Full Solutions
MATH102 December 2012

April 4, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

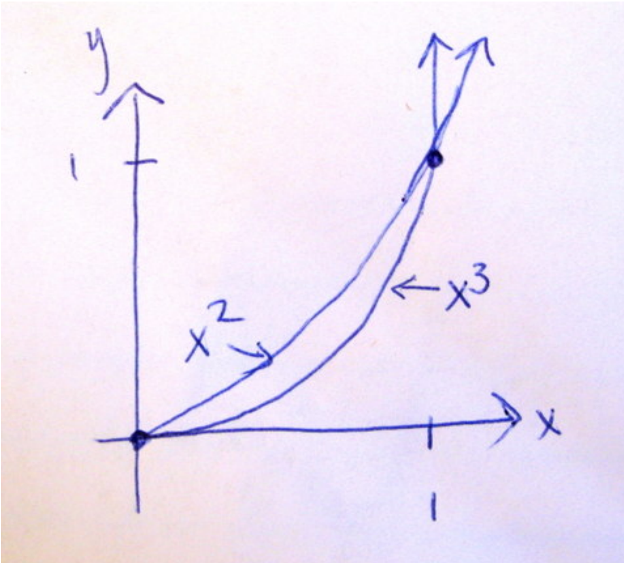
1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

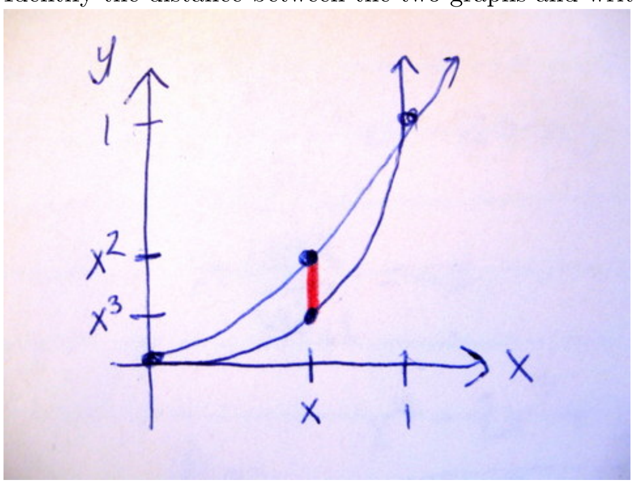
This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question A 01

SOLUTION. As stated in the hint, sketching a picture of the two graphs between 0 and 1 is a good first step.



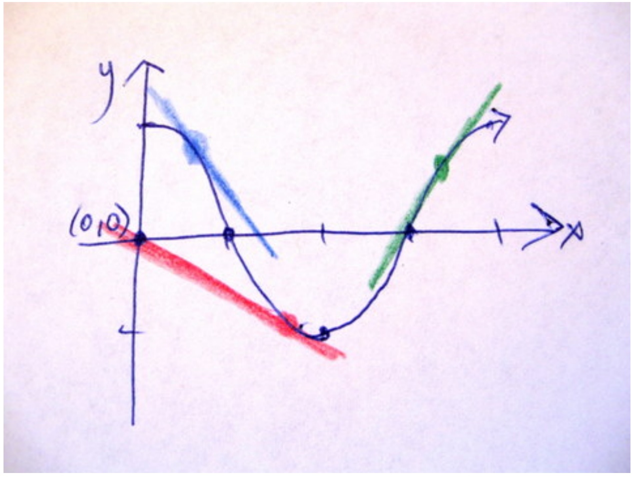
Identify the distance between the two graphs and write a formula for it.



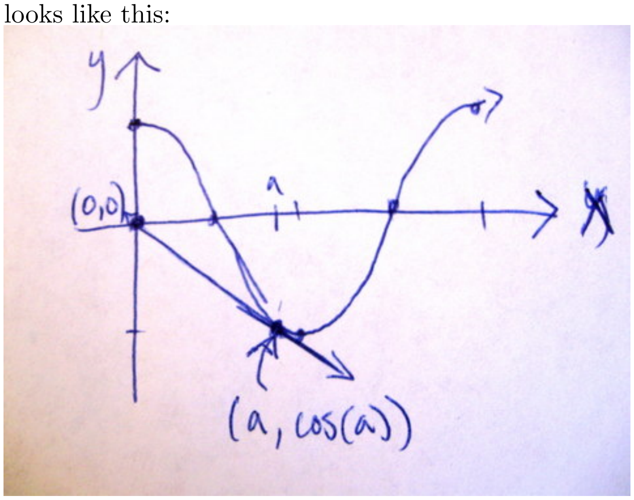
The distance between the two graphs at a point x can be given by
$D(x) = x^2 - x^3$
Now this is simply a max/min problem (see the keyword "furthest apart" in the question): we are looking for the maximum of $D(x)$.
First we find critical points by differentiating, setting equal to zero, and solving for x:
$D'(x) = 2x - 3x^2$
$0 = x(2 - 3x)$
So we have critical points $x = 0$ and $x = 2/3$. We know $x = 0$ isn't the correct answer as we already know that the graphs are equal there. Thus, the point of furthest distance must be $x = 2/3$ or **(c)**.

