Full Solutions
MATH101 April 2008

April 4, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

SOLUTION. As suggested by the hint, we have

$$\begin{aligned}\int \frac{x^3-2x}{\sqrt{x}}\,dx &= \int \frac{x^3}{\sqrt{x}}\,dx - \int \frac{2x}{\sqrt{x}}\,dx\\ &= \int x^{5/2}\,dx - 2\int x^{1/2}\,dx\\ &= \frac{2x^{7/2}}{7} - \frac{4x^{3/2}}{3} + C.\end{aligned}$$

Question 1 (b)

SOLUTION. Proceeding directly, we have

$$\begin{aligned}\int_0^{\pi} (4\sin(\theta) - 3\cos(\theta))\,d\theta &= \Big(- 4\cos(\theta) - 3\sin(\theta)\Big)\Big|_0^{\pi}\\ &= -4\cos(\pi) - 3\sin(\pi) - (-4\cos(0) - 3\sin(0))\\ &= -4(-1) - 3(0) - (-4(1) - 3(0))\\ &= 4 - 0 + 4 + 0\\ &= 8\end{aligned}$$

Question 1 (c)

SOLUTION. From the question, we immediately see that

$$\Delta x = \frac{1}{n}$$

Now, to determine x_i, notice that we have that the function is of the form

$$f(x_i) = \frac{1}{1+(\frac{i}{n})^2}.$$

The only composition of functions is underneath the square sign. So we must have that

$$f(x) = \frac{1}{1+x^2}$$

and hence that

$$x_i = \frac{i}{n}$$

As $x_i = a + i\Delta x$, we have that $a = 0$ and hence since

$$\Delta x = \frac{1}{n} = \frac{b-a}{n} = \frac{b}{n}$$

we have that $b = 1$. Combining the above gives

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1+(\frac{i}{n})^2} = \int_0^1 \frac{1}{1+x^2}\, dx$$

which completes the question.

Question 1 (d)

SOLUTION. Recall that when $n = 3$, $a = 1$, and $b = 4$ we have
$\Delta x = \dfrac{b-a}{n} = \dfrac{4-1}{3} = 1$
and that
$x_i = a + i\Delta x = 1 + i(1) = 1 + i.$
We plug in these values to see that

$$\begin{aligned} T_3 &= \frac{\Delta x}{2}\Big(f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)\Big) \\ &= \frac{1}{2}\Big(f(1+0) + 2f(1+1) + 2f(1+2) + f(1+3)\Big) \\ &= \frac{1}{2}\Big((1\cos(\pi/(1))) + 2(2\cos(\pi/2)) + 2(3\cos(\pi/3)) + (4\cos(\pi/4))\Big) \\ &= \frac{1}{2}\Big(\cos(\pi) + 4\cos(\pi/2) + 6\cos(\pi/3) + 4\cos(\pi/4)\Big) \end{aligned}$$

completing the question. Note that if we actually compute the value we get that, $T_3 = 2.414213562$ and that,

$$\int_1^4 x\cos\left(\frac{\pi}{x}\right) = 2.265092039$$

which already to this low approximation shows fairly good accuracy.

Question 1 (e)

SOLUTION. **(a)** Using the hint we see that, first of all, k needs to be a positive constant. We can calculate k explicitly by integrating:

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\, dx &= \int_{-\infty}^{0} 0\, dx + \int_0^1 kx^2(1-x)\, dx + \int_1^{\infty} 0\, dx \\ &= k\int_0^1 (x^2 - x^3)\, dx \\ &= k\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 \\ &= k\left(\frac{1}{3} - \frac{1}{4}\right) \\ &= \frac{k}{12}. \end{aligned}$$

The only way that this integral has value 1 is when $\boldsymbol{k = 12}$.

Question 1 (f)

SOLUTION. Recall the following Maclaurin series expansion

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... = \sum_{n=1}^{\infty} x^n$$

Now plugging in $x = t^8$ gives

$$\frac{1}{1-t^8} = 1 + t^8 + t^{16} + t^{24} + ... = \sum_{n=1}^{\infty} t^{8n}$$

and then multiplying by t gives

$$\frac{t}{1-t^8} = t + t^9 + t^{17} + t^{25} + ... = \sum_{n=1}^{\infty} t^{8n+1}$$

Hence, the first three nonzero terms are

$$t, t^9, t^{17}.$$

Therefore,

$$\begin{aligned}\int_0^x \frac{t}{1-t^8} \mathrm{d}t &\approx \int_0^x \left(t + t^9 + t^{17}\right) \mathrm{d}t \\ &= \left(\frac{t^2}{2} + \frac{t^{10}}{10} + \frac{t^{18}}{18}\right)\Bigg|_0^x \\ &= \frac{x^2}{2} + \frac{x^{10}}{10} + \frac{x^{18}}{18}\end{aligned}$$

produces the first three non-zero terms of the desired integral.

