Full Solutions
MATH100 December 2010

April 4, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

SOLUTION 1. Directly plugging in x=1 into the limit gives 0/0. We try to factor to cancel out the terms making the fraction zero.

$$\begin{aligned}
&\lim_{x\to 1} \frac{x^2+2x-3}{x-1} \\
&= \lim_{x\to 1} \frac{(x-1)(x+3)}{x-1} \\
&= \lim_{x\to 1} (x+3) \\
&= (1)+3 \\
&= 4
\end{aligned}$$

SOLUTION 2. Directly plugging in x=1 into the limit gives 0/0. Since this is an indeterminate form, we can use L'Hospital's rule:

$$\begin{aligned}
&\lim_{x\to 1} \frac{x^2+2x-3}{x-1} \\
&= \lim_{x\to 1} \frac{(x^2+2x-3)'}{(x-1)'} \\
&= \lim_{x\to 1} \frac{2x+2}{1} \\
&= \lim_{x\to 1} 2x+2 \\
&= 2(1)+2 \\
&= 4
\end{aligned}$$

Question 1 (b)

SOLUTION. Rationalizing the numerator gives

$$\begin{aligned}
&\lim_{x\to\infty} \sqrt{9x^2+x}-3x \\
&= \lim_{x\to\infty} (\sqrt{9x^2+x}-3x)\cdot\frac{\sqrt{9x^2+x}+3x}{\sqrt{9x^2+x}+3x} \\
&= \lim_{x\to\infty} \frac{9x^2+x-9x^2}{\sqrt{9x^2+x}+3x} \\
&= \lim_{x\to\infty} \frac{x}{\sqrt{9x^2+x}+3x}
\end{aligned}$$

Now we can factor the denominator. Remembering that $\sqrt{x^2}=|x|$, we have that

$$\begin{aligned}
&\lim_{x\to\infty} \frac{x}{\sqrt{9x^2+x}+3x} \\
&= \lim_{x\to\infty} \frac{x}{|x|\sqrt{9+1/x}+3x}
\end{aligned}$$

As the limit goes to positive infinity, $|x| = x$ and so

$$\begin{aligned}
&\lim_{x\to\infty} \frac{x}{|x|\sqrt{9+1/x}+3x} \\
&= \lim_{x\to\infty} \frac{x}{x\sqrt{9+1/x}+3x} \\
&= \lim_{x\to\infty} \frac{x}{x(\sqrt{9+1/x}+3)} \\
&= \lim_{x\to\infty} \frac{1}{\sqrt{9+1/x}+3} \\
&= \frac{1}{\sqrt{9+0}+3} \\
&= \frac{1}{6}
\end{aligned}$$

Question 1 (c)

SOLUTION. Using the Product Rule, we get:

$$\begin{aligned}
f'(x) &= (x\cos x)' \\
&= (x)'(\cos x) + x(\cos x)' \\
&= \cos x + x(-\sin x) \\
&= \cos x - x\sin x
\end{aligned}$$

Question 1 (d)

SOLUTION. To use the quotient rule we first identify the numerator $f(x) = e^x$ and the denominator $g(x) = x^2 - 3$. Then we calculate $f'(x) = e^x$ and $g'(x) = 2x$. The rest is a walk in the park, just plug it in:

$$\begin{aligned}
&\frac{d}{dx}\left(\frac{e^x}{x^2-3}\right) \\
&= \frac{e^x(x^2-3) - 2x(e^x)}{(x^2-3)^2} \\
&= \frac{e^x(x^2-3-2x)}{(x^2-3)^2} \\
&= \frac{e^x(x-3)(x+1)}{(x^2-3)^2}
\end{aligned}$$

Question 1 (e)

