Full Solutions
MATH257 December 2011

April 16, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1

Easiness: 5.0/5

SOLUTION. Following the comments in the hints, we see that $x = 0$ is a regular singular point. Therefore, we should look for series solutions of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}.$$

We have that

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \; y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}, \; y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}.$$

(Please note that the both of the sums for y' and y'' start at $n = 0$.)
Substitute these into the ODE

$$2x^2 y'' + (3x + x^2) y' - y = 0,$$

we get that

$$\sum_{n=0}^{\infty} 2a_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} 3a_n (n+r) x^{n+r} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r+1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

We should shift the index n of $\sum_{n=0}^{\infty} a_n (n+r) x^{n+r+1}$ so the order of x matches with the other terms. To do so, we should replace every n with $n - 1$. Then

$$\sum_{n=0}^{\infty} a_n (n+r) x^{n+r+1} = \sum_{n=1}^{\infty} a_{n-1} (n+r-1) x^{n+r}.$$

Then we have

$$\sum_{n=0}^{\infty} 2a_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} 3a_n (n+r) x^{n+r} + \sum_{n=1}^{\infty} a_{n-1} (n+r-1) x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

which, after separating the $n = 0$ terms, can be rewritten as

$$[2a_0 r(r-1) + 3a_0 r - a_0]\, x^r + \sum_{n=1}^{\infty} [2a_n (n+r)(n+r-1) + 3a_n (n+r) + a_{n-1}(n+r-1) + a_n]\, x^{n+r} = 0.$$

Look at the above equation, because the right hand side is 0, the coefficient of the x^r term must be 0 as well. Hence,

$$[2a_0 r(r-1) + 3a_0 r - a_0] = a_0 (2r(r-1) + 3r - 1) = 0.$$

Now, if $a_0 = 0$, then all the subsequent a_n will be zero as well and we only get the trivial solution $y = 0$. (Why? You can verify that yourself after we finish the question. Set $a_0 = 0$ and you will see the other terms will be forced to be 0 by the recursion relation.) In other words, we must have

$$2r(r-1) + 3r - 1 = 2r^2 + r - 1 = (2r-1)(r+1) = 0$$

giving us two roots $r = \frac{1}{2}$ and $r = -1$. (Note: In the above, $2r(r-1)+3r-1=0$ is known as the indicial equation.)
By the same reason as $[2a_0 r(r-1)+3a_0 r - a_0] = 0$, we have that

$$[2a_n(n+r)(n+r-1)+3a_n(n+r)+a_{n-1}(n+r-1)+a_n] = 0 \text{ foreach } n \geq 1.$$

If we Isolate a_n, we get the recurrence relation

$$a_n = \frac{-(n+r-1)a_{n-1}}{2(n+r)(n+r-1)+3(n+r)-1} \text{ for } n \geq 1.$$

After simplifying and factoring the expression in the denominator, the above expression can be rewritten as

$$a_n = \frac{-(n+r-1)a_{n-1}}{2(n+r)^2+(n+r)-1} = \frac{-(n+r-1)a_{n-1}}{(2(n+r)-1)((n+r)+1)}$$

which may be easier to do calculations with.
Now recall that from the indicial equation, we found $r = \frac{1}{2}$ and $r = -1$.
In the case $r = -1$, $y = \sum_{n=0}^{\infty} a_n x^{n-1} = a_0 x^{-1} + a_1 + a_2 x + \cdots$.
Note that the above equation does not satisfy $\lim_{x \to 0^+} y(x) = 0$ due to the $a_0 x^{-1}$ term, so the expression corresponding to $r = -1$ is not what the question is asking for.
In the case $r = \frac{1}{2}$, $y = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}} = a_0 x^{\frac{1}{2}} + a_1 x^{\frac{3}{2}} + a_2 x^{\frac{5}{2}} + \cdots$.
Note that the above equation satisfies $\lim_{x \to 0^+} y(x) = 0$. In other words, this is the one the question is asking for.
It remains to find a_1 and a_2 in terms of a_0. To find a_1, substitue $n = 1$ and $r = \frac{1}{2}$ into the recurrence relation. We get that

$$a_1 = \frac{-(1+\frac{1}{2}-1)a_0}{(2(1+\frac{1}{2})-1)(1+\frac{1}{2}+1)} = \frac{-1}{10}a_0.$$

To find a_2, substitue $n = 2$ and $r = \frac{1}{2}$ into the recurrence relation. We get that

$$a_2 = \frac{-(2+\frac{1}{2}-1)a_1}{(2(2+\frac{1}{2})-1)(2+\frac{1}{2}+1)} = \frac{-3}{28}a_1 = \frac{-3}{28}(\frac{-1}{10}a_0) = \frac{3}{280}a_0.$$

Therefore, we have

$$y(x) = a_0 x^{\frac{1}{2}} + a_1 x^{\frac{3}{2}} + a_2 x^{\frac{5}{2}} + \cdots = a_0 x^{\frac{1}{2}} - \frac{1}{10}a_0 x^{\frac{3}{2}} + \frac{3}{280}a_0 x^{\frac{5}{2}} + \cdots$$

Question 2 (a)

Easiness: 4.0/5

SOLUTION. For the one dimensional wave equation

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

where $-\infty < x < \infty$ and $t > 0$, the solution $u(x,t)$ is given by d'Alembert's formula

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds.$$

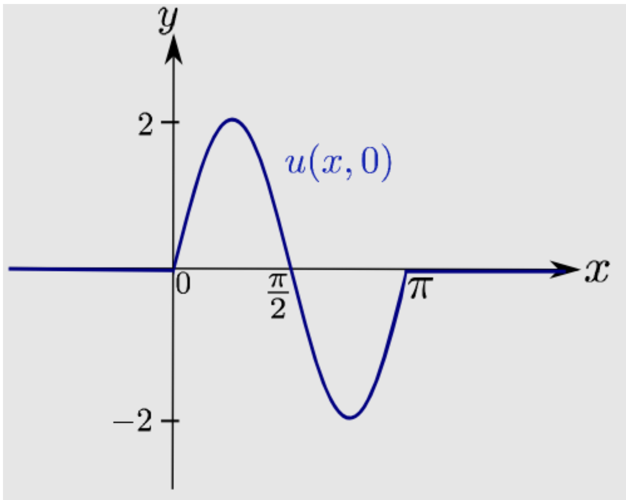
For this question, we have $c = 1$,

$$f(x) = \begin{cases} 2\sin(2x) & 0 < x < \pi \\ 0 & \text{otherwise} \end{cases}$$