Question 1 (g)

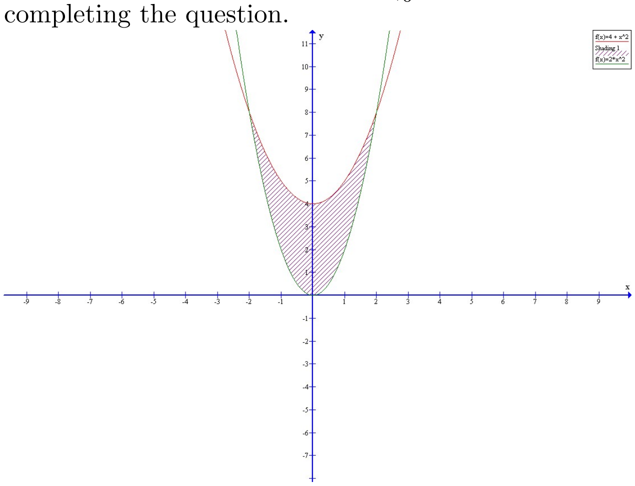
SOLUTION. Following the hint, we have

$$\begin{aligned}f(x) &= \int_x^{x^3} \sqrt{t}\sin t\, dt \\ &= \int_x^0 \sqrt{t}\sin t\, dt + \int_0^{x^3} \sqrt{t}\sin t\, dt \\ &= -\int_0^x \sqrt{t}\sin t\, dt + \int_0^{x^3} \sqrt{t}\sin t\, dt\end{aligned}$$

Now, taking derivatives by the Fundamental Theorem of Calculus I and using the chain rule on the second integral, we have
$f'(x) = -\sqrt{x}\sin x + 3x^2\sqrt{x^3}\sin(x^3)$
and this completes the question.

Question 2 (a)

SOLUTION. A picture is included below.
First, we try to find the points of intersection. To do this, make the two curves equal so
$2x^2 = 4 + x^2$
and solving for x gives
$x = \pm 2$.
Now, from the picture (or by plugging in $x = 0$ into both curves), it is clear that $4+x^2$ is the top most curve.
Hence, we need to compute
$A = \int_{-2}^{2}(4 + x^2 - 2x^2)\,dx = \int_{-2}^{2}(4 - x^2)\,dx = 2\int_0^2(4 - x^2)\,dx$
where the last equality holds above since the function is an even function over a symmetric interval. Thus,
$A = 2\int_0^2(4 - x^2)\,dx = (8x - \frac{2x^3}{3})\Big|_0^2 = 16 - \frac{16}{3} = \frac{32}{3}$
completing the question.



Question 2 (b)

SOLUTION. Let $f(x) = \frac{1}{x^p}$. Then, we wish to know when is
$V = \int_1^\infty \pi f(x)^2\,dx$
a finite number. To solve this, we compute directly that
$V = \int_1^\infty \pi f(x)^2\,dx = \pi\int_1^\infty \frac{1}{x^{2p}}\,dx = \pi\lim_{b\to\infty}\int_1^b \frac{dx}{x^{2p}}$
This integral has to be treated in two cases. One when $2p = 1$ and one when $2p \neq 1$. When $2p = 1$ we have
$V = \pi\lim_{b\to\infty}\int_1^b \frac{dx}{x^{2p}} = \pi\lim_{b\to\infty}\ln|x|\big|_1^b = \lim_{b\to\infty}\pi\ln|b|$
and this diverges. When $2p \neq 1$, we have

$$\begin{aligned}
V &= \pi\lim_{b\to\infty}\int_1^b \frac{dx}{x^{2p}}\\
&= \pi\lim_{b\to\infty}\frac{x^{1-2p}}{1-2p}\Big|_1^b\\
&= \lim_{b\to\infty}\frac{\pi b^{1-2p}}{1-2p} - \frac{\pi}{1-2p}
\end{aligned}$$

Now, the above limit converges when $1 - 2p < 0$ and diverges when $1 - 2p > 0$ (this is because the term will appear in the denominator in the first case and in the numerator in this second case). Thus, we get convergence only when $\frac{1}{2} < p$ as required.

Question 2 (c)

SOLUTION 1. The first step to solve this problem is to understand what volume we are trying to compute. The question states that there is a region bounded by the curves y=5 and y=x+4/x. For large values of x the second curve looks like the line y=x, but there is also a vertical asymptote at x=0. So it seems that it will indeed cross the horizontal line y=5 twice (as it dips down before going back up). We can compute where the intersections are by solving:

$$x + \frac{4}{x} = 5$$

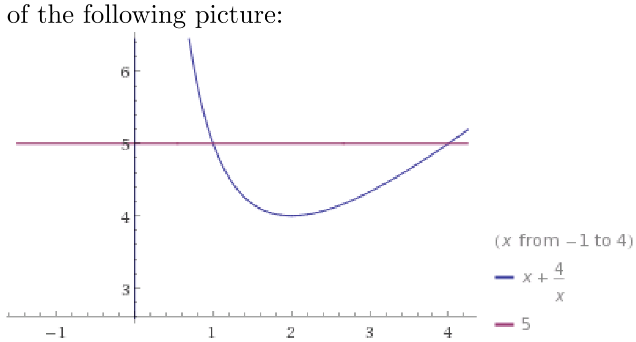
which is equivalent to

$$0 = x^2 - 5x + 4 = (x-1)(x-4)$$

and has solutions x=1 and x=4. Then we want to know how far down the curve goes, so we look for its local minimum. We compute its derivative:

$$\frac{d}{dx}\left(x + \frac{4}{x}\right) = 1 - \frac{4}{x^2}$$

and hence see that it has two critical points, one at x=-2 and one at x=2. We are only interested in the local minimum which is at x=2. This should be enough details to have you be able to obtain the equivalent of the following picture:



