Full Solutions
MATH103 April 2011

April 4, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

Solution.

1. $\sum_{n=1}^{\infty} \frac{2}{n}$ is divergent, by the integral test.
2. $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent. This is a geometric series, equal to $\frac{1}{1-\frac{1}{2}} - 1 = 1$.
3. $\sum_{n=1}^{\infty} 2^n$ is divergent, since $\lim_{n\to\infty} 2^n = \infty$.
4. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, by the integral test.
5. $\sum_{n=2}^{\infty} \frac{n}{1-n^3}$ is convergent. To show this, use the fact that the absolute value of the terms inside the sequence is $\sum_{n=2}^{\infty} \left|\frac{n}{1-n^3}\right| = \sum_{n=2}^{\infty} \frac{n}{n^3-1}$. As all the terms are positive, we can use the comparison test with $\sum_{n=1}^{\infty} \frac{2}{n^2}$ and show that since this series converges by the p-series test and

$\frac{n}{n^3-1} \leq \frac{2n}{n^3} = \frac{2}{n^2}$
valid since $2 \leq n^3$ for all $n \geq 2$ and thus $1 \leq \frac{n^3}{2} = n^3 - \frac{n^3}{2}$ (then rearrange this last inequality to get the above displayed equation). Therefore, the series
$\sum_{n=2}^{\infty} \left|\frac{n}{1-n^3}\right| = \sum_{n=2}^{\infty} \frac{n}{n^3-1}$
converges. Then, we know that $\sum_{n=2}^{\infty} \frac{n}{1-n^3}$ is absolutely convergent and hence convergent.

Question 1 (b)

Solution. The first 3 can easily be deduced just by looking at the pictures.
1. The pdf with the smallest mean is (1).
2. The pdf with the largest variance is (4).
3. The pdf with the smallest standard deviation is (3).
4. The pdf with median larger than the mean is (2).
By symmetry, both graphs (3) and (4) have the same mean and median. To decide between (1) and (2), we can compute the mean and median directly for (2). We can do this in the simple case where $a = 0$ and $b = 1$. Then $p(x) = Cx$ for some constant C. Since $1 = \int_0^1 p(x)\,dx = C\int_0^1 x\,dx = \frac{C}{2}$, $C = 2$. The mean is then $\int_0^1 xp(x)\,dx = 2\int_0^1 x^2\,dx = \frac{2}{3}$. The median m is determined by $\frac{1}{2} = \int_0^m p(x)\,dx = \int_0^m 2x\,dx = m^2$, hence, $m = \frac{1}{\sqrt{2}}$, which is larger than $\frac{2}{3}$.
5. The total area under the graph must be 1. Note that the area of a triangle is equal to half of its base length times its height; both triangles have the same area, with base length $\frac{b-a}{2}$ and height y. Therefore,

$$1 = 2 \cdot \frac{1}{2}\left(\frac{b-a}{2}\right) y = \frac{1}{2}(b-a)y,$$

where y is the maximal probability density.
Hence,

$$y = \frac{2}{b-a}.$$

Question 1 (c)

Solution. *It's probably the case that you've seen a general formula in Math 103 for this type of scenario and you are just being asked to recall it. The correct answer is 3.5. In any case, here is a full derivation, which might be useful for your review.*
This is a sequence of Bernoulli trials. We'll call rolling a 6 "success" and any other number "failure".
In a single roll, since 6 is twice as likely as all other numbers, the probability of rolling a 6 must be 2/7 and the probability of rolling any other number must be 1/7. That is, the probability of success in one trial is $p = 2/7$ and the probability of failure is $q = 5/7$.

Let N be the number of rolls until a 6 is obtained. For example, the probability that $N = 1$ is simply equal to the probability of success in one trial,

$$P(N = 1) = p.$$

$N = 2$ if and only the first trial resulted in failure **and** the second trial resulted in success. Hence,

$$P(N = 2) = qp.$$

In general, $N = k$ if and only if the first $k - 1$ trials resulted in failure and the k^{th} trial resulted in success. Hence,

$$P(N = k) = q^{k-1}p.$$

This is the probability distribution for the random variable N. Our goal is to compute the expected number of throws required to get a 6, which is precisely the mean value of N,

$$\overline{N} = \sum_{k=1}^{\infty} kP(N = k) = \sum_{k=1}^{\infty} kq^{k-1}p.$$

