Full Solutions
MATH103 April 2005

April 5, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

Easiness: 4.0/5

SOLUTION. Rewrite the integral as
$\int_0^1 e^{(x+e^x)}dx = \int_0^1 e^x e^{e^x} dx.$
Let $u = e^x$. Then $du = e^x dx$, and our new endpoints of integration are
$e^1 = e$ and $e^0 = 1.$
Thus, our integral becomes
$\int_1^e e^u du = e^u \mid_1^e = e^e - e.$
Therefore, the correct answer is (ii).

Question 1 (b)

Easiness: 4.0/5

SOLUTION. Let $u = 4 - x^2$. Then $du = -2xdx$
$\int \frac{x}{4-x^2} = \int \frac{-du}{2u}$
Take the constant $\frac{-1}{2}$ out of the integral
$\int \frac{x}{4-x^2} = \int \frac{-du}{2u} = \frac{-1}{2} \int \frac{-du}{2u} = \frac{-\ln|u|}{2}$
Substitute back $u = 4 - x^2$ to get
$\int \frac{x}{4-x^2} = \frac{-\ln|4-x^2|}{2}$
Hence, the correct answer is $-\dfrac{1}{2}\ln|4 - x^2|$, which is (iv).

Question 1 (c)

Easiness: 3.7/5

SOLUTION. For $\int_a^b f(x)$
The Riemann Sum approximation is

$$\sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n} f(a + k\Delta x)\Delta x$$

where n represents the total number of rectangles, k enumerates the particular rectangles, and Δx will be equal to $\frac{b-a}{n}$
In this case, $\Delta x = (2 - 0)/n = 2/n$ and $k\Delta x = 2k/n$
Therefore, the solution is (v).

Question 1 (d)

Easiness: 3.7/5

SOLUTION. Simple substitution does not work in this problem, so we will need to integrate by parts. Let $u = \ln x$ and $dv = \frac{dx}{x^2} = x^{-2}dx$. Then $du = \frac{dx}{x}$ and $v = -x^{-1} = -\frac{1}{x}$. Thus,
$\int \frac{\ln x}{x^2} dx = -\frac{\ln x}{x} - \int \left(-\frac{1}{x}\right)\left(\frac{dx}{x}\right) = -\frac{\ln x}{x} + \int \left(\frac{dx}{x^2}\right) = -\frac{\ln x}{x} - \frac{1}{x}.$
Evaluating the definite integral then gives
$\int_1^e \frac{\ln x}{x^2} dx = \left(-\frac{\ln e}{e} - \frac{1}{e}\right) - \left(-\frac{\ln 1}{1} - \frac{1}{1}\right) = \left(-\frac{2}{e}\right) - (0 - 1) = -\frac{2}{e} + 1.$
Therefore, the correct answer is (i).

Question 1 (e)

Easiness: 4.3/5

SOLUTION. Given the mass density function $d(x) = 4 - x^2$ over the interval $0 \le x \le 2$, we use the formula for the centre of mass to get

$$
\begin{aligned}
\overline{x} &= \frac{\int_0^L x\rho(x)dx}{\int_0^L \rho(x)dx} \\
&= \frac{\int_0^2 x(4-x^2)dx}{\int_0^2 (4-x^2)dx} \\
&= \frac{\int_0^2 (4x-x^3)dx}{\int_0^2 (4-x^2)dx} \\
&= \frac{I_1}{I_2}
\end{aligned}
$$

The integral on the top is calculated as

$$
\begin{aligned}
I_1 &= \int_0^2 (4x-x^3)dx \\
&= \left[2x^2 - \frac{x^4}{4}\right]_0^2 \\
&= \left[\left(2(2)^2 - \frac{(2)^4}{4}\right) - \left(2(0)^2 - \frac{(0)^4}{4}\right)\right] \\
&= 4
\end{aligned}
$$

and the integral on the bottom reduces to

$$
\begin{aligned}
I_2 &= \int_0^2 (4-x^2)dx \\
&= \left[4x - \frac{x^3}{3}\right]_0^2 \\
&= \left[\left(4(2) - \frac{(2)^3}{3}\right) - \left(4(0) - \frac{(0)^3}{3}\right)\right] \\
&= \frac{16}{3}
\end{aligned}
$$

Putting these together we finally obtain

$$
\overline{x} = \frac{I_1}{I_2} = \frac{4}{\frac{16}{3}} = \frac{3}{4}
$$

Therefore, the correct answer is (iv).

Question 2

Solution. Given the probability density *p(x)=1-x/2* over the interval $0 \leq x \leq 2$ we plug this into the formula for the mean to get

$$
\begin{aligned}
\mu &= \int_0^2 x\left(1-\frac{x}{2}\right) dx \\
&= \int_0^2 \left(x-\frac{x^2}{2}\right) dx \\
&= \left[\frac{x^2}{2}-\frac{x^3}{6}\right]_0^2 \\
&= \left(\frac{2^2}{2}-\frac{2^3}{6}\right)-\left(\frac{0^2}{2}-\frac{0^3}{6}\right) \\
&= \mathbf{\frac{2}{3}}
\end{aligned}
$$