This in particular shows that y=5 is bigger than y=x+4/x.
Now, consider a point x between 1 and 4. The distance from x to x=-1 is the radius of the cylindrical shell and is r=x+1. The height of the function at this point is the difference of the top function y=5 to the bottom function y=x+4/x which is 5-x-4/x. When we rotate, we travel a distance of $2\pi r$ (the circumference) and we are multiplying by $h\Delta x$. Thus, we have

$$
\begin{aligned}
V &= \int_1^4 2\pi(x+1)(5-x-4/x)\,dx \\
&= 2\pi\int_1^4 (5x - x^2 - 4 + 5 - x - 4/x)\,dx \\
&= 2\pi\int_1^4 (-x^2 + 4x + 1 - 4/x)\,dx \\
&= 2\pi(-x^3/3 + 2x^2 + x - 4\ln|x|)\Big|_1^4 \\
&= 2\pi((-(4)^3/3 + 2(4)^2 + 4 - 4\ln|4|) - (-(1)^3/3 + 2(1)^2 + 1 - 4\ln|1|)) \\
&= 2\pi((-64/3 + 32 + 4 - 4\ln|4|) - (-1/3 + 2 + 1 - 0)) \\
&= 2\pi(12 - 4\ln|4|) \\
&= 2\pi(12 - 8\ln(2)) \\
&= 8\pi(3 - 2\ln(2)) \\
&= 24\pi - 16\pi\ln(2)
\end{aligned}
$$

Where each of the last four answers are equivalent and would be accepted as a final answer.

Solution 2. The first step to solve this problem is to understand what volume we are trying to compute. The question states that there is a region delimited by the curves y=5 and y=x+4/x. The second curve looks like the line y=x except that there is a vertical asymptote at x=0. So it seems that it will indeed cross the horizontal line y=5 twice (as it dips down before going back up). We can compute where the intersections are by solving:

$$x + \frac{4}{x} = 5$$

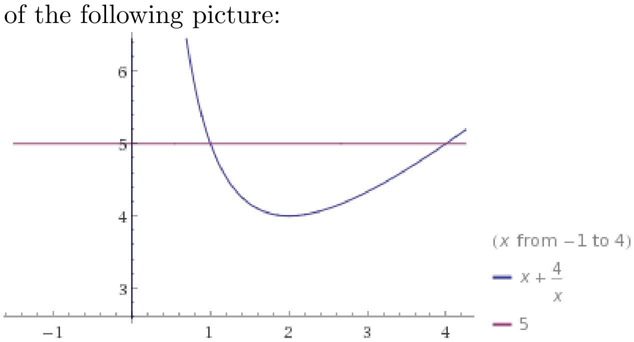
which is equivalent to

$$x^2 - 5x + 4 = 0$$

and has solutions x=1 and x=4. Then we want to know how far down the curve goes, so we look for its local minimum. We compute its derivative:

$$\frac{d}{dx}\left(x + \frac{4}{x}\right) = 1 - \frac{4}{x^2}$$

and hence see that it has two critical points, one at x=-2 and one at x=2. We are only interested in the local minimum which is at x=2. This should be enough details to have you be able to obtain the equivalent of the following picture:



Now we want to rotate this region around the vertical axis x=-1. To do this, we will use the method know as disk integration. The idea is fairly simple: imagine the solid viewed from the top (looking down at the axis), if we slice it along that axis, we will see a slice of a hollow cylinder. We will compute the area of that slice (the difference of area between the outer disk and the inner disk) and integrate.
The outer disc has a radius that starts at the point (2,4) and moves right towards the point (4,5) following the curve y=x+4/x. The inner disc has a radius that starts at the point (2,4) and moves left towards the point (1,5) following the other side of the curve. We need to compute that radius (starting at x=-1). For this, the first step is to express the curve in terms of y instead of x:

$$\begin{aligned} y = x + \frac{4}{x} &\iff yx = x^2 + 4 \\ &\iff x^2 - yx + 4 = 0 \\ &\iff x = \frac{y \pm \sqrt{y^2 - 16}}{2} \end{aligned}$$

If we look at this, it makes sense: This is only well defined if y is larger than 4 (see on the graph) and there are two components: the one with the + for the right side and the one with the - for the left side. Since the axis of rotation is at x=-1, we need to add 1 to each of the radius and obtain:

$$\begin{aligned} r_{outer}(y) &= 1 + \frac{y + \sqrt{y^2 - 16}}{2} \\ r_{inner}(y) &= 1 + \frac{y - \sqrt{y^2 - 16}}{2} \end{aligned}$$

And so we can write the volume by integrating over all the shells, which are obtained by taking the difference of the outer and inner disk:

$$V = \int_4^5 \left(\pi \left(1 + \frac{y}{2} + \frac{\sqrt{y^2 - 16}}{2}\right)^2 - \pi \left(1 + \frac{y}{2} - \frac{\sqrt{y^2 - 16}}{2}\right)^2 \right) dy$$

(since the area of a disk of radius r is π r^2).
Now we can do some algebra to rewrite this integral into something easier to handle. Notice the pattern here

$$(A + B)^2 - (A - B)^2 = 4AB$$

Hence we can rewrite this as

$$\begin{aligned} V &= \pi \int_4^5 4(1 + \frac{y}{2})(\frac{\sqrt{y^2 - 16}}{2})\, dy \\ &= \pi \int_4^5 (2 + y)(\sqrt{y^2 - 16})\, dy \end{aligned}$$

Now we need to compute that integral to obtain the desired volume. Unfortunately, this won't be straightforward since (2+y) isn't a multiple of the derivative of what is in the square root. There are many ways to compute this integral, here we will split it in 2 and deal with each term.