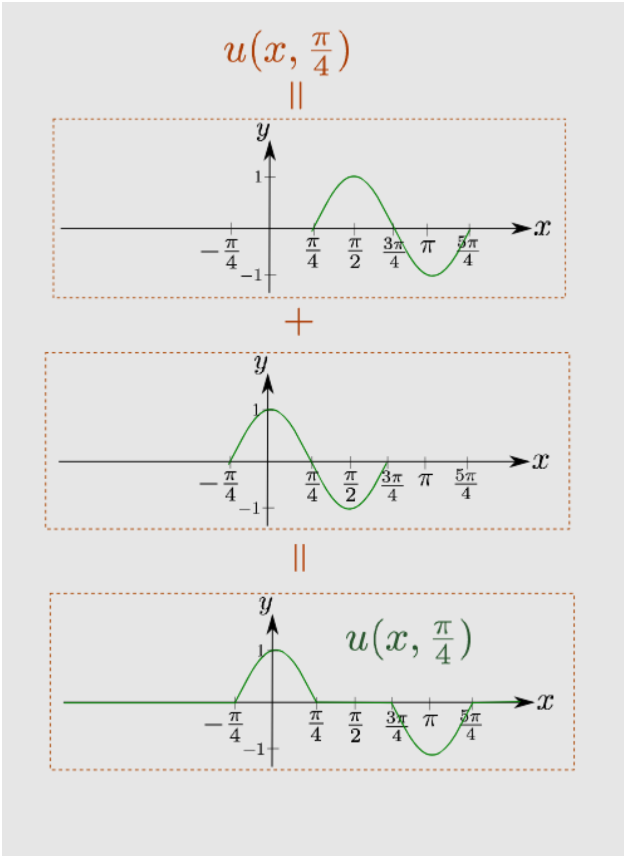
and $g(x) = 0$. Therefore, the solution is given by

$$u(x,t) = \frac{1}{2}[f(x-t) + f(x+t)].$$

This equation says that at time $t = 0$, the original wave form splits into two parts, each with half the amplitude of the original. One part moves to the left at speed 1 and the other moves to the right at the same speed. At any time $t > 0$, the solution of the wave equation is the sum of the left-traveling part and the right-traveling part.

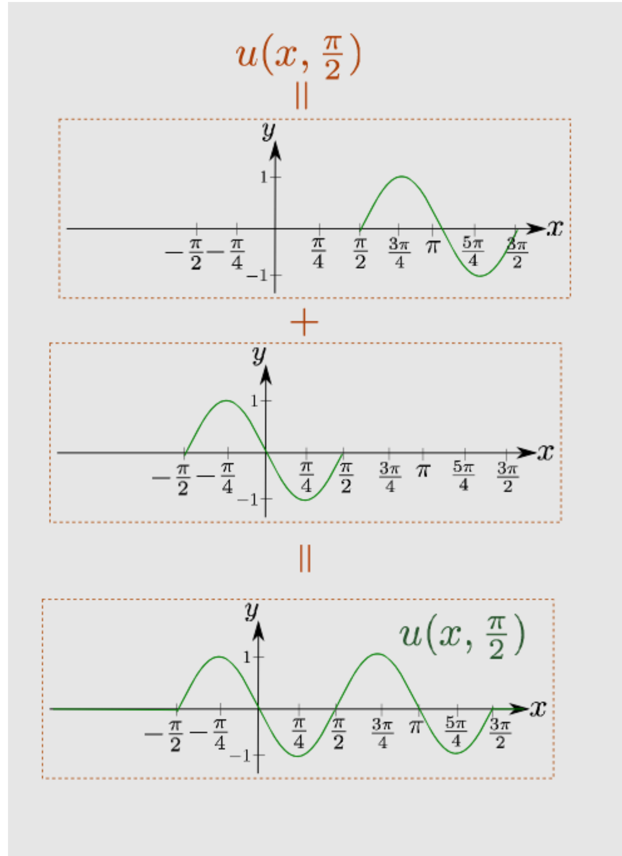


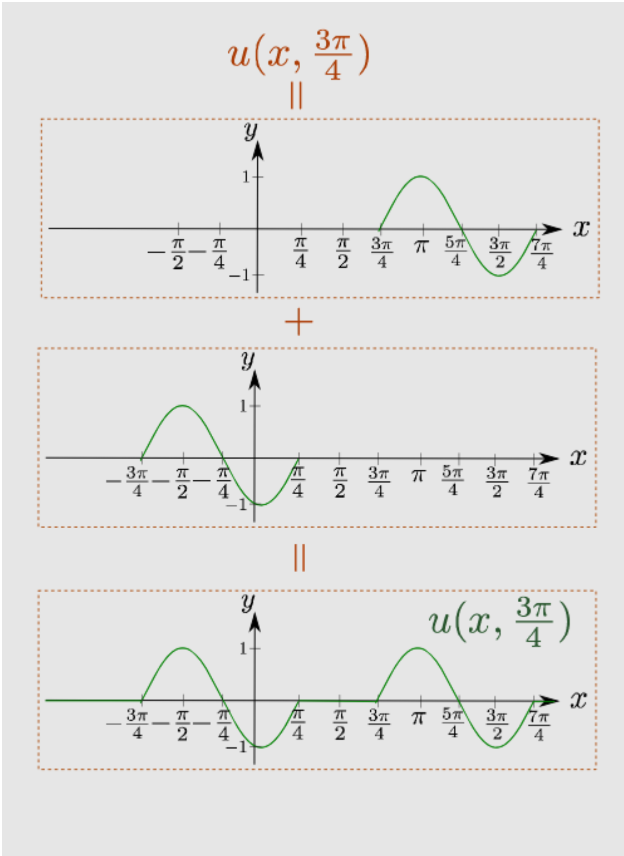
At $t = 0$, $u(x, 0)$ is the given function $f(x)$.
To find $u\left(x, \frac{\pi}{4}\right)$, one imagines that at $t = 0$, the original wave form $f(x)$ splits into two equal pieces but with half the original amplitude. Hence, in this case, each part will be like $\sin(2x)$. One part will move towards the left at speed 1 and the other moves to the right at speed 1. At $t = \frac{\pi}{4}$, one part has traveled $\frac{\pi}{4}$ units to the left and the other has traveled $\frac{\pi}{4}$ units to the right. The solution $u\left(x, \frac{\pi}{4}\right)$ is the sum of those two parts.



Notice that in the above, the left traveling wave cancels with the right travelling wave in the interval from $x = \frac{\pi}{4}$ to $x = \frac{3\pi}{4}$.

Similarly, one finds the solutions for $u\left(x, \frac{\pi}{2}\right)$ and $u\left(x, \frac{3\pi}{4}\right)$.





