# Full Solutions
# MATH101 April 2005

December 6, 2014

## How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Educational Resources.

## Tips for Using Previous Exams to Study: Exam Simulation

*Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.*

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Exam Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Educational Resources.



## Question 1 (a)

**Easiness: 96/100**

SOLUTION. $\int(x^2+e^{2x})\,dx = \frac{x^3}{3}+\frac{e^{2x}}{2}+C$

## Question 1 (b)

**Easiness: 84/100**

SOLUTION. Here $a=0$, $b=\pi$ and $f(x)=\sin x$. Thus

$$\begin{aligned} f_{av} &= \frac{1}{\pi-0}\int_0^\pi \sin x\,dx \\ &= \frac{1}{\pi}\left(-\cos x\right)\Big|_0^\pi \\ &= \frac{1}{\pi}(-\cos\pi+\cos 0) \\ &= \frac{2}{\pi}. \end{aligned}$$

## Question 1 (c)

**Easiness: 73/100**

SOLUTION. By the fundamental theorem of calculus, we know that there exists a function g(x) such that

$$F(x)=\int_0^{x^2} f(t)\,dt = g(x^2)-g(0)$$

where g(x) is the anti-derivative of the function f(x) (i.e: g'(x) = f(x)). Taking the derivative of F(x) gives

$$F'(x)=\frac{d}{dx}\left(g(x^2)-g(0)\right)=g'(x^2)\cdot 2x$$

by the chain rule. Using the anti-derivative property of g,

$$F'(x)=f(x^2)\cdot 2x.$$

Plugging in x = 2 and using the fact that f(4) = 1 gives us our answer

$$F'(2)=f(4)\cdot 4 = 4$$

## Question 1 (d)

SOLUTION. This is a 2nd-order differential equation with constant coefficients. So make the substitution of $y=e^{rx}$ into the equation and solve for $r$.

$$\begin{aligned} y''-2y'+y &= 0 \\ r^2e^{rx}-2re^{rx}+e^{rx} &= 0 \\ e^{rx}\left(r^2-2r+1\right) &= 0 \\ \rightarrow \quad e^{rx} &= 0 \quad \text{or} \quad r^2-2r+1=0 \end{aligned}$$

Solving for $r$ we find that there is a double root $r = 1$. Thus we have found that one solution is $y(x) = e^x$, but we need to find a second solution. We multiply the first solution by $x$ and confirm that it is indeed a solution of the above equation (i.e: Let $y = xe^x$ be another possible solution and substitute it into the differential equation).

$$\begin{aligned} y'' - 2y' + y &= (2e^x + xe^x) - 2\left(xe^x + e^x\right) + xe^x \\ &= 2e^x + xe^x - 2xe^x - 2e^x + xe^x \\ &= 0 \end{aligned}$$

Thus, the general solution is $y(x) = c_1 e^x + c_2 x e^x$, where $c_1$, $c_2$ are arbitrary constants.

## Question 1 (e)

**Solution.** To find the general solution for this problem, we need to find both the particular solution, $y_p$, and homogeneous solution, $y_h$.
The homogeneous solution satisfies

$$y_h'' + 2y_h' + y_h = 0,$$

which can be solved using the technique from part (d) of this exam to obtain the solution

$$y_h(x) = c_1 e^{-x} + c_2 x e^{-x}.$$

To find the particular solution, we look at the right hand side of the equation and notice that there is a term that is linear in $x$ (as opposed to quadratic, cubic, etc.). So we can assume a trial solution of $y_p = ax + b$ and plug this into the differential equation to find the constants $a, b$ such that the original differential equation is satisfied.

$$\begin{aligned} y_p'' + 2y_p' + y_p &= x \\ 0 + 2a + ax + b &= x \end{aligned}$$

By comparing the linear and constant terms on both sides of the equation, we can see that for the differential equation to be satisfied, we must satisfy $a = 1$ and $2a + b = 0$. Thus, $a = 1$, $b = 2$. So the general solution to the differential equation is the sum of the homogeneous and particular solutions:

$$y(x) = x + 2 + c_1 e^{-x} + c_2 x e^{-x}$$

where $c_1$, $c_2$ are arbitrary constants.

## Question 1 (f)

**Easiness: 82/100**

**Solution.**

$$\begin{aligned} \int_0^\infty \frac{dx}{(x+1)^3} &= -\frac{1}{2}\frac{1}{(x+1)^2}\bigg|_0^\infty \\ &= \lim_{x\to\infty} -\frac{1}{2}\frac{1}{(x+1)^2} + \frac{1}{2}\frac{1}{(0+1)^2} \\ &= 0 + \frac{1}{2} \\ &= \frac{1}{2} \end{aligned}$$

## Question 1 (g)

SOLUTION. We recall that the probability density function (PDF) for an exponential random variable is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}, \quad \lambda > 0.$$

The probability that $X$ is greater than or equal to 8 is given by integrating the PDF from $x = 8$ to infinity.