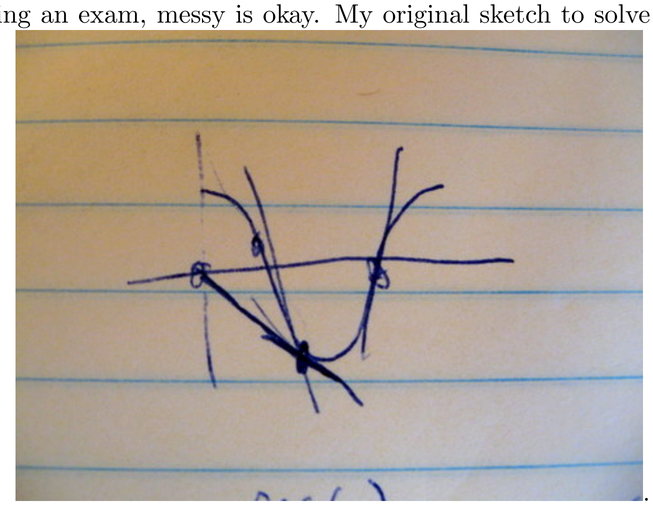
Question A 02

SOLUTION. As stated in the hint, we will begin by sketching a picture of $\cos x$ and a few tangent lines.



The scenario described in the question is shown by the red line line. Redrawing and relabeling the sketch looks like this:



(Solution writer's note: The sketches above are nice and neat for clarity, but in real life, i.e. when you're actually taking an exam, messy is okay. My original sketch to solve the problem looked like the thumbnail on the right. . The point of drawing a sketch is often just to understand what the problem is asking.)

Now there are two ways to compute the slope of the tangent line above. The first is our traditional slope formula:

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\cos(a) - 0}{a - 0} = \frac{\cos(a)}{a}$

The second is through the derivative, as this is the tangent line of $y = \cos x$ at $x = a$:

$y' = -\sin x$
$m = -\sin(a)$
As these two methods are giving the slope of the same line, they must be equal:
$\frac{\cos(a)}{a} = -\sin(a)$
Cross-multiplying gives:
$\frac{\cos(a)}{\sin(a)} = -a$
Or $\cot(a) = -a$, which is answer **(b)**.

Question A 03

SOLUTION. Let's test each answer one-by-one:

- **(a)** If $n < 3$ then $L = 0$.

If $n < 3$, try a test value of $n = 2$. Then our limit becomes:

$$\begin{aligned} L &= \lim_{x\to-\infty} \frac{x^2 - 2x^3 + 2}{x^3 + 4} \\ &= \lim_{x\to-\infty} \frac{-2 + x^2/x^3 + 2/x^3}{1 + 4/x^3} \\ &= -2 \end{aligned}$$

The answer does not match the conclusion $L = 0$ so this answer is not necessarily true.

- **(b)** If $n = 3$ then $L = -2$.

If $n = 3$ then our limit becomes:

$$\begin{aligned} L &= \lim_{x\to-\infty} \frac{x^3 - 2x^3 + 2}{x^3 + 4} \\ &= \lim_{x\to-\infty} \frac{-1 + 2/x^3}{1 + 4/x^3} \\ &= -1 \end{aligned}$$

The answer does not match the conclusion $L = -2$ so this answer is not true.

- **(c)** If $n > 3$ and n is odd then"L = "$-\infty$

If $n > 3$ and n is odd try a test value of $n = 5$. Then our limit becomes:

$$\begin{aligned} L &= \lim_{x\to-\infty} \frac{x^5 - 2x^3 + 2}{x^3 + 4} \\ &= \lim_{x\to-\infty} \frac{x^2 - 2 + 2/x^3}{1 + 4/x^3} \\ &= \infty \end{aligned}$$

The answer does not match the conclusion $L = -\infty$, so this answer is not necessarily true.

- **(d)** If $n > 3$ and n is even then"L = "$-\infty$.