We can compute the value of this sum by making the clever observation that it appears to be the derivative of a geometric series. For any x in the interval $[0, 1)$, define

$$f(x) = \sum_{k=0}^{\infty} x^k p.$$

Differentiating term-by-term, we see that

$$f'(x) = \sum_{k=1}^{\infty} kx^{k-1}p.$$

Hence, $\overline{N} = f'(q)$.
However, $f(x)$ is simply a geometric series; the result is

$$f(x) = \frac{p}{1 - x}.$$

Therefore,

$$f'(x) = \frac{p}{(1 - x)^2}.$$

Hence,

$$\overline{N} = f'(q) = \frac{p}{(1 - q)^2} = \frac{p}{p^2} = \frac{1}{p} = \frac{7}{2} = 3.5.$$

Question 2 (a)

SOLUTION 1. We first solve the indefinite integral, using the trigonometric identity $\sin(2x) = 2\sin(x)\cos(x)$:

$$\begin{aligned}\int \sin(\pi x)\cos(\pi x)\,dx &= \int \frac{1}{2}\sin(2\pi x)\,dx \\ &= -\frac{1}{4\pi}\cos(2\pi x) + C.\end{aligned}$$

Therefore

$$\begin{aligned}I_1 &= \int_{1/2}^{1} \sin(\pi x)\cos(\pi x)\,dx \\ &= \frac{-1}{4\pi}\cos(2\pi x)\Big|_{1/2}^{1} \\ &= \frac{-1}{4\pi}\cos(2\pi) + \frac{1}{4\pi}\cos(\pi) \\ &= -\frac{1}{4\pi} - \frac{1}{4\pi} = -\frac{1}{2\pi}.\end{aligned}$$

SOLUTION 2. Alternatively, substitute $u = \sin(\pi x)$. Then $du = \pi\cos(\pi x)dx$, when $x = 1/2$ then $u = \sin(\pi/2) = 1$, and when $x = 1$ then $u = \sin(\pi) = 0$. Therefore

$$\begin{aligned}I_1 &= \int_{1/2}^{1} \sin(\pi x)\cos(\pi x)\,dx \\ &= \int_{1}^{0} u\cos(\pi x)\frac{du}{\pi\cos(\pi x)} \\ &= \frac{1}{\pi}\int_{1}^{0} u\,du \\ &= \frac{1}{2\pi}u^2\Big|_{1}^{0} \\ &= -\frac{1}{2\pi}.\end{aligned}$$

Question 2 (b)

SOLUTION.

$$I_2 = \int \sin^3 x\cos^2 x\,dx = \int (1-\cos^2 x)\cos^2 x\sin x\,dx$$

Let $u = \cos x$ so that $du = -\sin x dx$. Then

$$\begin{aligned}I_2 &= -\int (1-u^2)u^2\,du = \int (u^4 - u^2)\,du \\ &= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C\end{aligned}$$

Question 2 (c)

Solution. Assume that the solution $y = y(x)$ has a Taylor series expansion about $x = 0$,

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = \sum_{n=0}^{\infty} a_nx^n.$$

We'll use the initial condition and the differential equation to solve for the coeffients.
From the initial condition,

$$1 = y(0) = a_0 + a_1(0) + a_2(0)^2 + a_3(0)^3 + \cdots = a_0.$$

Let's start with the left hand side dy/dx of the differential equation $dy/dx = y\text{-}x$. Differentiating the series for y term by term yields

$$\begin{aligned}\frac{dy}{dx} &= 0 + a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots \\ &= \sum_{n=1}^{\infty} a_nnx^{n-1} \\ &= \sum_{k=0}^{\infty} a_{k+1}(k+1)x^k \\ &= \sum_{n=0}^{\infty} a_{n+1}(n+1)x^n\end{aligned}$$

Here we substituted the dummy variable $n = k$+1, (i.e. $k = n$-1) in the third step, and then renamed k as n. (This substitution of dummy variable was done purely for <> to be able to compare the two Taylor series of the LHS and the RHS of the differential equation more easily. In general it helps with both series start with $n = 0$.)
The right hand side y-x of the differential equation $dy/dx = y$-x gives

$$\begin{aligned}y - x &= a_0 + (a_1 - 1)x + a_2x^2 + a_3x^3 + \dots \\ &= a_0 + (a_1 - 1)x + \sum_{n=2}^{\infty} a_nx^n.\end{aligned}$$

Now we can equate the left hand side with the right hand side of the differential equation to get

$$\begin{gathered}\frac{dy}{dx} = y - x \\ a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots = a_0 + (a_1 - 1)x + a_2x^2 + a_3x^3 + \dots \\ a_1 + 2a_2x + \sum_{n=2}^{\infty} a_{n+1}(n+1)x^n = a_0 + (a_1 - 1)x + \sum_{n=2}^{\infty} a_nx^n.\end{gathered}$$

From the differential equation, we can equate each of the coefficients of the above series.
For $n = 0$, (i.e. the coefficients in from of x^0) we have $a_1 = a_0 = 1$.
For $n = 1$, (i.e. the coefficients in from of x^1) $2a_2 = a_1 - 1 = 0$, hence, $a_2 = 0$.