$$\mathbb{P}[X \geq 8] = \int_8^\infty f(x)\,dx = \lambda \int_8^\infty e^{-\lambda x}\mathrm{d}x = -e^{-\lambda x}\Big|_8^\infty = e^{-8\lambda}$$

Clearly, we need the value of $\lambda$ if we are to evaluate the probability that $X$ is greater than or equal to 8. We can determine this by evaluating the mean of $f(x)$ and comparing to the given value. The mean of $f(x)$ is

$$\underbrace{\lambda \int_0^\infty x e^{-\lambda x}\mathrm{d}x}_{\text{integrate by parts}}$$
$$= -xe^{-\lambda x}\Big|_0^\infty + \int_0^\infty e^{-\lambda x} = \frac{1}{\lambda}.$$

Therefore the mean of $f(x)$ is $\lambda^{-1} = 4$, and thus $\lambda = 1/4$. Plugging this into the result above, we find that

$$\mathbb{P}[X \geq 8] = e^{-2} \quad (\approx 0.1353).$$

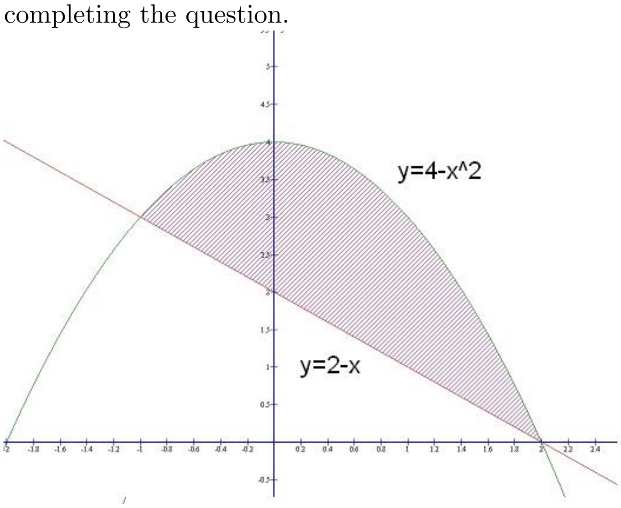
## Question 2 (a)

**Easiness: 80/100**

SOLUTION. A picture can be found below.
First, we find the points of intersection. Set the curves equal to each other so that
$4 - x^2 = 2 - x$
Then isolating to one side gives
$0 = x^2 - x - 2 = (x+1)(x-2)$
and so the $x$ coordinates of the points of intersection are $x = -1, 2$. Now, plugging in the point $x = 0$ into both curves shows that $4 - x^2$ is the upper most curve. Hence, the area we seek is

$$
\begin{aligned}
A &= \int_{-1}^{2} (4 - x^2 - (2 - x))\, dx \\
&= \int_{-1}^{2} (-x^2 + x + 2)\, dx \\
&= \left( \frac{-x^3}{3} + \frac{x^2}{2} + 2x \right) \Big|_{-1}^{2} \\
&= \left( \frac{-8}{3} + \frac{4}{2} + 2(2) \right) - \left( \frac{1}{3} + \frac{1}{2} + 2(-1) \right) \\
&= \frac{-8}{3} + 6 - \frac{1}{3} - \frac{1}{2} + 2 \\
&= -3 + \frac{15}{2} \\
&= \frac{9}{2}
\end{aligned}
$$

completing the question.




## Question 2 (b)

**Easiness: 100/100**

SOLUTION. We would use disc integration to evaluate this volume. First, we need to determine the $x$ values that define the endpoints of the interval that we need to integrate over. The endpoints are where the curves intersect, so we solve $4 - x^2 = 2 - x$ or

$$x^2 - x - 2 = 0 \quad \rightarrow \quad (x - 2)(x + 1) = 0$$

Thus, the endpoints of the interval $x = -1$ and $x = 2$. Since we are using the disc method, we recall that the volume of a single disc with radius $r$ and thickness $\Delta x$ is $\pi r^2 \Delta x$. Since we are rotating each curve about the $x$-axis, we can treat the value of the function as the height of each disc.
By taking a sample point in the interval $(-1, 2)$, we can see that $4 - x^2 > 2 - x$ in the domain of integration so we can get the volume of the defined solid by taking the volume of the solid with radius $4 - x^2$ and subtracting the volume with radius $2 - x$.

$$
\begin{aligned}
V &= \int_{-1}^{2} \pi(4-x^2)^2\,dx - \int_{-1}^{2} \pi(2-x)^2\,dx \\
&= \pi \int_{-1}^{2} (4-x^2)^2 - (2-x)^2\,dx
\end{aligned}
$$

(Note: If you wish to evaluate the integral, the volume of the solid obtained by rotating the region $R$ about the $x$-axis is $V = 108\pi/5$.)

