Full Solutions
MATH221 December 2009

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How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

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Question 1

SOLUTION. Step 1: Write the system of equations in matrix form $A\vec{x} = \vec{b}$

In this case $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & c^2-5 \end{bmatrix}$ $\quad \vec{b} = \begin{bmatrix} 2 \\ 3 \\ c \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Step 2: Write A and b as augmented matrix.

Augmented form :

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & c^2-5 & c \end{array}\right]$$

Step 3: Reduce the augmented matrix in row echelon form.

After performing the row operations row2 = row2 - row1 and row3 = row3 - row1, we get the following reduced row echelon matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & c^2-4 & c-2 \end{array}\right]$$

For **part (a)**, we can see that in order for the system of equations to have no solution, $c^2 - 4 = 0$ while $c - 2 \neq 0$ and thus for $c = -2$ there is no solution. With this value of c our augment matrix looks like

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & -4 \end{array}\right]$$

indiacting no solution

For **part (b)**, that is, for the system of equations to have unique solution, we do not want any row of our matrix to be zero entries. Thus we want $c^2 - 4 \neq 0$ and $c - 2 \neq 0$ Thus for all values of c except $c = 2$ and $c = -2$, we have unique solution.

For **part (c)**, For the system of equation to have an infinitely many solutions, we want at least one free variable, i.e we want at least one row to be all zero entries. This can be achieved if $c - 2 = 0$ and $c^2 - 4 = 0$. Thus with $c = 2$, we have the third row as all zero entries.

Thus our augmented matrix for $c = 2$ is

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We know that the first column represents entries of x_1, the second column represents entries of x_2 and the third column represents entries of x_3.

Thus looking at our augmented matrix, it can be said column 1 and 2 are pivots columns and so only x_3 is a free variable.

We solve for general solution as follows:

$x_2 + 2x_3 = 1 \Rightarrow x_2 = 1 - 2x_3$

$x_1 = 2 - x_2 + x_3 \Rightarrow x_1 = 2 - 1 + 2x_3 + x_3 \Rightarrow x_1 = 1 + 3x_3$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow \vec{x} = \begin{bmatrix} 1+3x_3 \\ 1-2x_3 \\ x_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \ t \in \mathbb{R}$$

Question 2 (a)

SOLUTION 1. The following proof can be generalized very easily to an n by n matrix and so we present this solution first. Let

$$J = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

and let I be the five by five identity matrix. Then

$A = J - I$. Next, let

$$e_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

be the standard basis vectors where the 1 above is in the ith position. Notice then that the vectors $e_{1,i}$ defined by

$$e_{1,i} = e_1 - e_i$$

for i from 2 to 5 (or in general, the size of the matrix J). A simple calculation shows us that

$$Je_{1,i} = 0$$

for each value of i from 2 to 5. Further, taking the vector $v = e_1 + e_2 + e_3 + e_4 + e_5$ (the all ones vector), we see that

$$Jv = 5v.$$

As every vector is an eigenvector of the identity matrix, we have

$$Ae_{1,i} = (J - I)e_{1,i} = Je_{1,i} - e_{1,i} = -e_{1,i}$$

for each i from 2 to 5 and

$$Av = (J - I)v = Jv - v = 5v - v = (5 - 1)v = 4v$$

Hence, we get $(5 - 1)$ eigenvalues of (-1) and one eigenvalue of $(5 - 1)$. Thus, we have

$$\det(A) = (-1)^{5-1}(5 - 1) = 4$$

To generalize the above, just change every 5 above to an n.

Solution 2. Use the following determinant rules: The determinant of upper triangular matrix (i.e a matrix in row echelon form in this case) is equal to the product of its diagonal entries. If i≠j , then subtracting some constant time row i from row j does not change the determinant.

Although there are multiple ways of finding determinants, this solution will use rule 2 when performing row operations so that the determinant of the original matrix is the same as the matrix in row echelon form and then use rule 1 to find the determinant.