Question 2 (b)

SOLUTION. We are on a finite domain, so we anticipate that separation of variables is a reasonable approach to this problem. With this in mind, we attempt a solution of the form $u(x,t) = \psi(x)\phi(t)$. Upon applying the boundary conditions at $x = 0$, we get,
$u(0,t) = \psi(0)\phi(t) = 0$
which implies for non-zero $\phi(t)$ that we require, $\psi(0) = 0$. Similarly at the other endpoint, $x = \pi$, we conclude that $\psi(\pi) = 0$. For the homogeneous initial condition on the derivative of u,
$u_t(x,0) = \psi(x)\phi_t(0) = 0$
we get that for non-zero $\psi(x)$ that $\phi_t(0) = 0$. Recall that we can only analyze homogeneous boundary conditions with the technique of separation of variables because they imply conditions on $\psi(x)$ and $\phi(t)$ to be non-trivial. We will deal with the last condition $u(x,0) = f(x)$ at the end. Now, substitute our form for $u(x,t)$ into the PDE, $u_{tt} = u_{xx}$ to get

$$\psi(x)\phi_{tt}(t) = \phi(t)\psi_{xx}(x)$$
$$\frac{\phi_{tt}(t)}{\phi(t)} = \frac{\psi_{xx}}{\psi} = -\lambda$$

with λ a constant. This constant comes from the fact that on the left side we have a function only containing the time variable t and on the right side, a function only containing the space variable x. Therefore, the only way a function of t can equal a function of x is if they are a constant. The choice of λ being negative is in general, arbitrary, but chosen for a reason we will see shortly. We now have the following ODE's to solve:

$$\psi_{xx}(x) = -\lambda\psi(x), \qquad \psi(0) = \psi(\pi) = 0$$
$$\phi_{tt}(t) = -\lambda\phi(t), \qquad \phi_t(0) = 0.$$

We will start by solving the equation for $\psi(x)$ because it has the two boundary conditions required for a complete solution. We start by analyzing different possible values of λ.

Case 1:$\lambda < 0$.
If $\lambda < 0$ then
$\psi_{xx}(x) = -\lambda\psi(x)$
has solution $\psi(x) = a_1 \exp(\sqrt{-\lambda}x) + b_1 \exp(-\sqrt{-\lambda}x)$ which after applying the conditions $\psi(0) = \psi(\pi) = 0$ results in $a_1 = b_1 = 0$. Therefore $\lambda < 0$ is **not** permitted if we want to have non-trivial solutions.

Case 2:$\lambda = 0$
If $\lambda = 0$ then
$\psi_{xx}(x) = -\lambda\psi(x)$
has solution $\psi(x) = a_2 x + b_2$ which after applying the conditions $\psi(0) = \psi(\pi) = 0$ results in $a_2 = b_2 = 0$. Therefore $\lambda = 0$ is **not** permitted if we want to have non-trivial solutions.

Case 3:$\lambda > 0$
If $\lambda > 0$ then
$\psi_{xx}(x) = -\lambda\psi(x)$
has solution $\psi(x) = a_3 \sin(\sqrt{\lambda}x) + b_3 \cos(\sqrt{\lambda}x)$. Applying the condition at $x = 0$,
$\psi(0) = a_2 \sin(0) + b_3 \cos(0) = b_3 = 0$
and so we conclude that $b_3 = 0$. At the other endpoint, $x = \pi$,
$\psi(\pi) = a_2 \sin(\sqrt{\lambda}\pi) = 0$.
In order for this condition to be satisfied we require that $\sqrt{\lambda}\pi = n\pi$ or that $\lambda = n^2$ with $n = 1, 2, 3\ldots$, an integer. We must start at $n = 1$ because we already concluded that $\lambda = 0$ is invalid. Therefore, we conclude (ignoring the constant out front) that,
$\psi(x) = \sin(nx)$.
Before, we continue, note that had I taken λ instead of $-\lambda$ for the original constant, I would have done the same analysis and recovered the same function, only with $\lambda < 0$ being the requirement. We choose the constant to be $-\lambda$ from the start so that we have positive values of λ but this is really just a matter of style.
Now for the other differential equation, knowing $\lambda = n^2$ we have
$\phi_{tt} = -n\phi(t)$
which, recalling $n > 0$, has solution, $\phi(t) = a_4 \cos(nt) + b_4 \sin(nt)$. Applying the condition that $\phi_t(0) = 0$ we conclude that $b_4 = 0$ so that,
$\phi(t) = \cos(nt)$
where we once again haven't included the constant out front of the solution. Finally, combining all this information we get that $u(x,t) = A_n \sin(nx)\cos(nt)$ but this must hold for all n and therefore, by the principle of superposition,
$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(nx)\cos(nt)$.
Now we know that $u(x,0) = \sum_{n=1}^{\infty} A_n \sin(nx) = f(x)$ and so we could apply the usual Fourier series orthogonality to get that,
$A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\mathrm{d}x$
which will work just fine. However, in this case, $f(x) = 2\sin(2x)$ is special because it is one of the eigenfunctions (i.e. it is of the form, $\sin(nx)$ with $n = 2$). Therefore, $\sin(2x)$ is orthogonal to every eigenfunction of the form $\sin(nx)$ except for when $n = 2$. Therefore, only the coefficient A_2 will be non-zero and is given by,
$A_2 = \frac{2}{\pi}\int_0^{\pi} 2\sin(2x)^2\mathrm{d}x = 2$.
If you are unconvinced of how this works, try getting the other coefficients using the formula above and you will quickly see that they are zero! Therefore, we conclude that the solution $u(x,t)$ to the PDE, $u_{tt} = u_{xx}$ is
$u(x,t) = 2\sin(2x)\cos(2t)$

Question 3 (a)

Easiness: 4.0/5

SOLUTION. We will start by computing the Fourier sine series. The Fourier sine series of $f(x)$ is given by

$Sf(x) = \sum_{n=0}^{\infty} b_n \sin(\frac{n\pi x}{L})$
where
$b_n = \frac{2}{L}\int_0^L f(x)\ \sin(\frac{n\pi x}{L})\ dx$
for $n = 1, 2, 3, \cdots$
Here, we have $f(x) = x$ and $L = 1$. So

$$b_n = 2\int_0^1 x\ \sin(n\pi x)\ dx.$$

Integrate by parts using

$$u = x \qquad dv = \sin(n\pi x)\ dx$$

$$du = dv \quad v = \frac{-1}{n\pi}\cos(n\pi x),$$

we get that

$$\begin{aligned} b_n &= 2[\frac{-x}{n\pi}\cos(n\pi x)|_0^1 + \frac{1}{n\pi}\int_0^1 \cos(n\pi x)\ dx] \\ &= 2[\frac{-x}{n\pi}\cos(n\pi x)|_0^1 + (\frac{1}{n\pi})^2 \sin(n\pi x)|_0^1] \\ &= 2[\frac{-1}{n\pi}\cos(n\pi) + 0 + (\frac{1}{n\pi})^2\sin(n\pi) - (\frac{1}{n\pi})^2\sin(0)]. \end{aligned}$$

Since $\sin(n\pi) = 0$ and$\sin(0) = 0$, we get that

$$b_n = \frac{-2}{n\pi}\underbrace{\cos(n\pi)}_{=(-1)^n} = \frac{-2}{n\pi}(-1)^n = \frac{2}{n\pi}(-1)^{n+1}$$

for $n = 1, 2, 3, \cdots$
Therefore, the Fourier sine series of $f(x)$ is

$$Sf(x) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L}) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(-1)^{n+1}\sin(n\pi x).$$