SOLUTION. Let $f_1(x) = x^5$ and let $f_2(x) = \sin(\cos(3x^2))$. Then

$$y = f_1(f_2(x)) = \sin^5(\cos(3x^2)).$$

So the derivative by the chain rule is

$$y' = f_1'(f_2(x))f_2'(x)$$

(Equation 1)
Now, $f_1'(x) = 5x^4$ and so

$$f_1'(f_2(x)) = 5(\sin(\cos(3x^2)))^4 = 5\sin^4(\cos(3x^2))$$

(Equation 2)
For $f_2'(x)$, we again use the chain rule. Let $f_3(x) = \sin(x)$ and $f_4(x) = \cos(3x^2)$ so we have that $f_2(x) = f_3(f_4(x))$. Using the chain rule, we have that

$$f_2'(x) = f_3'(f_4(x))f_4'(x)$$

(Equation 3)
Notice that $f_3'(x) = \cos(x)$ and thus

$$f_3'(f_4(x)) = \cos(\cos(3x^2))$$

(Equation 4)
Thus it suffices to find $f_4'(x)$. Guess what we're going to use... the chain rule! Let $f_5(x) = \cos(x)$ and $f_6(x) = 3x^2$. Then, the chain rule states that

$$f_4'(x) = f_5'(f_6(x))f_6'(x)$$

(Equation 5)
Now, $f_5'(x) = -\sin(x)$ and so

$$f_5'(f_6(x)) = -\sin(3x^2)$$

(Equation 6)
Here, the derivative of $f_6(x)$ is actually doable! We have that

$$f_6'(x) = \frac{d}{dx}(3x^2) = 6x$$

and hence by Equations 5 and 6, we have

$$f_4'(x) = f_5'(f_6(x))f_6'(x) = (-\sin(3x^2))(6x)$$

(Equation 7)
Combining Equations 1,2,3,4,6 and 7, we have

$$\begin{aligned}
y' &= f_1'(f_2(x))f_2'(x) && \text{(via Equation 1)}\\
&= 5\sin^4(\cos(3x^2))(f_3'(f_4(x))f_4'(x)) && \text{(via Equation 2 and 3)}\\
&= 5\sin^4(\cos(3x^2))(\cos(\cos(3x^2))(-\sin(3x^2))(6x))) && \text{(via Equation 4 and 7)}\\
&= -30x\sin^4(\cos(3x^2))\cos(\cos(3x^2))\sin(3x^2))
\end{aligned}$$

Phew!

Question 1 (f)

Solution. A quick application of the chain rule gives

$$\begin{aligned}\frac{d}{dx}(\sin^{-1}(\sqrt{x})) &= \frac{1}{\sqrt{1-(\sqrt{x})^2}}\frac{d}{dx}(\sqrt{x}) \\ &= \frac{1}{\sqrt{1-x}}\frac{1}{2\sqrt{x}} \\ &= \frac{1}{2\sqrt{x-x^2}}\end{aligned}$$

Question 1 (g)

Solution. First, we must find the derivative of the curve using implicit differentiation:

$$\begin{aligned}x^4 - x^2y + y^4 &= 1 \\ (x^4 - x^2y + y^4)' &= (1)' \\ 4x^3 - (x^2y)' + 4y^3y' &= 0 \\ 4x^3 - x^2y' - 2xy + 4y^3y' &= 0 \\ y'(-x^2 + 4y^3) &= 2xy - 4x^3 \\ y' &= \frac{2xy - 4x^3}{4y^3 - x^2}\end{aligned}$$

This gives us the equation for the slope. The question is asking us for the slope at the point (-1,1) so we plug in x=-1 and y=1:

$$\begin{aligned}y' &= \frac{2xy - 4x^3}{4y^3 - x^2} \\ &= \frac{2(-1)(1) - 4(-1)^3}{4(1)^3 - (-1)^2} \\ &= \frac{-2+4}{4-1} \\ &= \frac{2}{3}\end{aligned}$$

Therefore, the slope of the tangent line to the curve at (-1,1) is 2/3.

Question 1 (h)

Solution. To differentiate this, we should use logarithmic differentiation. First, we rewrite the function:

$$\ln(f(x)) = \ln\left((\tan x)^{\cos x}\right) = \cos(x)\ln(\tan x)$$

And then use implicit differentiation

$$
\begin{aligned}
(\ln(f(x)))' &= (\cos(x)\ln(\tan(x)))' \\
\frac{1}{f(x)}f'(x) &= \cos(x)(\ln(\tan(x)))' - \sin(x)\ln(\tan(x)) \\
&= \cos(x)\frac{\sec^2(x)}{\tan(x)} - \sin(x)\ln(\tan(x))
\end{aligned}
$$