If $n > 3$, and even try a test value of $n = 4$. Then our limit becomes:

$$\begin{aligned} L &= \lim_{x \to -\infty} \frac{x^4 - 2x^3 + 2}{x^3 + 4} \\ &= \lim_{x \to -\infty} \frac{x - 2 + 2/x^3}{1 + 4/x^3} \\ &= -\infty \end{aligned}$$

The answer matches the conclusion $L = -\infty$ so this answer is correct.
The final answer is **(d)**.

Question B 01

Solution. Field A:

- The ODE must be independent of t, because restricted to every vertical line (restricted to any t), the slope field looks equal.
- We have equilibriums at $x = -1$ and $x = 1$, because the slopes get more and more horizontal, the closer they are at $x = \pm 1$.
- For $x \in (-1, 1)$ the slope is positive.

This fits exactly to the ODE $\frac{dx}{dt} = 1 - x^2$ (d).
Field B:

- The ODE must be independent of t, because restricted to every vertical line (restricted to any t), the slope field looks equal.
- We have equilibriums at $x = -1$ and $x = 1$, because the slopes get more and more horizontal, the closer they are at $x = \pm 1$.
- For $x \in (-1, 1)$ the slope is negative.

This fits exactly to the ODE $\frac{dx}{dt} = x^2 - 1$ (c).
Field C:

- The ODE must be independent of x, because restricted to every horizontal line (restricted to any x), the slope field looks equal.
- We have equilibriums at $t = -1$ and $t = 1$, because the slopes get more and more horizontal, the closer they are at $t = \pm 1$.
- For $t \in (-1, 1)$ the slope is positive, for $t < -1$ and $t > 1$ the slope is negative.

This fits exactly to the ODE $\frac{dx}{dt} = 1 - t^2$ (a).
ODE (b) and Field D:
This ODE causes a slope field where the slopes are horizontal for every x and $t = 0$. This is not the case for the Field D. Hence, ODE (b) and Field D cannot fit.

Question B 02

Solution. Before taking derivatives, we will fill in the first table, by testing values in the two intervals listed.
Note that $x^2 + 1$ will always be positive, so when calculating whether the function/derivatives are positive or negative, it is sufficient to look at the numerator as the denominator will always be positive.

- $(-\infty, 0)$, test point $x = -1$

$$f(-1) = \frac{-1}{(-1)^2 + 1} = \frac{-1}{2} < 0$$

- $(0, \infty)$, test point $x = 1$

$$f(1) = \frac{1}{(1)^2 + 1} = \frac{1}{2} > 0$$

So the first table should be filled in as:

	$(-\infty, 0)$	0	$(0, \infty)$
f (x)	-	0	$+$

Proceeding to the second table, we will calculate the first derivative using the quotient rule:
$f'(x) = \frac{1*(x^2+1)-x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$
Setting equal to zero and solving for x gives:
$0 = 1 - x^2$
That is, $x = -1$, $x = 1$. These are then our values for c_1 and c_2. We can now test points in the three intervals labeled in the table.

- $(-\infty, -1)$, test point $x = -2$

$$f'(-2) = \frac{1 - (-2)^2}{((-2)^2 + 1)^2} = \frac{-3}{25} < 0$$

- $(-1, 1)$, test point $x = 0$

$$f'(0) = \frac{1 - (0)^2}{((0)^2 + 1)^2} = \frac{1}{1} > 0$$

- $(1, \infty)$, test point $x = 2$

$$f'(2) = \frac{1 - (2)^2}{((2)^2 + 1)^2} = \frac{-3}{25} < 0$$

Filling in the table should look like this:

	$(-\infty, c_1)$	*c1*	(c_1, c_2)	*c2*	(c_2, ∞)
f '(x)	-	0	$+$	0	-

Finally, we compute the second derivative, again using the quotient rule:

$$\begin{aligned} f''(x) &= \frac{-2x(x^2+1)^2 - (1-x^2)(2)(x^2+1)(2x)}{(x^2+1)^4} \\ &= \frac{-2x(x^2+1) - (1-x^2)(4x)}{(x^2+1)^3} \\ &= \frac{-2x^3 - 2x - 4x + 4x^3}{(x^2+1)^3} \\ &= \frac{2x^3 - 6x}{(x^2+1)^3} \\ &= \frac{2x(x^2-3)}{(x^2+1)^3} \end{aligned}$$

Again solving for where $f''(x) = 0$ we have
$0 = 2x(x^2 - 3)$
which yields critical points of $x = -\sqrt{3}, x = 0$ and $x = \sqrt{3}$, which will be r_1, r_2, r_3 respectively. When testing points in each of the four intervals of the table, it is worth noting that $\sqrt{3}$ is somewhere between 1 and 2.