$$V = \pi \int_4^5 2\sqrt{y^2 - 16}\, dy + \pi \int_4^5 y\sqrt{y^2 - 16}\, dy$$

Let us call the first integral V$_1$ and the second one V$_2$. The first one is the tougher one. Indeed:

$$
\begin{aligned}
V_2 &= \pi \int_4^5 y\sqrt{y^2-16}\,dy \\
&= \frac{\pi}{2}\int_4^5 2y\sqrt{y^2-16}\,dy \\
&= \frac{\pi}{2}\left[\frac{2}{3}(y^2-16)^{3/2}\right]_4^5 \\
&= \frac{\pi}{2}\frac{2}{3}(25-16)^{3/2} \\
&= \frac{\pi}{3}\cdot 27 \\
&= 9\pi
\end{aligned}
$$

The first integral is actually quite complicated. At the same time it is a classic. As of today (2012) you are not expected to be able to integrate this by hand and/or know the formula from memory. For completion sake, here is the formula:

$$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln(x+\sqrt{x^2-a^2}) + c$$

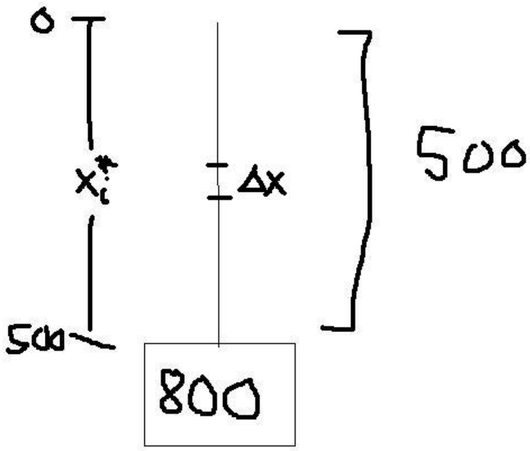
Which in this case gives us:

$$
\begin{aligned}
V_1 &= \pi \int_4^5 2\sqrt{y^2-16}\,dy \\
&= 2\pi\left[\frac{y}{2}\sqrt{y^2-16} - 8\ln(y+\sqrt{y^2-16})\right]_4^5 \\
&= 2\pi\left[(\frac{5}{2}\sqrt{9} - 8\ln(5+\sqrt{9}) - (0 - 8\ln(4+0))\right] \\
&= 2\pi(\frac{15}{2} - 8\ln(8) + 8\ln(4) \\
&= \pi(15 - 48\ln(2) + 32\ln(2)) \\
&= \pi(15 - 16\ln(2))
\end{aligned}
$$

Summing $V = V_1 + V_2$ gives $V = 24\pi - 16\pi\ln(2)$ which is the same as above.
... This is why using the correct method of solids of revolution is very important.

Question 2 (d)

SOLUTION. A picture is included below.
Let x_i^* be a sample point. Then the work done on this ith part in foot-pounds is
$W_i = 2\Delta x \cdot x_i^*$
where the force is the $2\Delta x$ part and comes from the fact that since the cable weights 2 pounds per foot and the delta x is how much of the cable we are using in the ith part. The x_i^* part is the distance. Hence, the work done is
$W_{cable} = \lim_{n\to\infty}\sum_{i=1}^n 2x_i^*\Delta x = \int_0^{500} 2x\,dx = x^2\big|_0^{500} = 250000$
However, this does not include the work done to lift the coal itself. This undergoes
$W_{coal} = 800 * 500 = 400000$
and so the total work done is
$W_{total} = 400000 + 250000 = 650000$
where this is measured in foot pounds. This completes the proof.



Question 3 (a)

SOLUTION. We follow the hints and first rewrite the integral.

$$\int_1^2 \frac{e^{1/x}}{x^2}\,dx = \int_1^2 x^{-2}e^{x^{-1}}\,dx.$$

Now we substitute

$$u = x^{-1}, \quad \frac{du}{dx} = -x^{-2}, \quad dx = -x^2\,du.$$

Further, for $x = 1$ we have $u = 1$, and for $x = 2$ we have $u = 1/2$. Therefore,

$$\begin{aligned}\int_1^2 x^{-2}e^{x^{-1}}\,dx &= \int_1^{1/2} x^{-2}e^u(-x^2\,du)\\ &= -\int_1^{1/2} e^u\,du\\ &= \int_{1/2}^1 e^u\,du\\ &= e^u\big|_{1/2}^1\\ &= e - e^{1/2}.\end{aligned}$$

Question 3 (b)

SOLUTION. As in the hint, let $u = \sqrt{x}$ so that $du = \frac{dx}{2\sqrt{x}} = \frac{dx}{2u}$. This gives
$\int \cos\sqrt{x}\,dx = \int 2u\cos(u)\,du$
To solve this last integral, we use integration by parts. Let

$$\begin{aligned} w &= u & v &= \sin(u)\\ dw &= du & dv &= \cos(u)du\end{aligned}$$

Then, we have

$$\begin{aligned}\int 2u\cos(u)\,du &= 2\int u\cos(u)\,du\\ &= 2u\sin(u) - 2\int \sin(u)\,du\\ &= 2u\sin(u) + 2\cos(u)\end{aligned}$$

and hence

$$\begin{aligned}\int \cos(\sqrt{x})\,dx = \int 2u\cos(u)\,du &= 2u\sin(u) + 2\cos(u)\\ &= 2\sqrt{x}\sin(\sqrt{x}) + 2\cos(\sqrt{x})\end{aligned}$$