Next, we will find the Fourier cosine series. The Fourier cosine series of $f(x)$ is given by

$$Cf(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L})$$

where
$a_n = \frac{2}{L}\int_0^L f(x)\ \cos(\frac{n\pi x}{L})\ dx$
for $n = 0, 1, 2, \cdots$
With $f(x) = x$ and $L = 1$,

$$a_n = 2\int_0^1 x\ \cos(n\pi x)\ dx.$$

When $n = 0$ we have that $\cos(n\pi x)$=1 and so we have two distinct cases for evaluating the integral; when $n >= 1$ and when $n = 0$. For the case where $n \geq 1$, integrate by parts using

$$u = x \qquad dv = \cos(n\pi x)\ dx$$

$$du = dv \quad v = \frac{1}{n\pi}\sin(n\pi x),$$

we get that

$$\begin{aligned} a_n &= 2[\frac{x}{n\pi}\sin(n\pi x)|_0^1 - \frac{1}{n\pi}\int_0^1 \sin(n\pi x)\ dx] \\ &= 2[\frac{x}{n\pi}\sin(n\pi x)|_0^1 + (\frac{1}{n\pi})^2\cos(n\pi x)|_0^1] \\ &= 2[\frac{x}{n\pi}\sin(n\pi) + 0 + (\frac{1}{n\pi})^2\cos(n\pi) - (\frac{1}{n\pi})^2\cos(0)]. \end{aligned}$$

Since $\sin(n\pi) = 0$, we get that

$$a_n = 2[(\frac{1}{n\pi})^2\underbrace{\cos(n\pi)}_{=(-1)^n} - (\frac{1}{n\pi})^2\underbrace{\cos(0)}_{=1}] = 2(\frac{1}{n\pi})^2[(-1)^n - 1]$$

for $n = 1, 2, 3, \cdots$
Now for the case where $n = 0$, we get

$$a_0 = 2\int_0^1 x\ \cos(0)dx = 2\int_0^1 xdx = 2(\frac{1}{2}x^2)|_0^1 = 1.$$

Therefore, the Fourier cosine series of $f(x)$ is

$$Cf(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(n\pi x) = \frac{1}{2} + \sum_{n=1}^{\infty} 2(\frac{1}{n\pi})^2[(-1)^n - 1]\cos(n\pi x).$$

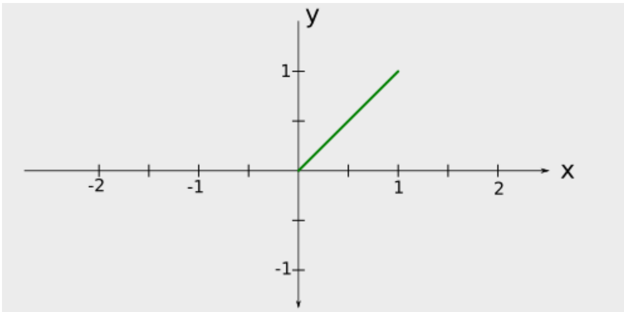
Question 3 (b)

Easiness: 3.0/5

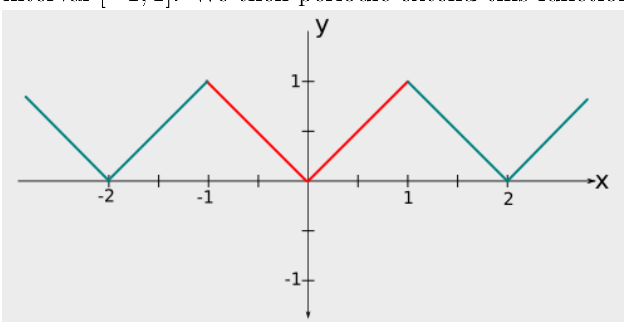
SOLUTION. In short, the Fourier convergence theorem states the following.
If f is a periodic function such that f and f' are piecewise continuous, then the Fouriner series converges to $f(x)$ at all points where f is continuous and converges to $\frac{1}{2}[f(x+) + f(x-)]$ at all points where f is discontinuous.
In the above, $f(x+)$ denotes the right hand limit of f at x, while $f(x-)$ denotes the left hand limit of f at x. In other words, for a real number a,

$$f(a+) = \lim_{x\to a^+} f(x) \ \text{ and } \ f(a-) = \lim_{x\to a^-} f(x).$$

With the Fourier convergence theorem in mind, let's find out the point of convergence of the Fourier cosine series $Cf(x)$ at each point x in the interval $[0, 1]$. To do so, we first need to understand the origin of the Fourier cosine series. Recall that the Fourier cosine series is the Fourier series of f_{even} where f_{even} is the even extension of f. Following the Fourier convergence theorem, to find the point of convergence of the Fourier cosine series, we need to know where the points of continuity and discontinuity of f_{even} are. We will do so by sketching the graph of f_{even}. To begin, we sketch the graph of $f(x)$.



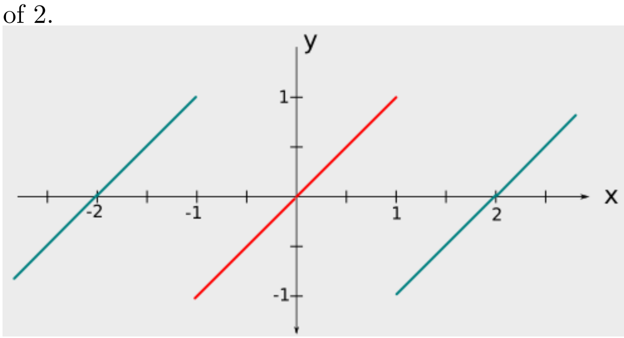
The graph of f_{even} is obtained by first reflecting f over the y-axis and hence obtaining a function over the interval $[-1, 1]$. We then periodic extend this function to get f_{even} which will have a period of 2.



From the sketch of f_{even}, we see that f_{even} is continuous everywhere. Hence, by the Fourier convergence theorem, the Fourier cosine series $Cf(x)$ converges to f_{even} for every real number x. Now, since f agrees with f_{even} at every point in the interval $[0, 1]$,

$Cf(x)$ **converges to** $f(x) = x$ **at every point** x **in the interval** $[0, 1]$.

Similarly, let f_{odd} be the odd extension of f. Then the Fourier sine series $Sf(x)$ is the Fourier series of f_{odd}. We will sketch f_{odd} to find its points of continuity and discontinuity.
The graph of f_{odd} is obtained by first rotating f about the origin by 180 degree and hence obtaining a function over the interval $[-1, 1]$. We then periodic extend this function to get f_{odd} which will have a period of 2.



From the sketch of f_{odd} is discontinuous at points $x = -1 + 2k$ with k an integer. Hence, by the Fourier convergence theorem, the Fourier sine series Sf converges to f_{odd} at every point in the interval $[0, 1)$, but at $x = 1$, Sf converges to

$$\frac{1}{2}[f_{\text{odd}}(1-) + f_{\text{odd}}(1+)] = \frac{1}{2}[1 + (-1)] = 0.$$

Since f_{odd} agrees with f in the interval $[0, 1)$,

'$Sf(x)$ converges to $f(x) = x$ at every point in the interval $[0, 1)$ and at $x = 1$, $Sf(1)$ converges to 0'.

