Full Solutions
MATH101 April 2009

December 6, 2014

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Educational Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Exam Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Educational Resources.

Question 1 (a)

Easiness: 91/100

SOLUTION. If we simply spit the fraction in two, we obtain simple terms that we an easily integrate.

$$\begin{aligned}\int \frac{3+x^5}{\sqrt{x}}\,dx &= \int 3x^{-1/2} + x^{9/2}\,dx \\ &= 6x^{1/2} + \frac{2}{11}x^{11/2} + C\end{aligned}$$

Question 1 (b)

Easiness: 75/100

SOLUTION. We recognize that this has the form of a Riemann sum:

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

We identify

$$\Delta x = \frac{b-a}{n} = \frac{\pi}{4n}$$

and

$$x_i = a + i\Delta x = a + \frac{i\pi}{4n}.$$

This gives us that

$$b - a = \frac{\pi}{4}$$

Now, notice that the function in question is $f(x) = \tan(x)$ and so

$$\begin{aligned}f(x_i) &= \tan(x_i) \\ &= \tan\left(a + \frac{i\pi}{4n}\right) \\ &= \tan\left(\frac{i\pi}{4n}\right)\end{aligned}$$

and so we see that we can choose $a = 0$. This gives us that

$$b = b - 0 = b - a = \frac{\pi}{4}.$$

Hence, the integral is

$$\int_0^{\frac{\pi}{4}} \tan(x)\,dx$$

completing the question.

Question 1 (c)

Easiness: 100/100

SOLUTION. If we use the substitution

$$u = 2y + y^2$$

then

$$du = (2 + 2y)\,dy = 2(1 + y)\,dy$$

and

$$\begin{aligned}\int_0^1 (y+1)\sqrt{2y+y^2}\,dy &= \frac{1}{2}\int_0^3 \sqrt{u}\,du \\ &= \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_0^3 \\ &= \frac{1}{3}3^{3/2} \\ &= \sqrt{3}.\end{aligned}$$

Question 1 (d)

Easiness: 85/100

SOLUTION. Integrating by parts,

$$\begin{aligned}\int x^2 \ln x\,dx &= \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\frac{1}{x}\,dx \\ &= \frac{x^3}{3}\ln x - \int \frac{x^2}{3}\,dx \\ &= \frac{x^3}{3}\ln x - \frac{x^3}{9} + C.\end{aligned}$$

Question 1 (e)

SOLUTION. To solve this question, the question itself hints that we should be thinking as x as a function of y.

$$x = 100 + 2y^{3/2}$$

Now, $\frac{dx}{dy} = 2(3/2)y^{3/2-1} = 3\sqrt{y}$
Thus, the length is given by
$\int_0^{11} \sqrt{1+(3\sqrt{y})^2}\,dy = \int_0^{11} \sqrt{1+9y}\,dy$
Let $u = 1 + 9y$ so that $du = 9dy$. Changing the endpoints gives $u(0) = 1 + 9(0) = 1$ and $u(11) = 1 + 9(11) = 100$. Thus,

$$
\begin{aligned}
\int_0^{11} \sqrt{1+(3\sqrt{y})^2}\,dy &= \int_0^{11} \sqrt{1+9y}\,dy \\
&= \frac{1}{9}\int_1^{100} \sqrt{u}\,du \\
&= \frac{1}{9}\left(2u^{3/2}/3\right)\Big|_1^{100} \\
&= \frac{2}{27}\left(u^{3/2}\right)\Big|_1^{100} \\
&= \frac{2}{27}(100^{3/2}-1^{3/2}) \\
&= \frac{2}{27}(10^3-1) \\
&= \frac{2}{27}(999) \\
&= 74
\end{aligned}
$$

Question 1 (f)

Easiness: 71/100

SOLUTION 1. (The following solution does not take advantage of the symmetry about $\theta = \pi/4$. Solution 2 does.)

The average value of $|\sin\theta - \cos\theta|$ over the interval $0 \le \theta \le \pi/2$ is

$$f_{ave} = \frac{1}{\pi/2}\int_0^{\pi/2} |\sin\theta - \cos\theta|\,d\theta.$$

In order to simplify the absolute value, we split the domain of integration into regions where $\sin\theta - \cos\theta$ is positive and negative. Then

$$
\begin{aligned}
f_{ave} &= \frac{2}{\pi}\left(\int_0^{\pi/4}(\cos\theta - \sin\theta)d\theta + \int_{\pi/4}^{\pi/2}(\sin\theta - \cos\theta)d\theta\right) \\
&= \frac{2}{\pi}\left([\sin\theta + \cos\theta]_0^{\pi/4} + [-\cos\theta - \sin\theta]_{\pi/4}^{\pi/2}\right) \\
&= \frac{2}{\pi}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1 - 0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) \\
&= \frac{2}{\pi}\left(\frac{4}{\sqrt{2}} - 2\right) \\
&= \frac{4}{\pi}\left(\sqrt{2} - 1\right)
\end{aligned}
$$