Question 3 (c)

SOLUTION. We recognize a denominator of the form $x^2 + a^2$ and so we attempt a trigonometric substitution of the form

$$x = 2\tan\theta, \quad dx = 2\sec^2\theta\,d\theta.$$

Then the integral becomes

$$\begin{aligned}\int \frac{dx}{x(x^2+4)} &= \int \frac{2\sec^2\theta\,d\theta}{2\tan\theta\,\underbrace{(4\tan^2\theta + 4)}_{=4\sec^2\theta}}\\ &= \int \frac{d\theta}{4\tan\theta} = \frac{1}{4}\int \frac{\cos\theta\,d\theta}{\sin\theta}.\end{aligned}$$

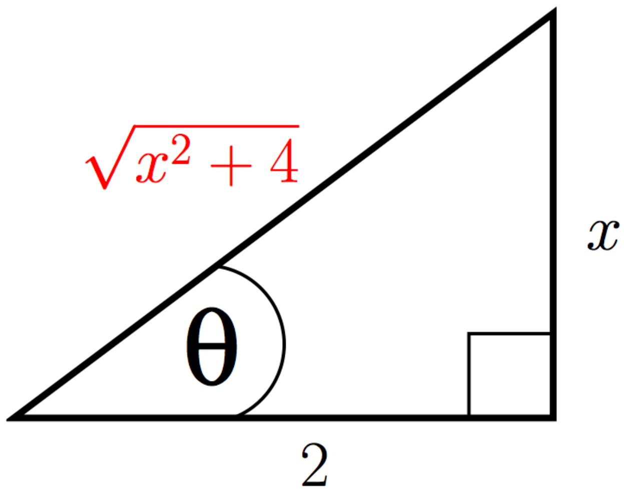
To solve this integral we use another substitution:

$$u = \sin\theta, \quad du = \cos\theta\,d\theta,$$

to obtain

$$\begin{aligned}\frac{1}{4}\int \frac{\cos\theta\,d\theta}{\sin\theta} &= \frac{1}{4}\int \frac{\cos\theta\,du}{u\cos\theta}\\ &= \frac{1}{4}\int \frac{du}{u} = \frac{1}{4}\ln|u| + C\\ &= \frac{1}{4}\ln|\sin\theta| + C.\end{aligned}$$

To revert back to an expression with x we have to calculate $\sin\theta$.



So we use a reference triangle. The sides x and 2 are given from the substitution $\tan\theta = x/2$. With Pythagoras we can then calculate the length of the remaining side of the triangle, which is $\sqrt{x^2+4}$. Hence our final answer is

$$\int \frac{dx}{x(x^2+4)} = \frac{1}{4}\ln|\sin\theta| + C = \frac{1}{4}\ln\left|\frac{x}{\sqrt{x^2+4}}\right| + C.$$

Question 3 (d)

SOLUTION. We use the trigonometric substitution

$$x = 4\tan\theta, \quad dx = 4\sec^2\theta\, d\theta$$

and obtain

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2+16}} &= \int \frac{4\sec^2\theta\, d\theta}{\sqrt{16\tan^2\theta+16}}\\ &= \int \frac{4\sec^2\theta\, d\theta}{4\sec\theta}\\ &= \int \sec\theta\, d\theta.\end{aligned}$$

To solve this integral we substitute

$$u = \sec\theta + \tan\theta, \quad du = (\tan\theta + \sec\theta)\sec\theta\, d\theta,$$

and obtain

$$\begin{aligned}\int \sec\theta\, d\theta &= \int \frac{\sec\theta\, du}{(\tan\theta+\sec\theta)\sec\theta}\\ &= \int \frac{du}{u} = \ln|u|\\ &= \ln|\sec\theta + \tan\theta| + C.\end{aligned}$$

We use the first substitution $\tan\theta = x/4$ to find that $\cos\theta = \frac{4}{\sqrt{x^2+16}}$ so that our final answer becomes

$$\int \frac{dx}{\sqrt{x^2+16}} = \ln\left|\frac{\sqrt{x^2+16}}{4} + \frac{x}{4}\right| + C.$$

Note that $\ln\left|\frac{\sqrt{x^2+16}}{4} + \frac{x}{4}\right| + C$ *can also be written as* $\sinh^{-1}\left(\frac{x}{4}\right) + C$*, the inverse hyperbolic sine function.*

Question 4 (a)

SOLUTION. Setting up the auxiliary polynomial and computing the roots gives
$2x^2 + 5x + 3 = (2x+3)(x+1)$
and so the roots are $x = -\frac{3}{2}$ and $x = -1$. Thus, the general solution to this differential equation is

$$y = c_1 e^{-(3/2)x} + c_2 e^{-x}$$

Now, we plug in the initial conditions. Finding the derivative, we have
$y' = -\frac{3c_1}{2}e^{-3/2x} - c_2 e^{-x}$
At the initial condition $y'(0) = -4$, we have
$-4 = y'(0) = -\frac{3c_1}{2}e^{-3/2(0)} - 3c_2 e^{-(0)} = -\frac{3c_1}{2} - c_2$
Clearing denominators and simplifying yields
$8 = 3c_1 + 2c_2$
At the initial condition $y(0) = 3$, we have
$3 = y(0) = c_1 e^{-3/2(0)} + c_2 e^{-3(0)} = c_1 + c_2$
We solve for the constants. Taking -3 times the second equation and adding to the first yields
$-1 = -c_2$
and so $c_2 = 1$. Taking -2 times the second equation and adding it to the first yields
$2 = c_1$
and so $c_1 = 2$. Thus, the solution is

$$y = 2e^{-3/2x} + e^{-3x}$$

completing the question.

