Full Solutions
MATH104 December 2014

April 16, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

Factoring gives us $\frac{x^2-9}{3-x} = \frac{(x-3)(x+3)}{-(x-3)} = -(x+3)$ where the last equality comes from cancelling out the factors of $(x-3)$ from the numerator and denominator. Hence, we obtain

$$\lim_{x\to 3} \frac{x^2-9}{3-x} = \lim_{x\to 3} -(x+3) = -6$$

Question 1 (b)

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

Factoring gives us $\frac{\sqrt{x}-4}{x-16} = \frac{\sqrt{x}-4}{(\sqrt{x}-4)\sqrt{x}+4)} = \frac{1}{\sqrt{x}+4}$ where the last equality comes from cancelling out the factors of $(\sqrt{x}-4)$ from the numerator and denominator. Hence, we obtain

$$\lim_{x\to 16} \frac{\sqrt{x}-4}{x-16} = \lim_{x\to 16} \frac{1}{\sqrt{x}+4} = \frac{1}{8}$$

Question 1 (c)

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

We note first that at $x = 2$, $f(2) = 3(2) = 6$. If $\lim\limits_{x\to 2^+} f(x) = 6$, then f is continuous from the right at 2. $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^2 - 2x = 2^2 - 2(2) = 0 \neq 6$ So $f(x)$ is not continuous from the right at $x = 2$.

Question 1 (d)

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

$f'(x) = x^2 + 2x - 7$. The derivative of f at x is the slope of the tangent line at $(x, f(x))$. So we set $f'(x) = -8$ and solve the quadratic equation using the quadratic formula/ factoring. $x^2 + 2x - 7 = -8$ $x^2 + 2x + 1 = (x+1)(x+1)$ So the f has tangent line of slope -8 when $x = -1$. The point which we get is $(-1, f(-1)) = (-1, \frac{107*3+2}{3}) = (-1, \frac{323}{3})$, where $f(-1) = \frac{(-1)^3}{3} + (-1)^2 - 7(-1) + 100 = \frac{107*3+2}{3} = \frac{323}{3}$.

Question 1 (e)

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

Note first that $h(4) = \frac{f(4)}{4-5} = -3$. Rewrite $h(x) = f(x)(x-5)^{-1}$ and take the derivative of h with respect to x using the product rule. $h'(x) = f'(x)(x-5)^{-1} + f(x)(-1)(x-5)^{-2}$ If you're more comfortable with the quotient rule:

$$\begin{aligned} h'(x) &= \frac{f'(x)(x-5) - f(x)(1)}{(x-5)^2} \\ &= \frac{f'(x)}{(x-5)} - \frac{f(x)}{(x-5)^2} \end{aligned}$$

Evaluating h' at $x = 4$ gives the slope of the tangent line at $x = 4$

$$h'(4) = f'(4)(4-5)^{-1} + f(4)(-1)(4-5)^{-2} = -2 - 3 = -5$$

We know that $(4, h(4))$ is a point on the tangent line of $h(x)$ at $x = 4$; so we obtain $y - h(4) = -5(x-4)$ $y = -5x + 20 + h(4) = -5x + 17$

Question 1 (f)

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

Apply product rule to $y(x) = f(x)g(x)$ where $f(x) = e^{9x}$ and $g(x) = \cos x$. Note that $f'(x) = 9e^{9x}$ and $g'(x) = -\sin x$ We obtain:

$$\begin{aligned} y'(x) &= f'(x)g(x) + g'(x)f(x) \\ &= 9e^{9x}\cos x - e^{9x}\sin x \end{aligned}$$

and we are done.

Question 1 (g)

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

If the position of an object is given by $s = f(t)$, then the acceleration is $a = f''(t)$, the second derivative of the position function. So, we determine what the acceleration function is. $a = f''(t) = (f'(t))' = (4t^3 - 2)' = 12t^2$ and evaluating the acceleration at $t = 2$, we obtain $f''(2) = 12(2)^2 = 12(4) = 48$.