SOLUTION 2. (The following solution takes advantage of symmetry about $\theta = \pi/4$.)
The average value of $|\sin\theta - \cos\theta|$ over the interval $0 \le \theta \le \pi/2$ is

$$f_{ave} = \frac{1}{\pi/2}\int_0^{\pi/2} |\sin\theta - \cos\theta|\,d\theta.$$

If we draw a picture of the integrand, we can see that it is symmetric about $\theta = \pi/4$. Motivated by this symmetry, we make a change of variables $\theta = \pi/4 + x$.

$$f_{ave} = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} \left|\sin(\pi/4+x) - \cos(\pi/4+x)\right| dx.$$

Now, this integrand is symmetric about the origin since

$$\begin{aligned}
\left|\sin(\pi/4+x) - \cos(\pi/4+x)\right| &= \left|\cos(\pi/2-(\pi/4+x)) - \sin(\pi/2-(\pi/4+x))\right| \\
&= \left|\cos(\pi/4-x) - \sin(\pi/4-x)\right| \\
&= \left|\sin(\pi/4-x) - \cos(\pi/4-x)\right|.
\end{aligned}$$

Hence,

$$f_{ave} = \frac{2}{\pi}\cdot 2\int_{0}^{\pi/4} \left|\sin(\pi/4+x) - \cos(\pi/4+x)\right| dx.$$

Since $\sin(\pi/4+x) - \cos(\pi/4+x)$ is positive for $0 \le x \le \pi/4$,

$$\begin{aligned}
f_{ave} &= \frac{4}{\pi}\int_{0}^{\pi/4} \left(\sin(\pi/4+x) - \cos(\pi/4+x)\right) dx \\
&= \frac{4}{\pi}\left[-\cos(\pi/4+x) - \sin(\pi/4+x)\right]_0^{\pi/4} \\
&= \frac{4}{\pi}\left(-\cos(\pi/2) - \sin(\pi/2) + \cos(\pi/4) + \sin(\pi/4)\right) \\
&= \frac{4}{\pi}\left(0 - 1 + \frac{2}{\sqrt{2}}\right) \\
&= \frac{4}{\pi}\left(\sqrt{2} - 1\right)
\end{aligned}$$

Question 1 (g)

Easiness: 77/100

SOLUTION. Since the Maclaurin series for e^y is
$\sum_{n=0}^{\infty} \frac{y^n}{n!}$,
the Maclaurin series for e^{-x^2} is
$\sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} = 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \ldots$
Hence, the first few nonzero terms of the Maclaurin series for the integrand $\frac{e^{-x^2}-1}{x}$ are

$$\frac{1}{x}\left(\left(1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \ldots\right) - 1\right) = -x + \frac{1}{2}x^3 - \frac{1}{6}x^5 + \ldots.$$

Anti-differentiating these, we obtain the first few nonzero terms of the indefinite integral

$$\int \frac{e^{-x^2}-1}{x}\, dx,$$

$C - \frac{x^2}{2} + \frac{1}{8}x^4 - \frac{1}{36}x^6 + \ldots$

Question 1 (h)

Solution. When $y = x^r$, we know that $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Hence, we have

$$\begin{aligned} 0 &= x^2y'' + 4xy' + 2y \\ &= x^2(r(r-1)x^{r-2}) + 4x(rx^{r-1}) + 2x^r \\ &= (r^2 - r)x^r + 4rx^r + 2x^r \\ &= (r^2 + 3r + 2)x^r \end{aligned}$$

Now, since $x > 0$ always, we must have that the coefficient of x^r above has to be 0. Thus, we solve

$$0 = r^2 + 3r + 2 = (r+1)(r+2)$$

and so $r = -1$ and $r = -2$ completing the question.

Question 1 (i)

Easiness: 82/100

Solution. Using the fundamental theorem of calculus, we have

$$\begin{aligned} f'(x) &= \frac{d}{dx}\int_x^{e^x} \sqrt{\cos t}\, dt \\ &= \frac{d}{dx}\left(\int_x^0 \sqrt{\cos t}\, dt + \int_0^{e^x} \sqrt{\cos t}\, dt\right) \\ &= \frac{d}{dx}\left(-\int_0^x \sqrt{\cos t}\, dt + \int_0^{e^x} \sqrt{\cos t}\, dt\right) \\ &= -\sqrt{\cos x} + e^x\sqrt{\cos e^x} \end{aligned}$$

where in the last equality on the right most sum, we used the chain rule to compute the derivative. This completes the question.