(Remember that the derivative of $\tan(x)$ is $\sec^2(x)$). Now, simply solve for $f'(x)$ to obtain the answer:

$$
\begin{aligned}
f'(x) &= f(x)\left(\frac{\cos(x)\sec^2(x)}{\tan(x)} - \sin(x)\ln(\tan(x))\right) \\
&= (\tan(x))^{\cos(x)}\left(\frac{\sec(x)}{\tan(x)} - \sin(x)\ln(\tan(x))\right) \\
&= (\tan(x))^{\cos(x)}\left(\frac{1}{\sin(x)} - \sin(x)\ln(\tan(x))\right) \\
&= (\tan(x))^{\cos(x)}(\csc(x) - \sin(x)\ln(\tan(x)))
\end{aligned}
$$

(Note that $\cos(x)\sec^2(x) = \sec(x)$.)

Question 1 (i)

Solution. Let f (x)=x$^{1/3}$. First we need a number a as close to 30 as possible but whose cube root we know. A good candidate would be 27, because 3^3=27 and so $27^{1/3} = 3$.
We know that the general formula for a linear approximation is given by

$$L(x) = f(a) + f'(a)(x - a)$$

We can find the derivative:

$$f'(x) = \frac{x^{-2/3}}{3}$$

Knowing these values, we can plug them in to the linear approximation equation and find the approximation for x=30, using a=27:

$$
\begin{aligned}
L(x) &= (27)^{1/2} + \frac{27^{-2/3}}{3}(x - 27) \\
L(30) &= (27)^{1/3} + \frac{27^{-2/3}}{3}(30 - 27) \\
&= 3 + \frac{1/9}{3}(3) \\
&= 3 + \frac{1}{9} \\
&= \frac{28}{9}
\end{aligned}
$$

So using linear approximation, $(30)^{1/3} \approx 28/9$

Question 1 (j)

SOLUTION. To find the second degree Taylor polynomial, we use the equation

$$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

We know that a =4 and that f(x)=$\sqrt{x}$. We can easily compute the derivatives to get

$$f'(x) = \frac{1}{2}x^{-1/2}$$
$$f''(x) = -\frac{1}{4x^{3/2}}$$

Plugging this all in to the Taylor equation, we get:

$$\begin{aligned}
T_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 \\
&= f(4) + f'(4)(x-4) + \frac{f''(4)}{2}(x-4)^2 \\
&= \sqrt{4} + \frac{1}{2}4^{-1/2}(x-4) + \frac{-\frac{1}{4\times 4^{3/2}}}{2}(x-4)^2 \\
&= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 \\
&= 2 + \frac{x-4}{4} - \frac{(x-4)^2}{64}
\end{aligned}$$

Question 1 (k)

SOLUTION. Notice that $f(x)$ is differentiable on $[0,4]$ and hence it is continuous there. Thus, by the mean value theorem, there exists a point $c \in [0,4]$ such that
$f'(c) = \frac{f(4)-f(0)}{4-0}$.
We know that on $[0,4]$ that $f'(x) \geq 3$. Since $c \in [0,4]$ and $f(0) = 10$, we have that
$3 \leq f'(c) = \frac{f(4)-f(0)}{4-0} = \frac{f(4)-10}{4}$.
Solving gives $22 \leq f(4)$ and hence the minimum $f(4)$ could be is 22.

Question 1 (l)

SOLUTION. We first take the derivative to see that
$f'(x) = e^x(\sin x - \cos x) + e^x(\cos x + \sin x) = 2e^x \sin x$
Setting this to zero, we see that the function is zero whenever $\sin x = 0$ (the exponential funciton is always bigger than 0) and on our interval $[\frac{\pi}{2}, 2\pi]$, this occurs at π and at 2π.
Checking the endpoints and values at the critical points, we have
$f(\frac{\pi}{2}) = e^{\frac{\pi}{2}}(\sin(\frac{\pi}{2}) - \cos(\frac{\pi}{2})) = e^{\frac{\pi}{2}}$
$f(\pi) = e^{\pi}(\sin(\pi) - \cos(\pi)) = e^{\pi}$
$f(2\pi) = e^{2\pi}(\sin(2\pi) - \cos(2\pi)) = -e^{2\pi}$
Hence, our function obtains its absolute maximum at π and the value is e^{π}

Question 1 (m)

Solution. Let us first compute the derivative

$$f'(x) = e^x - 2x$$

and now crank up the algorithm.