Question 1 (h)

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

By the chain rule, we get that $h'(x) = f'(g(x))g'(x)$. From the tangent line at $(4, 7)$ on the graph of g, we obtain that $g(4) = 7$ and $g'(4) = 3$, and likewise, from the tangent line at $(7, 9)$ on the graph of f, we obtain that $f'(7) = -2$. So, evaluating h' at $x = 4$ gives:

$$\begin{aligned} h'(4) &= f'(g(4))g'(4) \\ &= f'(7)g'(4) \\ &= (-2)(3) \\ &= -6 \end{aligned}$$

Question 1 (i)

Solution. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

Given $y = x^{\ln x}$, take ln of both sides and obtain:
$\ln y = \ln (x^{\ln x}) = (\ln x)^2$
Now we differentiate with respect to x and obtain:

$$\begin{aligned}
\frac{1}{y}\frac{dy}{dx} &= 2\ln x \frac{1}{x} \\
\frac{dy}{dx} &= \frac{2}{x}(\ln x) y \\
&= \frac{2}{x}(\ln x) x^{\ln x}
\end{aligned}$$

Question 1 (j)

Solution. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

Since the annual interest rate is 5% compounded continuously, the model that we use is $P(t) = P_0 e^{0.05t}$, where $P(t)$ is the value of the investment after time t in years, and P_0 is the initial amount invested. We want to find $P'(t)$ when $P(t) = 500$. Take ln of both sides in the model and then differentiate with respect to t:

$$\begin{aligned}
\ln P(t) &= \ln P_0 + 0.05t \\
\frac{1}{P(t)} P'(t) &= 0.05 \\
P'(t) &= 0.05 P(t)
\end{aligned}$$

When the investment is $P(t) = 500$, $P'(t) = 0.05(500) = 25$.

Question 1 (k)

Solution 1. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

Assuming that $-\pi/2 < f(x) < \pi/2$, take tan of both sides of f and obtain that $\tan f(x) = 5x$. Now $\tan f(0) = 0$ if and only if $f(0) = 0$ (assuming that $-\pi/2 < f(x) < \pi/2$). Now we take the derivative of $\tan f(x)$ with respect to x using Chain Rule and obtain

$$\sec^2 f(x)(f'(x)) = 5 \Rightarrow f'(0) = \frac{5}{\sec^2 f(0)} = \frac{5}{\sec^2(0)} = 5$$

and we are done.

Solution 2. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

Here is an alternative method to solve the problem. Let us take tan of both sides to yield

$$\tan f(x) = 5x.$$

Next, we take the derivative of both sides with respect to the variable x and obtain

$$\sec^2(f(x))f'(x) = 5 \Rightarrow f'(x) = \frac{5}{\sec^2(f(x))}$$

What we want to do now is express the right hand side, namely the $\sec^2(f(x))$ explicitly in terms of x. Recall that $\sec(f(x)) = \frac{1}{cos(f(x))}$ and hence $f'(x) = 5\cos^2(f(x))$. Recall that taking the cosine of a right angle triangle with angle $f(x)$ is taking the ratio of adjacent and hypotenuse. We know that $\tan(f(x)) = 5x$ and since $\tan(f(x)) = \frac{\sin(f(x))}{\cos(f(x))} = \frac{5x}{1}$, we obtain that $\sin(f(x)) = 5x$ and $\cos(f(x)) = 1$, implying that $\cos^2(f(x)) = 1^2 = 1$. Putting it all together, we obtain that $f'(x) = \frac{5}{\sec^2(f(x))} = 5\cos^2(f(x)) = 5 \cdot 1 = 5 \Rightarrow f'(0) = 5$ and we are done.

Question 1 (l)

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

$h(x)$ has a local extreme value at $x = 2$, if $h'(2) = 0$. Since f has a local extrema at $x = 2$, $f'(2) = 0$. First let us find $h'(x)$

$$h'(x) = f(x) + xf'(x) + 2$$

Substituting in $x = 2$ we obtain that $h'(2) = f(2) + 2f'(2) + 2 = 0 + 2(0) + 2 = 2 \neq 0$. Since $h'(2) \neq 0$, h does not have a local extrema at $x = 2$.

Question 1 (m)

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

Given a function $f(x)$, the linearization of f at a point a is given by

$$L(x) = f(a) + f'(a)(x - a).$$

Take $f(x) = \sin x$ and $a = 0$, we obtain that $L(x) = \sin(0) + \cos(0)(x - 0) = 0 + 1(x) = x$; then $|\sin 0.12 - 0.12| \approx |L(0.12) - 0.12| = 0.12 - 0.12 = 0$

Question 1 (n)

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

The second order Taylor expansion of a function $f(x)$ at point a is given by

$$f_2(x) = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2$$

Here we take $f(x) = x^{2/3}$, and $a = 1$. Using the power rule, find the first and second derivatives of f:

$$\begin{aligned} f(x) &= x^{2/3} \Rightarrow f(1) = 1 \\ f'(x) &= \frac{2}{3}x^{-1/3} \Rightarrow f'(1) = \frac{2}{3} \\ f''(x) &= \frac{-2}{9}x^{-4/3} \Rightarrow f''(1) = \frac{-2}{9} \end{aligned}$$

Putting all the values into the second order Taylor expansion formula, we obtain

$$f_2(x) = 1 + \frac{2}{3}(x-1) - \frac{1}{9}(x-1)^2$$

Question 2

SOLUTION. THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!