Question 1 (j)

Easiness: 100/100

Solution. The formula for the Simpson's rule is given by

$$S_n = \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + \ldots + 4f(x_{n-1}) + f(x_n)\right)$$

When $n = 4$, we have that

$$x_i = a + i\Delta x = 0 + \frac{(2-0)i}{4} = \frac{i}{2}$$

and thus, the Simpson's rule approximation that we need is

$$\begin{aligned} S_n &= \frac{1}{2\cdot 3}\left(f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + f(2)\right) \\ &= \frac{1}{6}(\sin(e^0) + 4\sin(e^{0.5}) + 2\sin(e^1) + 4\sin(e^{1.5}) + \sin(e^2)) \end{aligned}$$

Question 2

Easiness: 100/100

SOLUTION. First, write this integral as
$\int_e^\infty \frac{dx}{x(\ln x)^p} = \lim_{b\to\infty} \int_e^b \frac{dx}{x(\ln x)^p}$
Let $u = \ln(x)$. Then $du = dx/x$, $u(e) = \ln(e) = 1$ and $u(b) = \ln(b)$. Then

$$\begin{aligned}\int_e^\infty \frac{dx}{x(\ln x)^p} &= \lim_{b\to\infty} \int_e^b \frac{dx}{x(\ln x)^p}\\ &= \lim_{b\to\infty} \int_1^{\ln(b)} \frac{du}{u^p}\end{aligned}$$

At this point, we see that we need to break this up into two cases. The integral above is different when $p = 1$ and when $p \neq 1$. So suppose $p \neq 1$. Then we have

$$\begin{aligned}\int_e^\infty \frac{dx}{x(\ln x)^p} &= \lim_{b\to\infty} \left.\frac{u^{-p+1}}{1-p}\right|_1^{\ln(b)}\\ &= \lim_{b\to\infty} \frac{\ln(b)^{-p+1}}{1-p} - \frac{1}{1-p}\end{aligned}$$

This first term will tend to 0 provided that $1 - p < 0$ so whenever $p > 1$. This term will tend to infinity when $p < 1$ since the logarithmic term stays in the numerator. To take care of the $p = 1$ case that we have left out, we have

$$\begin{aligned}\int_e^\infty \frac{dx}{x(\ln x)^1} &= \lim_{b\to\infty} \int_e^b \frac{dx}{x\ln x}\\ &= \lim_{b\to\infty} \int_1^{\ln(b)} \frac{du}{u}\\ &= \lim_{b\to\infty} \ln(u)|_1^{\ln(b)}\\ &= \lim_{b\to\infty} \ln\ln(b) - \ln(1)\end{aligned}$$

and this diverges. So our integral converges when $p > 1$ and diverges when $p \leq 1$ as required.

Question 3 (a)

Easiness: 100/100

SOLUTION. The first point with $x > 0$ where $\cos(x)$ crosses the x-axis is at $x = \frac{\pi}{2}$. Hence we have that $a = 0$ and $b = \frac{\pi}{2}$. Thus, we have

$$\begin{aligned}A &= \int_a^b f(x)\,dx\\ &= \int_0^{\frac{\pi}{2}} \cos(x)\,dx\\ &= \sin(x)|_0^{\frac{\pi}{2}}\\ &= \sin(\tfrac{\pi}{2}) - \sin(0)\\ &= 1\end{aligned}$$

The x-coordinate of the centroid are computed using the formulas

$$\begin{aligned}\bar{x} &= \frac{1}{A}\int_a^b x f(x)\,dx \\ &= \frac{1}{1}\int_0^{\frac{\pi}{2}} x\cos(x)\,dx\end{aligned}$$

To solve this, we use integration by parts. Let

$$\begin{aligned} u &= x \qquad & v &= \sin(x) \\ du &= dx \qquad & dv &= \cos(x)dx \end{aligned}$$

Then,

$$\begin{aligned}\bar{x} &= \int_0^{\frac{\pi}{2}} x\cos(x)\,dx \\ &= x\sin(x)|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin(x)\,dx \\ &= x\sin(x)|_0^{\frac{\pi}{2}} + \cos(x)_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{2}\sin(\frac{\pi}{2}) - (0)\sin(0) + \cos(\frac{\pi}{2}) - \cos(0) \\ &= \frac{\pi}{2} - 1\end{aligned}$$