Finally, the question asks us to verify our conclusions for the Fourier sine series at x = 0 and x = 1. Evaluate the sine series found at part a at $x = 0$, we get

$$Sf(0) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(-1)^{n+1} \underbrace{\sin(n\pi\ 0)}_{=0} = \sum_{n=1}^{\infty} 0 = 0$$

and evaluating at $x = 1$, we get

$$Sf(1) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(-1)^{n+1} \underbrace{\sin(n\pi)}_{=0} = \sum_{n=1}^{\infty} 0 = 0$$

as expected.

Question 4 (a)

SOLUTION. By the Taylor series expansion,

$$u(x, t + \Delta t) = u(x, t) + \Delta t\ u_t(x, t) + O((\Delta t)^2).$$

If we isolate $u_t(x, t)$ in the above, we get that

$$u_t(x, t) = \frac{u(x, t + \Delta t) - u(x, t)}{\Delta t}.$$

Similarly, by Taylor series expansion, we have

$$u(x + \Delta x, t) = u(x, t) + \Delta x\ u_x(x, t) + \frac{(\Delta x)^2}{2!} u_{xx}(x, t) + \frac{(\Delta x)^3}{3!} u_{xxx}(x, t) + O((\Delta x)^4)$$

and

$$u(x - \Delta x, t) = u(x, t) - \Delta x\ u_x(x, t) + \frac{(\Delta x)^2}{2!} u_{xx}(x, t) - \frac{(\Delta x)^3}{3!} u_{xxx}(x, t) + O((\Delta x)^4).$$

If we add the two equations above together, we get that

$$u(x + \Delta x, t) + u(x - \Delta x, t) = 2u(x, t) + (\Delta x)^2 u_{xx}(x, t) + O((\Delta x)^4).$$

If we isolate $u_{xx}(x, t)$ in the above, we get that

$$u_{xx}(x, t) = \frac{u(x + \Delta x, t) - 2u(x, t) + u(x - \Delta x, t)}{(\Delta x)^2} + O((\Delta x)^2).$$

From these, we see that we can approximate $u_t(x, t)$ by

$$u_t(x, t) \approx \frac{u(x, t + \Delta t) - u(x, t)}{\Delta t}$$

and $u_{xx}(x, t)$ by

$$u_{xx}(x,t) \approx \frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{(\Delta x)^2}.$$

Hence, we can approximate the equation

$$u_t = u_x x + t$$

by

$$\frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} = \frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{(\Delta x)^2} + t.$$

If we isolate $u(x, t + \Delta t)$ in the above, we get that

$$u(x,t+\Delta t) = u(x,t) + \frac{\Delta t}{(\Delta x)^2}[u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)] + t\Delta t. \quad (1)$$

Next we will put $N + 1$ mesh points in the interval $[0, 1]$. Let

$$x_n = n\Delta x \;\text{ for }\; n = 0, 1, 2, \cdots, N$$

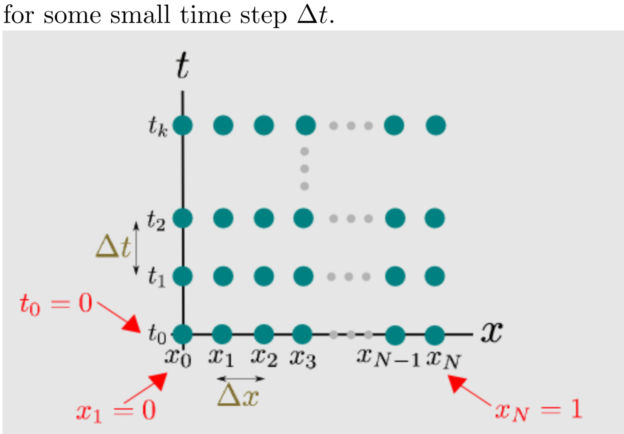
where

$$\Delta x = \frac{1}{N}.$$

Here, $x_0 = 0$ and $x_N = 1$. Similarly, we will let the mesh points in t be

$$t_k = k\Delta t \;\text{ for }\; k = 0, 1, 2, \cdots$$

for some small time step Δt.



At $x = x_n$ and $t = t_k$, equation (1) becomes

$$u(x_n, t_{k+1}) = u(x_n, t_k) + \frac{\Delta t}{(\Delta x)^2}[u(x_{n+1}, t_k) - 2u(x_n, t_k) + u(x_{n-1}, t_k)] + t_k\Delta t.$$

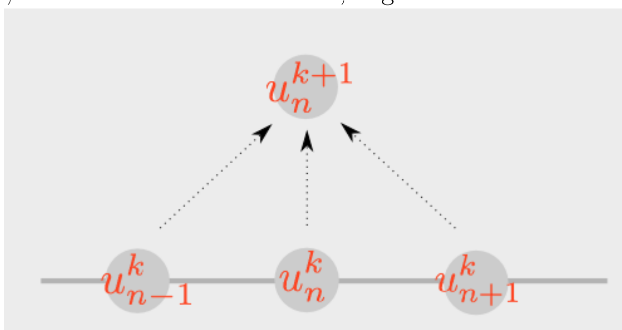
To get the above, we used

$$t + \Delta t = t_{k+1}, x + \Delta x = x_{n+1} \text{ and } x - \Delta x = x_{n-1}.$$

Using the notation $u_n^k \approx u(x_n, t_k)$, we can write the above as

$$u_n^{k+1} = u_n^k + \frac{\Delta t}{(\Delta x)^2}[u_{n+1}^k - 2u_n^k + u_{n-1}^k] + t_k \Delta t. \quad (2)$$

If we look at the above equation, we see that for each fixed k, it gives us a scheme to compute u_n^{k+1} once u_n^k



is known for $n = 0, 1, 2, \cdots, N$.
For the initial condition $u(x, 0) = 0$, we set

$$u_n^0 = 0 \text{ for } n = 0, 1, 2, \cdots, N.$$

For the boundary condition $u(1, t) = 0$, we set

$$u_N^k = 0 \text{ for } k = 1, 2, 3, \cdots .$$

Now, for the boundary condition $u_x(0, t) = 0$, there is more than one way to handle that. One possible way is to set

$$\frac{u(0,t) - u(0 - \Delta x, t)}{\Delta x} = 0$$

which implies

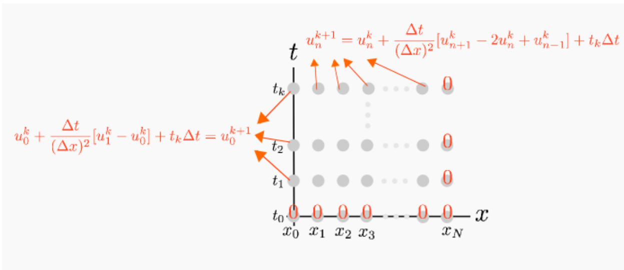
$$u(0, t) = u(-\Delta x, t)$$

If we allow an extra mesh point to the left of x_0, the above gives

$$u_0^k = u_{-1}^k.$$

Hence, for $n = 0$, equation (2) can be rewritten as

$$\begin{aligned} u_0^{k+1} &= u_0^k + \frac{\Delta t}{(\Delta x)^2}[u_1^k - 2u_0^k + u_{-1}^k] + t_k \Delta t \\ &= u_0^k + \frac{\Delta t}{(\Delta x)^2}[u_1^k - u_0^k] + t_k \Delta t. \end{aligned}$$