Recall that given a function $f(x)$, the derivative of f at x is the following limit (if it exists)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

We use this to evaluate the derivative of the given function:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{(x+h)^2}{(x+h)^2+1} - \frac{x^2}{x^2+1}}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2(x^2+1) - x^2((x+h)^2+1)}{h((x+h)^2+1)(x^2+1)} \\
&= \lim_{h\to 0} \frac{(x+h)^2x^2 + (x+h)^2 - x^2(x+h)^2 - x^2}{h((x+h)^2+1)(x^2+1)} \\
&= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h((x+h)^2+1)(x^2+1)} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h((x+h)^2+1)(x^2+1)} \\
&= \lim_{h\to 0} \frac{h(2x+h)}{h((x+h)^2+1)(x^2+1)} \\
&= \lim_{h\to 0} \frac{(2x+h)}{((x+h)^2+1)(x^2+1)} \\
&= \frac{2x}{(x^2+1)^2}
\end{aligned}$$

Lastly, $f'(1) = \frac{2(1)}{(1^2+1)} = 1$.

Question 3 (a)

SOLUTION. THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!

Recall that the price elasticity of a product is given by $e(p,q) = \frac{dq}{dp}\frac{p}{q}$, where p and q, are price and demand respectively. We implicitly differentiate the given relation with respect to p

$$3p^2 + \frac{dq}{dp} + 3q^2\frac{dq}{dp} = 0$$

$\frac{dq}{dp} = \frac{-3p^2}{1+3q^2}$ Substitute into the formula for elasticity and obtain

$$e(p,q) = \frac{-3p^2}{1+3q^2}\frac{p}{q} = \frac{-3p^3}{q+3q^3}$$

Question 3 (b)

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

Substituting $q = 3$ in the relation given, we obtain that $p^3 = 38 - 3 - 3^3 = 35 - 27 = 8 \Rightarrow p = 2$. Next, we determine the value of q' when $p = 2, q = 3$. From the calculation in the above, we know that $q' = \frac{-3p^2}{1+3q^2}$, substituting $p = 2, q = 3$ yields

$$q' = \frac{-3}{7}.$$

Last, we check what the change in revenue is when $p = 2, q = 3$. Recall that $R = pq$ and hence, $R' = q + pq'$ using the product rule when differentiating with respect to price, p. Substituting $p = 2, q = 3, q' = \frac{-3}{7}$, we obtain that $R' = 3 + 2(\frac{-3}{7}) = 3 - \frac{6}{7} > 0$. Since $R' > 0$ when price $p = 2$, slightly raising the price will increase the revenue.

Question 3 (c)

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

From the above we know that when $q = 3$, by the relation give, $p = 2$. Now, we take that above relation and differentiate it with respect to time

$$3p^2\frac{dp}{dt} + \frac{dq}{dt} + 3q^2\frac{dq}{dt} = 0$$

We know that $\frac{dp}{dt} = 7, p = 2, q = 3$. Substitute into the newfound relation and isolate for $\frac{dq}{dt}$

$$\frac{dq}{dt} = \frac{-3(2)^2(7)}{1 + 3(3)^2} = \frac{-84}{28} = -3$$

So the demand is decreasing at a rate of 3 units/month.

Question 4

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

We know that if we sell x bus tour tickets, we sell $y = 100 - x$ train tour tickets. Let us now make the revenue function:

$$\begin{aligned} R &= x(30 - \frac{x}{4}) + y(70 - \frac{y}{2}) \\ &= x(30 - \frac{x}{4}) + (100 - x)(70 - \frac{100 - x}{2}) \\ &= 30x - \frac{x^2}{4} + 70(100 - x) - \frac{(100 - x)^2}{2} \end{aligned}$$

Taking the derivative of R with respect to x, we obtain:

$$\begin{aligned}R'(x) &= 30 - \frac{x}{2} - 70 + (100 - x)\\ &= 60 - \frac{3x}{4}\end{aligned}$$

Now we want to maximize the revenue with respect to the number of bus tickets we sell; hence we let $R'(x) = 0$ and determine x. We obtain that $x = 40$. This means that if we sell 40 bus tour tickets and 60 train tour tickets, we maximize the revenue.