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ &= 0 - \frac{1}{1} = -1 \\ x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} \\ &= -1 - \frac{e^{-1} - 1}{e^{-1} + 2} \\ &= \frac{-1 - 2e^{-1}}{2 + e^{-1}} \end{aligned}$$

Question 1 (n)

Solution. If $f''(t) = \frac{3}{\sqrt{t}}$, then antidifferentiating gives
$f'(t) = 6\sqrt{t} + C$. Since $f'(4) = 7$, we have that
$7 = f'(4) = 6\sqrt{4} + C = 12 + C$
and hence C=-5. Thus
$f'(t) = 6\sqrt{t} - 5$
and antidifferentiating once more gives
$f(t) = 4\sqrt{t}^3 - 5t + D$
Since $f(4) = 2$, we see that
$2 = f(4) = 4\sqrt{4}^3 - 5(4) + D = 12 + D$
and so D=-10. Thus, the function we seek is
$f(t) = 4\sqrt{t}^3 - 5t - 10$

Question 2 (a)

Solution. This one you just have to know by heart for the exam.

$$\frac{dT}{dt} = k(T - T_s) = k(T - 19)$$

Question 2 (b)

Solution. Proceeding as in the hints, let
$y(t) = T(t) - 19$
Then, Newton's law of cooling reads

$\frac{dy}{dt} = \frac{dT}{dt} = k(T(t) - 19) = ky(t)$
Solving this differential equation gives
$y(t) = y(0)e^{kt}$
In terms of T and using the initial condition $T(0) = 3$, we have
$T(t) - 19 = (T(0) - 19)e^{kt} = (3 - 19)e^{kt} = -16e^{kt}$
After 30 minutes, we know $T(30) = 11$ and so
$-16e^{30k} = T(30) - 19 = 11 - 19 = -8$
Solving for k by isolating and taking logarithms gives
$k = \frac{\ln(1/2)}{30} = -\frac{\ln(2)}{30}$
We want to find the temperature after 90 minutes and so

$$\begin{aligned}
T(90) - 19 &= -16e^{90\cdot(-\ln(2))/30} \\
&= -16e^{-3\ln(2)} \\
&= -16e^{\ln(2^{-3})} \\
&= -16(1/8) \\
&= -2
\end{aligned}$$

and thus $T(90) = 19 - 2 = 17$ degrees celsius

Question 2 (c)

SOLUTION. From the previous question, we know that

$$T(t) - 19 = -16e^{kt}$$

where
$k = \frac{-\ln(2)}{30}$
Searching for when $T(t) = 16$ yields

$$\begin{aligned}
16 - 19 &= -16e^{kt} \\
-3 &= -16e^{kt} \\
\frac{3}{16} &= e^{kt} \\
\ln(3/16) &= kt \\
\ln(3/16) &= \frac{-\ln(2)t}{30} \\
t &= -30\frac{\ln(3/16)}{\ln(2)} \\
t &= 30\frac{\ln(16/3)}{\ln(2)}
\end{aligned}$$

completing the question.

Question 3

SOLUTION. Let V_1, r_1, and h_1 be the volume, radius, and depth of the water in the large pool and let V_2,

r_2, and h_2 be the volume, radius, and depth of the water in the small pool. We know that $r_1 = 8$m and $r_2 = 5$m. Since the pools are being filled at the same rate (in m^3/min),

$$\frac{dV_1}{dt} = \frac{dV_2}{dt}.$$

We know that $\frac{dh_2}{dt} = 0.5m/min$ and we want to find $\frac{dh_1}{dt}$.
By the volume formula for a cylinder,

$$V_1 = \pi\, r_1^2 h_1$$

and $V_2 = \pi\, r_2^2 h_2$.
Note that r_1 and r_2 are constants. The variables of the problem are V_1, V_2, h_1, and h_2.
Since $\frac{dV_1}{dt} = \frac{dV_2}{dt}$,

$$\pi\, r_1^2 \frac{dh_1}{dt} = \pi\, r_2^2 \frac{dh_2}{dt}.$$

Hence, the water depth in the the larger pool is increasing at a rate of

$$\frac{dh_1}{dt} = \frac{r_2^2}{r_1^2}\frac{dh_2}{dt} = \frac{(5m)^2}{(8m)^2}\frac{1}{2}m/min = \frac{25}{128}m/min.$$