- $(-\infty, -\sqrt{3})$, test point $x = -2$

$$f''(-2) = \frac{2(-2)((-2)^2-3)}{(-2^2+1)^3} = \frac{-4}{5^3} < 0$$

- $(-\sqrt{3}, 0)$, test point $x = -1$

$$f''(-1) = \frac{2(-1)((-1)^2-3)}{(-1^2+1)^3} = \frac{4}{2^3} > 0$$

- $(0, \sqrt{3})$, test point $x = 1$

$$f''(1) = \frac{2(1)((1)^2-3)}{(1^2+1)^3} = \frac{-4}{2^3} < 0$$

- $(\sqrt{3}, \infty,)$, test point $x = 2$

$$f''(2) = \frac{2(2)((2)^2-3)}{(2^2+1)^3} = \frac{4}{5^3} > 0$$

Filling in the table yields:

	$(-\infty, r_1)$	r_1	(r_1, r_2)	r_2	(r_2, r_3)	r_3	(r_3, ∞)
f(x)"	-	0	+	0	-	0	+

Question B 03

Solution. Let us consider each row of the table:

- Sample Mean

The sample mean should stay roughly the same. This is because the mean length of adult wombats is a well-defined quantity, and we have no information to suggest that we have been sampling adult wombat length in a biased manner.

- Sample Maximum

The sample maximum is likely to increase. This is simply because the larger the set sampled, the more likely you are to include extreme outliers, which will increase the maximum and decrease the minimum of your set. In more technical notation, this is the same as saying $\max_A x \geq \max_B x$ when B is a subset of A.

- Standard Error of the Mean

The standard error of the mean will decrease. Recall that the standard error of the mean estimates how far away your sample mean is from the "true" population mean. More and more samples will improve the accuracy of your sample mean, decreasing the error. This can also be seen in the formula for standard error of the mean:
SE $= \frac{s}{\sqrt{n}}$
where s is the standard deviation of the sample and n is the number of objects sampled. As n increases, the overall fraction will decrease.
So the table should be filled in as follows:

	decrease	stay roughly the same	increase
The sample mean will		X	
The sample maximum will			X
The standard error of the mean will	X		

Question B 04

Solution. Since we have three possible outcomes here (selling 1, 2, or 3 cars), we will use a formula for expected value that looks like this:
$E[X] = x_1p_1 + x_2p_2 + x_3p_3$
Where x_i is the money earned (or value) of selling i cars in a month and p_i is the corresponding probability of selling i cars. Plugging in the appropriate probabilities and amount earned per car (plus the \$1000 dollar bonus if 3 cars are sold!), we get:
$E[X] = 2000 * .7 + 4000 * .2 + 7000 * .1$
Simplifying gives:
$E[X] = 2900$
So the sales person's expected monthly income is \$2,900.

Question B 05

Solution. As stated in the second hint, we will first use the information about half-life to solve for the rate of decay, or k in the equation:

$$y(t) = Ce^{kt}$$

If $y(t)$ is the percentage remaining after t years, we know that 50% of ^{90}Sr remains after 29 years. At $t = 0$ the amount is 100 percent, hence choose $C = 100$. Plugging this into the formula gives:

$$50 = 100e^{k29}$$

Solving for k, we get
$\ln(\frac{1}{2}) = 29k$
$k = \ln(1/2)/29$
Plugging this back into our formula for $y(t)$, we can now solve for the percentage remaining on April 26, 2012, when $t = 26$.

$$y(t) = 100e^{(\ln(1/2)/29)*26}$$

Simplifying will give:

$$y(t) = 100 * (\frac{1}{2})^{26/29}$$

or 53% remaining.

Question B 06

SOLUTION. (In what follows, the notation *P(outcome)* should be read as "the probability that'outcome' occurs.")
In order to solve this problem, we need to break it down into pieces. We are asked to find:

P(muscle contracts)

A muscle contracts if 4 or more neurons fire. Since there are 6 neurons total we can rewrite this probability as:

P(muscle contracts) = P(4, 5, or 6 neurons fire)

When you have different outcomes joined by an "or", you can separate the outcomes into individual probabilities and add them together. So

P(4, 5, or 6 neurons fire) = P(4 neurons fire) + P(5 neurons fire) + P(6 neurons fire)

Hopefully the problem is more manageable now - we're just computing three individual probabilities and then add them up. Let's start with the last one, because it's the easiest. (For ease, we will be calling the six neurons **a, b, c, d, e** and **f**.)

