Full Solutions
MATH307 December 2010

April 5, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

SOLUTION 1. Since the matrix norm of a diagonal matrix with diagonal entries $\lambda_1, \lambda_2, ..., \lambda_n$ is the largest value of $|\lambda_k|$, if $|a| \le 2$, then $\|A\| = 2$.

SOLUTION 2. The matrix norm is given by
$\| A \| = \max_{\vec{x}, \|\vec{x}\|=1} \| A\vec{x} \|$
Let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, where $\| \vec{x} \| = 1$
For $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 2 \end{bmatrix}$
$A\vec{x} = \begin{pmatrix} x_1 \\ ax_2 \\ 2x_3 \end{pmatrix}$
So,

$$\begin{aligned} \| A\vec{x} \| &= (x_1^2 + a^2x_2^2 + 4x_3^2)^{0.5} \leqslant (\max\{1, a^2, 4\}(x_1^2 + x_2^2 + x_3^2))^{0.5} \\ &= \max\{1, |a|, 2\}(x_1^2 + x_2^2 + x_3^2)^{0.5} \\ &= \max\{1, |a|, 2\} \end{aligned}$$

Notice that each of these values is obtained by taking our vector x to be one of the unit basis vectors. Thus for any unit vector $\vec{x}$ we have established that $\| A \| = \max\{1, |a|, 2\}$.
Going by this, $\| A \| = 2 \Leftrightarrow |a| \leqslant 2$

The vector $\vec{x}$ that will maximize the magnitude of $||A\vec{x}||$ in this example is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ if $|a| \le 2$

Question 1 (b)

SOLUTION. Please refer to part (a) regarding the computation of $\| A \|$
Recall that
$\text{cond}(A) = \| A \| \cdot \| A^{-1} \|$
With A being a diagonal matrix, its inverse is given by:
$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{a} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$
valid for all values where a is nonzero. When a is zero, this inverse, and hence cond(A) is undefined. Similar to $\| A \|$, we have $\| A^{-1} \| = \max\{\| 1 \|, \| \frac{1}{a} \|, \| \frac{1}{2} \|\}$ and these are defined for all nonzer values of a.
So cond(A) is defined for all values of a, except a = 0.

Question 1 (c)

SOLUTION. Recall that
$\text{cond}(A) = \|A\| \left\|A^{-1}\right\|$
Now, since A is diagonal matrix, it is easy to find A^{-1}.
$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{a} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$
If $|a| > 1$, $\left\|A^{-1}\right\| = 1$ since $\max\left\{|1|, \left|\frac{1}{a}\right|, \left|\frac{1}{2}\right|\right\} = 1$.
And $\|A\| \ge 2$ since $\max\{|1|, |a|, |2|\} \ge 2$.

Thus, $\left\|A^{-1}\right\| \|A\| \geq 2$.
Therefore, it is impossible to get $\text{cond}(A) = \frac{1}{2}$.
If $|a| \leq 1$, $\|A\| = 2$ since $\max\{|1|, |a|, |2|\} = 2$.
And $\left\|A^{-1}\right\| \geq 1$ since $\max\left\{|1|, \left|\frac{1}{a}\right|, \left|\frac{1}{2}\right|\right\} \geq 1$.
Thus, $\left\|A^{-1}\right\| \|A\| \geq 2$.
Hence, again it is impossible to get $\text{cond}(A) = \frac{1}{2}$.
Therefore, there is no such value a which gives $\text{cond}(A) = \frac{1}{2}$

Question 1 (d)

SOLUTION. For convenience, let's recall our results from part (b) and part(c):

$$\begin{aligned}\text{cond}(A) &= \|A\|\|A^{-1}\| = \max\{1, |a|, 2\} \max\{1, 1/|a|, 1/2\} \\ &= \max\{|a|, 2\} \max\{1, 1/|a|\}\end{aligned}$$

and *cond(A)* is not defined when a = *0*. This suggests considering three cases for $|a|$:

- Case 1: If $0 < |a| \leq 1$ then

$$\text{cond}(A) = (2)(1/|a|) = 2/|a|$$

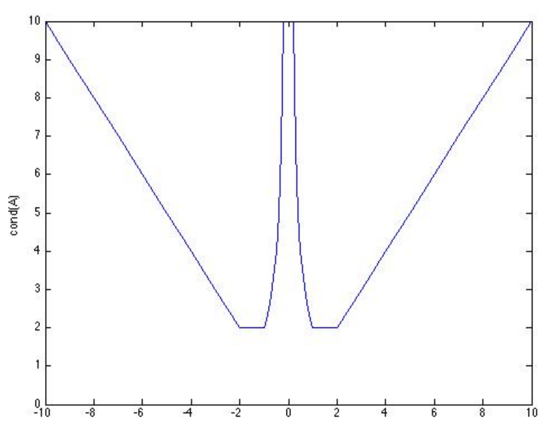
- Case 2: If $1 \leq |a| \leq 2$ then

$$\text{cond}(A) = (2)(1) = 2$$

- Case 3: If $2 \leq |a|$ then

$$\text{cond}(A) = |a|(1) = |a|$$

So, the sketch looks like this:



Question 1 (e)

SOLUTION. We use the three regions from part (d).

- Case 1: If $0 < |a| \leq 1$ then

$$4 = \text{cond}(A) = 2/|a|$$

has the solutions $|a|$ = *1/2*, that is, a = *-1/2* and a = *1/2*.

- Case 2: If $1 \leq |a| \leq 2$ then

$$4 = \text{cond}(A) = 2$$

has no solutions.

- Case 3: If $2 \leq |a|$ then

$$4 = \text{cond}(A) = |a|$$

has the solutions a = *-4* and a = *4*.
Overall we have that *cond(A)* = *4* holds if a = *-4*, a = *-1/2*, a = *1/2* or a = *4*.

Question 2 (a)

SOLUTION. Given rref(A) all we will need to find are the pivot columns in $R(A)$ and find those same columns in the matrix A, giving $R(A)$.
Columns 1, 3, and 5 correspond to the pivot columns in rref(A) so columns 1, 3, and 5, give the basis for $R(A)$.
Hence, a basis of $R(A)$ is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Question 2 (b)

SOLUTION. Considering $A\vec{x} = \vec{0}$ with $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$. From the above rref, with x_2 and x_4 free, the following equations can be derived:

$$\begin{aligned} x_1 &= -x_2 - x_4 \\ x_2 &= x_2 \\ x_3 &= -x_4 \\ x_4 &= x_4 \\ x_5 &= 0 \end{aligned}$$

Therefore $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

If we for example let $x_2 = x_4 = 1$, then a basis for N(A) is given by

span $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

Question 2 (c)

SOLUTION. For $R(A^T)$ take the pivot rows of *rref(A)* as columns. Hence, a basis for $R(A^T)$ is given by

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Question 2 (d)

SOLUTION. First, *rank(A)* is the number of pivot columns in A (or *rref(A)*), which is 3.
From the Rank-nullity theorem, since A^T is a 5×4 matrix: $dim(N(A^T)) = 4 - rank(A^T)$. Since the rank of A equals the rank of A^T we obtain $dim(N(A^T)) = 4 - 3 = 1$.

Question 2 (e)

SOLUTION. To begin with, we define the matrix A:

```
A = [1 1 0 1 0; 2 2 1 3 0; 3 3 1 4 1; 4 4 1 5 1];
```

Then, use the formula for the projection matrix to define P:

```
P = A*inv(A'*A)*A';
```

Finally, compute the projection of the given vector:

```
result = P*[1; 0; 0; 0]
```

Question 3 (a)