Question 4 (a)

Solution. To have continuity we need that:
[1] The function f(x) is defined at x = 1.
[2] $\lim_{x\to 1} f(x)$ exists and is equal to f(1).
Clearly we can evaluate $f(1) = \frac{4}{\pi}\tan^{-1}(1) = 1$. So f(1) is defined and [1] is satisfied.
To check condition 2), we need to evaluate the limit as x goes to 1 of f(x), but since the function is piecewise defined, we need to evaluate the left and right-hand limits and confirm that they are equal.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1}\left(\frac{4}{\pi}\tan^{-1}(x)\right) = \frac{4}{\pi}\tan^{-1}(1) = 1$$
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1}\left(2 - x^4\right) = 2 - (1)^4 = 1$$

From this we can see the left and right-hand limits are equal and that they equal f(1), so [2] is also satisfied. Thus f(x) is continuous at x = 1.

Question 4 (b)

Solution. By inspection of the function, we can see that it is defined everywhere, hence there are no possibilities for vertical asymptotes.
To check if there are any horizontal asymptotes, we must check the behaviour of the function $f(x)$ as x gets very large (both positively and negatively). So we evaluate

$$\lim_{x\to+\infty} f(x), \quad \lim_{x\to-\infty} f(x).$$

Evaluating the limits gives

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} 2 - x^4 = -\infty$$
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{4}{\pi}\tan^{-1}(x) = \frac{4}{\pi}\cdot\frac{\pi}{2} = 2$$

Note: that

$$\lim_{x\to+\infty} \tan^{-1}(x) = \frac{\pi}{2}$$

since

$$\lim_{x\to\pi/2} \tan(x) = +\infty$$

(Or draw a trigonometric circle to convince yourself of this).
Therefore, there is a horizontal asymptote on the right, given by the line $y = 2$. It describes the behaviour of the function for large positive x.
Finally, we check for slant asymptotes. Having already found an asymptote on the right (a slant asymptote of slope 0 if you will) we have to check on the left. Since on the left, the function is a polynomial of degree 4, we can directly conclude there is no slant asymptote.
More formally, a function f has a slant asymptote of equation $y = mx + h$ if and only if

$$\lim_{x\to\infty} \frac{f(x)}{x} = m \qquad \lim_{x\to\infty} f(x) - mx = h$$

(For a polynomial of degree 2 or more, the first limit never converges).
So all in all, there are no vertical asymptotes, no asymptotes (horizontal or slant) on the left and a horizontal asymptote on the right at $y = 2$.

Question 4 (c)

SOLUTION. The critical points of a function f(x) are all the values x where
1. $f'(x) = 0$ or
2. $f'(x)$ does not exist and $f(x)$ is defined.

First, we look at the part of the domain of $f(x)$ where $x > 1$. In this domain:

$$f(x) = \frac{4}{\pi}\tan^{-1}(x) \quad \rightarrow \quad f'(x) = \frac{4}{\pi}\left(\frac{1}{1+x^2}\right).$$

We see that there are no values of x such that the derivative is 0 or undefined and hence there are no critical points there.
We now move on to the part of the domain where $x < 1$:

$$f(x) = 2 - x^4 \quad \rightarrow \quad f'(x) = -4x^3.$$

Here we see that $f'(x) = 0$ for $x = 0$. And since $x = 0$ is in the part of the domain we are looking at, this is indeed a critical point.
Finally, we need to look at $x = 1$. Remember from part (a) that $f(x)$ is defined and even continuous at $x = 1$. To see if the function is differentiable at $x = 1$we compute the limit as x approaches 1 from either side. First from the left:

$$\lim_{x\to 1^-} f'(x) = \lim_{x\to 1^-} -4x^3 = -4$$

and then from the right:

$$\lim_{x\to 1^+} f'(x) = \lim_{x\to 1^-} \frac{4}{\pi}\left(\frac{1}{1+x^2}\right) = \frac{4}{\pi}\cdot\frac{1}{2} = \frac{2}{\pi}$$