Let $A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$

Step 1: Do the following row operations on the matrix A in the following order:

row 1 = row 1 + row 2
row 2 = row 2 - row 1
row 3 = row 3 - row 1
and we get

$$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & -2 & -1 & -1 \\ 0 & 0 & -1 & -2 & -1 \\ 0 & 0 & -1 & -1 & -2 \end{bmatrix}$$

The matrix is still not in upper triangular form. Then perform the following row operations:
row 4 = row 4 - $\frac{1}{2}$row 3
row 5 = row 5 - $\frac{1}{2}$row 3
and we get :

$$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & -2 & -1 & -1 \\ 0 & 0 & 0 & -3/2 & -1/2 \\ 0 & 0 & 0 & -1/2 & -3/2 \end{bmatrix}$$

After doing one more row operation, row 5 = row 5 - $\frac{1}{3}$row 4 , we get the following upper triangular matrix:

$$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & -2 & -1 & -1 \\ 0 & 0 & 0 & -3/2 & -1/2 \\ 0 & 0 & 0 & 0 & -4/3 \end{bmatrix}$$

Step 2:

$$\text{let } U = \begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & -2 & -1 & -1 \\ 0 & 0 & 0 & -3/2 & -1/2 \\ 0 & 0 & 0 & 0 & -4/3 \end{bmatrix}$$

Then $\det(A) = \det(U) =$ product of diagonal entries $= 1 \times -1 \times -2 \times -\frac{3}{2} \times -\frac{4}{3} = 4$
Thus $\det(A) = 4$

Question 2 (b)

SOLUTION. Strategy:
Use the following determinant properties.
The determinant of upper triangular matrix (i.e a matrix in row echelon form in this case) is equal to the product of its diagonal entries. If i$\neq$j , then subtracting some constant time row i from row j does not change the determinant.
Step 1:
Perform the following row operations on B:
row 2 = row 2 - 2 row 1
row 3 = row 3 - 2 row 1
and we get

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & -1 \\ 0 & 2 & -5 & 0 \\ 0 & 3 & -1 & 1 \end{bmatrix}$$

Since the above matrix is still not in upper triangular form, perform the following row operations to it:
row 3 = row 3 - 2 row 1
row 4 = row 4 - 2 row 1
Then we get:

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 5 & 4 \end{bmatrix}$$
After performing one more row operation,
row 4 = row 4 + 5 row 5,
we get the following upper triangular matrix:
$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 14 \end{bmatrix}$$
Step 2 :
let $U' = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 14 \end{bmatrix}$
Then $det(A) = det(U') = 1 \times 1 \times -1 \times 14 = -14$
Thus $det(A) = -14$

Question 3 (a)

Solution. For our incidence matrix we'll assume that incoming flow is equal to outgoing flow at each of the 3 nodes. To setup our incidence matrix, imagine that flow leaving is negative and flow incoming is positive. The system is then:
$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -100 \\ 350 \\ -100 \end{bmatrix}$$

Question 3 (b)

Solution. See Q3 (a) for the flow diagram and system of equations.
The least squares equation takes the form

$$D^T D\mathbf{x} = D^T \mathbf{b}$$

In general, $D^T D$, has a diagonal of the number of edges connected to the respective node and has off-diagonals entries of -1 if the nodes are connected and 0 if not. It turns out that all of the nodes are connected to 2 other nodes by some edge.
Simplifying the least squares equation we have

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 450 \\ -450 \\ 0 \end{bmatrix}$$

Solving this system of equations through row reduction yields the following least squares solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x_3 + \begin{bmatrix} 150 \\ -150 \\ 0 \end{bmatrix}$$

Question 4

Solution. A vector normal to the plain is given by $\vec{w_1} \times \vec{w_2}$ which is $\vec{w_1} \times \vec{w_2} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$

To find the T matrix we need to find how the basis vectors $e_1 = [1, 0, 0], e_2 = [0, 1, 0], e_3 = [0, 0, 1]$ are transformed.

It is easy to see that $e_2 = [0, 1, 0]$ gets transformed to itself because it lies on the plane. e_1 and e_3 are more difficult. For e_1, imagine that we were looking directly on the x,z plane, the plane would then be squished into the line x = z. Then it is clear that the vector $e_1 = [1, 0, 0]$ transforms into $[0, 0, 1]$. Similarly, we can say that $e_3 = [0, 0, 1]$ transforms into $[1, 0, 0]$

Hence, our transformation matrix is $T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Question 5 (a)

Solution. We begin by forming

$W = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 2 \end{bmatrix}$

First thing that we need to do to matrix W is find the rref: The steps needed to take to obtain the rref are as follows: 1.) Add -3 times the 1st row to the 3rd row ($R_3 = R_3 - 3R_1$) 2.) Add -4 times the 2nd row to the 3rd row ($R_3 = R_3 - 4R_2$) 3.) Add the 2nd row to the 4th row ($R_4 = R_2 + R_2$) 4.) Swap row 3 and row 4 ($R_3 \Leftrightarrow R_4$) 5.) Add 3rd row to 2nd row ($R_2 = R_2 + R_3$) 6.) And finally add -4 times the 3rd row to the 1st row ($R_1 = R_1 - 4R_3$)

We should obtain that:

$rref(W) = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Lastly to obtain the basis all we need to do is to interpret $rref(W)$. The columns that are linearly independent in $rref(W)$ show us that those columns in the original equation are the vectors that form the basis for the span of W. We can see that vectors

$$v_1, v_2, v_4$$

are the basis vectors.