Referring to the diagram above, to find an approximate solution to this problem, we first set the initial conditions

$$u_n^0 = 0 \text{ for } n = 0, 1, 2, \cdots, N$$

and the boundary condition

$$u_N^k = 0 \text{ for } k = 1, 2, 3, \cdots$$

Then, we compute u_n^1 from u_n^0 by

$$u_0^1 = u_0^0 + \frac{\Delta t}{(\Delta x)^2}[u_1^0 - u_0^0] + t_0 \Delta t$$

and

$$u_n^1 = u_n^0 + \frac{\Delta t}{(\Delta x)^2}[u_{n+1}^0 - 2u_n^0 + u_{n-1}^0] + t_0 \Delta t \text{ for } n = 1, 2, 3, \cdots, N.$$

Once we have u_n^1, we can iterate the process to find $u_n^2, u_n^3, \cdots$

Question 4 (b)

Solution. We must solve this eigenfunction expansion. First, following the hints, we consider the problem without a source term

$$u_t = u_{xx}$$

and attempt a separation of variables $u = \Phi(t)\Psi(x)$:

$$\Psi\Phi_t = \Phi\Psi_{xx}, \quad \text{which can only be true if} \quad \frac{\Phi_t}{\Phi} = \frac{\Psi_{xx}}{\Psi} = -\lambda,$$

for some fixed constant λ.
Let's start with the x-equation.

$$\Psi_{xx} = -\lambda\Psi, \qquad \Psi_x(0) = \Psi(1) = 0.$$

From the boundary conditions we require λ to be positive. Therefore

$$\Psi = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x).$$

To plug in the boundary conditions we first calculate the derivative of Ψ:

$$\Psi_x = A\sqrt{\lambda}\cos(\sqrt{\lambda}x) - B\sqrt{\lambda}\sin(\sqrt{\lambda}x),$$

so that $\Psi_x(0) = A\sqrt{\lambda}$. From the boundary condition $\Psi_x(0) = 0$, we get $A = 0$. (The possibility that $\lambda = 0$ is ruled out by the other boundary condition).
Similarly, from $0 = \Psi(1) = B\cos(\sqrt{\lambda})$, we get

$$\sqrt{\lambda} = \frac{(2n+1)\pi}{2}, \quad n = 0, 1, 2, \ldots$$

Hence, the eigenfunction for

$$\Psi_{xx} = -\lambda\Psi, \quad \Psi_x(0) = \Psi(1) = 0$$

is given (after suppressing the arbitrary constant B) by

$$\Psi_n = \cos\left(\frac{(2n+1)\pi x}{2}\right).$$

At this point we have $u = \Phi(t)\Psi(x)$ where we normally would find Φ for this problem. This would lead us to a solution of the form:
$u = \sum_{n=0}^{\infty} A_n \cos\left(\frac{(2n+1)\pi x}{2}\right)\Phi_n(t)$.
However, recall that the problem we really want to solve is $u_t = u_{xx} + t$, not just $u_t = u_{xx}$.
We use the same eigenfunctions for x and now allow our coefficients A_n to depend on t (which subsumes the $\Phi_n(t)$ terms as well). Thus

$$u(x,t) = \sum_{n=0}^{\infty} A_n(t)\Psi_n(x)$$

so that $u_t - u_{xx} = t$ becomes

$$\sum_{n=0}^{\infty} A_n'\Psi_n - A_n(\Psi_n)_{xx} = t.$$

Recall that $(\Psi_n)_{xx} = -\lambda_n\Psi_n$ so we get

$$\sum_{n=0}^{\infty}(A_n' + \lambda_n A_n)\Psi_n = t$$

Further, recall that our eigenfunctions are orthogonal

$$\int_0^1 \Psi_n\Psi_m\,dx = \begin{cases} 0, & n \neq m, \\ \frac{1}{2}, & n = m, \end{cases}$$

so we can multiply both sides by Ψ_n and then integrate over x to get

$$(A_n' + \lambda_n A_n)\frac{1}{2} = t\int_0^1 \Psi_n\,dx = \frac{t}{\sqrt{\lambda}}\sin\left(\sqrt{\lambda}x\right)\Big|_0^1 = \frac{t}{\sqrt{\lambda}}\sin(\sqrt{\lambda}).$$

Also recall that $\sqrt{\lambda_n} = \frac{(2n+1)\pi}{2}$ so that $\sin\sqrt{\lambda_n} = (-1)^n$. Therefore,

$$A_n' + \lambda_n A_n = \frac{2(-1)^n}{\sqrt{\lambda}} t.$$

We can solve this by integrating factor,

$$\frac{d}{dt}\left(A_n e^{\lambda_n t}\right) = \frac{2(-1)^n}{\sqrt{\lambda}} t e^{\lambda_n t}.$$

Recall that $\int t e^{\lambda_n t}\, dt = (t-1)e^{\lambda_n t}/\lambda_n + C$ and hence

$$A_n e^{\lambda_n t} = \frac{2(-1)^n}{\sqrt{\lambda}} \left(\left(\frac{t-1}{\lambda_n}\right) e^{\lambda_n t} + C\right)$$

so that

$$A_n(t) = \tilde{C} e^{-\lambda_n t} + \frac{2(-1)^n}{\sqrt{\lambda}} \left(\frac{t-1}{\lambda_n}\right),$$

where we have redefined the constant. To solve for the arbitrary constant recall that

$$u(x,0) = \sum_{n=0}^{\infty} A_n(0)\Psi_n(x) = 1.$$

We use orthogonality once more to obtain $A_n(0) = 2\int_0^1 \Psi_n\, dx = \frac{2\sin\sqrt{\lambda_n}}{\sqrt{\lambda_n}} = \frac{2(-1)^n}{\sqrt{\lambda_n}}$. Using the above we get

$$A_n(0) = \tilde{C} - \frac{2(-1)^n}{\sqrt{\lambda_n}\lambda_n} = \frac{2(-1)^n}{\sqrt{\lambda_n}},$$

which implies $\tilde{C} = \frac{2(-1)^n}{\sqrt{\lambda_n}}(1 + 1/\lambda_n)$. Hence, we finally conclude that

$$u(x,t) = \sum_{n=0}^{\infty} \frac{2(-1)^n}{\sqrt{\lambda_n}} \left[\left(1 + \frac{1}{\lambda_n}\right) e^{-\lambda_n t} + \frac{t-1}{\lambda_n}\right] \cos\left(\sqrt{\lambda_n} x\right),$$

for $\lambda_n = \frac{(2n+1)\pi}{2}$.