## Question 2 (c) **Easiness: 70/100**

SOLUTION. To evaluate this volume, we would use the method of cylindrical shells. We must determine first where the endpoints for the bounded area are. So we solve the following equation:

$$
\begin{aligned}
4 - x^2 &= 2 - x \\
0 &= x^2 - x - 2 \\
0 &= (x+1)(x-2) \quad \rightarrow \quad x = -1, 2.
\end{aligned}
$$

For the method of cylindrical shells, we are basically summing up terms of the form $2\pi rh\Delta r$ where $r$ is the distance of each shell to the axis of rotation and $h$ is the height of the shell at radius $r$. The distance from the point $x$ to the axis of rotation is $r = |x-2| = 2-x$ (since we are integrating over $[-1, 2]$), and the height $h$ is just the difference of the functions that define the boundaries. Thus, the volume of the described object can be evaluating by the following integral:

$$
V = \int_{-1}^{2} 2\pi(2-x)(4-x^2-(2-x))\,dx
$$

(Note: Evaluating the volume gives $V = 27\pi/2$.)

## Question 2 (d)

SOLUTION. From part c), we found that the integral needed to be taken from $x = -1$ to $x = 2$. Thus we need to evaluate the arclength of the curve $y = 4 - x^2$ in between those two points. We recall the formula for arclength is given by

$$
l = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.
$$

Making the appropriate substitutions for our situation we get:

$$
l = \int_{-1}^{2} \sqrt{1 + (-2x)^2}\,dx = \int_{-1}^{2} \sqrt{1 + 4x^2}\,dx
$$

Now we need to choose the appropriate trigonometric substitution. Making the substitution $\tan(\theta) = 2x$, allows the integral to be more easily evaluated in terms of trig functions:

$$\begin{aligned} l &= \int_{-1}^{2} \sqrt{1+4x^2}\, dx \\ &= \int_{\arctan(-2)}^{\arctan(4)} \sqrt{1+\tan^2(\theta)} \cdot \frac{1}{2}\sec^2(\theta)\, d\theta \\ &= \int_{\arctan(-2)}^{\arctan(4)} \sec(\theta) \cdot \frac{1}{2}\sec^2(\theta)\, d\theta \\ &= \frac{1}{2}\int_{\arctan(-2)}^{\arctan(4)} \sec^3(\theta)\, d\theta. \end{aligned}$$

## Question 3 (a)

SOLUTION. We will use the method of partial fractions to solve this integral.
First, we will rewrite the integrand:

$$\frac{4x+4}{x(x+1)^2} = \frac{4x}{x(x+1)^2} + \frac{4}{x(x+1)^2}.$$

We can now cancel a factor of $x$, leaving:

$$\frac{4}{(x+1)^2} + \frac{4}{x(x+1)^2}.$$

We have made a little progress. We have shown that

$$\frac{4x+4}{x(x+1)^2} = \frac{4}{(x+1)^2} + \frac{4}{x(x+1)^2}.$$

We want to compute

$$\int \frac{4x+4}{x(x+1)^2} dx.$$

By the above computation, this integral is equal to

$$\int \frac{4}{(x+1)^2} + \frac{4}{x(x+1)^2} dx,$$

which is equal to

$$\int \frac{4}{(x+1)^2} dx + \int \frac{4}{x(x+1)^2} dx,$$

by basic properties of the integral. Now we can see why breaking the integrand up into pieces was helpful: the integral to the left of the "+" can be evaluated by making the substitution $u = x+1, du = dx$. Thus,

$$\int \frac{4}{(x+1)^2} dx = \int \frac{4}{u^2} du = -4u^{-1} + C = -4(x+1)^{-1} + C.$$

Now we will have to use partial fractions to compute the next integral. This means we need to find constants A, B, and C to make the following equality true:

$$\frac{4}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

Multiply both sides by $x(x+1)^2$ to cancel the denominators, yielding

$$4 = A(x+1)^2 + Bx(x+1) + Cx.$$

Now we expand the right hand side.

$$4 = A(x^2 + 2x + 1) + Bx^2 + Bx + Cx$$

$$4 = Ax^2 + 2Ax + A + Bx^2 + Bx + Cx$$

$$4 = (A+B)x^2 + (2A+B+C)x + A$$

Now since we have an equality of polynomials, we know that their coefficients of $x^2$ and $x$ must be equal, as well as the constant coefficient. The polynomial on the left hand side has coefficient of $x^2$ and $x$ equal to zero, and constant coefficient equal to 4. Thus we get three equations:
For the coefficient of $x^2$:

$$0 = A + B$$

For the coefficient of $x$:

$$0 = 2A + B + C$$

For the constant coefficient:

$$4 = A$$

Since A=4, we get from the $x^2$ coefficient equation that

$$0 = 4 + B$$

Thus, B=-4. Now we can solve for C in the $x$ coefficient equation,

$$0 = 2 * 4 + (-4) + C$$

Thus, C=-4. Putting all of this together, we can conclude that

$$\frac{4}{x(x+1)^2} = \frac{4}{x} + \frac{-4}{x+1} + \frac{-4}{(x+1)^2}.$$

So:

$$\int \frac{4}{x(x+1)^2}dx = \int \frac{4}{x}dx + \int \frac{-4}{x+1}dx + \int \frac{-4}{(x+1)^2}dx.$$