Since the left and right limits are different we can conclude that the function is not differentiable at $x = 1$, which makes that point a critical point. In conclusion, $x = 0$ and $x = 1$ are the critical points of the function. To determine the intervals of increase and decrease we need to take test points of the function on the intervals,

$$(-\infty, 0), (0, 1), (1, \infty)$$

Plugging a value from each interval of x into our expressions for the derivative, we get the following

$$f'(-1) = 4 > 0, \quad f'(0.5) = -4(0.5)^3 < 0, \quad f'(2) = \frac{4}{5\pi} > 0$$

Therefore, $f(x)$ is increasing on the intervals $(-\infty, 0)$ and $(1, +\infty)$ and decreasing on the interval $(0, 1)$. By the first derivative test, we can see that the function has a local maximum at $x = 0$ and a local minimum at $x = 1$.
Therefore the function f has a local maximum of 2 (at $x = 0$) and a local minimum of 1 (at $x = 1$).

Question 4 (d)

SOLUTION. To determine the intervals of concavity, we must evaluate $f''(x)$ and determine where it is positive and negative:

$$f(x) = \begin{cases} \frac{4}{\pi}\tan^{-1}(x) & \text{if } x \geq 1 \\ 2 - x^4 & \text{if } x < 1 \end{cases}$$

Evaluating the second derivative of f gives

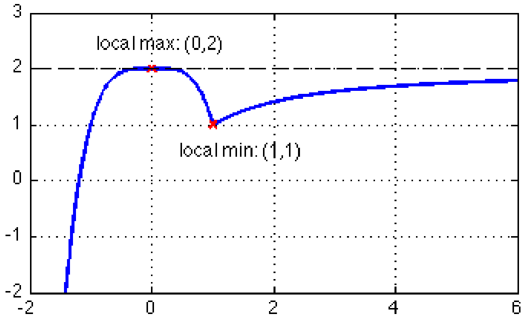
$$f''(x) = \begin{cases} \frac{4}{\pi}\frac{-2x}{(1+x^2)^2} & \text{if } x > 1 \\ -12x^2 & \text{if } x < 1 \end{cases}$$

From this we see that the possible inflection points are $x = 0$. We need only confirm that the sign of the concavity changes across $x = 0$. We must also consider concavity changes across $x = 1$ since f is not differentiable there.
When $x < 1$, the second derivative is always negative since -$12x^2$ is always negative.
When $x > 1$, the second derivative is always negative as well since in that case, -$2x$ is always negative.
From these results we can see that $f(x)$ is concave down for all x except at the critical points. In other words, $f(x)$ is concave down on $(-\infty, 0), (0, 1), (1, \infty)$ and so, there are no inflection points.

Question 4 (e)

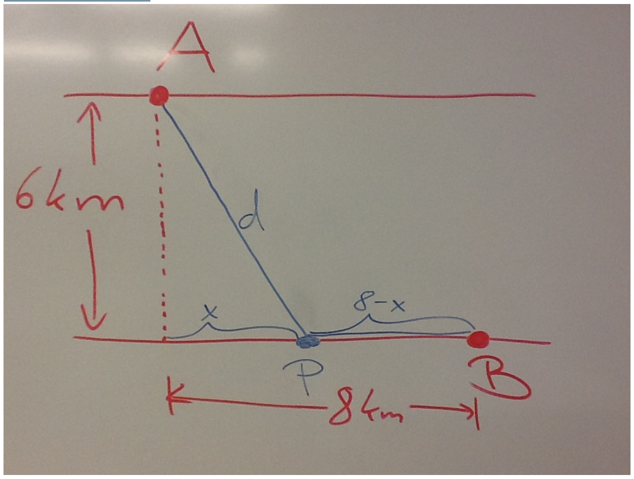
SOLUTION. From parts a) - d), we found that
[1] f is continous everywhere

[2] There is a horizontal asymptote of $y = 2$ for $x \to \infty$.
[3] There is a local max at (0,2) and a local min at (1,1)
[4] There are no inflection points at f is concave down everywhere except at x = 0, x = 1.
From this information we can obtain an image that looks similar to this:



Question 5

SOLUTION. According to the picture,



Thus, the cost equation is
$C = 40d + 20(8 - x)$
To get this in terms of one variable, note by the Pythagorean theorem, we have
$d^2 = 6^2 + x^2$
Hence, we have
$C = 40\sqrt{36 + x^2} + 20(8 - x)$
Differentiating yields

$$\begin{aligned} C' &= \frac{40(2x)}{2\sqrt{36 + x^2}} - 20 \\ &= \frac{40x - 20\sqrt{36 + x^2}}{\sqrt{36 + x^2}} \end{aligned}$$