Question 5 (a)

SOLUTION. To use the method of separation of variables, we write u as

$$u(r,\theta) = R(r)\Theta(\theta).$$

We compute that

$$u_r(r,\theta) = R'(r)\Theta(\theta),\ u_{rr}(r,\theta) = R''(r)\Theta(\theta),$$

and

$$u_{\theta\theta}(r,\theta) = R(r)\Theta''(\theta).$$

Substitute the above into our differential equation

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0,$$

we get that

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0.$$

Next, we would like to group all the R terms onto one side and all the Θ terms onto another, so divide the entire equation by $R\Theta$ to get

$$\frac{R''\Theta}{R\Theta} + \frac{1}{r}\frac{R'\Theta}{R\Theta} + \frac{1}{r^2}\frac{R\Theta''}{R\Theta} = 0$$

which gives

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = 0.$$

To get rid of the factor of $\frac{1}{r^2}$ in the last term on the left, we will multiply the equation by r^2 to get

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} = 0$$

which gives

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta}.$$

Here, r and θ are independent of each other. However, the right hand side of the equation is a function of r while the right hand side a function of θ, so both sides of the equation must be equal to a constant which we will denote μ. In other words, we have

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \mu.$$

This gives us two equations

$$r^2R'' + rR' = \mu R \text{ and } -\Theta'' = \mu\Theta.$$

We will first solve the equation for R. Before we go ahead to do so, we would like to figure out the boundary conditions for R. The boundary conditions

$$u(1,\theta) = 0 \text{ and } u(2,\theta) = 0$$

imply that

$$R(1)\Theta(\theta) = 0 \text{ and } R(2)\Theta(\theta) = 0.$$

Since we do not want Θ to be 0 since this will give the trivial solution (i.e. $u \equiv 0$) which is not of interest to us. So the boundary conditions give us

$$R(1) = 0 \text{ and } R(2) = 0.$$

We will now solve the equation for R which says

$$r^2R'' + rR' - \mu R = 0.$$

This is the Euler's equation and we recall that we should look for solutions in the form

$$R(r) = r^\alpha$$

for an unknown constant α to be determined. We compute that

$$R' = \alpha r^{\alpha-1} \text{ and } R'' = \alpha(\alpha-1)r^{\alpha-2}.$$

Substituting these into

$$r^2R'' + rR' - \mu R = 0$$

we get that

$$\alpha(\alpha-1) + \alpha = \mu$$

which after simplifying gives

$$\alpha^2 = \mu.$$

Here, the sign of μ will affect the form of the solution, so we have three cases to consider. They are

case 1: $\mu = -\lambda^2 < 0$
case 2: $\mu = 0$
case 3: $\mu = \lambda^2 > 0$

Further calculations show that **case 2** and **case 3** only give trivial solutions. If you are not sure, you should try verifying that yourself. For completeness of the solution, I will also include the computation in the appendix of this solution.
Now, **case 1:** says that

$$\alpha^2 = \mu = -\lambda^2.$$

This gives

$$\alpha = \pm\lambda i.$$

Recall that for the Euler's equation, if α is in the form of $\alpha = a \pm bi$, the solution of the Euler's equation can be written as

$$R(r) = C_1 r^a \cos(b\ln(r)) + C_2 r^a \sin(b\ln(r))$$

for some arbitrary constants C_1 and C_2. For us, we have $a = 0$ and $b = \lambda$, so our solution for R is

$$R(r) = C_1 \cos(\lambda \ln(r)) + C_2 \sin(\lambda \ln(r)).$$

Next, we need to match the boundary conditions $R(1) = 0$ and $R(2) = 0$.

$$R(1) = 0 \Rightarrow 0 = C_1 \cos(\lambda \underbrace{\ln(1)}_{=0}) + C_2 \sin(\lambda \underbrace{\ln(1)}_{=0}) = C_1 \underbrace{\cos(0)}_{=1} + C_2 \underbrace{\sin(0)}_{=0} = C_1.$$

Hence, $C_1 = 0$ and

$$R(r) = C_2 \sin(\lambda \ln(r)).$$

Now,

$$R(2) = 0 \Rightarrow C_2 \sin(\lambda \ln(2)) = 0.$$

If $C_2 = 0$, we will have the trivial solution. To get non-trivial solutions, we need

$$\sin(\lambda \ln(2)) = 0 \Rightarrow \lambda \ln(2) = n\pi \Rightarrow \lambda = \frac{n\pi}{\ln(2)}.$$

In other words, we have an infinite numbers of eigenvalues λ_n give by

$$\lambda_n = \frac{n\pi}{\ln(2)} \quad \text{for} \quad n = 1, 2, 3, \cdots$$

with the corresponding eigenfunctions

$$R_n(r) = C_n \sin(\lambda_n \ln(r)).$$

Now, we will go back to solve the θ equation. Recall that we have $\mu = -\lambda^2$ so the θ equation $-\Theta'' = \mu\Theta$ becomes

$$-\Theta'' = -\lambda^2\Theta$$
$$\Theta'' = \lambda^2\Theta$$

which gives solutions of the form

$$\Theta = A\cosh(\lambda\theta) + B\sinh(\lambda\theta)$$

for some arbitrary constants A and B. Since we found that an infinite number of $\lambda_n = \frac{n\pi}{\ln(2)}$, we have an infinite number of Θ_n given by

$$\Theta_n = A_n \cosh(\lambda_n \theta) + B_n \sinh(\lambda_n \theta).$$

Putting everything together, we can express u as

$$u(r,t) = \sum_{n=1}^{\infty} C_n \sin(\lambda_n \ln(r)) \left(A_n \cosh(\lambda_n \theta) + B_n \sinh(\lambda_n \theta)\right).$$

We can absorb the arbitrary constant C_n into A_n and B_n to get

$$u(r,t) = \sum_{n=1}^{\infty} \sin(\lambda_n \ln(r)) \left(A_n \cosh(\lambda_n \theta) + B_n \sinh(\lambda_n \theta)\right).$$

Finally, we will use the other two boundary conditions $u(r,0) = 0$ and $u(r, \frac{\pi}{2}) = f(r)$ to find A_n and B_n.

$$\begin{aligned}
&u(r,0) = 0 \\
&\sum_{n=1}^{\infty} \sin(\lambda_n \ln(r))(A_n \underbrace{\cosh(0)}_{=1} + B_n \underbrace{\sinh(0)}_{=0}) = 0 \\
&\sum_{n=1}^{\infty} A_n \sin(\lambda_n \ln(r)) = 0. \\
&A_n = 0 \ \text{ for each } \ n = 1, 2, 3, \cdots
\end{aligned}$$

Hence

$$\begin{aligned}
u(r,t) &= \sum_{n=1}^{\infty} B_n \sin(\lambda_n \ln(r)) \sinh(\lambda_n \theta) \\
&= \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{\ln(2)} \ln(r)\right) \sinh\left(\lambda_n \theta\right)
\end{aligned}$$

Finally, to find B_n, we use the last boundary condition

$$u\left(r, \frac{\pi}{2}\right) = \sin\left(\frac{2\pi}{\ln(2)} \ln(r)\right).$$

This boundary condition gives us that

$$u\left(r, \frac{\pi}{2}\right) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{\ln(2)} \ln(r)\right) \sinh\left(\lambda_n \frac{\pi}{2}\right) = \sin\left(\frac{2\pi}{\ln(2)} \ln(r)\right).$$

