# Full Solutions
# MATH307 April 2012

December 4, 2014

### How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Educational Resources.

## Tips for Using Previous Exams to Study: Exam Simulation

*Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.*

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Exam Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Educational Resources.



## Question 4 (b)

SOLUTION. From part (a) of the question, we determined that $N(D) = \vec{1}$ so the dimension of $N(D)$ is equal to 1, $dim(N(D)) = 1$.
To find $N(D^T)$, we can use the rank-nullity theorem which states

$$dim(N(A)) = m - r(A)$$

Where m is the number of columns in the matrix.
Remember that $r(A) = dim(R(A))$ so

$$dim(N(A)) = m - dim(R(A))$$

So, if we solve for $dim(R(A))$, we get

$$dim(R(A)) = m - dim(N(A))$$

$dim(R(A)) = 3 - 1$ $dim(R(A)) = r(A) = 2$
We can now use $dim(R(A)) = 2$ to solve for $dim(N(D^T))$ using the rank-nullity again. Finding $dim(N(D^T))$ does not directly give a solution to the question, but it does tell us how many vectors there are in $N(D^T)$.
The rank-nullity theorem states that

$$dim(N(D^T)) = n - r(D)$$

where n is the number of rows. Plugging in for what we solved above $r(A) = 2$, we get

$$dim(N(D^T)) = 3 - 2$$

$dim(N(D^T)) = 1$
This shows that there is only 1 vector in $N(D^T)$
Now solving for what the question asks, lets look at $D^T$

$$D^T = \begin{bmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

To find the nullspace of $D^T$, we have to find vectors $\vec{v}$ such that $D^T\vec{v} = 0$. Looking at $D^T$, we can easily see that the only solution to this problem is

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

This also matches our expectation that there is only 1 vector in N(D$^{\mathrm{T}}$).
Proof:
$rref(D) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
$\begin{bmatrix} x_1 \\ x_2 \\ x_3 7 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ $N(D) = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$
FINAL ANSWER $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

## Question 4 (c)

SOLUTION. Yes, it is true that the nullspace of the incidence matrix (D) is equal to the nullspace of the

Laplacian matrix. The reason for this is that the Laplacian matrix represents the total current flowing into or out of a node

$$L = D^T R^{-1} D$$

According to Kirchoff′s Current Law, current flowing into a node must equal to the current flowing out of the node, there cannot be a buildup of current at any node if there are no batteries attached, which there isn′t in this question. This can be expressed as

$$L\vec{v} = 0$$

For this to be true, $\vec{v} = \vec{1}$ because current flow has to be 0. For current flow to be equal to 0, you would have to have a voltage difference of 0, meaning that the voltage at each node is exactly the same,
Therefore

$$N(L) = N(D) = span(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix})$$

## Question 4 (d)

SOLUTION. The formula for the Laplacian matrix L is

$$L = D^T R^{-1} D$$

But we can simply calculate the Laplacian matrix by computing the following:
For the diagonal entries ($L_{ii}$)

$$L_{ii} = \sum_{connected\ edges,i} \gamma_i$$

For the non-diagonal edges ($L_{ij}$)

$$L_{ij} = \sum_{shared\ edgesj} \gamma_j$$

if there are shared edges and $L_{ij} = 0$ if no shared edges
$\gamma = \frac{1}{R}$ Where R is the resistance of the edge. which in this question $R = 1$ so $\gamma = 1$.
Following the equations given above, the Laplacian matrix is

$$L = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 & 0 \\ -1 & 4 & -1 & 0 & -1 & -1 \\ -1 & -1 & 4 & -1 & 0 & -1 \\ -1 & 0 & -1 & 4 & -1 & -1 \\ -1 & -1 & 0 & -1 & 4 & -1 \\ 0 & -1 & -1 & -1 & -1 & 4 \end{bmatrix}$$