SOLUTION. Since the piece-wise function above has inclusive restrictions, we can see that both polynomials, p_1 and p_2, have the same value $f(x_2)$. The same goes for polynomials p_2 and p_3.
Thus, we need to satisfy

$$p_i(x) = p_{i+1}(x)$$

Therefore,

$$p_1(x_2) = p_2(x_2)$$
$$p_2(x_3) = p_3(x_3)$$

Hence these equations imply that $f(x)$ is continuous as they show that both polynomials surrounding the given point equal each other at x_i.
As well, to show that the function goes through the given points, we need to write the following six equations:

$$\begin{aligned} p_1(x_1) &= y_1 \\ p_1(x_2) &= y_2 \\ p_2(x_2) &= y_2 \\ p_2(x_3) &= y_3 \\ p_3(x_3) &= y_3 \\ p_3(x_4) &= y_4 \end{aligned}$$

Question 3 (b)

SOLUTION. The function $f'(x)$ being continuous implies that the slopes of the adjacent polynomials must be the same at x_i, for $i = 2, 3$. In our case this implies

$$\begin{aligned} p_1'(x_2) &= p_2'(x_2) \\ p_2'(x_3) &= p_3'(x_3) \end{aligned}$$

Question 3 (c)

SOLUTION. The function $f''(x)$ is continuous if the second derivative of adjacent polynomials coincide at x_2 and x_3. In our case this means

$$\begin{aligned} p_1''(x_2) &= p_2''(x_2) \\ p_2''(x_3) &= p_3''(x_3) \end{aligned}$$

Question 3 (d)

SOLUTION. In part (a) we were asked to find the different equations that can be drawn from the condition that $f(x)$ goes through all the given points. This yielded the following 6 equations:

$$\begin{aligned} p_1(x_1) &= y_1 \\ p_1(x_2) &= y_2 \\ p_2(x_2) &= y_2 \\ p_2(x_3) &= y_3 \\ p_3(x_3) &= y_3 \\ p_3(x_4) &= y_4 \end{aligned}$$

In part (b) we were then asked to find the equations that can be found from the condition that the function, $f(x)$, was also first order differentiable. This yielded 2 more equations:

$$p_1'(x_2) = p_2'(x_2)$$
$$p_2'(x_3) = p_3'(x_3)$$

This same method can be applied in part (c) to find the conditions that the function is second order differentiable yielding 2 more equations:

$$p_1''(x_2) = p_2''(x_2)$$
$$p_2''(x_3) = p_3''(x_3)$$

Therefore, we were able to find 10 equations in total using the conditions set forth in (a), (b), and (c). However, we have three polynomials and each of these polynomials have 4 unknowns (a_i, b_i, c_i, and d_i ; $i = 1, 2, 3$), so we have a total of 12 unknowns that we need to solve for. It follows that we would need 2 more equations in order to have the same number of equations as unknowns.
These equations are usually prescribed depending on the context of the problem. The usual condition is that the second derivative values at the endpoints should be 0. Thus, the last 2 equations that we need are:

$$p_1''(x_1) = 0$$
$$p_3''(x_4) = 0$$

Question 4 (a)

SOLUTION. *If* the quadratic polynomial could fit all four points, then these conditions would be satisfied:

$$y_1 = ax_1^2 + bx_1 + c$$
$$y_2 = ax_2^2 + bx_2 + c$$
$$y_3 = ax_3^2 + bx_3 + c$$
$$y_4 = ax_4^2 + bx_4 + c$$

this is equivalent to:

$$\begin{pmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \\ x_4^2 & x_4 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$
$$A\mathbf{a} = \mathbf{b}$$

.
However, we only have 3 unknowns and 4 equations, hence we ask for the best solution (in the least square sense), which is found by solving $A^T A\mathbf{a} = A^T\mathbf{b}$. Therefore,

$$A = \begin{pmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \\ x_4^2 & x_4 & 1 \end{pmatrix}, \quad \mathbf{a} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}.$$

Question 4 (b)

SOLUTION. First, let's define the matrix A and b:

```
A = [X1^2 X1 1; X2^2 X2 1; X3^2 X3 1; X4^2 X4 1];
b = [Y1; Y2; Y3; Y4];
```

Next, we solve $A^T A\boldsymbol{a} = A^T\boldsymbol{b}$ for $\mathbf{a}$:

```
a = (A'*A)\(A'*b);   % or a = inv(A'*A)*A'*b;
```