$\therefore B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 8 \\ 2 \end{bmatrix} \right\}$ Where B forms a basis for the span of W

Question 5 (b)

Solution. $W = R(A)$ where $A = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 2 \end{bmatrix}$

That means that the orthogonal component of W, $W^{\perp} = N(A^T)$

First we need to find $N(A^T)$:

$A^T = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 4 & -1 \\ 2 & -3 & -6 & 3 \\ 4 & -1 & 8 & 2 \end{bmatrix}$

$rref(A^T) = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
If (x_1, x_2, x_3, x_4) is in $N(A^T)$, then

$$x_1 = -3x_3,$$

$x_2 = -4x_3$, $x_4 = 0$ and x_3 is free.
A basis for $N(A^T)$, and hence a basis for $W^\perp$ is

$$\begin{Bmatrix} -3 & -4 & 1 & 0 \end{Bmatrix}$$

Question 5 (c)

SOLUTION. No content found.

Question 6

SOLUTION. The diagonal matrix D will be the eigenvalues of A, such that:
$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$
where λ_i are the eigenvalues, and $\lambda_1 \geq \lambda_2 \geq \lambda_3$.
The invertible matrix P will be the corresponding eigenvectors, such that:
$P = \begin{bmatrix} | & | & | \\ \vec{v_1} & \vec{v_2} & \vec{v_3} \\ | & | & | \end{bmatrix}$
where $\vec{v_i}$ is the eigenvector corresponding to the eigenvalue λ_i. The first thing that we need to do is to determine the eigenvalues of A. To do this, we take:
$\det(A - \lambda I) = 0$.
From this, we get

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 1 & 1-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = 0$$

Computing the determinant, we get that $\lambda(\lambda - 2)(\lambda + 1) = 0$ and therefore the eigenvalues are:
$\lambda_1 = 2, \lambda_2 = 0, \lambda_3 = -1$
Next, we must compute the eigenvectors corresponding to these eigenvalues. To do this we must solve: $A\vec{v} = \lambda\vec{v}$ for each eigenvalue/eigenvector combination.
For λ_1

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \vec{v_1} = 2 * \vec{v_1}$$

Solving this, we get that $\vec{v_1} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

For λ_2

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \vec{v_2} = 0 * \vec{v_2}$$

Solving this, we get that $\vec{v_2} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

For λ_3

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \vec{v_3} = -1 * \vec{v_3}$$

Solving this, we get that $\vec{v_3} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Now that we have the eigenvalues and eigenvectors, we can put these into matrices to get P and D!

$P = \begin{bmatrix} 2 & -2 & 1 \\ 3 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$

$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

To check that these matrices are correct, we can compute PDP^{-1}. When we do this calculation, we see that

$PDP^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} = A$

Question 7 (a)

Solution. $\begin{bmatrix} Hk+1 \\ Sk+1 \end{bmatrix} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} Hk \\ Sk \end{bmatrix}$

$\begin{bmatrix} 0.55 \\ 0.45 \end{bmatrix} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$

Question 7 (b)

Solution. Solving for eigenvalues $\lambda_1 = 1, \lambda_2 = 0.5$. We have n linearly independent eigenvectors so we can use Power Method. The eigenvector for $\lambda_1 = 1$ is $\begin{bmatrix} 1.5 \\ 1 \end{bmatrix}$. We need to scale the vector into a market share vector

$ANS = \begin{bmatrix} .6 \\ .4 \end{bmatrix}$

Question 8

Solution. We can first let $A = \begin{bmatrix} 17 & 9 \\ -30 & -16 \end{bmatrix}$ such that A solves the equation $\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = A \begin{bmatrix} x_n \\ y_n \end{bmatrix}$ In this problem, we are asked to find x_{30} and y_{30} based on x_0 and y_0. Since

$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = A \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$,

we can write

$\begin{bmatrix} x_{30} \\ y_{30} \end{bmatrix} = A^{30} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$

Now we need to solve for A^{30} This would be easier if we do the diagonalization $A = PDP^{-1}$ First we need to find the eigenvalues of A Let