If we expand out the summation, we see that

$$\begin{aligned}
&B_1 \sin\left(\frac{\pi}{\ln(2)} \ln(r)\right) \sinh\left(\lambda_1 \frac{\pi}{2}\right) + B_2 \sin\left(\frac{2\pi}{\ln(2)} \ln(r)\right) \sinh\left(\lambda_2 \frac{\pi}{2}\right) \\
&+ B_3 \sin\left(\frac{3\pi}{\ln(2)} \ln(r)\right) \sinh\left(\lambda_3 \frac{\pi}{2}\right) + \cdots = \sin\left(\frac{2\pi}{\ln(2)} \ln(r)\right)
\end{aligned}$$

Here, we see that $n = 2$ is the only term on the left hand side that matches with the right hand side. Hence, we have

$$B_n \sinh\left(\lambda_n \frac{\pi}{2}\right) = 0 \text{ if } n \neq 2$$

and

$$B_2 \sinh\left(\lambda_2 \frac{\pi}{2}\right) = 1$$

$$\Rightarrow B_2 = \frac{1}{\sinh(\lambda_2 \frac{\pi}{2})} = \frac{1}{\sinh(\frac{2\pi}{\ln(2)} \frac{\pi}{2})}.$$

In other words, the solution is

$$u(r, \theta) = \frac{1}{\sinh\left(\frac{2\pi}{\ln(2)} \frac{\pi}{2}\right)} \sin\left(\frac{2\pi}{\ln(2)} \ln(r)\right) \sinh\left(\frac{2\pi}{\ln(2)} \theta\right).$$

Appendix:
We will address two final questions in this appendix.
Question 1: Refer to the solution above, why doesn't $\mu = 0$ lead to non-trivial solutions for the Euler's equation?
Answer to question 1: Suppose $\mu = 0$, then

$$\alpha^2 = \mu \Rightarrow \alpha = 0.$$

The solution to the Euler's equation is

$$R(r) = C_1 \ln(r) + C_2$$

for some arbitrary constants C_1 and C_2. The boundary condition $R(1) = 0$ implies

$$R(1) = C_1 \underbrace{\ln(1)}_{=0} + C_2 = 0 \Rightarrow C_2 = 0.$$

So

$$R(r) = C_1 \ln(r).$$

The boundary condition $R(2) = 0$ implies

$$R(2) = C_1 \ln(2) = 0 \Rightarrow C_1 = 0.$$

Hence, we only get the trivial solution.
Question 2: Refer to the solution above, why doesn't $\mu = \lambda^2 > 0$ lead to non-trivial solutions for the Euler's equation?
Answer to question 2: Suppose $\mu = \lambda^2 > 0$, then

$$\alpha^2 = \mu \Rightarrow \alpha = \pm\lambda.$$

The solution to the Euler's equation is

$$R(r) = C_1 r^{\lambda} + C_2 r^{-\lambda}$$

for some arbitrary constants C_1 and C_2. The boundary condition $R(1) = 0$ implies

$$R(1) = C_1 + C_2 = 0 \Rightarrow C_1 = -C_2.$$

So

$$R(r) = C_1 r^{\lambda} - C_1 r^{-\lambda}.$$

The boundary condition $R(2) = 0$ implies

$$R(2) = C_1 2^{\lambda} - C_1 2^{-\lambda} = 0.$$

If $C_1 = 0$, we get a trivial solution, so suppose $C_1 \neq 0$, then we must have

$$2^{\lambda} = 2^{-\lambda} \Rightarrow 2^{2\lambda} = 1 \Rightarrow \lambda = 0.$$

However, this contradicts with our original assumption that $\lambda^2 > 0$.

Question 5 (b)

SOLUTION. In part (a) we found a solution for a specific f(r) that happened to be one of the original eigenfunctions and so we were able to compare just by analyzing the coefficient of one term. However now we have a general f(r) and if we continue from part (a) we have,

$$u\left(r, \frac{\pi}{2}\right) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{\ln(2)} \ln(r)\right) \sinh\left(\lambda_n \frac{\pi}{2}\right) = f(r).$$

We wish to impose an orthogonality argument so that we can isolate each individual coefficient, B_n. Recall that if we have a problem in Sturm-Liouville form,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(p(r)\frac{\mathrm{d}\psi}{\mathrm{d}r}\right) - q(r)\psi + \sigma(r)\mu\psi = 0$$

then the eigenfunctions ψ have the orthogonality relationship

$$\int_a^b \psi_n \psi_m \sigma(r)\mathrm{d}r = 0, \quad n \neq m.$$

Our eigenfunction problem for r was

$$r^2 R'' + rR' + \left(\frac{n\pi}{\ln 2}\right)^2 R = 0$$

which we can write in Sturm-Liouville form to get

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right)+\frac{1}{r}\left(\frac{n\pi}{\ln 2}\right)^2 R=0$$

so our orthogonality relation is

$$\begin{aligned}&\int_a^b \frac{R_n R_m}{r}\mathrm{d}r\\&=\int_1^2 \sin\left(\frac{n\pi}{\ln 2}\ln r\right)\sin\left(\frac{m\pi}{\ln 2}\ln r\right)\frac{1}{r}\mathrm{d}r\end{aligned}$$

If we compute the integrals we can use a substitution of

$$\begin{aligned}x&=\frac{\ln r}{\ln 2}\\ \mathrm{d}x&=\frac{1}{\ln 2}\frac{1}{r}\mathrm{d}r\end{aligned}$$

to get

$$\ln 2\int_0^1 \sin(n\pi x)\sin(m\pi x)\,\mathrm{d}x.$$

We immediately recognize this new integral as the standard sine orthogonality relation and so we get

$$\begin{aligned}&\int_1^2 \sin\left(\frac{n\pi}{\ln 2}\ln r\right)\sin\left(\frac{m\pi}{\ln 2}\ln r\right)\frac{1}{r}\mathrm{d}r\\&=\ln 2\int_0^1 \sin(n\pi x)\sin(m\pi x)\,\mathrm{d}x=\begin{cases}0 & ,n\neq m\\ \frac{\ln 2}{2} & ,n=m\end{cases}.\end{aligned}$$

Returning to

$$\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi}{\ln(2)}\ln(r)\right)\sinh\left(\lambda_n\frac{\pi}{2}\right)=f(r)$$

we can use our orthogonality relation to get

$$\begin{aligned}B_n\sinh\left(\lambda_n\frac{\pi}{2}\right)\frac{\ln 2}{2}&=\int_1^2 f(r)\sin\left(\frac{n\pi}{\ln(2)}\ln(r)\right)\frac{1}{r}\mathrm{d}r\\ B_n&=\frac{2}{\ln 2}\frac{1}{\sinh\left(\lambda_n\frac{\pi}{2}\right)}\int_1^2 f(r)\sin\left(\frac{n\pi}{\ln(2)}\ln(r)\right)\frac{1}{r}\mathrm{d}r.\end{aligned}$$

Therefore we now have the constants B_m for any $f(r)$ and therefore we have the solution to the partial differential equation,

$$u(r,\theta)=\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi}{\ln(2)}\ln(r)\right)\sinh(\lambda_n\theta).$$

Good Luck for your exams!