The vector a now holds the coefficients of *q(x)*, so we can define the function *q(x)*

```
q = @(x) a[1]*x^2 + a[2]*x + a[3];
```

and finally plot this anonymous function:

```
fplot(q, [X1 X4])  % or plot(linspace(X1, X4), q(linspace(X1, X4)))
```

Note: We assume that the x-values are well-ordered, $x_1 \leq x_2 \leq x_3 \leq x_4$. If this is not the case, replace *X1* with *min([X1, X2, X3, X4])* and *X4* with *max([X1, X2, X3, X4])*.

Question 5 (a)

SOLUTION. As suggested in the hint we first add two trivial equations to help us rewrite the recursion relation in matrix form:

$$\begin{aligned} x_{n+3} &= x_{n+2} + x_{n+1} + x_n \\ x_{n+2} &= x_{n+2} \\ x_{n+1} &= \qquad\quad x_{n+1} \end{aligned}$$

This can be written as:

$$\begin{bmatrix} x_{n+3} \\ x_{n+2} \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{n+2} \\ x_{n+1} \\ x_n \end{bmatrix}$$

So define $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $X_n = \begin{bmatrix} x_{n+2} \\ x_{n+1} \\ x_n \end{bmatrix}$. Plugging in $n = 0$ we see that the initial vector X_0 contains the initial conditions

$$X_0 = \begin{bmatrix} c \\ b \\ a \end{bmatrix}.$$

Question 5 (b)

SOLUTION. First, we notice that the recursion relation $X_{n+1} = AX_n$ implies that

$$X_{n+1} = AX_n = A(AX_{n-1}) = A(A(AX_{-2})) = \ldots = A^{n+1}X_0$$

Since A is a 3×3 matrix with three distinct eigenvalues $\lambda_1 = 1.83929$, $\lambda_2 = -0.41964 + 0.60629i$, $\lambda_3 = -0.41964 + 0.60629i$ there is a basis of corresponding eigenvectors v_1, v_2, v_3. Let us rewrite X_0 with respect to this basis:

$$X_0 = \alpha v_1 + \beta v_2 + \gamma v_3$$

Then

$$\begin{aligned} X_{n+1} &= A^{n+1} X_0 = A^{n+1}(\alpha v_1 + \beta v_2 + \gamma v_3) \\ &= \alpha A^{n+1} v_1 + \beta A^{n+1} v_2 + \gamma A^{n+1} v_3 \\ &= \alpha \lambda_1^{n+1} v_1 + \beta \lambda_2^{n+1} v_2 + \gamma \lambda_3^{n+1} v_3 \end{aligned}$$

Now we see that the last two terms will vanish as n goes to infinity, because $|\lambda_2| < 1$ and $|\lambda_3| < 1$. However, since $|\lambda_1| > 1$ the norm of the first vector will blow up as n approaches infinity, **unless** $\boldsymbol{\alpha = 0}$.

This is the key to this question: As long as the vector with the initial conditions, $X_0 = \begin{bmatrix} c \\ b \\ a \end{bmatrix}$ lies in the plane that is spanned by v_2 and v_3 but has no component in direction of v_1, then the sequence will converge to zero: $x_n \to 0$ as $n \to \infty$. Eigenvectors v_2 and v_3 can be obtained in MATLAB/Octave with

```
   [V,D] = eig(A);  % V is a matrix with eigenvectors in its columns
v_2 = V[:,2];
v_3 = V[:,3];
```

Then the initial values that imply $x_n \to 0$ as $n \to \infty$ are

$$\left\{ \begin{bmatrix} c \\ b \\ a \end{bmatrix} \text{ such that } \begin{bmatrix} c \\ b \\ a \end{bmatrix} = s v_2 + t v_3 \text{ for some } s, t \in \mathbb{C} \right\}$$

Question 5 (c)