Lastly, for the other coordinate, we have

$$\begin{aligned}\bar{y} &= \frac{1}{A}\int_a^b \frac{1}{2}(f(x))^2\,dx \\ &= \frac{1}{1}\int_0^{\frac{\pi}{2}} \frac{1}{2}(f(x))^2\,dx \\ &= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^2(x)\,dx\end{aligned}$$

Now we use the double angle formula
$\cos^2(x) = \frac{1+\cos(2x)}{2}$
to see that

$$\begin{aligned}\bar{y} &= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^2(x)\,dx \\ &= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1+\cos(2x)}{2}\,dx \\ &= \frac{1}{4}x\Big|_0^{\frac{\pi}{2}} + \frac{1}{4}\int_0^{\frac{\pi}{2}} \cos(2x)\,dx \\ &= \frac{1}{4}x\Big|_0^{\frac{\pi}{2}} + \frac{1}{8}\sin(2x)\Big|_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{8} + \frac{1}{8}(\sin(\pi) - \sin(0)) \\ &= \frac{\pi}{8}\end{aligned}$$

and so the centroid is
$(\bar{x}, \bar{y}) = \left(\frac{\pi}{2} - 1, \frac{\pi}{8}\right)$
completing the proof.

Question 3 (b)

Easiness: 75/100

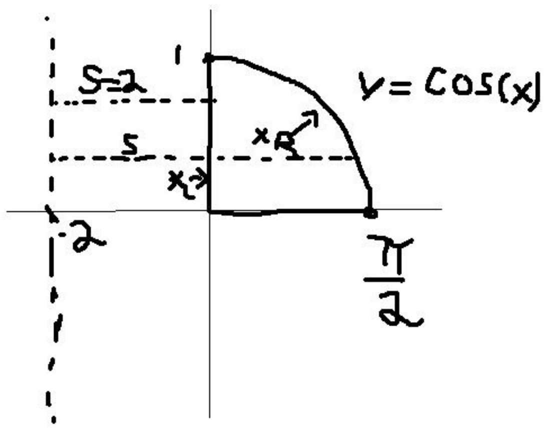
SOLUTION. A picture is included below. To solve this question, we need to use the method of left and right side integration. Recall the formula

$$V = \pi \int_a^b \left((x_R + s)^2 - (x_L + s)^2\right) dy$$

where we think of x_R and x_L as functions of y and the s factor is the shifting factor (so basically this factor accounts for what line we are rotating over - the picture should make this clear). Here, the right most curve is $x_R = \arccos(y)$ and the left most curve is $x_L = 0$ and the s factor in this case is *2*. The last thing is the limits of integration. we know on the x-axis, from the previous problem, we were integrating from 0 to $\frac{\pi}{2}$. Here however we are integrating from the y-axis so we want to know when $\cos(x)$ crosses the y-axis. This occurs when $y = 1$. Thus, we have

$$V = \pi \int_0^1 \left((\arccos(y) + 2)^2 - (2)^2\right) dy = \pi \int_0^1 \left((\arccos(y))^2 + 4\arccos(y)\right) dy$$

To challenge yourself, try showing the above is actually equal to $\pi^2 + 2\pi$.



Question 4 (a)

Easiness: 62/100

SOLUTION. Following the hint, we let

$$x = 2\sin\theta$$

Then

$$dx = 2\cos\theta d\theta$$

and so we have

$$
\begin{aligned}
\int (4-x^2)^{-3/2}\,dx &= \int \frac{dx}{\sqrt{4-x^2}^3} \\
&= \int \frac{2\cos\theta d\theta}{\sqrt{4-4\sin^2\theta}^3} \\
&= \int \frac{2\cos\theta d\theta}{\sqrt{4-4\sin^2\theta}^3} \\
&= \int \frac{2\cos\theta d\theta}{\sqrt{4\cos^2\theta}^3} \\
&= \int \frac{2\cos\theta d\theta}{8\cos^3\theta} \\
&= \int \frac{d\theta}{4\cos^2\theta} \\
&= \frac{1}{4}\int \sec^2\theta \\
&= \frac{\tan\theta}{4} + C
\end{aligned}
$$

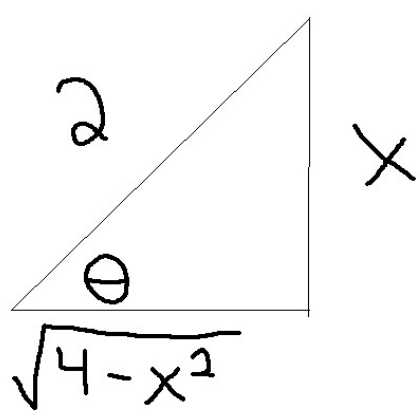
To substitute back to an expression with x you need to undo the original substitution $\sin\theta = x/2$. With the help of a little diagram (included below) and the Pythagorean theorem to see that

$$\tan\theta = \frac{x}{\sqrt{4-x^2}}$$

and hence

$$\int (4-x^2)^{-3/2}\,dx = \frac{\tan\theta}{4} + C = \frac{x}{4\sqrt{4-x^2}} + C$$

completing the question.