Question 4 (b)

SOLUTION. We proceed as in the hint. Multiplying both sides by e^{-x} gives
$e^{-x}y'' - e^{-x}y' = e^{-x}\sin(2x)$
Now, the left hand side is just $(e^{-x}y')'$ and so
$(e^{-x}y')' = e^{-x}\sin(2x)$
Integrating both side, we see that the integral of the right hand side above is difficult - so we compute it separately. We proceed by integration by parts. Let

$$\begin{aligned} u &= e^{-x} & v &= -\frac{\cos(2x)}{2} \\ du &= -e^{-x}dx & dv &= \sin(2x)dx \end{aligned}$$

and so the integral becomes

$$\int e^{-x}\sin(2x)\,dx = -e^{-x}\frac{\cos(2x)}{2} - \frac{1}{2}\int e^{-x}\cos(2x)\,dx$$

We use parts again. Let

$$u = e^{-x} \qquad v = \frac{\sin(2x)}{2}$$
$$du = -e^{-x}dx \qquad dv = \cos(2x)dx$$

Then, we have

$$\begin{aligned}\int e^{-x}\sin(2x)\,dx &= -e^{-x}\frac{\cos(2x)}{2} - \frac{1}{2}\int e^{-x}\cos(2x)\,dx \\ &= -e^{-x}\frac{\cos(2x)}{2} - \frac{1}{2}\left(\frac{e^{-x}\sin(2x)}{2} + \frac{1}{2}\int e^{-x}\sin(2x)\,dx\right) \\ &= -e^{-x}\frac{\cos(2x)}{2} - \frac{e^{-x}\sin(2x)}{4} - \frac{1}{4}\int e^{-x}\sin(2x)\,dx\end{aligned}$$

This last integral on the right is the same as the integral on the left (up to a constant). Hence, we have

$$\begin{aligned}\int e^{-x}\sin(2x)\,dx + \frac{1}{4}\int e^{-x}\sin(2x)\,dx &= -e^{-x}\frac{\cos(2x)}{2} - \frac{e^{-x}\sin(2x)}{4} + C \\ \frac{5}{4}\int e^{-x}\sin(2x)\,dx &= -e^{-x}\frac{\cos(2x)}{2} - \frac{e^{-x}\sin(2x)}{4} + C \\ \int e^{-x}\sin(2x)\,dx &= -\frac{2e^{-x}\cos(2x)}{5} - \frac{e^{-x}\sin(2x)}{5} + D\end{aligned}$$

where $D = \frac{4C}{5}$.
After this gigantic diversion, we return to the point. We had
$(e^{-x}y')' = e^{-x}\sin(2x)$
and integrating both sides, we get
$e^{-x}y' = \int e^{-x}\sin(2x)\,dx = -\frac{2e^{-x}\cos(2x)}{5} - \frac{e^{-x}\sin(2x)}{5} + D$
Dividing both sides by e^{-x}, we have
$y' = -\frac{2\cos(2x)}{5} - \frac{\sin(2x)}{5} + De^{x}$
Again integrating both sides gives

$$\begin{aligned}y &= \int -\frac{2\cos(2x)}{5} - \frac{\sin(2x)}{5} + De^{x}\,dx \\ &= \frac{-\sin(2x)}{5} + \frac{\cos(2x)}{10} + De^{x} + E\end{aligned}$$

where E is another constant. This is the general solution.

Question 5 (a)

SOLUTION. As suggested in the hint, we know that
$\cos(y) = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!}\ldots = \sum_{n=0}^{\infty}\frac{(-1)^n y^{2n}}{(2n)!}$
Plugging in $y = x^2$ gives
$\cos(x^2) = 1 - \frac{x^4}{2} + \frac{x^8}{24} - \frac{x^{12}}{720} + \ldots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{4n}}{(2n)!}$
Now, integrating gives
$\int\cos(x^2) = C + x - \frac{x^5}{(2)(5)} + \frac{x^9}{(24)(9)} + \frac{x^{13}}{(720)(13)} + \ldots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+1}}{(4n+1)(2n)!}$

Now, using just the first three terms and integrating from 0 to 1 gives
$\int_0^1 \cos(x^2) = 1 - \frac{1}{10} + \frac{1}{216}$
This expansion is correct to within 0.001 which we can show by using the alternating series estimation theorem. We know that
$b_n = \frac{1}{(4n+1)(2n)!}$
is decreasing. The terms also limit to 0 and clearly this is an alternating series. Hence, the error above using three terms is no worse than what the fourth term tells us it is. Hence, the error is bounded above by
$Error \leq \frac{1}{(720)(13)} < \frac{1}{(100)(10)} = \frac{1}{1000} = 0.001$
and this completes the question.