We have succeeded in splitting up a difficult integral into three simpler ones!

$$\int \frac{4}{x}dx = 4\ln|x| + c_1,$$

$$\int \frac{-4}{x+1}dx = -4\ln|x+1| + c_2,$$

$$\int \frac{-4}{(x+1)^2}dx = 4(x+1)^{-1} + c_3,$$

where $c_1, c_2, c_3$ are constants.
Thus

$$\int \frac{4}{x(x+1)^2}dx = 4\ln|x| - 4\ln|x+1| + 4(x+1)^{-1} + C$$

Finally, we are able to conclude that

$$\begin{aligned}\int \frac{4x+4}{x(x+1)^2}dx = &-4(x+1)^{-1} + 4\ln|x| \\ &- 4\ln|x+1| + 4(x+1)^{-1} + C\end{aligned}$$

where the term $-4(x+1)^{-1}$ comes from the integral we did at the very beginning. Notice two terms cancel out to get as our final answer

$$\int \frac{4x+4}{x(x+1)^2}\mathrm{d}x = 4\ln|x| - 4\ln|x+1| + C.$$

## Question 3 (b)

**Easiness: 75/100**

SOLUTION. We want to evaluate this integral:

$$\int (x+1)\ln(x)\,dx$$

We start by making the following substitution:

$$y = \ln(x).$$

Then

$$dy = \frac{1}{x}\,dx$$

and thus

$$dx = x\,dy$$

So, after making the substitution, the integral becomes

$$\int (x+1)yx\,dy.$$

But because

$$y = \ln(x),$$

it follows that

$$x = e^y.$$

So the integral reduces to

$$\int e^y(e^y+1)y\,dy$$

$$= \int (e^{2y}+e^y)y\,dy$$

Now, we apply integration by parts with

$$u = y$$

$$du = dy$$

$$dv = e^{2y}+e^y$$

$$v = \frac{1}{2}e^{2y}+e^y$$

So,

$$\int u\,dv = uv - \int v\,du$$

$$= y(\frac{1}{2}e^{2y}+e^y) - \int \frac{1}{2}e^{2y}+e^y\,dy$$

$$= y(\frac{1}{2}e^{2y}+e^y) - \frac{1}{4}e^{2y} - e^y + C.$$

$$= y(\frac{1}{2}(e^y)^2 + e^y) - \frac{1}{4}(e^y)^2 - e^y + C$$

Finally, we substitute y = ln(x) back in:

$$= \frac{1}{2}x^2 \ln(x) + x \ln x - \frac{1}{4}x^2 - x + C$$

So the integral is given by

$$\int (x+1)\ln(x)dx = \frac{1}{2}x^2 \ln(x) + x \ln x - \frac{1}{4}x^2 - x + C$$

*This is not required in a complete solution, but it is recommended to check your answer by differentiating:*

$$\begin{aligned}
&\frac{d}{dx}\left(\frac{1}{2}x^2 \ln(x) + x\ln(x) - \frac{1}{4}x^2 - x + C\right) \\
&= x\ln(x) + \frac{1}{2}x + \ln(x) + 1 - \frac{1}{2}x - 1 \\
&= x\ln(x) + \ln(x) \\
&= (x+1)\ln(x).
\end{aligned}$$

## Question 3 (c)

**Easiness: 6/100**

SOLUTION. We want to integrate:

$$\int \frac{dx}{(5-4x-x^2)^{3/2}}.$$

The first step is to complete the square in the denominator:

$$\begin{aligned}
\int \frac{dx}{(9-4-4x-x^2)^{3/2}} &= \int \frac{dx}{(9-(x+2)^2)^{3/2}} \\
&= \int \frac{1}{27}\,\frac{dx}{(1-\frac{(x+2)^2}{9})^{3/2}} \\
&= \frac{1}{27}\int \frac{dx}{(1-(\frac{x+2}{3})^2)^{3/2}}
\end{aligned}$$

Now we substitute:

$$y = \frac{x+2}{3}$$

$$dy = \frac{dx}{3}$$

$$3dy = dx,$$

So the integral simplifies to

$$\frac{3}{27}\int \frac{dy}{(1-y^2)^{3/2}} = \frac{1}{9}\int \frac{dy}{(1-y^2)^{3/2}}$$

We now perform a trigonometric substitution (this step is only valid for |y| smaller than 1, which we know is true because 1-y^2 must be positive):

$$y = \sin(t)$$

$$dy = \cos(t)dt$$

So the above integral becomes:

$$\begin{aligned}
\frac{1}{9}\int \frac{\cos(t)dt}{(1-\sin^2(t))^{3/2}} &= \frac{1}{9}\int \frac{\cos(t)dt}{(\cos^2(t))^{3/2})} \\
&= \frac{1}{9}\int \frac{dt}{\cos^2(t)} \\
&= \frac{1}{9}\int \sec^2(t)dt \\
&= \frac{1}{9}\tan(t) + C \\
&= \frac{1}{9}\tan(\arcsin(y)) + C \\
&= \frac{1}{9}\tan(\arcsin(\frac{x+2}{3})) + C
\end{aligned}$$