For all $n \geq 2$, $a_{n+1}(n+1) = a_n$.
That is, $a_{n+1} = \frac{a_n}{n+1}$. For example, $a_3 = a_2/3 = 0$. Similarly, $a_4 = a_3/3 = 0$. We can see that for all $n \geq 3$, $a_n = 0$.
Hence, the solution is

$$y = 1 + x.$$

Note: We can easily check that this is indeed a solution to the initial value problem

$$y(0) = 1 + 0 = 1$$

and *dy*/*dx*=1=*y*-*x*.

Question 2 (d)

SOLUTION 1. This is a Bernoulli trial, where each multiple-choice question is a trial. For each trial, "success" means that the student answered the question correctly and "failure" means that he answered the question incorrectly. If the student randomly checks answers (with the same probability for each of the 4 responses), then his probability of success on each question is $p = 1/4$ and his probability of failure is $q = 3/4$. Passing the test means that he obtained either 5 or 6 successes in 6 trials. Let X be his total score. The probability that he passes is

$$\begin{aligned} P(\text{pass}) &= P(X = 5 \textit{ or } X = 6) \\ &= P(X = 5) + P(X = 6) \\ &= C(6,5)p^5 q + C(6,6)p^6 \\ &= 6(1/4)^5(3/4) + (1/4)^6 \\ &= \frac{6 \cdot 3 + 1}{4^6} \\ &= \frac{19}{4^6} \end{aligned}$$

Note: here $C(n, k)$ is the binomial coefficient,

$$C(n,k) = C_k^n = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

(This is the number of ways to choose k objects among n total objects.)

SOLUTION 2. The probability of getting a perfect score is p^6, where p is the probability of success on a single question. In order for p^6 to be at least 80%=4/5, p must be at least $(4/5)^{1/6}$. (This is approximately 96%, but it would be fine to leave your answer in the form $(4/5)^{1/6}$ on the exam.)

Question 3 (a)

SOLUTION. We will preface this question by noting that the original intent in the exam was to understand the sketch well enough to proceed to part (b). However, we have decided to present the solution as detailed as possible for those who would like to improve their curve sketching techniques.
Now we will complete this question by following a checklist that is fairly standard for these types of problems.

1. Find roots
2. Determine any asymptotes
3. Determine critical points
4. Determine intervals of increase and decrease and local max/min
5. Determine intervals of concavity and points of inflection
6. Sketch the graph

1: Find roots:
Luckily we have a polynomial that is already written in factored form so we can read, as the roots, the factors. Therefore if

$$f(x) = x(x-2)(x+1) = 0$$

then the roots are at $x = 0$, $x = 2$, $x = -1$.
2: Asymptotes:
Now polynomials are continuous everywhere (their domain is all the real numbers) and so we do not expect any vertical asymptotes. When looking for horizontal asymptotes we typically take the limit as x goes to ∞. We don't need to be so formal here because it's clear that as x gets large then $f(x)$ also gets large as their is nothing (such as a denominator) to slow its growth. What's to be determined is whether $f(x)$ gets large and positive or large and negative. If we were to put a very large positive value of x into $f(x)$ we would get that

$$f(x) = x(x-2)(x+1) \approx x^3$$

since adding or subtracting numbers to something large is unaffected. Now as x goes to ∞ then $f(x)$ will go to ∞ as well. If we put in a large negative number, once again adding or subtracting small numbers will have no affect and once again,

$$f(x) \approx x^3$$

which approaches $-\infty$ as x goes to $-\infty$. Therefore,

$$\begin{aligned}\lim_{x\to\infty} f(x) &= +\infty\\ \lim_{x\to-\infty} f(x) &= -\infty.\end{aligned}$$

3. Critical Points:
In order to find critical points we first need the derivative. In order to do this we need to apply the product rule three times. Therefore,

$$f'(x) = (x-2)(x+1) + x(x+1) + x(x-2) = 3x^2 - 2x - 2.$$

Critical points will occur when $f'(x) = 0$. Therefore,

$$\begin{aligned}3x^2 - 2x - 2 &= 0\\ x = \frac{2 \pm \sqrt{4+24}}{6} &= \frac{1}{3} \pm \frac{\sqrt{7}}{3}\end{aligned}$$

are our critical points.
4. Intervals of Increase and Decrease and Local Max/Min:
We have that our derivative is,

$$f'(x) = 3x^2 - 2x - 2$$

with critical points

$$x = \frac{1}{3} \pm \frac{\sqrt{7}}{3}.$$

To find intervals of increasing and decreasing we look at the derivative between each critical point over the entire domain. If the derivative is positive then the function is increasing and if the derivative is negative, the function is decreasing.