*P(6 neurons fire) = P(**a** and **b** and **c** and **d** and **e** and **f** fire)*

Recall that when probabilities are joined by an "and", you multiply them. Since the probability of each neuron firing is *.3*, this means that

P(6 neurons fire) = $(.3)^6$

Now onto the next probability, *P(5 neurons fire)*. Suppose I had a list of 5 specific neurons I wanted to fire, say **a** and **b** and **c** and **d** and **e**. Since only 5 are firing, this also means the sixth neuron **f** does not fire, which has a probability of *.7*. The probability of this specific combination occurring involves multiplying the probability for each neuron firing/not firing, which gives $(.3)^5(.7)$. However, this is not the only combination of 5 neurons that can fire - we need to include all possible combinations of 5 neurons. This can be done using the binomial $\binom{6}{5}$. Since the same $(.3)^5(.7)$ probability will occur for each list of 5 neurons, our overall probability is

P(5 neurons fire) = $\binom{6}{5}$*(.3)*^{5}*(.7)*

Finally, computing *P(4 neurons fire)*, we use the same method as the previous probability. The probability that a specific group of 4 neurons will fire is *(.3)*^{4}, the probability that the other two neurons do not fire is *(.7)*^{2}, and the number of groups of 4 neurons is found by the binomial $\binom{6}{4}$, giving

P(4 neurons fire) = $\binom{6}{4}$*(.3)*^{4}*(.7)*^{2}

Since we don't need to simplify any further, we can simply add our probabilities together as shown above to get a final answer of:

$$P(\text{muscle contracts}) = (.3)^6 + \binom{6}{5}(.3)^5(.7) + \binom{6}{4}(.3)^4(.7)^2$$

Question C 01

Solution. First we will solve for $f'(x)$ using the chain rule on the left and the product rule on the right.
$\frac{1}{f(x)}f'(x) = \ln(x) + x\frac{1}{x}$
$f'(x) = (\ln(x) + 1)f(x)$
In order to complete the question, we must solve for $f(x)$ in terms of x. We will do this by using the equation

$$\ln(f(x)) = x\ln(x)$$

a second time, together with the log rule $a\ln(b) = \ln(b^a)$. So we can rewrite the right side of the the equation above as:
$\ln(f(x)) = \ln(x^x)$
Applying the exponential function to both sides causes the logarithm to disappear, leaving:
$e^{\ln(f(x))} = e^{\ln(x^x)}$
$f(x) = x^x$
Substituting this into our derivative, we arrive at the final answer:
$f'(x) = (\ln(x) + 1)(x^x)$

Question C 02

Solution. Using the definition of the derivative from the hint, we set up:
$g'(x) = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$
Now we use the hint from the statement of the question. If we multiply by a conjugate, we can move the square roots and make our limit easier to solve.

$$\begin{aligned}
g'(x) &= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} * \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\
&= \lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \lim_{h\to 0}\frac{1}{(\sqrt{x+h}+\sqrt{x})} \\
&= \frac{1}{(\sqrt{x+0}+\sqrt{x})} \\
&= \frac{1}{2\sqrt{x}}
\end{aligned}$$

Question C 03 (a)

SOLUTION. The number 48 is between perfect squares 36 and 49, so we have:

$$\begin{array}{ccc} 36 & < 48 & < 49 \\ \sqrt{36} & < \sqrt{48} & < \sqrt{49} \\ 6 & < b & < 7 \end{array}$$

Question C 03 (b)

SOLUTION. Using the linear approximation formula
$f(x) \approx f'(a)(x-a) + f(a)$
where our function is $f(x) = \sqrt{x}$ and our value of a will be a number near 48. Choosing $a = 49$ is sensible because you can take the square root easily. We can either calculate $f'(x)$ again or note that we already found it in question C2 as $f'(x) = 1/(2\sqrt{x})$. Plugging all this in gives:
$f(x) \approx \frac{1}{2\sqrt{49}}(x-49) + \sqrt{49} = \frac{1}{14}(x-49) + 7$
To approximate $b = \sqrt{48}$, we say
$\sqrt{48} = f(48) \approx \frac{1}{14}(48-49) + 7 = \frac{-1}{14} + 7 = 6\frac{13}{14}$
Whether the linear approximation is an over or under estimate depends on the concavity of the function - if $f(x)$ is concave up, the tangent line approximation lies below the graph and is an under-estimate; the opposite is true if $f(x)$ is concave down. Because $f(x) = \sqrt{x}$ is concave down, our estimate in this problem is an over-estimate.