Setting the derivative to 0 yields

$$\begin{aligned}
0 &= \frac{40x - 20\sqrt{36+x^2}}{\sqrt{36+x^2}} \\
0 &= 40x - 20\sqrt{36+x^2} \\
\sqrt{36+x^2} &= 2x \\
36 + x^2 &= 4x^2 \\
36 &= 3x^2 \\
12 &= x^2 \\
\pm 2\sqrt{3} &= x
\end{aligned}$$

Notice that the negative answer is inadmissable since it corresponds to a side length (and x should be positive to minimize cost). Now, the endpoints are $x = 0$ and $x = 8$. Then,
$C(0) = 40\sqrt{36+0^2} + 20(8-0) = 240 + 160 = 400$

$$\begin{aligned}
C(2\sqrt{3}) &= 40\sqrt{36 + (2\sqrt{3})^2} + 20(8 - 2\sqrt{3}) \\
&= 40\sqrt{36+12} + 160 - 40\sqrt{3} \\
&= 160\sqrt{3} + 160 - 40\sqrt{3} \\
&= 120\sqrt{3} + 160 < 400
\end{aligned}$$

$C(8) = 40\sqrt{36+8^2} + 20(8-8) = 40\sqrt{100} = 400$
and so the minimum occurs at $x = 2\sqrt{3}$

Question 6

Solution 1. By the definition of a derivative, we have that

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{x+h}{x+h+5} - \frac{x}{x+5}}{h}
\end{aligned}$$

The key step to simplifying this expression is here: we find a common denominator for the fractions on top and combine them.

$$\begin{aligned}
&= \lim_{h\to 0} \frac{\frac{(x+h)(x+5)}{(x+h+5)(x+5)} - \frac{x(x+h+5)}{(x+h+5)(x+5)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{x^2+xh+5x+5h-x^2-xh-5x}{(x+h+5)(x+5)}}{h}
\end{aligned}$$

Now we simplify the expression into a single fraction and cancel the h on top and bottom.

$$\begin{aligned}
&= \lim_{h\to 0} \frac{5h}{h(x+h+5)(x+5)} \\
&= \lim_{h\to 0} \frac{5}{(x+h+5)(x+5)}
\end{aligned}$$

Finally taking the limit gives:

$$
\begin{aligned}
&= \frac{5}{(x+5)(x+5)} \\
&= \frac{5}{(x+5)^2}
\end{aligned}
$$

SOLUTION 2. Using the alternative definition,

$$
\begin{aligned}
f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\
f'(a) &= \lim_{x \to a} \frac{\frac{x}{x+5} - \frac{a}{a+5}}{x - a}
\end{aligned}
$$

Here we find a common denominator to simplify the fractions on top of the large fraction.

$$
\begin{aligned}
f'(a) &= \lim_{x \to a} \frac{\frac{x(a+5)}{(x+5)(a+5)} - \frac{a(x+5)}{(a+5)(x+5)}}{x - a} \\
f'(a) &= \lim_{x \to a} \frac{\frac{xa+5x-ax-5a}{(x+5)(a+5)}}{x - a}
\end{aligned}
$$

We then simplify the fraction, factoring out a 5 on top and then canceling the *(x - a)* term.

$$
\begin{aligned}
f'(a) &= \lim_{x \to a} \frac{5x - 5a}{(x+5)(a+5)(x-a)} \\
f'(a) &= \lim_{x \to a} \frac{5(x-a)}{(x+5)(a+5)(x-a)} \\
f'(a) &= \lim_{x \to a} \frac{5}{(x+5)(a+5)}
\end{aligned}
$$

Finally we take the limit.

$$
\begin{aligned}
f'(a) &= \frac{5}{(a+5)(a+5)} \\
f'(a) &= \frac{5}{(a+5)^2}
\end{aligned}
$$

Which completes the question.