But for any $u$, we have

$$\tan(\arcsin(u)) = \frac{u}{\sqrt{1-u^2}},$$

So the above quantity is equal to

$$= \frac{1}{9}\,\frac{\frac{x+2}{3}}{\sqrt{1-(\frac{x+2}{3})^2}}$$

$$= \frac{1}{9}\,\frac{x+2}{\sqrt{5-4x-x^2}}.$$

So we have

$$\int \frac{dx}{(5-4x-x^2)^{3/2}} = \frac{1}{9}\,\frac{x+2}{\sqrt{5-4x-x^2}}.$$

*This is not required in a test, but we can check our answer by differentiating:*

$$\frac{d}{dx}\,\frac{1}{9}\,\frac{x+2}{\sqrt{5-4x-x^2}}$$

$$
\begin{aligned}
&= \frac{1}{9}\,\frac{\sqrt{5-4x-x^2}-\frac{1}{2}(x+2)(5-4x-x^2)^{-1/2}(-2x-4)}{(5-4x-x^2)} \\
&= \frac{1}{9}\,\frac{5-4x-x^2+x^2+2x+2x+4}{(5-4x-x^2)^{3/2}} \\
&= \frac{1}{9}\,\frac{9}{(5-4x-x^2)^{3/2}} \\
&= \frac{1}{(5-4x-x^2)^{3/2}}
\end{aligned}
$$

## Question 4

SOLUTION. Since this is a non-homogeneous problem, we must find a particular solution and a homogeneous solution. Let $x_p$ and $x_h$ be the particular and homogeneous solutions, resp.
__NOTOC__

### Particular solution

We will first find the particular solution $x_p$ by assuming a oscillatory solution of the same frequency as the right hand side of the differential equation:

$$x_p(t) = A\cos(3t) + B\sin(3t)$$

where $A, B$ are constants to be determined.

$$
\begin{aligned}
x_p'' + 4x_p' + 3x_p &= 60\cos(3t) \\
-9A\cos(3t) - 9B\sin(3t)& \\
-12A\sin(3t) + 12B\cos(3t)& \\
+3A\cos(3t) + 3B\sin(3t) &= 60\cos(3t)
\end{aligned}
$$

Separating the parts of the equation that are multiplied by $\sin(3t)$ and $\cos(3t)$, we find that we must solve the following equations for $A, B$:

$$\left.\begin{aligned} -9A + 12B + 3A &= 60 \\ -9B - 12A + 3B &= 0 \end{aligned}\right\} \quad \to \quad A = -2,\ B = 4.$$

Thus the particular solution is:

$$x_p(t) = -2\cos(3t) + 4\sin(3t)$$

### Homogeneous solution

Now will find the solution to the homogeneous equation, $x_h(t)$, which satisfies:

$$x_h'' + 4x_h' + 3x_h = 0$$

As is standard in homogeneous constant coefficient problems, we substitute $x_h(t) = e^{mt}$ into the above equation and get

$$m^2e^{mt} + 4me^{mt} + 3e^{mt} = e^{mt}\left(m^2 + 4m + 3\right) = 0$$

Factoring and solving for $m$ gives

$$(m+3)(m+1) = 0 \quad \to \quad m = -1, -3$$

Thus, the homogeneous solution is given by

$$x_h(t) = c_1e^{-t} + c_2e^{-3t}$$

where the constants $c_1$, $c_2$ will be chosen to satisfy the initial condition.

**Initial conditions**

Putting both the particular and homogeneous solution together we obtain the full solution:

$$\begin{aligned} x(t) &= x_p(t) + x_h(t) \\ &= -2\cos(3t) + 4\sin(3t) + c_1e^{-t} + c_2e^{-3t} \end{aligned}$$

Now, apply the initial conditions

$$x(0) = x'(0) = 0.$$

$$\left.\begin{aligned} x(0) &= -2 + c_1 + c_2 = 0 \\ x'(0) &= 12 - c_1 - 3c_2 = 0 \end{aligned}\right\} \quad \to \quad c_1 = -3,\ c_2 = 5$$

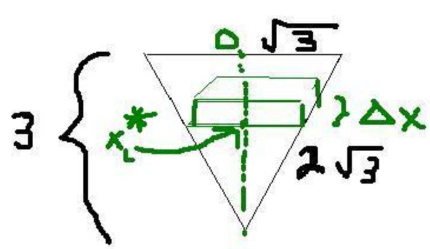
**Final answer**

Thus, the solution to the initial value problem is

$$x(t) = -2\cos(3t) + 4\sin(3t) - 3e^{-t} + 5e^{-3t}$$

## Question 5 (a)

SOLUTION. Consider the following diagram:



In the picture, we take the perpendicular bisector of the top base and draw the line. This splits our equilateral triangle into two equal parts. The base is $\sqrt{3}$ and the height is 3 and the other side length is $2\sqrt{3}$. Choose a sample point say $x_i^*$. Let's call the length of the small triangle point downward $l$. By similar triangles, we have

$\frac{\sqrt{3}}{3} = \frac{l/2}{3 - x_i^*}$

and hence

$l = \frac{2(3 - x_i^*)\sqrt{3}}{3}$

The volume of the thin prism at this height is

$V_i = l(12)\Delta x = \frac{2(3 - x_i^*)\sqrt{3}}{3}(12\Delta x) = 8(3 - x_i^*)\sqrt{3}\Delta x$

Next, the force acting on this prism is

$F_i = V_i(density) = (62)8(3 - x_i^*)\sqrt{3}\Delta x$

Lastly, the work done on this piece, noting that the displacement is 1 added to the distance to the bottom of the triangle (which is $(3 - x_i^*)$)

$W_i = F_i(displacement) = (62)8(3 - x_i^*)\sqrt{3}\Delta x(4 - x_i^*)$

Summing over all pieces, we have

$W = \lim_{n\to\infty} \sum_{i=1}^{n} W_i = \lim_{n\to\infty} \sum_{i=1}^{n} (62)8(3 - x_i^*)\sqrt{3}\Delta x(4 - x_i^*)$

and this last value is an integral given by

$$\begin{aligned}
W &= \int_0^3 (62)8(3 - x)\sqrt{3}(4 - x)\,dx \\
&= 62(8)\sqrt{3}\int_0^3 (12 - 7x + x^2)\,dx \\
&= 62(8)\sqrt{3}(12x - 7x^2/2 + x^3/3)\Big|_0^3 \\
&= 62(8)\sqrt{3}(12(3) - 7(3)^2/2 + (3)^3/3 - 0) \\
&= 62(8)\sqrt{3}(36 - 63/2 + 9) \\
&= 62(4)(27)\sqrt{3}
\end{aligned}$$

## Question 5 (b)

SOLUTION. We proceed as in part (a). Here though we will make the bottom of the trough our origin. Letting $l(y)$ be the length of hte regtangle at height $y$, we see that

$\frac{\sqrt{3}}{3} = \frac{l(y)/2}{y}$

and solving gives

$l(y) = \frac{2y\sqrt{3}}{3}$.

Next, since the width of the rectangle is fixed and given in the question, we see that

$A(y) = (12)l(y) = 8\sqrt{3}y$

Using Torcelli's law, we have

$$\begin{aligned}
8\sqrt{3}y\frac{dy}{dt} &= -k\sqrt{y}\\
\int 8\sqrt{3}\sqrt{y}\,dy &= \int -k\,dt\\
\frac{16\sqrt{3}y^{3/2}}{3} &= -kt + C
\end{aligned}$$

for some constant $C$. Since $y(0) = 3$ (the trough started full), we have
$C = -k(0) + C = \frac{16\sqrt{3}y(0)^{3/2}}{3} = \frac{16\sqrt{3}^4}{3} = 48$
Using $y(2) = 1$, we have
$C - 2k = 48 - 2k = \frac{16\sqrt{3}(y(2))^{3/2}}{3} = \frac{16\sqrt{3}}{3}$
and thus
$k = 24 - \frac{8\sqrt{3}}{3}$
Here is a good sanity check spot. Notice that $k$ is positive which is good since the question told us this. Now, we are looking for the value of $t$ where $y(t) = 0$ and so, we with to find the $t$ value where
$0 = \frac{16\sqrt{3}y(t)^{3/2}}{3} = -kt + C$
and hence we want the value where $t = C/k$. Plugging in the values above gives
$t = \frac{C}{K} = \frac{48}{24 - \frac{8\sqrt{3}}{3}} = \frac{48(3)}{72 - 8\sqrt{3}} = \frac{18}{9 - \sqrt{3}}$
as required. Another good sanity check, notice that this number is bigger than 2 which is good since at 2 we had that the depth was 1 metre.

## Question 6 (a)

**Solution.** This is a straightforward application of Simpson's Rule to the integral

$$V = \int_0^{60} A(y)\,dy.$$

Notice that
$\frac{\Delta x}{3} = \frac{\frac{60-0}{6}}{3} = \frac{10}{3}$
By Simpson's Rule, the integral can be approximated by the sum

$$V \approx \frac{10}{3}(A(0) + 4A(10) + 2A(20) + 4A(30) + 2A(40) + 4A(50) + A(60))$$

Remembering the given information about the diameter $D(y)$ of the tree trunk at various heights, $y$, we make the appropriate substitutions to determine the cross sectional area at each height:

$$A(y) = \pi\left(\frac{D(y)}{2}\right)^2 = \frac{\pi}{4}D(y)^2$$

Plugging into our sum we get the approximate volume for the tree trunk.

$$V \approx \frac{10\pi}{12}(10^2 + 4\times 8^2 + 2\times 7^2 + 4\times 6^2 + 2\times 5^2 + 4\times 4^2 + 3^2)$$