	$\left(-\infty, \frac{1}{3} - \frac{\sqrt{7}}{3}\right)$	$\left(\frac{1}{3} - \frac{\sqrt{7}}{3}, \frac{1}{3} + \frac{\sqrt{7}}{3}\right)$	$\left(\frac{1}{3} + \frac{\sqrt{7}}{3}, \infty\right)$
$f'(x)$	Positive	Negative	Positive
$f(x)$	Increasing	Decreasing	Increasing

From the table we see that the critical point

$$\frac{1}{3} - \frac{\sqrt{7}}{3}$$

is a local maximum since the derivative is increasing on the left and decreasing on the right. By a similar logic we have that the other critical point

$$\frac{1}{3} + \frac{\sqrt{7}}{3}$$

is a local minimum.

5. Intervals of Concavity and Points of Inflection:

In order to determine points of inflection we have to compute the second derivative. Since the first derivative is just a polynomial we can easily compute term by term with the power rule to get

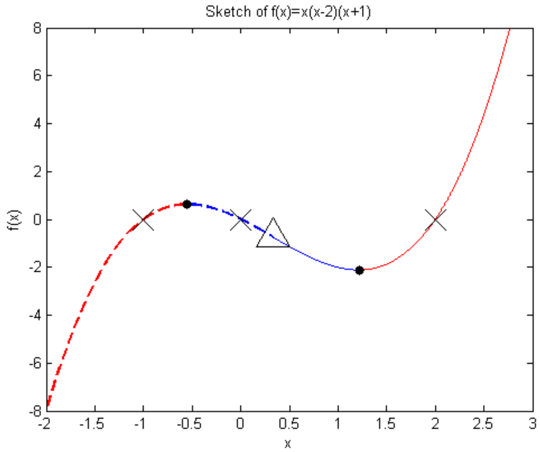
$$f''(x) = 6x - 2.$$

Possible points of inflection occur when $f''(x) = 0$, i.e. there is a possible point of inflection as $x = 1/3$. In order for it to be a point of inflection we require that the second derivative change sign as we cross $x = 1/3$. If the second derivative is positive, the graph is concave up; if it is negative, the graph is concave down.

	$\left(-\infty, \frac{1}{3}\right)$	$\left(\frac{1}{3}, \infty\right)$
$f''(x)$	Negative	Positive
$f(x)$	Concave Down	Concave Up

Notice that the change in concavity, confirms that we do indeed have an inflection point at $x = 1/3$.

6. Sketch:



The picture to the right shows a sketch of the graph. We have labelled intervals of increasing with red and decreasing with blue. Concave up is a thin solid line while concave down is a thick dashed line. We have placed x's to mark the roots, filled in circles to mark the local maximum and minimum, and triangles to indicate the point of inflection.

Question 3 (b)

SOLUTION. The area of the region bounded by the graph of f and the x- axis is

$$\begin{aligned} A &= \int_{-1}^{2} |f(x)|\, dx \\ &= \int_{-1}^{0} f(x)\, dx + \int_{0}^{2} (-f(x))\, dx \\ &= \int_{-1}^{0} (x^3 - x^2 - 2x)\, dx + \int_{0}^{2} (-x^3 + x^2 + 2x)\, dx \\ &= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} + \left[-\frac{x^4}{4} + \frac{x^3}{3} + x^2\right]_{0}^{2} \\ &= -\frac{(-1)^4}{4} + \frac{(-1)^3}{3} + (-1)^2 - \frac{2^4}{4} + \frac{2^3}{3} + 2^2 \\ &= -\frac{1}{4} - \frac{1}{3} + 1 - 4 + \frac{8}{3} + 4 \\ &= -\frac{1}{4} + 1 + \frac{7}{3} \\ &= -\frac{1}{4} + 3 + \frac{1}{3} \\ &= 3 + \frac{1}{12} = \frac{37}{12}. \end{aligned}$$