The columns of matrix L are labeled as node 1 to 6 from left to right.
FINAL ANSWER

$$L = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 & 0 \\ -1 & 4 & -1 & 0 & -1 & -1 \\ -1 & -1 & 4 & -1 & 0 & -1 \\ -1 & 0 & -1 & 4 & -1 & -1 \\ -1 & -1 & 0 & -1 & 4 & -1 \\ 0 & -1 & -1 & -1 & -1 & 4 \end{bmatrix}$$

## Question 1 (a)

SOLUTION. For $f(x)$ to pass through all the data points, we must have $p_n(x_n) = y_n$ and for $f(x)$ to be continuous, we must have $p_n(x_{n+1}) = y_{n+1}$. Each of these give us *N-1* equations because there are *N-1* polynomials. So in total, *(N-1) + (N-1) = 2N - 2* equations.

## Question 1 (b)

SOLUTION. For $f(x)$ to have continuous first derivatives, we must have $p_i'(x_{i+1}) = p_{i+1}'(x_{i+1})$ . This gives us N-2 equations because we cannot fit the endpoints to anything.
For $f(x)$ to have continuous second derivatives, we must have $p_i''(x_{i+1}) = p_{i+1}''(x_{i+1})$ . This gives us N-2 equations because again, we cannot fit the endpoints to anything.
So in total, $(N-2)+(N-2) = 2N-4$ equations

## Question 1 (c)

SOLUTION. No content found.

## Question 1 (d)

SOLUTION. No content found.

## Question 2 (a)

SOLUTION. $F'' = (\Delta x)^{-2} D_{N-1} D_N F = (\Delta x)^{-2} \begin{bmatrix} 1 & -2 & 1 & 0 & ... & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & ... & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & ... & 0 & 0 & 0 \\ : & : & : & : & & : & : & : \\ 0 & 0 & 0 & 0 & ... & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & ... & 0 & 1 & -2 \end{bmatrix}$

Thus, $f''(x_i) \approx \frac{f_{i+1} - 2f_i + f_{i-1}}{\Delta x^2}$ $i = 1, 2, 3, ...N$

## Question 2 (b)

SOLUTION. No content found.

## Question 2 (c)

SOLUTION. No content found.

## Question 3

SOLUTION. (i) The rank of A is determined by the number of pivots in rref(A). Therefore, the rank of A is 2.
(ii) A basis for N(A) is determined by inspecting the free variables of rref(A), which are 2 and 4.
Let $x_2 = s$ and $x_4 = t$. So, $x_1 = -2x_2 - 2x_4$ and $x_3 = -2x_4$. Then we can write:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

Therefore, a basis for N(A) is $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$.

(iii) A basis for R(A) are the columns of A corresponding to the pivot columns of rref(A). Since we don′t know A, we cannot determine a basis for R(A).

(iv) A basis for $\mathrm{N(A^T)}$ is determined by inspecting the free variables of $\mathrm{rref(A^T)}$. Since we don′t know $\mathrm{rref(A^T)}$, we cannot determine a basis for $\mathrm{N(A^T)}$.

(v) A basis for $\mathrm{R(A^T)}$ are the rows of rref(A) that correspond to the pivots. Therefore a basis for $\mathrm{R(A^T)}$ is $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$.

## Question 4 (a)

SOLUTION. Looking at the resistor network, we can make the incidence matrix where the columns are the nodes and the rows are the edges.

$$D = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

The first column represents node 1, the second column represents node 2, and the third column represents node 3. An entry of -1 represents an edge leaving the corresponding node column, and an entry of 1 represents an edge entering the corresponding node column.

To find the nullspace of D, we have to find all vectors $\vec{v}$ that are solutions to $D\vec{v} = \vec{0}$. For matrix D, it is pretty easy to see that $\vec{v} = \vec{1}$ or any multiple of this as only solution to $D\vec{v} = \vec{0}$.