Question 4 (b)

Easiness: 80/100

SOLUTION. Following the hint, we start off by setting
$u = \sqrt{1+e^x}$

Then we have that
$u^2 - 1 = e^x$
and more importantly that
$du = \frac{e^x}{2\sqrt{1+e^x}}\,dx = \frac{u^2-1}{2u}\,dx$
Thus, substituting these values in yield
$\int \sqrt{1+e^x}\,dx = \int u\left(\frac{2u}{u^2-1}\,du\right) = \int \frac{2u^2}{u^2-1}\,du$
Via the third hint, we can simplify this last integral to
$\int \frac{2u^2}{u^2-1}\,du = \int \frac{2u^2-2+2}{u^2-1}\,du = \int 2\,du + \int \frac{2}{u^2-1}\,du$
(one could find this expression by explicitly doing the long division though on an exam you probably want to try to save some time). The first integral is
$\int 2\,du = 2u = 2\sqrt{1+e^x}$
(we'll add the constant in later) and the second integral we evaluate by partial fractions.
$\frac{2}{u^2-1} = \frac{2}{(u+1)(u-1)} = \frac{A}{u+1} + \frac{B}{u-1} = \frac{A(u-1)+B(u+1)}{(u+1)(u-1)}$
This gives the expression
$2 = A(u-1) + B(u+1)$
Plugging in $u = 1$ yields
$2 = A(1-1) + B(1+1) = 2B$
and so $B = 1$. Plugging in $u = -1$ yields
$2 = A((-1)-1) + B((-1)+1) = -2A$
and so $A = -1$. Hence
$\int \frac{2}{u^2-1}\,du = \int -\frac{1}{u+1}\,du + \int \frac{1}{u-1}\,du = -\ln|u+1| + \ln|u-1|$
(we'll add the constant in later). Hence, applying everything, we have

$$\begin{aligned}
\int \sqrt{1+e^x}\,dx &= \int \frac{2u^2}{u^2-1}\,du \\
&= \int 2\,du + \int \frac{2}{u^2-1}\,du \\
&= 2u - \ln|u+1| + \ln|u-1| + C \\
&= 2\sqrt{1+e^x} - \ln|\sqrt{1+e^x}+1| + \ln|\sqrt{1+e^x}-1| + C
\end{aligned}$$

completing the question.

Question 5

Solution 1. **(a)** Using hint 1 we see that k needs to be a positive constant. We can calculate k directly by integrating:

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{0} 0\,dx + \int_0^1 kx(1-x^4)\,dx + \int_1^{\infty} 0\,dx \\
&= k\int_0^1 (x - x^5)\,dx \\
&= k\left(\frac{x^2}{2} - \frac{x^6}{6}\right)\Big|_0^1 \\
&= k\left(\frac{1}{2} - \frac{1}{6}\right) \\
&= \frac{k}{3}.
\end{aligned}$$

The only way that this integral has value 1 is when $\boldsymbol{k = 3}$.

Solution 2. **(b)** Using the second hint and the fact that $k = 3$ the calculation is straight forward

$$\begin{aligned}
\mu &= \int_{-\infty}^{\infty} x f(x)\, dx \\
&= \int_0^1 (x) 3x(1-x^4)\, dx \\
&= 3\int_0^1 (x^2 - x^6)\, dx \\
&= 3\left(\frac{x^3}{3} - \frac{x^7}{7}\right)\Big|_0^1 \\
&= 3\left(\frac{1}{3} - \frac{1}{7}\right) \\
&= \frac{4}{7}.
\end{aligned}$$

Solution 3. **(c)** Following the third hint we need to find that value m which satisfies

$$\int_{-\infty}^{m} f(x)\, dx = \frac{1}{2} = \int_m^{\infty} f(x)\, dx.$$

Since $\int_{-\infty}^{0} f(x)\, dx = 0 = \int_1^{\infty} f(x)\, dx$ it is clear that m needs to be a number between 0 and 1. Therefore the algebraic equation that m satisfies is the following;

$$\begin{aligned}
\frac{1}{2} &= \int_{-\infty}^{m} f(x)\, dx \\
&= \int_0^m 3x(1-x^4)\, dx \\
&= 3\left(\frac{x^2}{2} - \frac{x^6}{6}\right)\Big|_0^m \\
&= 3\left(\frac{m^2}{2} - \frac{m^6}{6}\right).
\end{aligned}$$