In the third line, we've expanded the factors of $f(x)$ into the sum

$$f(x) = x(x-2)(x+1) = x(x^2 - x - 2) = x^3 - x^2 - 2x.$$

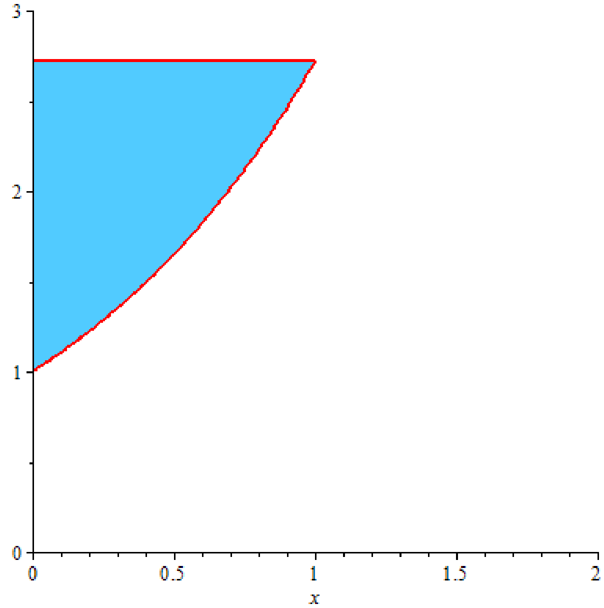
Question 4 (a)

SOLUTION. Following the hints, we slice this "horn" into circular pieces. The volume of each slice is equal to $\pi r^2\ \Delta x$ where r is the radius of the circle and Δx is its thickness. Hence, in the limit $\Delta x \to 0$,

$$\begin{aligned}
V_1 &= \int_0^1 \pi(f(x)-1)^2 dx \\
&= \pi \int_0^1 (e^x - 1)^2 dx \\
&= \pi \int_0^1 (e^{2x} - 2e^x + 1) dx \\
&= \pi \left[\frac{e^{2x}}{2} - 2e^x + x \right]_0^1 \\
&= \pi \left(\frac{e^2}{2} - 2e + 1 - \frac{e^0}{2} + 2e^0 - 0 \right) \\
&= \pi \left(\frac{e^2}{2} - 2e + 1 - \frac{1}{2} + 2 \right) \\
&= \pi \left(\frac{e^2}{2} - 2e + \frac{5}{2} \right)
\end{aligned}$$

Question 4 (b)

SOLUTION 1. **Disc Method.** In this solution, we consider the volume as a sum of slices, cut orthogonal to the y-axis. Each slice has volume $\pi r^2 \Delta y$, where r is its radius and Δy is its thickness. The bowl will be made by rotating the shaded blue region in the 2D figure



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If the slice is at height y, then its radius is equal to x, where $y = e^x$. Hence, $r = \ln y$. In the limit $\Delta y \to 0$,

$$V_2 = \int_1^e \pi(\ln y)^2\, dy$$

Note that the limits of integration are $f(0) = e^0 = 1$ and $f(1) = e^1 = e$, since x ranges from 0 to 1.

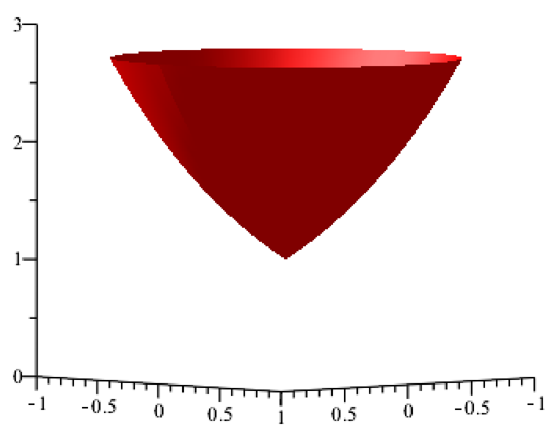
We use integration by parts to evaluate this integral, letting $f(y) = g(y) = \ln y$. So in order to perform the integration by parts we need to recall the antiderivative of $\ln y$. To calculate this antiderivative we again use integration by parts:

$$\begin{aligned}\int \ln y dy = \int 1 \cdot \ln y\, dy &= y \ln y - \int y \cdot \frac{1}{y}\, dy \\ &= y \ln y - \int 1\, dy = y \ln y - y + C.\end{aligned}$$

We now use this antiderivative to perform the integration by parts in the integral for V_2. Hence,