Question 5 (b)

Solution. The question tells us that we can choose $K = 60$ in our solution. Plugging this in, it suffices to find an n so that
$\frac{K(b-a)^5}{180n^4} = \frac{60}{180n^4} = \frac{1}{3n^4} \leq 0.001 = \frac{1}{1000}$
Cross multiplying yields
$1000 \leq 3n^4$
dividing both sides by *3* gives
$333.33333... = \frac{1000}{3} \leq n^4$
To find the smallest value for n, we start enumerating the first few even integers to the power of *4*.
$2^4 = 16 \qquad 4^4 = 256 \qquad 6^4 = 1296$
so we see that the first even integer n so that $n^4 \geq \frac{1000}{3}$ is $n = 6$ completing the proof.

Question 6

Solution. We will follow the strategy suggested in the hints above.
Step 1. First, we compute $A(y)$. If you look at the cup from the side, you can see a triangle with a base of 6 and a height of 10. The coffee level is a subtriangle (that is similar one) at height y and base length say $b(y)$. Since these are similar triangles, we obtain that

$$\frac{y}{b(y)} = \frac{10}{6}$$

And so the area at height y is

$$A(y) = (b(y))^2 = \left(\frac{6y}{10}\right)^2 = \frac{9y^2}{25}$$

Step 2. Now, we can rewrite Toricelli's Law and go at solving the differential equation:

$$A(y)\frac{dy}{dt} = k\sqrt{y}$$

becomes

$$\frac{9y^2}{25}\frac{dy}{dt} = k\sqrt{y}$$

We rearrange the terms to obtain

$$\frac{9y^{3/2}}{25}\frac{dy}{dt} = k$$

and we integrate with respect of t

$$\int \frac{9y^{3/2}}{25}\frac{dy}{dt}\,dt = \int k\,dt$$

On the left side, it is a change of variable, so we can integrate with respect to y

$$\int \frac{9y^{3/2}}{25}\,dy = \int k\,dt$$

And now, computing each integral separately, we obtain the equation:

$$\frac{9}{25}\frac{2}{5}y^{5/2} = kt + c$$

for some constant c to determine. Rearranging the equation to write it as a function of y we obtain

$$y(t) = \left(\frac{125}{18}(kt + c)\right)^{2/5}$$

Step 3. Now, we can use the given information, that can be described as:

$$y(0) = 10 \quad \text{and} \quad y(10) = 5$$

The first one yields:

$$10 = \left(\frac{125}{18}(0 + c)\right)^{2/5}$$

And rearranging we can solve for c and find:

$$c = \frac{18}{125}10^{5/2}$$

This allows us to rewrite the function for y as:

$$y(t) = \left(\frac{125}{18}kt + 10^{5/2}\right)^{2/5}$$

*(We all would love to have that 10 come out with the powers that look like they would nicely simplify, but this is **really not** possible, so we continue.)*
Now, the second information $y(10) = 5$ will allow us to compute the value of k

$$5 = \left(\frac{125}{18}10k + 10^{5/2}\right)^{2/5}$$

We do the algebra and get

$$\begin{aligned} k &= \frac{18(5^{5/2} - 10^{5/2})}{(10)(125)} = \frac{9(5^{5/2} - (2^{5/2})(5^{5/2}))}{5(5^3)} \\ &= 9 \cdot 5^{-3/2}(1 - 2^{5/2}) \end{aligned}$$

"(Yes, it is a negative number, but it does make a lot of sense, since we expect dy/dt to be negative.)"
And so we have finally solved the differential equation and have now a "beautiful" expression for $y(t)$ (after doing a little algebra to simplify it):

$$y(t) = \left(\frac{5^{3/2}}{2}(1 - 2^{5/2})t + 10^{5/2}\right)^{2/5}$$

Step 4. And we can solve the problem which is just to find which value of t yields y=0, so we just have to solve the equation:

$$0 = \left(\frac{5^{3/2}}{2}(1 - 2^{5/2})t + 10^{5/2}\right)^{2/5}$$

And by raising to the power 5/2

$$0 = \frac{5^{3/2}}{2}(1 - 2^{5/2})t + 10^{5/2}$$

And obtain

$$t = -\frac{10^{5/2}}{\frac{5^{3/2}}{2}(1 - 2^{5/2})} = \frac{5 \cdot 2^{7/2}}{2^{5/2} - 1} = \frac{40\sqrt{2}}{4\sqrt{2} - 1}$$

This last answer is as good as you can get without a calculator (you won't have one in the exam). Stopping anywhere earlier would work too, so if your simplification skills aren't great, it isn't a worry here, though it might make your life really hard earlier in the problem.
And since you asked gently, the numerical value of that thing is around 12.147, which makes a bunch of sense given the way y decrease.

Question 7 (a)

Solution 1. We proceed as in the hints. First, we isolate for y to see that
$y^2 = b^2 - \frac{b^2x^2}{a^2} = \frac{b^2}{a^2}\left(a^2 - x^2\right)$
Taking the square roots yields
$y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$
Now, we want to know the aera fo the ellipse. First, we set $y = 0$ to find the points of intersection with the x-axis. This gives
$\frac{x^2}{a^2} - 0 = 1$
So $x = \pm a$. Thus, we wish to integrate
$y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$
from $-a$ to a. By symmetry, we need only to integrate the positive function above and then double our resulting value (pictorially, the ellipse has a symmetry that is reflection about the x-axis. So to get the full area, compute half of it then double your result). Thus, we have that
$A = 2\int_{-a}^{a} \frac{b}{a}\sqrt{a^2 - x^2}\, dx = \frac{2b}{a}\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx$

This last integral can be interpreted as the area of a half circle of radius a. This has area $\dfrac{\pi a^2}{2}$. Thus, our area is
$A = \frac{2b}{a}\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx = \frac{2b}{a} \cdot \frac{\pi a^2}{2} = \pi ab$
completing the question.

