + b_1 b_2)X_2, X_3 - X_2] \\ &= corr(X_1X_3) - (b_1 + b_1 b_2)corr(X_2X_3) - corr(X_2X_1) + (b_1 + b_1 b_2)corr(X_2^2) \\ &= \rho_2 - (b_1 + b_1 b_2 - 1)\rho_1 + b_1 + b_1 b_2 \\ &= \frac{b_2}{b_1 + b_1 b_2} - (b_1 + b_1 b_2 - 1)\frac{1 + b_1^2 + b_2^2}{b_1 + b_1 b_2} + (b_1 + b_1 b_2) \end{aligned} \tag{13}$$

$$\begin{aligned} a_{3,3} &= corr[X_1 - L(X_1|X_2,X_3), X_4 - L(X_3|X_2,X_3)] \\ &= corr[X_1 - \frac{(\gamma_0\gamma_1 - \gamma_2\gamma_1)X_2 - (\gamma_1^2 - \gamma_2\gamma_0)X_3}{\gamma_0^2 - \gamma_1^2}, X_4 - X_3] \\ &= \rho_3 - \rho_2 - \frac{\gamma_0\gamma_1 - \gamma_2\gamma_1}{\gamma_0^2 - \gamma_1^2}\rho_2 + \frac{\gamma_1^2 - \gamma_2\gamma_0}{\gamma_0^2 - \gamma_1^2}\rho_1 + \frac{\gamma_0\gamma_1 - \gamma_2\gamma_1}{\gamma_0^2 - \gamma_1^2}\rho_1 - \frac{\gamma_1^2 - \gamma_2\gamma_0}{\gamma_0^2 - \gamma_1^2}\rho_0 \\ &= -\frac{b_2}{b_1 + b_1 b_2} - \frac{b_1 + b_1^3 + b_1^3 b_2 + b_1 b_2^3}{(1 + b_1^2 + b_2^2)^2 - (b_1 + b_1 b_2)^2}\frac{1 + b_1^2 + b_2^2 - b_2}{b_1 + b_1 b_2} \\ &+ \frac{(b_1 + b_1 b_2)^2 - b_2(1 + b_1^2 + b_2^2)}{(1 + b_1^2 + b_2^2)^2 - (b_1 + b_1 b_2)^2}(\frac{1 + b_1^2 + b_2^2}{b_1 + b_1 b_2} - 1) \end{aligned} \tag{14}$$

显然该 *MA(2)* 序列不具有 *p(某一整数值)* 后截尾性, 因为 $a_{n,n}=\frac{D_n}{D}$, 其中

$$\mathbf{D}=\begin{pmatrix} 1 & \rho_1 & \rho_2 & 0 & \dots & 0 \\ \rho_1 & 1 & \rho_1 & \rho_2 & \dots & 0 \\ \rho_2 & \rho_1 & 1 & \rho_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

$$\mathbf{D_n}=\begin{pmatrix} 1 & \rho_1 & \rho_2 & 0 & \dots & \rho_1 \\ \rho_1 & 1 & \rho_1 & \rho_2 & \dots & \rho_2 \\ \rho_2 & \rho_1 & 1 & \rho_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

都满秩, 即分子分母都不可能为 *0.*