Question 5 (a)

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

First, we determine $f'(x)$; rewrite $f(x) = e^x x^{-2}$ and use product rule on $h(x) = e^x, g(x) = x^{-2}$:

$$\begin{aligned}f'(x) &= h(x)g'(x) + h'(x)g(x)\\ &= e^x x^{-2} - 2x^{-3}e^x\\ &= \frac{e^x}{x^2} - \frac{2e^x}{x^3}\\ &= \frac{e^x(x-2)}{x^3}\end{aligned}$$

The critical points of f are at x where $f'(x) = 0$ and at values x where $f'(x)$ does not exist. We observe that at $x = 0$, $f'(x)$ does not exist and hence, there is a veritcal asymptote at $x = 0$. Let $f'(x) = 0$ and find x

$$0 = \frac{e^x(x-2)}{x^3}$$

Thus $e^x = 0$ or $(x - 2) = 0$.
Since $e^x > 0$ for all $x \in \mathbb{R}$, we have that $x - 2 = 0 \Rightarrow x = 2$. So the critical point of f occurs at $x = 2$.

Question 5 (b)

SOLUTION. Remark first that f does not exist at $x = 0$; hence, we have three intervals on which $f'(x)$ exists and is non-zero, namely $(-\infty, 0), (0, 2)$, and $(2, \infty)$. We create a table with these intervals and the function that make up f' to determine when f' is positive or negative, hence when f is increasing or decreasing.

	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
e^x	+	+	+
$(x-2)$	-	-	+
x^3	-	+	+
f'	+	-	+
f	increasing	decreasing	increasing

So f is increasing on the intervals $(-\infty, 0)$ and $(2, \infty)$ and decreasing on $(0, 2)$.

Question 5 (c)

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

Recall from part a that $f'(x) = \frac{e^x(x-2)}{x^3}$. We take the derivative of f' with respect to x using the quotient rule. Let

$$\begin{aligned} g(x) &= e^x(x-2) & g'(x) &= e^x(x-2) + e^x = e^x(x-1) \\ h(x) &= x^3 & h'(x) &= 3x^2 \end{aligned}$$

By the quotient rule:

$$\begin{aligned} f''(x) &= \frac{g'(x)h(x) - g(x)h'(x)}{(h(x))^2} \\ &= \frac{e^x(x-1)x^3 - 3e^x(x-2)x^2}{x^6} \\ &= \frac{e^x x^2(x(x-1) - 3(x-2))}{x^6} \\ &= \frac{e^x(x^2-4x+6)}{x^4} \end{aligned}$$

Question 5 (d)

SOLUTION. Recall that when $f'' > 0$, f is concave up; likewise, when $f'' < 0$, f is concave down. So what we need to do is find the intervals on which f'' exists and is non-zero. From above, we know that $f''(x) = \frac{e^x(x^2-4x+6)}{x^4}$, and indeed, $f''(x) = 0$ if and only if $e^x = 0$ or $x^2 - 4x + 6 = 0$. Since $e^x > 0$ for all x, we have that $x^2 - 4x + 6 = 0$. Using the quadratic formula, we find that this quadratic equation has no real roots. Therefore, there is no x where $f''(x) = 0$. Now, we make a table as before for f', noting that f'' does not exist at $x = 0$. This gives us two intervals, namely $(-\infty, 0)$ and $(0, \infty)$ where f'' is nonzero and defined:

	$(-\infty, 0)$	$(0, \infty)$
e^x	$+$	$+$
$x^2 - 4x + 6$	$+$	$+$
x^4	$+$	$+$
f''	$+$	$+$
f	Concave Up	Concave Up

So on both $(-\infty, 0)$ and $(0, \infty)$, f is concave up.