$\det \begin{bmatrix} 17-\lambda & 9 \\ -30 & -16-\lambda \end{bmatrix} = (17-\lambda)(-16-\lambda) + 270 = 0$

Solving the equation above and we get $\lambda_1 = 2$ and $\lambda_2 = -1$ Then we need to find the corresponding eigenvectors When $\lambda = 2$, we have
$\begin{bmatrix} 17-2 & 9 \\ -30 & -16-2 \end{bmatrix} v = 0$
Solve this to get
$v_1 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$
When $\lambda = -1$, we have
$\begin{bmatrix} 17+1 & 9 \\ -30 & -16+1 \end{bmatrix} v = 0$
Solve this to get
$v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$
So now we can write the diagonal matrix
$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$
and matrix
$P = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -5 & -2 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 2 & 1 \\ -5 & -3 \end{bmatrix}$
So finally,

$$\begin{aligned} A^{30} &= PD^{30}P^{-1} \\ &= \begin{bmatrix} 3 & 1 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} 2^{30} & 0 \\ 0 & (-1)^{30} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -5 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 6 \cdot 2^{30} - 5 & 3 \cdot 2^{30} - 3 \\ -10 \cdot 2^{30} + 10 & -5 \cdot 2^{30} + 6 \end{bmatrix} \end{aligned}$$

Thus

$$\begin{aligned} \begin{bmatrix} x_{30} \\ y_{30} \end{bmatrix} &= A^{30} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \\ &= \begin{bmatrix} 6 \cdot 2^{30} - 5 & 3 \cdot 2^{30} - 3 \\ -10 \cdot 2^{30} + 10 & -5 \cdot 2^{30} + 6 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 3 \cdot 2^{30} - 2 \\ -5 \cdot 2^{30} + 4 \end{bmatrix} \end{aligned}$$

So $x_{30} = 3 \cdot 2^{30} - 2, y_{30} = -5 \cdot 2^{30} + 4$

Question 9

SOLUTION. First we can build a sub-Vandermonde matrix, even though we know that this matrix probably won't fit all the points Since we have 4 points and we want a parabola, the dimension of the sub-Vandermonde matrix would be 4 x 3
$V = \begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \\ x_4^2 & x_4 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix}$
Since we want to find the least square, we would like to solve the equation
$V^T V \underline{w} = V^T \underline{y}$ for $\underline{w} = \begin{bmatrix} c \\ b \\ a \end{bmatrix}$
where

$\underline{y} = \begin{bmatrix} 1 \\ 0 \\ 5 \\ 6 \end{bmatrix}$

The system then becomes

$$\begin{bmatrix} 0 & 1 & 4 & 9 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} 0 & 1 & 4 & 9 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 5 \\ 6 \end{bmatrix}$$

Solve the equation to get

$c = 1.3161, b = 0.4884, a = 0.1955$

So the parabola is $y = 0.1955 + 0.4884x + 1.3161x^2$

Question 10 (a)

Solution. For an n by n matrix to be diagonalizable, the matrix must have n distinct eigenvectors. First let's find the eigenvalues of a

$$\det\left(\begin{bmatrix} 0-\lambda & 1 & 2 \\ 0 & 3-\lambda & a \\ 0 & 0 & 0-\lambda \end{bmatrix}\right) = \lambda^2(3-\lambda) = 0$$

Solve the equation above to get $\lambda_1 = 3, \lambda_2 = \lambda_3 = 0$

We got a double root here, so we need to have 2 corresponding eigenvectors for $\lambda = 0$. When

$$\lambda = 0, A - \lambda I = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & a \\ 0 & 0 & 0 \end{bmatrix}$$

In order to have 2 eigenvectors, we need to have 2 free variables here i.e. we need to have $\frac{1}{3} = \frac{2}{a}$ Solve this to get $a = 6$.

Question 10 (b)

Solution. Now let's find the eigenvalues of b

$$\det\left(\begin{bmatrix} 2-\lambda & 1 & b & 3 \\ 0 & 3-\lambda & -1 & c \\ 0 & 0 & 2-\lambda & 2 \\ 0 & 0 & 0 & 3-\lambda \end{bmatrix}\right) = -(\lambda-3)^2(\lambda-2)^2 = 0$$