Solution. For real matrices, just as complex eigenvalues, also complex eigenvectors come in conjugate pairs. Hence

$$v_2 = \overline{v_3}$$

Hence $v_2 + v_3$ is a real vector and therefore we ensure that the solution vector $[c, b, a]T$ in part (b) is real by choosing real coefficients s and t and setting $s = t$:

$$\left\{ \begin{bmatrix} c \\ b \\ a \end{bmatrix} \text{ such that } \begin{bmatrix} c \\ b \\ a \end{bmatrix} = s(v_2 + v_3) \text{ for some } s \in \mathbb{R} \right\}$$

Question 6 (a)

Solution. Let

$$e_n(x) = e^{2\pi i n x}$$

so that we want c_i such that

$$x = \sum_{i=-\infty}^{\infty} c_i e_i(x).$$

From Hint 1 (and setting L=1) we have that e_n are orthogonal i.e.,

$$\int_0^1 e_n(x)\overline{e_m}(x)\mathrm{d}x = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$$

where the overbar indicates the complex conjugate

$$\overline{e_n}(x) = e^{-2\pi i n x}.$$

We can get the coefficients of x using this orthogonality. Take the inner product with each $e_n(x)$

$$\langle e_n, x\rangle = \sum_{i=-\infty}^{\infty} c_i \langle e_n, e_i\rangle = c_n\langle e_n, e_n\rangle = c_n$$

$$c_n = \int_0^1 e^{-2\pi i n x} x dx.$$

If $n \neq 0$, we use integration by parts

$$\int u dv = uv - \int v du$$

with $u = x$, $dv = e^{-2\pi i n x}dx$ to obtain

$$\begin{aligned}
c_n &= \left[\frac{x}{-2\pi i n}e^{-2\pi i n x}\right]_0^1 + \int_0^1 \frac{1}{2\pi i n}e^{-2\pi i n x}dx \\
&= \left[-\frac{x}{2\pi i n}e^{-2\pi i n x} - \frac{1}{4\pi^2 i^2 n^2}e^{-2\pi i n x}\right]_0^1 \\
&= \left[-\frac{x}{2\pi i n}e^{-2\pi i n x} + \frac{1}{4\pi^2 n^2}e^{-2\pi i n x}\right]_0^1 \\
&= \left[\left(\frac{1}{4\pi^2 n^2} - \frac{x}{2\pi i n}\right)e^{-2\pi i n x}\right]_0^1 \\
&= \left[\left(\frac{1}{4\pi^2 n^2} - \frac{1}{2\pi i n}\right)e^{-2\pi i n(1)}\right] - \left[\left(\frac{1}{4\pi^2 n^2} - 0\right)e^{-2\pi i n(0)}\right] \\
&= \left(\frac{1}{4\pi^2 n^2} - \frac{1}{2\pi i n}\right)e^{-2\pi i n} - \frac{1}{4\pi^2 n^2} \\
&= \left(\frac{1}{4\pi^2 n^2} - \frac{1}{2\pi i n}\right)(1) - \frac{1}{4\pi^2 n^2} \\
&= -\frac{1}{2\pi i n} \\
&= \frac{i}{2\pi n}
\end{aligned}$$

Here we see the importance of having $n \neq 0$, otherwise the coefficient diverges using this formula. If $n = 0$,

$$c_0 = \int_0^1 f(x)dx = \left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2}$$

Therefore, we have all the coefficients for the series.

Question 6 (b)

SOLUTION.

$$\begin{aligned}\langle f(x), f(x)\rangle &= \int_a^b f(x)\overline{f(x)}dx \\ &= \int_0^1 x \cdot x\, dx \\ &= \frac{1}{3}\end{aligned}$$

Question 6 (c)