Solution 2. We proceed as in solution one until the line

$A = \frac{2b}{a}\int_{-a}^{a}\sqrt{a^2-x^2}\,dx$
Now we do a trig substitution. Let $x = a\sin\theta$ so that $dx = a\cos\theta d\theta$. Changing the bounds, we see that $a = a\sin(\theta)$ so $\theta = \frac{\pi}{2}$ and $-a = a\sin(\theta)$ so $\theta = -\frac{\pi}{2}$. Then the above becomes

$$\begin{aligned} A &= \frac{2b}{a}\int_{-a}^{a}\sqrt{a^2-x^2}\,dx \\ &= \frac{2b}{a}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{a^2-a^2\sin^2\theta}a\cos\theta\,d\theta \\ &= 2ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2\theta\,d\theta \end{aligned}$$

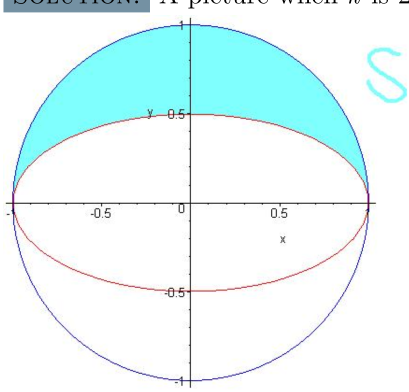
Now we use the double angle formula
$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$
to see that

$$\begin{aligned} A &= 2ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2(\theta)\,d\theta \\ &= 2ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1+\cos(2\theta)}{2}\,d\theta \\ &= ab\theta|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos(2\theta)\,d\theta \\ &= \frac{\pi ab}{2} + \frac{\pi ab}{2} + \frac{ab}{2}\sin(2\theta)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= \pi ab + \frac{ab}{2}(\sin(\pi) - \sin(-\pi)) \\ &= \pi ab \end{aligned}$$

completing the question.

Question 7 (b)

Solution. A picture when k is 2 is drawn below.



To compute the area of S, we actually don't need to do any more work. Notice that the area of the blue circle when y is greater than 0 is half the area of the unit circle which is $\frac{\pi}{2}$. Now, the area of the ellipse when y is greater than 0 is again half the area of the ellipse, which by the previous problem is $\frac{\pi(1)}{2}(\frac{1}{k}) = \frac{\pi}{2k}$. Hence, the area of S is
$A = \frac{\pi}{2} - \frac{\pi}{2k} = \frac{\pi}{2}\left(1 - \frac{1}{k}\right)$.
Now, notice that when x is 0, then the two points on the ellipse are $y = \pm\frac{1}{k}$. So the one lying in the upper half of the plane is $y = \frac{1}{k}$. The question asks when is the centroid outside of E (so inside S). By symmetry, the x coordinate of the centroid is 0. So all we need to do is figure out when $\bar{y} \geq \frac{1}{k}$ (NOTE: There is a bit of ambiguity in the sense that is something on the boundary of S and E inside S or outside of E - I suspect students using greater than or equal to vs just greater than would earn full marks but it is something we should be a bit careful with). In any case, we compute when

$$\frac{1}{k} \leq \bar{y} = \frac{1}{2A}\int_a^b (f(x)^2 - g(x)^2)\,dx$$

Here, the upper curve is the positive half of the circle so
$f(x) = \sqrt{1-x^2}$
and the lower curve is the upper half of the ellipse so
$g(x) = \sqrt{\frac{1-x^2}{k^2}}$.
The picture also clearly shows that we are integrating from $-1 \leq x \leq 1$ (we could have explicitly found these points of intersection as well). Thus, we have

$$\begin{aligned}\frac{1}{k} \leq \bar{y} &= \frac{1}{2A}\int_a^b (f(x)^2 - g(x)^2)\,dx \\ &= \frac{1}{2\cdot\frac{\pi}{2}\left(1-\frac{1}{k}\right)}\int_{-1}^1 (1 - x^2 - \frac{1-x^2}{k^2})\,dx \\ &= \frac{1}{\pi\cdot\frac{k-1}{k}}\int_{-1}^1 (1-x^2)\left(1 - \frac{1}{k^2}\right))\,dx\end{aligned}$$

where in the last step we factored out a $1 - x^2$. Pulling out the terms depending only on k in the integral and using the fact that the integral is the integral of an even function over a symmetric interval yields

$$\begin{aligned}\frac{1}{k} \leq \bar{y} &= \frac{1}{\pi\cdot\frac{k-1}{k}}\left(1-\frac{1}{k^2}\right)2\int_0^1 (1-x^2))\,dx \\ &= \frac{2k}{\pi(k-1)}\cdot\frac{k^2-1}{k^2}\left(x - \frac{x^3}{3}\right)\Big|_0^1 \\ &= \frac{2k}{\pi(k-1)}\cdot\frac{(k+1)(k-1)}{k^2}\left(1-\frac{1}{3}\right) \\ &= \frac{4(k+1)}{3\pi k}\end{aligned}$$

Using the inequality derived above, we have

$$\begin{aligned}\frac{1}{k} &\leq \frac{4(k+1)}{3\pi k} \\ 3\pi &\leq 4(k+1) \\ \frac{3\pi-4}{4} &\leq k\end{aligned}$$

and this completes the proof.

Good Luck for your exams!