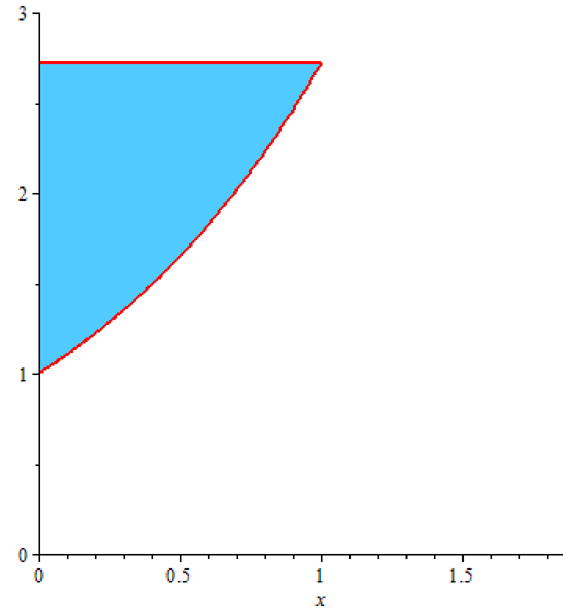
$$\begin{aligned}V_2 &= \pi \int_1^e (\ln y)^2\, dy \\ &= \pi \left[\ln y(y \ln y - y)\right]_1^e - \pi \int_1^e \frac{1}{y}(y \ln y - y)\, dy \\ &= \pi \left[1(e \cdot 1 - e) - 0\right] - \pi \int_1^e (\ln y - 1)\, dy \\ &= -\pi \left[y \ln y - y - y\right]_1^e \\ &= -\pi \left[e \ln e - e - e - 1 \ln 1 + 1 + 1\right] \\ &= -\pi \left(-e + 2\right) \\ &= \pi \left(e - 2\right).\end{aligned}$$

Reality check: Note that we obtain the same value for V_2 with both methods. To visualize the full bowl we



can look at the 3D figure .

SOLUTION 2. **Shell method.** In this solution, we consider the solid to be a sum of cylindrical shells. Each cylindrical shell has volume $2\pi r(x)h(x)\Delta r$, where $r(x)$ is its radius, $h(x)$ is its height and Δr is its thickness.

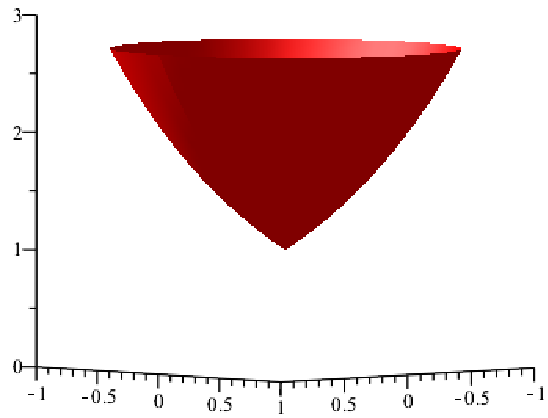


The bowl will be made up by rotating the blue shaded region in the 2D figure
From there we have $r(x) = x$, $h(x) = f(1) - f(x) = e - e^x$, and $\Delta r = \Delta x$. In the limit $\Delta x \to 0$, we obtain the volume by integrating along the x-axis:

$$\begin{aligned}
V_2 &= \int_0^1 2\pi r(x)h(x)\,dx \\
&= \int_0^1 2\pi x(e - e^x)\,dx \\
&= 2\pi e \int_0^1 x\,dx - 2\pi \int_0^1 xe^x\,dx \\
&= 2\pi e \left[\frac{x^2}{2}\right]_0^1 - 2\pi \left([xe^x]_0^1 - \int_0^1 e^x\,dx \right) \\
&= 2\pi e \frac{1}{2} - 2\pi \left(e - [e^x]_0^1 \right) \\
&= \pi e - 2\pi\,(e - (e-1)) \\
&= \pi e - 2\pi \\
&= \pi(e-2).
\end{aligned}$$

(We used integration by parts in the second integral to get from the third to the fourth line.)

Reality check: Note that we obtain the same value for V_2 with both methods. To visualize the full bowl we



can look at the 3D figure .

Question 5 (a)

SOLUTION. The probability density function is the derivative of the cumulative distribution function,

$$p_A(t) = F_A'(t) = \frac{d}{dt}(1 - e^{-t}) = 0 - (-e^{-t}) = e^{-t}.$$

Question 5 (b)

SOLUTION. Since p_B is a probability density function, $\int_0^\infty p_B(t)\,dt = 1$. Hence,

$$1 = \int_0^\infty p_B(t)\,dt = \int_0^\infty Ce^{-2t}\,dt = C\left[\frac{e^{-2t}}{-2}\right]_0^\infty = \frac{C}{2}.$$

Therefore, $C = 2$.