## Question 6 (b)

SOLUTION. If the error associated with Simpson's rule is bounded by 1, then we would get that the approximation is precise to 1 cubic foot. Hence we compute when

$$\frac{M(b-a)^5}{180n^4} = \frac{30000(60-0)^5}{180n^4} \leq 1$$

Solving and using [prime] factorization techniques, we have

$$\begin{aligned}
\frac{30000(60-0)^5}{180n^4} &\leq 1 \\
\frac{3\cdot 10^4\cdot 3^5\cdot 2^5\cdot 10^5}{2\cdot 3^2\cdot 10} &\leq n^4 \\
\frac{3^6\cdot 2^5\cdot 10^9}{2\cdot 3^2\cdot 10} &\leq n^4 \\
3^4\cdot 2^4\cdot 10^8 &\leq n^4 \\
3\cdot 2\cdot 10^2 &\leq n \\
600 &\leq n
\end{aligned}$$

Thus $n$ should be at least 600.

## Question 7 (a)

SOLUTION. First, note that $cos(x/10) \geq 0$ and $sin(x/10) \geq 0$ for any $0 \leq \text{x} \leq 5\pi$. Hence we need $k \geq 0$ as well.
To satisfy the second condition we require

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_0^{5\pi} k\cos\left(\frac{x}{10}\right)\sin^2\left(\frac{x}{10}\right)\,dx.$$

The integral can easily be solved using the substitution $u = sin(x/10)$, $du = 1/10\ cos(x/10)$, so that $u(0) = 0$ and $u(5\pi) = sin(\pi/2) = 1$. Hence we need to choose $k$ such that

$$\begin{aligned}
1 &= \int_0^{5\pi} k\cos\left(\frac{x}{10}\right)\sin^2\left(\frac{x}{10}\right)\,dx \\
&= \int_0^1 10ku^2\,du \\
&= \left.\frac{10ku^3}{3}\right|_0^1 \\
&= \frac{10}{3}k
\end{aligned}$$

Hence we choose"' $k = 3/10$. "'

## Question 7 (b)

Solution. A student takes at most $5\pi/2$ minutes to complete the question if his completion time is anywhere between 0 and $5\pi/2$. Hence the probability we are looking for is

$$\begin{aligned}
\text{Prob}(a \leq X \leq b) &= \int_a^b f(x)\,dx \\
&= \int_0^{5\pi/2} \frac{3}{10}\cos\left(\frac{x}{10}\right)\sin^2\left(\frac{x}{10}\right)\,dx \\
&= \int_0^{1/\sqrt{2}} 3u^2\,du \\
&= u^3\Big|_0^{1/\sqrt{2}} \\
&= \frac{1}{2\sqrt{2}}.
\end{aligned}$$

We used the substitution $u = sin(x/10)$, $du = 1/10\ cos(x/10)$, so that $u(0) = 0$ and $u(5\pi/2) = sin(\pi/4) = 1/\sqrt{2}$ from part (a).

## Question 7 (c)

**Easiness: 1/100**

Solution. From the definition of the mean we know that we have to calculate the mean $\mu$ by

$$\mu = \int_0^{5\pi} x\frac{3}{10}\cos\left(\frac{x}{10}\right)\sin^2\left(\frac{x}{10}\right)\,dx.$$

As suggested by the hint we choose $u = x$, $du = dx$, $dv = 3/10\ cos(x/10)sin^2(x/10)$, $v = ??$
To find $v$ here we use the same procedure that we already used in part (a) and part (b), i.e. the substitution $y = sin(x/10)$ (let's use $y$ instead of $u$, since $u$ is already in use at the integration by parts) to find

$$\begin{aligned}
\int \frac{3}{10}\cos\left(\frac{x}{10}\right)\sin^2\left(\frac{x}{10}\right)\,dx &= \int 3y^2\,dy \\
&= y^3 + C \\
&= \sin^3\left(\frac{x}{10}\right) + C.
\end{aligned}$$

With this intermediate result we can now go to work at calculating the mean $\mu$:

$$\begin{aligned}
\mu &= \int_0^{5\pi} \underbrace{x}_{=u}\underbrace{\frac{3}{10}\cos\left(\frac{x}{10}\right)\sin^2\left(\frac{x}{10}\right)\,dx}_{=dv} \\
&= x\sin^3\left(\frac{x}{10}\right)\Big|_0^{5\pi} - \int_0^{5\pi} \sin^3\left(\frac{x}{10}\right)\,dx \\
&= 5\pi - \int_0^{5\pi} \sin^3\left(\frac{x}{10}\right)\,dx
\end{aligned}$$

Let's compute the last integral separately, and only then come back to this equation. To do so, we need the trigonometric identity

$$\cos^2\theta + \sin^2\theta = 1.$$

With this we obtain

$$\begin{aligned}\int_0^{5\pi} \sin^3\left(\frac{x}{10}\right)\,dx &= \int_0^{5\pi} \sin\left(\frac{x}{10}\right)\left(1-\cos^2\left(\frac{x}{10}\right)\right)\,dx\\ &= \int_0^{5\pi} \sin\left(\frac{x}{10}\right)\,dx + \int_0^{5\pi} -\sin\left(\frac{x}{10}\right)\cos^2\left(\frac{x}{10}\right)\,dx\end{aligned}$$