SOLUTION. According to Parseval's identity it holds that

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \int_0^1 |f(x)|^2 dx$$

where c_n are the Fourier coefficients of *f (x)*. In part (b) we calculated the right hand side and found that the above equals to *1/3*.
In part (a) we found that the Fourier coefficients are $c_0 = \frac{1}{2}$ and $c_n = -\frac{1}{2\pi in}$. Hence

$$\begin{aligned}\frac{1}{3} &= \sum_{n=-\infty}^{\infty} |c_n|^2 \\ &= |c_0|^2 + 2\sum_{n=1}^{\infty} |c_n|^2 \\ &= \left|\frac{1}{2}\right| + 2\sum_{n=1}^{\infty}\left|-\frac{1}{2\pi in}\right|^2 \\ &= \frac{1}{4} + \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\end{aligned}$$

where we have noted that $|c_{-n}|^2 = |c_n|^2$ in splitting up the sum. Solving for the remaining series yields the final answer

$$\sum_{n=1}^{\infty} n^{-2} = \frac{\pi^2}{6}$$

Question 6 (d)

SOLUTION. From part (a), we figured out that $c_n = -\frac{1}{2\pi in}$. The modulus of this complex number, $|c_n|$, is the amplitude of oscillation with frequency $\omega_n = n$ since L is equal to 1. We have that

$$|c_n| = \left|-\frac{1}{2\pi in}\right| = \frac{1}{2\pi n}\left|\frac{-1}{i}\right| = \frac{1}{2\pi n}$$

for $n \neq 0$. When $n = 0$, the amplitude is equal to $|c_0|$ which is $c_0 = \frac{1}{2}$ from part (a). Therefore the points for a frequency amplitude plot are $(n, |c_n|)$ given by

$$\begin{gathered}(0,1/2)\\(1,1/2\pi)\\(2,1/4\pi)\\\vdots\\(n,1/2n\pi)\end{gathered}$$

Question 6 (e)

SOLUTION. The true Fourier coefficients as in part (a) are given by

$$c_n = \int_0^1 e^{-2\pi inx} x \mathrm{d}x$$

which we can approximate with a left Riemann sum over N points to get

$$c_n \approx \frac{1}{N}\sum_{j=0}^{N-1} e^{-2\pi inx_j} x_j$$

where $x_j = \frac{j}{N}$. Compare this to the discrete Fourier transform coefficients defined by

$$\tilde{c}_n = \sum_{j=0}^{N-1} e^{-2\pi inx_j} x_j$$

and comparing the expressions we see that $c_n \approx \frac{1}{N}\tilde{c}_n$. The discrete Fourier coefficients can be computed in Matlab using the command *fft*. Therefore, in Matlab, we need to sample x over say 100 points and use *fft*,

```
N = 100;
x = linspace(0,1,N);
C = fft(x);
```

Since $C = [\tilde{c_0}, \tilde{c_1}, \tilde{c_2}, ...]$ are the discrete Fourier coefficients, then the actual Fourier coefficients are given by $c_n \approx \frac{1}{N}\tilde{c_n}$ (where in Matlab we arbitrarily chose that N=100).

The discrete Fourier coefficients are periodic with period N and so frequencies w_n and $w_n + N$ are indistinguishable. Particularly this means that modes $-N/2 \leq n < 0$, will be indistinguishable to modes $N/2 < n < N$ since they differ by period N. These negative modes are relevant because they are the complex conjugates to modes $0 < n \leq N/2$ and will therefore have the same amplitudes meaning that if a mode on $1 < n < N/2$ has a high amplitude then so too will its complex conjugate affecting the result on $N/2 \leq n < N$. From part (a) we see that for the function $f(x) = x$, the amplitudes satisfied $|c_n| = \frac{1}{2\pi n}$ and that the highest amplitudes are for integers closest to zero. Therefore, negative modes slightly less than zero have the opportunity to create the biggest errors on modes slightly greater than $N/2$. Conversely, the modes $0 \le n < N/2$ will only be affected by modes smaller than $-N/2$ which for functions that behave like $f(x) = x$ will have small amplitudes on that region and thus create small errors. The amplitudes are therefore most accurate for frequencies $0 \le n < \frac{N}{2}$.