Question 5 (e)

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

We have already determined what the vertical asymptote is in part a, by seeing where f is undefined. To see the behaviour of f as it approaches $x = 0$ from the right and left, we have the following calculations:

$$\begin{aligned} \lim_{x\to 0^+} \frac{e^x}{x^2} &= \frac{\lim_{x\to 0^+} e^x}{\lim_{x\to 0^+} x^2} \\ &= \frac{1}{\lim_{x\to 0^+} x^2} = \infty \end{aligned}$$

$$\lim_{x\to 0^-}\frac{e^x}{x^2} = \frac{\lim\limits_{x\to 0^-} e^x}{\lim\limits_{x\to 0^-} x^2}$$
$$= \frac{1}{\lim\limits_{x\to 0^-} x^2} = \infty$$

where we obtain the last equality since the square of any negative number is a positive number. This happens when $x = 0$ so that is our vertical asymptote. To find the horizontal asymptote, we find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$. Essentially, what we are determining is the behaviour of $f(x)$ as x gets very large and very small. We will apply L′Hopital′s rule when finding the two limits.

$$\lim_{x\to\infty}\frac{e^x}{x^2} = \lim_{x\to\infty}\frac{e^x}{2x}$$
$$= \frac{1}{2}\lim_{x\to\infty}\frac{e^x}{x}$$

Since $\lim\limits_{x\to\infty}\frac{e^x}{x} = \infty$, we have that $\lim\limits_{x\to\infty}\frac{e^x}{x^2} = \infty$ (the limit does not exist). So as x gets very large, $f(x)$ also gets very large and proceeds towards ∞.

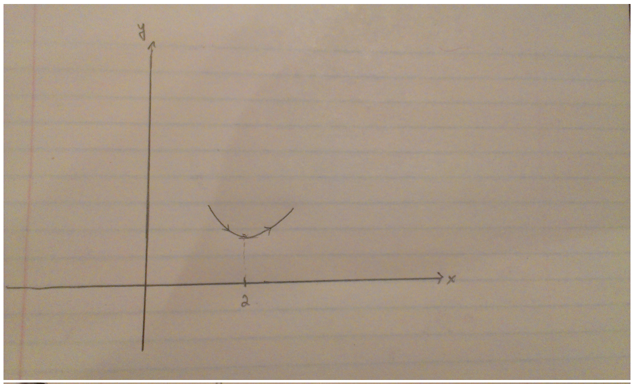
$$\lim_{x\to-\infty}\frac{e^x}{x^2} = \lim_{x\to-\infty}\frac{e^x}{2x}$$
$$= \frac{1}{2}\lim_{x\to-\infty}\frac{e^x}{x}$$
$$= \frac{1}{2}\lim_{x\to-\infty}\frac{e^x}{1}$$
$$= 0$$

In the second limit calculation, we applied L′Hopital′s rule twice. We learn from the second calculation that as x gets very small and speeds towards $-\infty$, $f(x)$ approaches 0.

Question 5 (f)