Question C 03 (c)

SOLUTION. The question is telling us to use Newton's method, which can only be used to estimate roots of functions. Therefore, in order to use Newton's method to estimate b, we must create a function with $b = \sqrt{48}$ as a root. Then estimating the root of this function will be the same as estimating the value of b.
One way to construct a function with specific roots is to place the roots into linear factors and then multiply them out to get a polynomial. If we want one of our roots to be $\sqrt{48}$, this means we should have at least one linear factor of the form $(x - \sqrt{48})$.
Ideally, we want our function to not actually contain the value we're trying to estimate. In order to make that $\sqrt{48}$ go away, we are going to use the difference of squares, which says:
$(a-b)(a+b) = a^2 - b^2$
or,
$(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}) = a - b$
Putting these two together, we can say:
$f(x) = (x - \sqrt{48})(x + \sqrt{48}) = x^2 - 48$
which has a root at $\sqrt{48}$ like we wanted.
Now we can finally use Newton's method to iterate and find x_1. The formula for the first iteration of Newton's method is as follows:
$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$
We have $f(x)$, $f'(x) = 2x$, and our starting point should be a value close to b. Based on the previous parts of the question, it makes the most sense to choose $x_0 = \sqrt{49} = 7$.
Putting this all together, we finally get:
$x_1 = 7 - \frac{(7)^2 - 48}{2(7)} = 7 - \frac{1}{14} = \frac{97}{14}$

Question C 04 (a)

SOLUTION. We want to find the differential equation of the mass Q of dead leaves. This means we find the derivative of Q with respect to time, dependent on Q.
Therefore, we separate $\frac{d}{dt}Q$ into $\frac{d}{dt}Q_+$ the leaves that are added every year and $\frac{d}{dt}Q_-$, that are removed every year. Such that
$\frac{d}{dt}Q = \frac{d}{dt}Q_+ + \frac{d}{dt}Q_-$
The leaves (in grams) that are added every year per cm^2 are
$\frac{d}{dt}Q_+ = 5\frac{\text{grams}}{\text{year}\cdot cm^2}$.
The leaves that are removed every year depends on the amount of leaves: According to the mass of leaves, half of the mass is lost every year:
$\frac{d}{dt}Q_- = -\frac{0.5Q}{\text{year}}$.
Since leave mass is lost, we put a negative sign here. Hence, our final answer is
$\frac{d}{dt}Q = \frac{5\text{grams}}{\text{year}\cdot cm^2} - \frac{0.5Q}{\text{year}}$.
Note that this works out nicely so that the unit of Q is grams/cm^2. With units suppressed, the above is
$\frac{d}{dt}Q = 5 - 0.5Q$.

Question C 04 (b)

SOLUTION. To find the steady state, we set $\frac{d}{dt}Q = 0$.
Hence:

$$0 = 5\frac{\text{grams}}{\text{year}\cdot cm^2} - \frac{0.5}{\text{year}}Q$$

$$Q = 10\frac{\text{grams}}{cm^2}$$

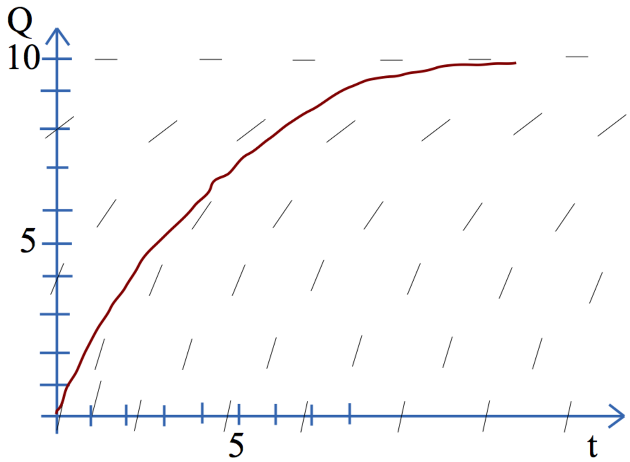
is the steady state.

Question C 04 (c)

SOLUTION. We want to sketch the solution without solving the equation. Hence, we draw the phase space for Q (the number of leaves) and dQ/dt (the daily rate of change of leaves). Therefore, we first make a table with Q and the corresponding dQ/dt.

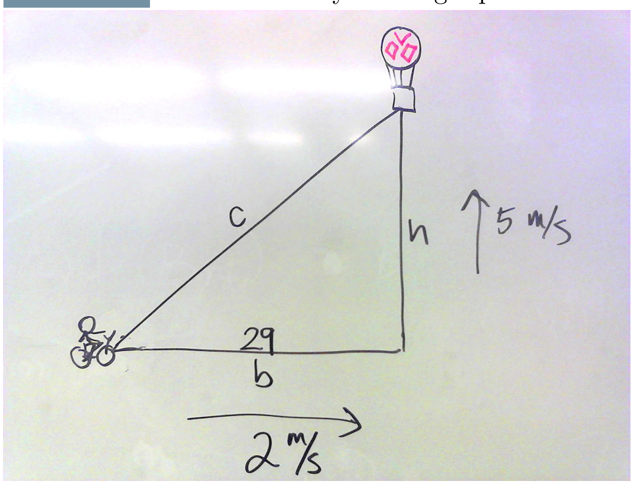
$Q = 0$	$Q = 2$	$Q = 4$	$Q = 6$	$Q = 8$	$Q = 10$
$dQ/dt = 5$	$dQ/dt = 4$	$dQ/dt = 3$	$dQ/dt = 2$	$dQ/dt = 1$	$dQ/dt = 0$

After drawing the phase space, the curve of the solution can be plotted by following the slopes.