Question 5 (c)

SOLUTION. The probability that lightbulb A is still working after t months is equal to the probability that it has **not** failed before t months, which is $1 - F_A(t) = e^{-t}$.
The probability that lightbulb B is still working after t months is equal to the probability that it fails at a time greater than t:

$$\int_t^\infty p_B(x)\,dx = 2\int_t^\infty e^{-2x}\,dx = 2\left[\frac{e^{-2x}}{-2}\right]_t^\infty = e^{-2t}.$$

The probability that **both** lightbulbs are still working after t months is equal to the product of the two probabilities found above, which is

$$e^{-t}e^{-2t} = e^{-3t}.$$

Question 5 (d)

SOLUTION. The expected lifetime of lightbulb A is

$$\bar{t}_A = \int_0^\infty t p_A(t) dt = \int_0^\infty t e^{-t} dt$$

Integrating by parts,

$$\bar{t}_A = \left[-te^{-t}\right]_0^\infty - \int_0^\infty (-e^{-t}) dt = \int_0^\infty e^{-t}\, dt = -e^{-t}\big|_0^\infty = 1.$$

The expected lifetime of lightbulb B is

$$\bar{t}_B = \int_0^\infty t p_B(t) dt = 2\int_0^\infty t e^{-2t} dt$$

Again, integrating by parts,

$$\bar{t}_B = 2\left[-t\frac{e^{-2t}}{2}\right]_0^\infty - 2\int_0^\infty \frac{e^{-2t}}{(-2)} dt = \int_0^\infty e^{-2t} dt = \frac{e^{-2t}}{-2}\bigg|_0^\infty = \frac{1}{2}.$$

Hence, lightbulb A has the longer expected lifetime.

Question 6 (a)

SOLUTION. The steady state solutions are solutions to the differential equation that are constant. That is, $y(x) =$ constant. For this to be possible, the derivative of y has to be zero.
Therefore, to be consistent with the differential equation, we have that
$0 = (y^2 - 1)x$.
For this equation to be true, we require that $y = \pm 1$ or $x = 0$. Since $x = 0$ does not fit the definition of a steady state solution, we are left with the steady solutions $y = \pm 1$.

Question 6 (b)

SOLUTION. The differential equation is separable; that is,

$$\frac{dy}{dx} = (y^2 - 1)x \implies \int \frac{1}{y^2 - 1}\, dy = \int x\, dx.$$

On the right hand side,

$$\int x\, dx = \frac{x^2}{2} + C_1,$$

for some constant C_1. On the left hand side, we factor $y^2 - 1 = (y+1)(y-1)$ and use partial fractions.

$$\begin{aligned}\frac{1}{(y+1)(y-1)} &= \frac{A}{y+1} + \frac{B}{y-1}\\ \implies 1 &= A(y-1) + B(y+1).\end{aligned}$$

This must be true for all y. In particular, at $y = -1$, $1 = -2A$, hence, $A = -1/2$. At $y = 1$, $1 = 2B$, hence $B = 1/2$. Therefore,

$$\begin{aligned}\int \frac{1}{y^2-1}\,dy &= -\frac{1}{2}\int \frac{1}{y+1}\,dy + \frac{1}{2}\int \frac{1}{y-1}\,dy\\ &= -\frac{1}{2}\ln|y+1| + \frac{1}{2}\ln|y-1| + C_2\\ &= \frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| + C_2,\end{aligned}$$

for some constant C_2. The initial condition is $y(0) = 0$, so at least for small x , we know that $-1 < y(x) < 1$. This means that y-1 is negative and y+1 is positive. Hence

$$\left|\frac{y-1}{y+1}\right| = -\frac{y-1}{y+1} = \frac{1-y}{y+1} = \frac{1-y}{1+y}.$$

Therefore

$$\int \frac{1}{y^2-1}\,dy = \frac{1}{2}\ln\left(\frac{1-y}{1+y}\right) + C_2.$$

Equating the left and right sides of our first equation, we have

$$\begin{aligned}\frac{1}{2}\ln\left(\frac{1-y}{1+y}\right) &= \frac{x^2}{2} + C_1 - C_2\\ \implies \frac{1-y}{1+y} &= Ae^{x^2}\\ \implies 1 - y &= (1+y)Ae^{x^2}\\ \implies 1 - Ae^{x^2} &= y(1 + Ae^{x^2})\\ \implies y &= \frac{1-Ae^{x^2}}{1+Ae^{x^2}}.\end{aligned}$$

where $A = e^{2(C_1 - C_2)}$. From the initial condition, $y(0) = 0$, A must be equal to 1. Hence,

$$y(x) = \frac{1-e^{x^2}}{1+e^{x^2}}.$$

Note that this solution does satisfy $-1 < y(x) < 1$ for all x.
Remember: It is important (and usually easy) to check that your solution actually satisfies the differential equation.