Question 7

SOLUTION 1. Taylor's remainder formula says that

$$
|T_2(1) - f(1)| = \frac{|f'''(c)||x-a|^3}{3!} \leq \frac{M|1-0|^3}{6} = \frac{M}{6}
$$

where M is an upper bound for the absolute value of the third derivative on the interval $[0, 1]$. Computing the third derivative via the product rule gives

$$
\begin{aligned}
f(x) &= e^x(x^2 - 7x + 15) \\
f'(x) &= e^x(x^2 - 7x + 15) + e^x(2x - 7) \\
&= e^x(x^2 - 7x + 15 + 2x - 7) \\
&= e^x(x^2 - 5x + 8) \\
f''(x) &= e^x(x^2 - 5x + 8) + e^x(2x - 5) \\
&= e^x(x^2 - 5x + 8 + 2x - 5) \\
&= e^x(x^2 - 3x + 3) \\
f'''(x) &= e^x(x^2 - 3x + 3) + e^x(2x - 3) \\
&= e^x(x^2 - 3x + 3 + 2x - 3) \\
&= e^x(x^2 - x)
\end{aligned}
$$

Notice that e^x is increasing on $[0, 1]$ and thus obtains its maximum at $x = 1$, i.e.

$$|e^x| = e^x \le e^1 = e$$

for $x \in [0, 1]$.
The function $x^2 - x$ is a parabola with roots at 0 and 1. Hence its minimum occurs at the midpoint of the roots, namely at $x = 1/2$. Since the value at the endpoints $x = 0$ and $x = 1$ is zero, the minimum value of the parabola maximizes its absolute value. Thus,

$$|x^2 - x| \le |(1/2)^2 - (1/2)| = 1/4$$

Since we want to bound the absolute value of the third derivative, we know that

$$|f'''(x)| = |e^x||x^2 - x| \le e \cdot 1/4 = e/4 = M.$$

Hence, we have

$$|T_2(1) - f(1)| \le \frac{M}{6} = \frac{e}{24}$$

as required.

Solution 2. Proceed as in solution 1 to see that

$$f'''(x) = e^x(x^2 - x)$$

We now need a bound on this function on $[0, 1]$. Notice that the third derivative is neither monotonically increasing nor is it monotonically decreasing on the interval. Hence we are not guaranteed to find the maximum value by just looking at the endpoints. In fact, at the endpoints we have $f'''(0) = 0 = f'''(1)$, which is certainly not the maximum of the absolute value $|f'''(x)|$. Instead we do it the proper way: Take the derivative

$$f''''(x) = e^x(x^2 - x) + e^x(2x - 1) = e^x(x^2 + x - 1)$$

and set it to zero

$$0 = e^x(x^2 + x - 1)$$

Recall that $e^x \neq 0$. Using the quadratic formula, we see that

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} = \frac{-1 \pm \sqrt{5}}{2}$$

only the positive such root is in $[0, 1]$. Let

$$\phi := \frac{-1 + \sqrt{5}}{2}.$$

Then, our function $f'''(x)$ obtains an extreme value at ϕ. Since the function is nonzero at this point, we know that $|f'''(x)|$ is maximal at ϕ. Thus, we have that

$$|T_2(1) - f(1)| \leq \frac{M}{6} = \frac{|f'''(\phi)|}{6} = \frac{e^{\phi}|\phi^2 - \phi|}{6}$$

completing the solution.

Question 8

SOLUTION. Notice that the function $T(\theta)$ is 2π periodic since the equator is shaped like a circle. Mathematically this in particular means

$$T(2\pi) = T(0).$$

We exploit this fact by looking at

$$f(0) = T(\pi) - T(0)$$

and

$$\begin{aligned} f(\pi) &= T(\pi + \pi) - T(\pi) \\ &= T(2\pi) - T(\pi) \\ &= T(0) - T(\pi) \\ &= -f(0). \end{aligned}$$

At this point we want to distinguish two cases:
Case 1. *f(0) = 0.* Then

$$0 = f(0) = T(\pi) - T(0)$$

and hence $T(\pi) = T(0)$, which is what we wanted to show.
Case 2. *f(0)* $\neq$ *0.* Then we have that one of $f(0)$ and $f(\pi)$ is positive and one is negative. As T is continuous, we have that f is continuous and hence we may invoke the intermediate value theorem to see that there exists a point $c \in (0, \pi)$ such that $0 = f(c) = T(c + \pi) - T(c)$. Thus, we have $T(c + \pi) = T(c)$ and this completes the proof.

Good Luck for your exams!