Question 6 (a)

Solution. We proceed as in the hint
$\frac{dy}{dx} = xy^2$
becomes
$\frac{dy}{y^2} = xdx$
Integrating both sides gives
$-y^{-1} = \frac{x^2}{2} + C$
Plugging in $y(0) = 1$ to solve for the constant yields
$-1 = C$
Therefore,
$-\frac{1}{y} = \frac{x^2}{2} - 1$
and isolating for y, we have

$y = \frac{2}{2-x^2}$
completing the question.

Question 6 (b)

SOLUTION. Following the hint, we multiply both sides by the integrating factor e^{-2t} to get

$$e^{-2t}\frac{dy}{dt} - 2ye^{-2t} = 4e^{-2t} + e^t$$

The point of the integrating factor is that we can now express the left hand side as simple derivative:

$$e^{-2t}\frac{dy}{dt} - 2ye^{-2t} = (ye^{-2t})'$$

Plugging this into the initial equation above we find

$$(ye^{-2t})' = 4e^{-2t} + e^t$$

and by integrating both sides we obtain

$$ye^{-2t} = -2e^{-2t} + e^t + C$$

Solving for y yields

$$y = -2 + e^{3t} + Ce^{2t}$$

completing the question.

Question 7 (a)

SOLUTION. *Hydrostatic Force was not covered in the 2012 offering of this course. Time might better be spent solving other problems given how soon the exam is coming.*
Consider the triangular side pictured to the right 300px|thumbnail|right. It has height 2 and length 1. Also consider a little rectangular sliver in the picture of length L(y) and height dy. The reason we consider rectangular pieces with constant height is because the pressure depends on height and so we would like the effect from height to be the same on any given rectangular piece. We will use the labelling in the diagram with the bottom vertex being (0,0), the right vertex being (1/2,2) and the left vertex being (-1/2,2). With our origin at the bottom, we will label y from there. This means that the height (as measured from the top of the surface) is (2-y). We then have that the force on any given rectangle, dF, is,

$$dF = \rho g(2-y)L(y)\mathrm{d}y.$$

We could then get the total force by integrating dF but first we need an expression for the length in terms of the height, y. The length of any rectangle will start at the left edge of the triangle and end at the right edge. We can use the coordinates that we listed to find equations for these lines. The right edge is given by

$$y = 4x$$

which written in terms of y is,

$$x_R = \frac{y}{4}$$

where we have added a subscript R to indicate it's the right edge. Notice we compute the line by finding the slope using the right vertex and bottom vertex and then notice that the line goes through our origin to its y-intercept is zero. We can similarly get that the left edge is

$$x_L = -\frac{y}{4}.$$

This can also be argued with symmetry of the right edge. We then have that

$$L(y) = x_R - x_L = \frac{y}{4} - \left(-\frac{y}{4}\right) = \frac{y}{2}.$$

Therefore we can now get F via integration where we see that y starts at zero and ends at y=2. Therefore,

$$F = \int_0^2 \rho g(2-y)L(y)\mathrm{d}y = \int_0^2 \rho g \frac{2y-y^2}{2}\mathrm{d}y = \rho g \left(\frac{y^2}{2} - \frac{y^3}{6}\right)\Bigg|_0^2 = \frac{2\rho g}{3}.$$

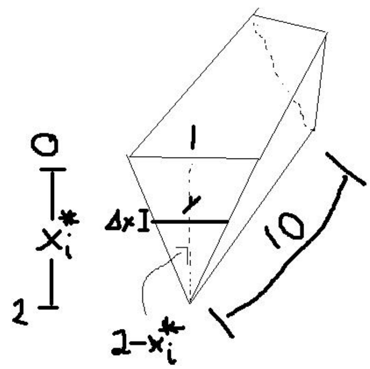
We are told in the problem that the density is 1000kg/m^3 and we take the gravitational constant as 9.8N/kg. Therefore,

$$F = \frac{2}{3}980 = \frac{1960}{3}N.$$

This is the hydrostatic force on each of the triangular sides.