Question 7 (a)

SOLUTION. The first three derivatives of $f(x) = \ln(1+x)$ are
$f'(x) = \frac{1}{1+x}$,
$f''(x) = \frac{-1}{(1+x)^2}$,
and
$f'''(x) = \frac{2}{(1+x)^3}$.
When we evaluate these at $x = 0$ we get $f'(0) = 1$, $f''(0) = -1$, $f'''(0) = 2$. Hence, the first few terms of the Taylor series for $f(x)$ about $x = 0$ are

$$\begin{aligned}
&f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots \\
&= 0 + x - \frac{1}{2}x^2 + \frac{2}{6}x^3 + \dots \\
&= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots
\end{aligned}$$

These are the first three non-zero terms.
(Note: It is actually pretty easy to obtain the entire Taylor series anyway. For any $n \geq 1$, we can show that $f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$. Hence, $f^{(n)}(0) = (-1)^{n-1}(n-1)!$ and therefore, $f(x) = \ln(x+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$.)

Question 7 (b)

SOLUTION. Dividing the first three nonzero terms in the Taylor series for $\sin x$ by x,

$$\frac{\sin x}{x} \approx 1 - \frac{x^2}{3!} + \frac{x^4}{5!}.$$

Hence,

$$\begin{aligned}
\int_0^1 \frac{\sin x}{x}\,dx &\approx \int_0^1 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!}\right) dx \\
&= x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!}\bigg|_0^1 \\
&= 1 - \frac{1}{18} + \frac{1}{5 \cdot 120} = 1 - \frac{1}{18} + \frac{1}{600} = \frac{1800 - 100 + 3}{1800} = \frac{1703}{1800}.
\end{aligned}$$

(Note that this is a pretty good approximation. $\frac{1703}{1800} \approx 0.946111$, *while WolframAlpha approximates the full integral* $\int_0^1 \frac{\sin x}{x}\,dx$ *with* 0.946083.*)*

Question 7 (c)

SOLUTION 1. Using the fact that $\cos(x)$ is the derivative of $\sin(x)$, we can use the Taylor series of $\sin(x)$ to obtain the Taylor series of $\cos(x)$. Specifically,
$\cos(x) = \frac{d}{dx}\sin(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}$.
Therefore, to find $x\cos(x)$ we multiply the taylor series for $\cos(x)$ by the factor x. That is
$x\cos(x) = x\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!} = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k)!}$.

SOLUTION 2. The first solution assumes that the Taylor series for $\cos(x)$ is not known in advance. The Taylor series for $\sin(x)$ is known since it was given in the statement of a previous question.

If the Taylor series for $\cos(x)$ is known, then we can start the solution from the starting point
$\cos(x) = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}$.
At this point, we multiply the sum by x to obtain
$x\cos(x) = x\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!} = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k)!}$.

Question 8

SOLUTION. We define $r(t)$ to be the radius (in [mm]) of the area $A(t)$ (in [mm^2]) at time t. Then the circumference of A is $C = 2\pi r$ and we know that

$$\frac{dA}{dt} = k \cdot C = k \cdot 2\pi r(t).$$

The area A expressed in terms of r is $A = r^2\pi$. We take the derivative on both sides and get

$$\frac{d}{dt}A = \frac{d}{dt}(r^2\pi) = 2r(t)\frac{dr}{dt}\pi.$$

We combine the two equations $dA/dt = 2k\pi r(t)$ and $dA/dt = 2r(t)dr/dt\pi$ above and cancel $2\pi r(t)$ to get

$$\frac{dr}{dt} = k.$$

Integrating leads to

$$r(t) \ = kt + const.$$

To find $A(t)$ again we get back to the formula

$$A(t) \ = r(t)^2\pi = (kt + const)^2\pi.$$

Now all that's left to do is find the constants k and $const$. For that we use the information given

$$A(t) = 1.$$

So, we get

$$A(0) = (k \cdot 0 + const)^2\pi = const^2\pi = 1,$$

which leads to

$$const = \frac{1}{\sqrt{\pi}}.$$

The unit of constant is [mm]. We can take the positive solution of the root, since negative numbers don't make sense in this context. Now we use

$$A(1) = \left(1 \cdot k + \frac{1}{\sqrt{\pi}}\right)^2 \pi = 2,$$

which gives

$$k = \frac{(\sqrt{2}-1)}{\sqrt{\pi}}.$$

The unit of k is [mm/day].
Hence, the final equation for $A(t)$ is given by

$$A(t) = \left(\frac{(\sqrt{2}-1)}{\sqrt{\pi}} t + \frac{1}{\sqrt{\pi}} \right)^2 \pi,$$

which you can simplify to

$$A(t) = \left((\sqrt{2}-1)t + 1 \right)^2.$$

Good Luck for your exams!