Question C 05 1

Solution. We will start by drawing a picture relating the different facts given in the problem.



Now it is necessary to figure out what the question is asking. This is an optimization problem, where the thing being optimized is the rate at which θ is changing, or $d\theta/dt$. So our next step is to actually find an expression for this rate.

We start by relating all the information we already have in the question. The diagram shows a right triangle, with adjacent and opposite sides labeled. Thus, the best way to relate the information is the tangent function.

$\tan\theta = h/b$

We can express the height of the balloon as a function of time. Since it is rising at a rate of 5 m/s, the height of the balloon can be given as $h(t) = 5t$. Similarly, the distance of the girl from the ballon's liftoff point can be given as the initial distance, minus her rate of travel, or $b(t) = 29 - 2t$. Plugging these into the formula, we get:

$\tan\theta = \frac{5t}{29-2t}$

Differentiating with respect to time (t) gives:

$\sec^2\theta \frac{d\theta}{dt} = \frac{5(29-2t)-5t(-2)}{(29-2t)^2}$

We want our final expression to be in terms of t, so we want to change the $\sec^2\theta$ into an expression in terms of t. We will do this by going back to the definition of $\sec\theta$ as a trig ratio. We recall $\sec\theta$ is the ratio of hypotenuse over adjacent, or c/b in our notation. In particular, we have $\sec^2\theta = c^2/b^2$. The adjacent side b is given by $29 - 2t$ and we can use the Pythagorean theorem to find the hypotenuse c^2.

$c^2 = (5t)^2 + (29 - 2t)^2$

So we have

$\sec^2\theta = \frac{(5t)^2+(29-2t)^2}{(29-2t)^2}$
Replacing $\sec^2\theta$ in our original equation, we get:
$\frac{(5t)^2+(29-2t)^2}{(29-2t)^2}\frac{d\theta}{dt} = \frac{5(29-2t)-5t(-2)}{(29-2t)^2}$
We want to solve this expression for $d\theta/dt$. Simplifying the above expression, we get:
$\frac{d\theta}{dt} = \frac{145}{(5t)^2+(29-2t)^2}$
Now we finally have an expression for $d\theta/dt$. To complete the problem, we must find where it is at a maximum. To do so, we will use our usual method for finding a maximum: differentiating, finding critical points, and testing whether it is a maximum.
Taking the derivative gives with respect to t gives:

$$\begin{aligned}\frac{d^2\theta}{dt^2} &= \frac{-145(2(5t)(5)+2(29-2t)(-2))}{((5t)^2+(29-2t)^2)^2} \\ &= \frac{-145(50t-116+8t)}{((5t)^2+(29-2t)^2)^2} \\ &= \frac{-145(58t-116)}{((5t)^2+(29-2t)^2)^2} \\ &= \frac{-145\cdot 58(t-2)}{((5t)^2+(29-2t)^2)^2}\end{aligned}$$