Solution. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

We know from part a that the critical point of f occurs at $x = 2$; hence the graph of the function will change from decreasing to the left of $x = 2$ to increasing to the right of $x = 2$ (see first image below). Next, we know that in the interval to the left of the vertical asymptote at $x = 0$, the function is increasing. Moreover, in that interval, as x tends to $-\infty$, $f(x)$ tends to 0. From part d, we know that in this interval, the function is concave up (see second image below).
Next, we focus on the intervals to the right of the vertical asymptote. In the interval $(0, 2)$, we recall that the function is decreasing whilst in the interval $(2,\infty)$, the function is increasing (see third image below). From part d, we know that it is concave up in $(0,\infty)$, and flies off to infinity as x tends to ∞. Combining these facts we have the graph depicted in the fourth image below: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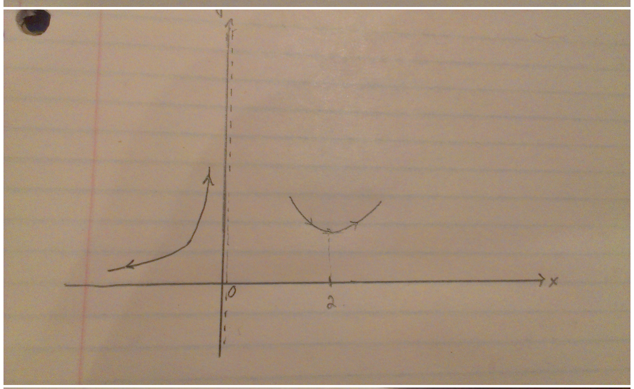
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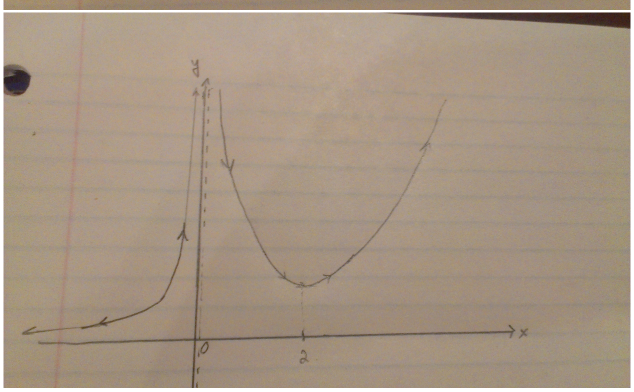
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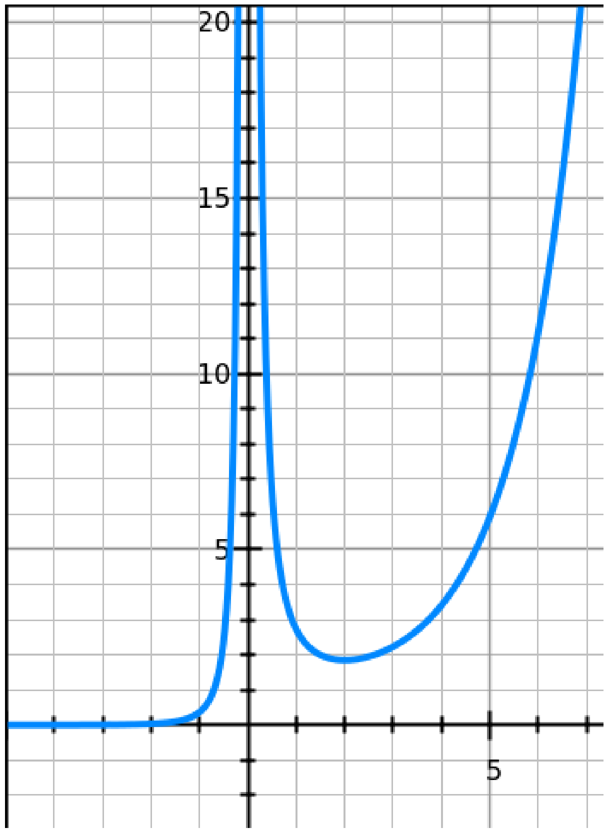

Question 6

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

The necessary condition for f to have an inflection point is that there exists a c in $(0, \infty)$ so that $f''(c) = 0$. A sufficient condition for f to have an inflection point is that f changes signs at c.
First, let us turn the disgusting looking function $\log_{(x+1)^2}(x+1)^{2014}$ into something nicer, using the change of base of logarithms formula

$$\log_d(a) = \log_d(b)\log_b(a).$$

Take $d = e$, $b = (x+1)^2$, and $a = (x+1)^{2014}$. Substituting into the formula, we obtain that $\log_{(x+1)^2}(x+1)^{2014} = \frac{\ln(x+1)^{2014}}{\ln(x+1)^2} = \frac{2014\ln(x+1)}{2\ln(x+1)} = 1007$ Then $f(x) = \frac{x^4}{12} - \frac{4(3^{-x})}{(\ln 3)^2} + 1007$. Now we take the first and second derivatives of f. Recall that if $g(x) = 3^{-x}$, then $g'(x) = -3^{-x}\ln 3$. Taking the first derivative of f, we get

$$f'(x) = \frac{4x^3}{12} + \frac{4(3^{-x})\ln 3}{(\ln 3)^2} = \frac{x^3}{3} + \frac{4(3^{-x})}{(\ln 3)}$$

Now, we take the second derivative of f

$$f''(x) = x^2 - \frac{4(3^{-x})\ln 3}{(\ln 3)} = x^2 - 4(3^{-x})$$

Remark next that $f''(x)$ is a continuous function of x on the interval $(0, \infty)$, since both x^2 and 3^{-x} are continuous functions of x on $(0, \infty)$. Now we want to show that there exists a $c \in (0, \infty)$ so that $f''(c) = 0$. Evaluate $f''(0)$

$$f''(0) = 0^2 - 4 = -4 < 0$$

Now evaluate $f''(2)$

$$f(2) = (2)^2 - 4(3^{-2}) = 2^2 - \frac{4}{9} > 0$$

Since $f''(x)$ is continuous on $(0, \infty)$, and $f''(0) < 0$ while $f''(2) > 0$, by the intermediate value theorem, there exists a point $c \in (0, 2)$ so that $f''(c) = 0$; as such there exists an $\epsilon > 0$ such that $f''(c + \epsilon)$ and $f''(c - \epsilon)$ have different signs. So indeed the given function f has an inflection point on $(0, \infty)$.

Good Luck for your exams!