Therefore:

$$N(D) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Proof:

$$rref(D) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x_3$$

$N(D) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ Check:

$$D\vec{v} = \vec{0}. \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1+1 \\ -1+1 \\ -1+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

FINAL ANSWER

$$N(D) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

## Question 4 (e)

**Solution.** To find the effective resistance between two nodes, you would have to swap the 1st and sixth columns to the 1 and 2 positions inside the Laplacian matrix. You would then break up this Laplacian matrix into a block matrix

$$L = \begin{bmatrix} A & B^T \\ B & C \end{bmatrix}$$

where $A$ will be a 2x2 matrix.
You would then find the voltage to current map (A.K.A Schur′s complement), matrix S whose equation is

$$S = A - B^T C^{-1} B$$

Then, after solving for S, the effective resistance would be the reciprocal of the 1st entry of matrix S.
MATLAB CODE:

```
   %nodes n and m position
n = 1;
m = 6;

%find the size of the matrix L
lenL = length(L);

%swap nodes n and m to the 1st and 2nd position in the Laplacian matrix
swap = [n, m, 1:(n-1), (n+1):(m-1), (m+1):lenL];
L = L(swap, swap);

%compute matrix S which is the voltage to current map (Schur's complement).
A = L((1:2), (1:2));
B = L((3:lenL), (1:2));
C = L((3:lenL), (3:lenL));
S = A - (B′ * C(-1) * B);

%finds the effective resistance which is the first entry of S.
r = 1/S(1,1);
```

You could run this for the 2 different Laplacian matrices to find the effective resistance for each and compare them to confirm the conjecture is correct.

## Question 5 (a)

**Solution.** When using Lagrange interpolation, we choose to fit the point with a polynomial of lowest degree that goes through all the points.
We have 100 points, meaning that we need a 99 degree polynomial:
$p(x) = a_1 x^{99} + a_2 x^{98} + ... + a_{99} x + a_{100}$

## Question 5 (b)

**Solution.** From part (a), we know that we require a polynomial of degree 99, of the form:
$p(x) = a_1 x^{99} + a_2 x^{98} + ... + a_{99} x + a_{100}$
Since $(x_i, y_i)$ for $1 \leq i \leq 100$, then there are 100 linear equations for the coefficient vector
$\underline{a} = [a_1, a_2, ..., a_{100}]^T$
Therefore, the matrix equation that is satisfied is:

$$\begin{bmatrix} x_1^{99} & x_1^{98} & ... & x_1^2 & x_1 & 1 \\ x_2^{99} & x_2^{98} & ... & x_2^2 & x_2 & 1 \\ \vdots & \vdots & & \ddots & & \vdots \\ x_{100}^{99} & x_{100}^{98} & ... & x_{100}^2 & x_{100} & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ ... \\ ... \\ ... \\ a_{99} \\ a_{100} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ ... \\ ... \\ ... \\ y_{99} \\ y_{100} \end{bmatrix}$$

However, since the Vandermonde matrix not very well conditioned (because Vandermonde matrices have large condition numbers, an error in the data points can cause a relatively large error in the solution), it is unlikely that the solution is accurate

## Question 5 (c)

SOLUTION. A= ones(100,1)
for n=1:99
A=[X.^n A];
end
a= A'*A\A'*Y
plot(X,Y,'or')
hold on
XX=linspace(X(1),X(100),1000);
plot(XX,a(1)*XX.^2+a(2)*XX=C3*ones(1,length(XX)))

## Question 6 (a)

SOLUTION. For the first half of the question, the set of functions $\{e^{2\pi i\omega_n t}\}$ form an orthonormal basis for the space $L^2([0,T])$.
Using Euler′s formula we can express $e^{2\pi i w_n t}$ as $\cos(2\pi w_n t) + i\sin(2\pi w_n t)$
So the temperature function y(t) can be viewed as a wave-like function with period T decomposed into a combination of sinusoidal waves. The Fourier series represents it as a purely oscillatory components with frequency $\omega_n$. Here the phase is 0.
For the Fourier coefficients $c_n$, since we have y(t) = $\sum_{n=-\propto}^{\propto} c_n e^{2\pi i w_n t}$ then we an wirte $c_n$ as
$c_n = \frac{1}{T-0}\int_0^T e^{-2\pi i\omega_n t} y(t)dt = \frac{1}{T}\int_0^T e^{-2\pi\ i\frac{n}{T}t} y(t)dt$