Using the first formula for the variance with $\mu = \frac{2}{3}$,

$$
\begin{aligned}
V &= \int_0^2 \left(x-\frac{2}{3}\right)^2 \left(1-\frac{x}{2}\right) dx \\
&= \int_0^2 \left(x^2-\frac{4}{3}x+\frac{4}{9}\right)\left(1-\frac{x}{2}\right) dx \\
&= \int_0^2 \left(\frac{-x^3}{2}+\frac{5}{3}x^2-\frac{14}{9}x+\frac{4}{9}\right) dx \\
&= \left[-\frac{x^4}{8}+\frac{5}{9}x^3-\frac{7}{9}x^2+\frac{4}{9}x\right]_0^2 \\
&= \left(-\frac{2^4}{8}+\frac{5}{9}(2)^3-\frac{7}{9}(2)^2+\frac{4}{9}(2)\right)-\left(-\frac{0^4}{8}+\frac{5}{9}(0)^3-\frac{7}{9}(0)^2+\frac{4}{9}(0)\right) \\
&= -\frac{16}{8}+\frac{40}{9}-\frac{28}{9}+\frac{8}{9}-0 = -2+\frac{20}{9} \\
&= \mathbf{\frac{2}{9}}
\end{aligned}
$$

Alternatively, using the other formula for the variance, we get the same value:

$$
\begin{aligned}
V &= \int_0^2 x^2\left(1-\frac{x}{2}\right) dx - \left(\frac{2}{3}\right)^2 \\
&= \int_0^2 \left(x^2-\frac{x^3}{2}\right) dx - \frac{4}{9} \\
&= \left[\frac{x^3}{3}-\frac{x^4}{8}\right]_0^2 - \frac{4}{9} \\
&= \left(\frac{2^3}{3}-\frac{2^4}{8}\right)-\left(\frac{0^3}{3}-\frac{0^4}{8}\right)-\frac{4}{9} = \frac{8}{3}-2-\frac{4}{9} \\
&= \mathbf{\frac{2}{9}}
\end{aligned}
$$

Question 4

Easiness: 5.0/5

SOLUTION. Given that the rate of change of volume is $10\sin(\pi t/10)$ cubic centimetres per day, the total change in the volume between *t=0* and *t=5*, in cubic centimetres, is

$$
\begin{aligned}
\int_0^5 10\sin(\pi t/10)dt &= \left[10\frac{-10}{\pi}\cos(\pi t/10)\right]_0^5 \\
&= \frac{-100}{\pi}\left(\cos(5\pi/10) - \cos(0/10)\right) \\
&= \frac{-100}{\pi}(0-1) \\
&= \frac{100}{\pi}
\end{aligned}
$$

Question 5 (a)

Easiness: 4.0/5

SOLUTION. There are actually two cases we have to consider here, if *p=1* and if $p \neq 1$. We will see why below:
First assume that $p \neq 1$. Given the function $y = f(x) = \frac{1}{x^p}$ on the interval [1,B], the area between this curve and the x-axis is

$$
\begin{aligned}
A &= \int_1^B \frac{1}{x^p}dx = \int_1^B x^{-p}dx \\
&= \left[\frac{x^{-p+1}}{-p+1}\right]_1^B = \frac{1}{1-p}\left(B^{1-p} - 1\right)
\end{aligned}
$$

Note that, if *p=1*, then the above does not work because we would divide by *1-p=0*.
Hence we treat *p=1* separately. The function becomes $y = f(x) = \frac{1}{x}$ and on the interval [1,B], the area between this curve and the x-axis is

$$
\begin{aligned}
A &= \int_1^B \frac{1}{x}dx \\
&= [\ln x]_1^B = \ln B
\end{aligned}
$$

So our final answer is

$$
A = \int_1^B \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{1-p}\left(B^{1-p} - 1\right), & p \neq 1 \\ \ln B, & p = 1 \end{cases}
$$

Question 5 (b)

SOLUTION. For similar reasons to part (a) we will have two cases because we will have to worry about the integral we are doing. There will be a unique situation if we are ever integrating *1/x* which occurs when *p=1/2* because

$$
V = \int_1^B \pi\left[\frac{1}{x^p}\right]^2 dx = \int_1^B \pi\frac{1}{x^{2p}}dx.
$$

If $p \neq 1/2$ then when we rotate the function $y = f(x) = \frac{1}{x^p}$ about the x-axis over the interval [1,B], the volume obtained is

$$V = \int_1^B \pi \left[\frac{1}{x^p}\right]^2 dx = \pi \int_1^B \frac{1}{x^{2p}} dx = \pi \int_1^B x^{-2p} dx$$
$$= \pi \left[\frac{x^{-2p+1}}{-2p+1}\right]_1^B = \frac{\pi}{1-2p}\left(B^{1-2p} - 1\right)$$

If $p=1/2$ then

$$V = \int_1^B \pi \left[\frac{1}{x^{1/2}}\right]^2 dx = \pi \int_1^B \frac{1}{x} dx$$
$$= \pi \left[\ln x\right]_{x=1}^{x=B} = \pi \ln B$$

Question 3 (a)

Solution. No content found.

Question 3 (b)

Solution. No content found.

Question 3 (c)

Solution. No content found.

Question 5 (c)

Solution. No content found.

Question 6 (a)

Solution. No content found.

Question 6 (b)

Solution. No content found.

Question 6 (c)

Solution. No content found.

Question 7 (a)

Solution. No content found.

Question 7 (b)

SOLUTION. No content found.

Question 7 (c)

SOLUTION. No content found.

Question 8 (a)

SOLUTION. No content found.

Question 8 (b)

SOLUTION. No content found.

Question 8 (c)

SOLUTION. No content found.

Good Luck for your exams!