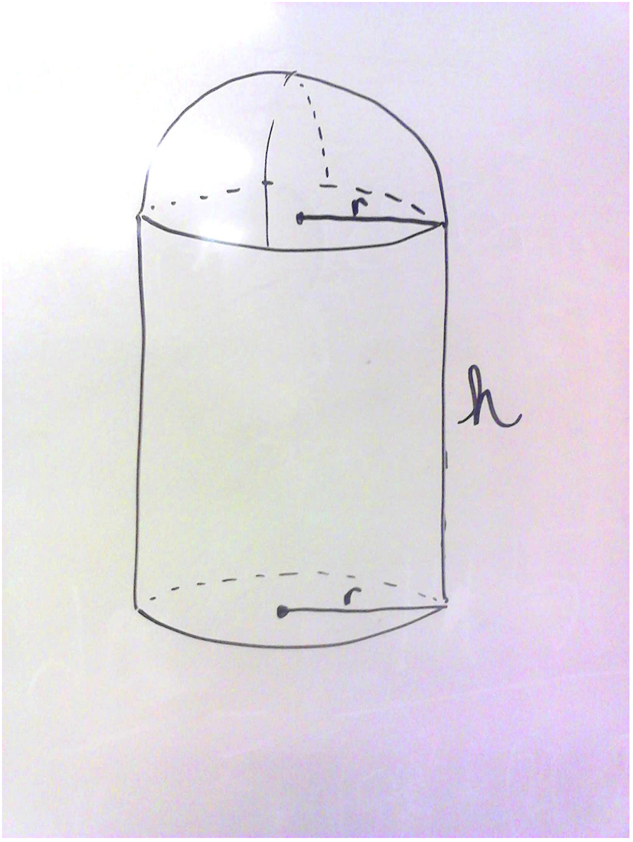
We are free to ignore the denominator, as it is always positive, thus never makes the derivative undefined. We can therefore restrict our attention to the numerator, set it equal to zero, and solve for t. This gives $t = 2$. It remains to show that this point is a maximum. The easiest way to do this is the first derivative test; that is choosing two points on either side of our critical point $t = 2$ and checking if the sign of the derivative changes. Again, the denominator of the derivative is always positive, so we need only consider the numerator:

	t < 2	t = 2	t > 2
$d^2\theta/dt^2$	+	0	-

This indicates that the original rate $d\theta/dt$ changes from increasing to decreasing at $t = 2$, meaning that it is indeed the maximum we're looking for. The time at which the angle θ is increasing the fastest is at $t = 2$ seconds after the balloon is released.

Question C 05 2

Solution. Let's start with a picture.



Let P be the price of the wall material per square area. Thus, the price of the roof is $2P$ per square area. Let V be the given volume. The formula for the volume of this house is
$V = \pi r^2 h + \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) = \pi r^2\left(h + \frac{2r}{3}\right)$
where the first term is the volume of a cylinder and the second term is half the volume of a sphere. We will need to isolate for h later so doing so now while the formula is fresh, we have
$h = \frac{V}{\pi r^2} - \frac{2r}{3}$
We're minimizing the cost of the house. The cost of this house is equal to the surface area of the wall multiplied by the price added to the surface area of the roof multiplied by $2P$. As part of our simplification process, we'll plug in the formula for h immediately above, to arrive at a final equation that's only in terms of one variable, r.

$$\begin{aligned} C &= 2\pi r h P + 0.5(4\pi r^2)(2P) \\ &= 2\pi r P(h + 2r) \\ &= 2\pi r P\left(\frac{V}{\pi r^2} - \frac{2r}{3} + 2r\right) \\ &= 2\pi r P\left(\frac{V}{\pi r^2} + \frac{4r}{3}\right) \end{aligned}$$

Following our usual procedure for optimization, we take the derivative by the product rule. Recall that V and P are constants - our only variable is r.

$$\begin{aligned} C' &= 2\pi P\left(\left(\frac{V}{\pi r^2} + \frac{4r}{3}\right) + r\left(\frac{-2V}{\pi r^3} + \frac{4}{3}\right)\right) \\ &= 2\pi P\left(\left(\frac{V}{\pi r^2} + \frac{4r}{3}\right) + \left(\frac{-2V}{\pi r^2} + \frac{4r}{3}\right)\right) \end{aligned}$$

This is where we'll replace V with our original volume formula above, and simplify.

$$\begin{aligned}
C' &= 2\pi P\left(\left(h+\frac{2r}{3}+\frac{4r}{3}\right)+\left(-2\left(h+\frac{2r}{3}\right)+\frac{4r}{3}\right)\right)\\
&= 2\pi P\left((h+2r)+(-2h)\right)\\
&= 2\pi P(2r-h)
\end{aligned}$$

Setting the derivative to zero and solving gives $\frac{r}{h}=\frac{1}{2}$. One more subtlety with this problem is that we should verify that this is indeed a minimum. Plugging back in terms of r in the first derivative gives

$$\begin{aligned}
C' &= 2\pi P(2r-h)\\
&= 2\pi P(2r-(V/(\pi r^2)-2r/3))\\
&= 2\pi P(8r/3-V/(\pi r^2))\\
&= 2\pi P((8\pi r^3-3V)/(3\pi r^2))
\end{aligned}$$

Setting to zero and solving gives $r^3 = 3V/(8\pi) > 0$ (notice that r equal 0 is not admissible since our house needs a radius, so we don't need to worry about this critical point). Taking the second derivative yields

$$C'' = 8/3 + \frac{2V}{\pi r^3}$$

and since r is positive in our critical value, we see that our second derivative is positive, hence our cost function is concave up and thus this ratio of r to h is a minimum as required.

Good Luck for your exams!