## Question 6 (b)

SOLUTION. Since $c_n$ is a complex number with real and imaginary part, we write $c_n$ in its complex exponential form
$c_n = |c_n|\, e^{2\pi i\phi_n}$
Therefore $c_n$ reflects the original temperature function′s frequency spectrum. If we put the above expression back into the Fourier series representation of y(t), we have
$y(t) = \sum_{n=-\propto}^{\propto} c_n e^{2\pi i\frac{n}{T}t} = \sum_{n=-\propto}^{\propto} |c_n|\, e^{2\pi i\phi_n} e^{2\pi i\frac{n}{T}t} = \sum_{n=-\propto}^{\propto} |c_n|\, e^{2\pi i(\omega_n t+\phi_n)}$
t=0 or t=T would make $c_n$ the largest. Because both t=0 and t=T would make $e^{2\pi i\frac{n}{T}t} = 1$. So $c_0$ and $c_T$ have the largest absolute value.
The values of $c_n$ might you expect to have the largest absolute value is the frequency for the maximum temperature swings over the course of years. This would probably be seasonal fluctuations, so it would be $c_n$ for about 12 months.

## Question 6 (c)

SOLUTION. T = 3650

```
N = linspace(-500,500,1000);
Omega = N/T;
amp = sqrt(2-2*cos(2*pi*T*(1-omega)))./(2*pi*T*abs(1-omega));
plot(Omega, amp);
```

## Question 6 (d)

SOLUTION. No content found.

## Question 7 (a)

SOLUTION. We can actually infer all the information we need to answer this question from the matrix A that was given. Let us first rewrite the matrix A in the context of how it is used:
$\begin{bmatrix} x_{n+1} \\ x_n \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ x_{n-1} \\ x_{n-2} \end{bmatrix}$
This equation yields the following three linear equations:

$$\begin{aligned} x_{n+1} &= 2x_{n-1} + x_{n-2} \\ x_n &= x_n \\ x_{n-1} &= x_{n-1} \end{aligned}$$

The first equation of this set of linear equations tells us the recursive relationship for this matrix and notice that to calculate the value for the $n^{th}$ value in the relationship where $n > 2$ and $n$ is an integer, we only need the first three values to be initially stated.
In summary, the recursive sequence that is inferred from A is:
$x_{n+1} = 2x_{n-1} + x_{n-2}$
and we would need the first three values of the sequence to calculate all the following values in the sequence $(x_0, x_1$, and $x_3)$.

## Question 7 (b)

SOLUTION. No content found.

## Question 7 (c)

SOLUTION. Let us begin by assuming that the cycle only repeats over a single value. It should be immediately obvious that the only possible value that this could be is 0 since if the initial values were less than 0, the values will become increasingly negative and if the initial values were greater than 0, the values will become increasingly positive. Since we want a non-zero periodic sequence, a cycle of 1 value is not possible. Therefore, we will look at a cycle which repeats over 2 values. This implies:

$$\begin{aligned} x_2 &= x_0 \\ x_3 &= x_1 \\ &\vdots \\ x_n &= x_{n-2} \end{aligned}$$

If we just take the first value that can use the recursion relation $(x_3)$, we see that we can arrive at the following through basic algebra (where in step two we use that $x_3 = x_1$ :

$$
\begin{aligned}
x_3 &= 2x_1 + x_0 \\
x_1 &= 2x_1 + x_0 \\
x_1 + x_0 &= 0 \\
x_1 &= -x_0 = -x_2
\end{aligned}
$$

We just showed that a recursive cycle of 2 values is possible! If we assign the first two values to be the negative of each other, we will have a repeating cycle of the two selected values. Therefore, the following are all possible choices for the initial three values (remember that we need three initial values!):
$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} -\pi \\ \pi \\ -\pi \end{bmatrix}, etc.$

## Question 8 (a)

**Solution.** If $\lambda$ is an eigenvalue of P with eigenvector $v$, then
$Pv = \lambda v$.
Multiplying the matrix with the vector,
$Pv = \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{3} \\ 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{4} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 3 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 3 \end{bmatrix} = 1 * \begin{bmatrix} 3 \\ 8 \\ 3 \end{bmatrix} = \lambda v.$
Therefore the eigenvalue is $\lambda = 1$.