Solve the equation above to get $\lambda_1 = \lambda_2 = 3, \lambda_3 = \lambda_4 = 2$ As similar to a, we need to make sure both eigenvalues have exactly 2 corresponding eigenvectors. When

$$\lambda = 3, A - \lambda I = \begin{bmatrix} -1 & 1 & b & 3 \\ 0 & 0 & -1 & c \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In order to get 2 free variables, we need to set $\frac{-1}{-1} = \frac{c}{2}$ Thus $c = 2$ When

$$\lambda = 2, A - \lambda I = \begin{bmatrix} 0 & 1 & b & 3 \\ 0 & 1 & -1 & c \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In order to get 2 free variables, we need to set $\frac{1}{1} = \frac{b}{-1}$ Thus $b = -1$

Question 11

Solution. First we can rearrange the equation. $X(A^TA + 2 - B) = B^T$

Let $C = A^TA + 2 - B$

$C = \begin{bmatrix} 1 & 4 & 5 \\ 3 & 1 & 4 \\ 2 & 3 & 2 \end{bmatrix}$

$C^{-1} = \begin{bmatrix} -\frac{10}{33} & \frac{7}{33} & \frac{1}{3} \\ \frac{2}{33} & -\frac{8}{33} & \frac{1}{3} \\ \frac{7}{33} & \frac{5}{33} & -\frac{1}{3} \end{bmatrix}$

$X = C^{-1}B^T$

$X = \begin{bmatrix} -\frac{10}{33} & \frac{7}{33} & \frac{1}{3} \\ \frac{2}{33} & -\frac{8}{33} & \frac{1}{3} \\ \frac{7}{33} & \frac{5}{33} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 7 & -3 & 3 \\ -2 & 16 & -2 \\ 6 & -1 & 10 \end{bmatrix}$

$X = \begin{bmatrix} -\frac{58}{33} & \frac{10}{3} & \frac{43}{33} \\ \frac{71}{33} & -\frac{14}{3} & \frac{130}{33} \\ \frac{1}{33} & \frac{8}{3} & -\frac{73}{33} \end{bmatrix}$

Question 12 (a)

Solution. **False** - defining "inconsistent" as related to linear dependence, if you have four equations with only three variables, there are only two possibilities: In the no solution case, the system is inconsistent
i.e. $x_1 + 2x_2 = 3$ and $x_1 + 3x_3 = 3$
However, in the case that
$x + y + z = 3$ $x = 1$ $y = 1$ $z = 1$
The system is consistent.

Question 12 (b)

Solution. **True**. After taking the row reduce of a matrix 4×3, at best, the lowest row will be entirely consisted of zeros. If d =/= 0, then the above has no solution. $\begin{bmatrix} 1 & 0 & 0 & |a \\ 0 & 1 & 0 & |b \\ 0 & 0 & 1 & |c \\ 0 & 0 & 0 & |d \end{bmatrix}$

Question 12 (c)

Solution. **True**. By the rank nullity theorem $dim(N(A)) = n - rank(A)$ $n = 9$ (columns)
The rank is the number of linearly independent columns of A, in this case is at the most 6, likewise dim(N(A)) ≥ 3.

Question 12 (d)

Solution. **False**. By the rank nullity theorem $dim(N(A)) = 6 - rank(A)$ $dim(N(A)) = 5 - 4$ $dim((N(A)) = 2$ Don't forget for the rank nullity theorem, its the number of columns minus the rank!

Question 12 (e)

Solution. **True**. For a matrix to be diagnonizable, it must be able to be written in this form

$$A = P^{-1}DP$$

where D is a matrix with the eigenvalues of A as its diagonal. All projection matrices to the same dimension are diagonizable. Since all vectors in W project onto itself with that transformation, the eigenvalues of T are 0 and 1, hence $P^{-1}D^nP = A$.

Question 12 (f)

SOLUTION. **False**. The rank is the number of non zero rows in the matrix. A counter example

$$rref(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

number of nonzeros in diagonal is 1 but the rank is 2.

Question 12 (g)

SOLUTION. **False**. If were linearly independent of each other then it can be said that n = 5 (i.e. is a basis), however, it is possible for n to be any number of vectors but still span R^5 given that some of the vectors are linearly dependent.

Question 12 (h)

SOLUTION. For $v_1, \ldots, v_m$ to be linearly independent it must hold that $m \leq n$. Hence they satisfy $k_1v_1 + \cdots + k_mv_m \in W$
Hence the statement is **True**.

Question 12 (i)

SOLUTION. True. Let x be a solution to $A^2x = b$. Take $y = Ax$ then $Ay = A(Ax) = A^2x = b$

Question 12 (j)

SOLUTION. **False**
In two dimensions every rotation matrix can be written in the form

$$\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix},$$

Row 1, Column 1 and Row 2, Column 2 don′t match.

Good Luck for your exams!