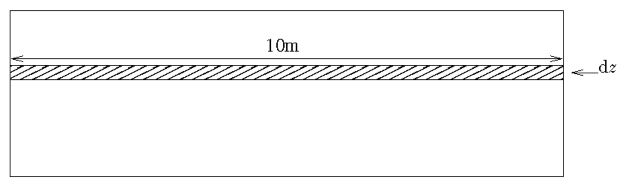
Question 7 (b)

SOLUTION. A picture is included at the bottom of the page. Let x_i^* be a sample point.
First, we use similar triangles as in the diagram to see that
$\frac{1}{2} = \frac{y}{2-x_i^*}$
Isolating yields that
$y = 1 - \frac{x_i^*}{2}$
Now we use hint 3 and write down the necessary formulas. For volume of the ith piece, we have
$V_i = y\Delta x \cdot 10 = 10(1 - \frac{x_i^*}{2})\Delta x = (10 - 5x_i^*)\Delta x$
For the mass of the ith piece, where $density = \rho = 1000$, we have
$m_i = V \cdot \rho = (10 - 5x_i^*)\Delta x(1000) = 1000(10 - 5x_i^*)\Delta x$
For the force, where $g = 9.8$, we have
$F_i = m_i a = 9.8(1000(10 - 5x_i^*)\Delta x) = 9800(10 - 5x_i^*)\Delta x$
For the work on the ith piece, we have
$W_i = F_i \cdot d = 9800(10 - 5x_i^*)\Delta x(x_i^*) = 9800x_i^*(10 - 5x_i^*)\Delta x$
and so the total work done is
$W = \lim_{n\to\infty} \sum_{i=1}^n W_i = \int_0^2 9800x(10-5x)\,dx$
completing the question.



Question 7 (c)

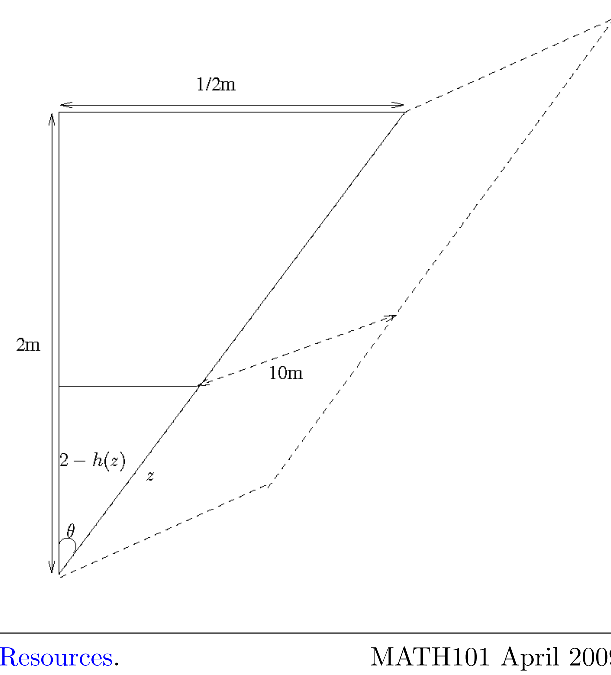
SOLUTION. *Hydrostatic Force was not covered in the 2012 offering of this course. Time might better be spent solving other problems given how soon the exam is coming.*



Consider the rectangular side picture right. We will define the coordinate along with width of the rectangle as z. Consider a thin rectangle at a fixed z with length 10 (as pictured). The hydrostatic force on this rectangle is,

$$dF = \rho g h(z) 10 \mathrm{d}z.$$

Notice that the depth here is a function of z since each little rectangle will correspond to a different depth in the tank. To determine the depth as a function of z consider the side view of the rectangular side where the



width is actually the side of a triangle (pictured left) .

We have only draw half of the triangle to highlight that we can use right triangle trigonometry. We know that the height of this triangle is 2 and the length is 1/2 (half of the base of the full triangle). From the diagram, we define θ to be the angle between the vertical line and the hypotenuse and we have that

$$\theta = \arctan\left(\frac{1}{4}\right).$$

Recall that the variable z indicates the distance along the hypotenuse. For a given z we can get the vertical length of the corresponding right triangle, y, as

$$y = 2 - h(z) = z\cos\theta = \frac{4}{\sqrt{17}}z.$$

Therefore we have that

$$h(z) = 2 - \frac{4}{\sqrt{17}}z$$

which we can substitute into our hydrostatic force to get,

$$dF = \rho g\left(2 - \frac{4}{\sqrt{17}}z\right)10\mathrm{d}z.$$

Now z starts at the bottom of the rectangle (which we will define as z=0) and ends at the full length of the triangular side. We can use Pythagorean theorem to get that the full length of that side is $\sqrt{2^2 + (1/2)^2} = \frac{\sqrt{17}}{2}$. Therefore we have that the hydrostatic force is,

$$F = \int_0^{\frac{\sqrt{17}}{2}} \rho g\left(2 - \frac{4}{\sqrt{17}}z\right)10\mathrm{d}z.$$

Since we are asked to set up the integral but not evaluate it, we are now done.