Question 6 (f)

SOLUTION. If we extend the domain from $[0, 1]$ to $[0, 2]$, we can write the new Fourier coefficients d_n using the techniques from part (a) as

$$d_n = \frac{1}{2}\int_0^2 xe^{-2\pi inx/2}\mathrm{d}x.$$

We could go and directly compute these integrals but we want to relate them to the coefficients c_n from part (a) computed on a domain $[0, 1]$. To do this, we can use a substitution and let $u = \frac{x}{2}$. Then $dx = 2du$. With this transformation we have $u(0) = 0$ and $u(2) = 1$ and so putting everything together,

$$d_n = 2\int_0^1 ue^{-2\pi inu}\mathrm{d}u = 2c_n$$

where we have recognized that the new integral is precisely what we computed to get the coefficients on a period $[0, 1]$. Therefore on $[0, 2]$ the Fourier coefficients double and

$$d_n = \begin{cases} 2\frac{1}{2} = 1, & n = 0 \\ 2\frac{i}{2\pi n} = \frac{i}{\pi n}, & n \ne 0 \end{cases}.$$

Therefore we have that $|d_n| = 2|c_n|$ and the points to plot for the frequency amplitude are $(n, 2|c_n|)$.

Question 7 (a)

SOLUTION. Using the hints, we know that $\mathbf{x}_\infty$ is the eigenvector associated to the largest eigenvalue of the matrix $(A - 3I)^{-1}$.
The eigenvalues of $A - 3I$ are the eigenvalues of A shifted by 3, with the same eigenvectors. We can see this by noting for each eigenvector v of A, we have

$$(A - 3I)v = Av - 3v = \lambda v - 3v = (\lambda - 3)v$$

Since A has the eigenvalues $\lambda_0 = 0$, $\lambda_1 = 1$, $\lambda_2 = 4$ and $\lambda_3 = 5$, the matrix $A - 3I$ has the eigenvalues $\eta_0 = -3$, $\eta_1 = -2$, $\eta_3 = 1$ and $\eta_4 = 2$. Next, the eigenvalues of the inverse $(A - 3I)^{-1}$ are the reciprocal of the original eigenvalues, with the same eigenvectors. Hence the eigenvalues of $(A - 3I)^{-1}$ are $\gamma_0 = -1/3$, $\gamma_1 = -1/2$, $\gamma_3 = 1$ and $\gamma_4 = 1/2$. We see that the dominating eigenvalue of $(A - 3I)^{-1}$ is $\gamma_3 = 1$, and therefore, by the power method, $\mathbf{x}_\infty$ is an eigenvector of $(A - 3I)^{-1}$ with eigenvalue 1.
This means that $\mathbf{x}_\infty$ is also an eigenvector of A with eigenvalue $\lambda_2 = 4$. In other words,

$$A\mathbf{x}_\infty = 4\mathbf{x}_\infty = [2, 2, 2, 2]^T$$

Question 7 (b)

SOLUTION. From part (a), we recall that $\lambda = 4$ was the eigenvalue associated to the eigenvector x_∞. We can use these to find the solution:

$$\begin{aligned} Ax_\infty &= \lambda x_\infty \\ Ax_\infty \cdot x_\infty &= \lambda x_\infty \cdot x_\infty \\ Ax_\infty \cdot x_\infty &= 4(1) \\ \langle \mathbf{x}_\infty, A\mathbf{x}_\infty \rangle &= 4 \end{aligned}$$

Question 7 (c)

SOLUTION. Taking the limit $n \to \infty$ on both sides yields

$$\begin{aligned} \lim_{n\to\infty} \mathbf{y}_n &= \lim_{n\to\infty} (A - 3I)^{-1}\mathbf{x}_{n-1} \\ &= (A - 3I)^{-1} \lim_{n\to\infty} \mathbf{x}_{n-1} \\ &= (A - 3I)^{-1}\mathbf{x}_\infty \\ &= \eta_3 \mathbf{x}_\infty \\ &= \mathbf{x}_\infty = [1/2, 1/2, 1/2, 1/2]^T \end{aligned}$$

We used, from part (a), that $\mathbf{x}_\infty$ is an eigenvector of $(A - 3I)^{-1}$ with eigenvalue $\eta_3 = 1$.

Good Luck for your exams!