## Question 8 (b)

**Solution.** We will begin by calculating what $P^2$ is:

$$
\begin{aligned}
P^2 &= \begin{bmatrix} 0 & 1/4 & 1/3 \\ 1 & 1/2 & 1/3 \\ 0 & 1/4 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & 1/4 & 1/3 \\ 1 & 1/2 & 1/3 \\ 0 & 1/4 & 1/3 \end{bmatrix} \\
&= \begin{bmatrix} 0(0) + 1/4(1) + 1/3(0) & 0(1/4) + 1/4(1/2) + 1/3(1/4) & 0(1/3) + 1/4(1/3) + 1/3(1/3) \\ 1(0) + 1/2(1) + 1/3(0) & 1(1/4) + 1/2(1/2) + 1/3(1/4) & 1(1/3) + 1/2(1/3) + 1/3(1/3) \\ 0(0) + 1/4(1) + 1/3(0) & 0(1/4) + 1/4(1/2) + 1/3(1/4) & 0(1/3) + 1/4(1/3) + 1/3(1/3) \end{bmatrix} \\
&= \begin{bmatrix} 1/4 & 5/24 & 7/36 \\ 1/2 & 7/12 & 11/18 \\ 1/4 & 5/24 & 7/36 \end{bmatrix}
\end{aligned}
$$

We know that one of the properties of a stochastic matrix is, if $P$ or some power $P^k$ has all positive entries (that is, no zero entries) then the other eigenvalues of $P$ besides $\lambda = 1$ will have $|\lambda_i| < 1$ (referenced from page 187 of the current 2014 Summer Math 307 Textbook). Since we just found that all the entries in $P^2$ are positive, we can see that the other eigenvalues of $P$ must follow

$$\lambda_i \in (-1, 1).$$

## Question 8 (c)

**Solution.** Regardless, of the initial vector ($\mathbf{x_0} = [1, 0, 0]^T$ in this case), the probability vector found from $\lim_{n\to\infty} P^n \mathbf{x_0}$ will converge to the eigenvector with the eigenvalue pair of 1 scaled by a constant $a$ such that the sum of the elements in the vector sums to 1. It was found in part (a) the following is this particular eigenvalue/vector pair:

$\mathbf{v} = \begin{bmatrix} 3 \\ 8 \\ 3 \end{bmatrix}, \lambda = 1$

Therefore, we want to scale the vector $[3, 8, 3]$ to have the three elements sum to 1. We can do that easily using basic algebra:

$$\begin{aligned} 3a + 8a + 3a &= 1 \\ 14a &= 1 \\ a &= 1/14 \end{aligned}$$

Therefore, the $\lim_{n\to\infty} P^n[1, 0, 0]^T$ would converge to the matrix:

$\mathbf{x}_\infty = \frac{1}{14} \begin{bmatrix} 3 \\ 8 \\ 3 \end{bmatrix}$

Note: Another way you can look at this is that we are using the power method to converge towards the eigenvector with the largest absolute eigenvalue. We already showed in (b) that there would be no eigenvalue greater than 1. Therefore, if we keep multiplying by P and normalize the vector every time, we would converge to the eigenvector $[3, 8, 3]^T$ (Notice that we don′t actually have to normalize it every time since a stochastic matrix will already cause the values of the matrix to sum to 1). Lastly, we know that any constant multiple of the eigenvector is still the same eigenvector, therefore we can scale $[3, 8, 3]^T$ to what the answer that we found above.

## Good Luck for your exams!