Question 8 (a)

Easiness: 100/100

SOLUTION. Cross multiplying our equation, we have that

$$\frac{1}{(0.1-x)(0.2-x)}\,dx = k\,dt$$

Now we integrate. The right hand side integrates to

$$\int k\,dt = kt$$

As for the left hand side, we integrate by partial fractions.

$$\begin{aligned}\frac{1}{(0.1-x)(0.2-x)} &= \frac{A}{(0.1-x)} + \frac{B}{(0.2-x)}\\ &= \frac{A(0.2-x)+B(0.1-x)}{(0.1-x)(0.2-x)}\\ &= \frac{(-A-B)x + A(0.2) + B(0.1)}{(0.1-x)(0.2-x)}\end{aligned}$$

Now we can either compare coefficients or plug in values for x. I will choose to plug in values for x. The above tells us that

$$1 = A(0.2 - x) + B(0.1 - x)$$

when $x = 0.2$, we have that

$$1 = A(0.2 - 0.2) + B(0.1 - 0.2) = B(-0.1)$$

and so $B = -10$. When $x = 0.1$, we have

$$1 = A(0.2 - 0.1) + B(0.1 - 0.1) = A(0.1)$$

and so $A = 10$. Hence, we have

$$\begin{aligned}\int \frac{1}{(0.1-x)(0.2-x)}\,dx &= \int \frac{10}{(0.1-x)}\,dx - \int \frac{10}{(0.2-x)}\,dx \\ &= -10\ln|0.1-x| + 10\ln|0.2-x|.\end{aligned}$$

(technically the above should add a constant but we can include this in a minute.) Hence

$$\int \frac{1}{(0.1-x)(0.2-x)}\,dx = \int k\,dt$$

becomes

$$10\ln|0.2-x| - 10\ln|0.1-x| = kt + C$$

Using the initial condition of $x(0) = 0$, we see that our constant is

$$10\ln|0.2| - 10\ln|0.1| = C$$

and simplifying, this becomes

$$C = 10(\ln \tfrac{0.2}{0.1}) = 10\ln(2)$$

Our equation simplifies to

$$10\ln\left|\frac{0.2-x}{0.1-x}\right| = 10\ln|0.2-x| - 10\ln|0.1-x| = kt + 10\ln(2)$$

Exponentiating the far left and the far right hand sides of the above gives

$$\left|\frac{0.2-x}{0.1-x}\right|^{10} = 2^{10}e^{kt}$$

Taking the tenth root of both sides yields

$$\left|\frac{0.2-x}{0.1-x}\right| = 2e^{\frac{kt}{10}}$$

Removing the absolute values, we have

$$\frac{0.2 - x}{0.1 - x} = \pm 2e^{\frac{kt}{10}}$$

Plugging in the initial condition of $x(0) = 0$, we see that only the positive value above works. Hence

$$\frac{0.2 - x}{0.1 - x} = 2e^{\frac{kt}{10}}$$

Now, we cross multiply to see that

$$0.2 - x = 2e^{\frac{kt}{10}}(0.1 - x) = 0.2e^{\frac{kt}{10}} - x(2e^{\frac{kt}{10}})$$

Bringing the x values to one side gives

$$x(2e^{\frac{kt}{10}}) - x = 0.2e^{\frac{kt}{10}} - 0.2$$

Lastly, isolating for x gives

$$x = \frac{0.2e^{\frac{kt}{10}} - 0.2}{2e^{\frac{kt}{10}} - 1}$$

completing the question.

Question 8 (b)

Easiness: 51/100

SOLUTION. From the previous part, we know that
$x(t) = \dfrac{0.2e^{\frac{kt}{10}} - 0.2}{2e^{\frac{kt}{10}} - 1}$
taking the limit as t tends to infinity yields
$\lim_{t\to\infty} \frac{0.2e^{\frac{kt}{10}} - 0.2}{2e^{\frac{kt}{10}} - 1} = \lim_{t\to\infty} \frac{0.2 - 0.2e^{\frac{-kt}{10}}}{2 - e^{\frac{-kt}{10}}} = \frac{0.2}{2} = \frac{1}{10}$
Thus, the concentration tends to $\frac{1}{10} = 0.1$ as t tends to infinity.

Good Luck for your exams!