The former integral is standard, for the latter we substitute $z = cos(x/10)$, "dz = -1/10 $sin(x/10)dx$, $z(0) = cos(0) = 1$, $z(5\pi) = cos(\pi/2) = 1$. Hence

$$\begin{aligned}\int_0^{5\pi} \sin^3\left(\frac{x}{10}\right)\,dx &= \int_0^{5\pi} \sin\left(\frac{x}{10}\right)\,dx + \int_0^{5\pi} -\sin\left(\frac{x}{10}\right)\cos^2\left(\frac{x}{10}\right)\,dx\\ &= -10\cos\left(\frac{x}{10}\right)\Big|_0^{5\pi} + \int_1^0 10z^2\,dz\\ &= 0+10+\frac{10}{3}z^3\Big|_1^0\\ &= 10+0-\frac{10}{3}\\ &= \frac{20}{3}.\end{aligned}$$

Alright. What did we want to do with the 20/3? Right, we needed to calculate this value to find $\mu$. Going back to the equation with $\mu$ we can now completely the question by plugging in the result:

$$\begin{aligned}\mu &= \int_0^{5\pi} x\frac{3}{10}\cos\left(\frac{x}{10}\right)\sin^2\left(\frac{x}{10}\right)\,dx\\ &= 5\pi - \int_0^{5\pi} \sin^3\left(\frac{x}{10}\right)\,dx\\ &= 5\pi - \frac{20}{3}.\end{aligned}$$

## Question 8 (a)

**Easiness: 59/100**

SOLUTION. The right hand Riemann sum is computed by
$R_{100} = \sum_{i=1}^{100} f(x_i)\Delta x$
whereas the left hand Riemann sum is computed by
$L_{100} = \sum_{i=0}^{99} f(x_i)\Delta x$
and hence, we have

$$\begin{aligned}R_{100} - L_{100} &= \sum_{i=1}^{100} f(x_i)\Delta x - \sum_{i=0}^{99} f(x_i)\Delta x\\ &= f(x_{100})\Delta x + \sum_{i=1}^{99} f(x_i)\Delta x - \sum_{i=1}^{99} f(x_i)\Delta x - f(x_0)\Delta x\\ &= f(4)\Delta x - f(0)\Delta x\\ &= 8\Delta x - (0)\Delta x\\ &= 8\Delta x\end{aligned}$$

By definition, we know
$\Delta x = \frac{b-a}{n} = \frac{4-0}{100} = \frac{1}{25}$
and so

$$R_{100} - L_{100} = 8\Delta x = \frac{8}{25}$$

completing the proof.

## Question 8 (b)

SOLUTION. The minimum is given by $g(x) = 2x$ and the maximum is given by
$h(x) = \begin{cases} 6x - x^2 & x \le 2 \\ 8 & 2 \le x \le 4 \end{cases}$
To show that $g(x)$ is the minimum, we note that our $f(x)$ is a nondecreasing function, $f(0) = 0$ and we also require that $2x \le f(x)$. This last condition means that the absolute smallest $f(x)$ can be is $f(x) = 2x$ and it turns out that this function also has the first two desired properties so this function works and is minimal. For the second function, we note that our $f(x)$ is a nondecreasing function; we also have $f(4) = 8$ and lastly that $f(x) \le 6x - x^2$. Again to make the function maximal and nondecreasing, we really would like $f(x) = 8$ however this breaks the upper bound condition. So we pick the function $6x - x^2$ until we reach the value of 8 and then pick the function $f(x) = 8$ until the end where we also need to hope that $8 \le 6x - x^2$ on this interval (so that we are not breaking the upper bound condition. Let $y = 6x - x^2$. Notice that
$y' = 6 - 2x > 0$ when $x < 3$.
This means that the function is non-decreasing until $x = 3$. Furthermore, notice that
$8 = 6x - x^2$ when $0 = x^2 - 6x + 8 = (x-2)(x-4)$ and so this occurs when $x = 2$ and $x = 4$. Notice that since this is a parabola, it has only one critical point (which we computed to be at $x = 3$ and so indeed between $2 \le x \le 4$, we have that
$8 \le 6x - x^2$.
Thus, the function $h(x)$ chosen above has the desired properties we require and thus gives the maximum value for $I$. Computing these values
$\int_0^4 g(x)\,dx = \int_0^4 2x\,dx = x^2|_0^4 = 16$
and
$\int_0^4 h(x)\,dx = \int_0^2 6x - x^2\,dx + \int_2^4 8\,dx = \left(3x^2 - \frac{x^3}{3}\right)\Big|_0^2 + 16 = 12 - \frac{8}{3} + 16 = \frac{76}{3}$
completing the question.

# Good Luck for your exams!