Full Solutions
MATH200 April 2012

April 4, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

SOLUTION. Let the line L be written in vector form as

$$[a_0, b_0, c_0] + [a_1, b_1, b_2]t.$$

If L is parallel to the plane 2x + y - z = 5, then L must be perpendicular to the normal vector of said plane, n = [2,1,-1]. i.e. The dot product between n and the direction of L must be equal to zero:

$$[2, 1, -1] \cdot [a_1, b_1, c_1] = 0.$$

Evaluating the dot product gives:

$$(\mathbf{1}) \quad [2, 1, -1] \cdot [a_1, b_1, c_1] = 2a_1 + b_1 - c_1 = 0$$

Remember that we also need the vector is perpendicular to the line,$[3 - t, 1 - 2t, 3t] = [3, 1, 0] + [-1, -2, 3]t$. Hence, the following equation must also be satisfied:

$$(\mathbf{2}) \quad [-1, -2, 3] \cdot [a_1, b_1, c_1] = -a_1 - 2b_1 + 3c_1 = 0$$

So we have two equations and three unknowns. This means there are infinitely many vectors that will satisfy the given conditions. We only need to find one. (Note: Clearly $a_1 = b_1 = c_1 = 0$ is a solution to the above equations (1), (2), but it is the trivial solution and is perpendicular to every vector, including the direction of the line L.)
Solving (1) for c_1 gives $c_1 = 2a_1 + b_1$. Subbing this into (2) gives

$$-a_1 - 2b_1 + 3(2a_1 + b_1) = 0 \quad \rightarrow \quad b_1 = -5a_1.$$

From this we get that $c_1 = 2a_1 - 5a_1 = -3a_1$.
Thus, any vector of the form k[1,-5,-3] where k is a constant (not equal to zero) will be parallel to L. For example, the vectors [1,-5,-3] and [-2,10,6] are acceptable solutions.

Question 1 (b)

SOLUTION. To find the line to the line L, we must find a, b, c such that the point $Q = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ has distance 2 to the xy-plane.
This means $|c| = 2$. Since $c > 0$, we have $c = 2$.
The distance from Q to the xz-plane is 3, hence $|b| = 3$. With $b > 0$, we have $b = 3$.
The distance from Q to the yz-plane is 4, hence $|a| = 4$. With $a < 0$, we have $a = -4$.
Hence $Q = \begin{pmatrix} -4 \\ 3 \\ 2 \end{pmatrix}$ and $L = \begin{pmatrix} -4 \\ 3 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix}$.

Question 2 (a)

SOLUTION. The linear approximation to a multivariable function f(x,y) at a point (x,y) = (a,b) is given by L(x,y) where

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

where the subscripts denote partial differentiation with respect to the given variable. In this case the partial derivatives of $f(x, y) = \ln(4x^2 + y^2)$ are

$$f_x(x, y) = \frac{8x}{4x^2 + y^2}, \quad f_y(x, y) = \frac{2y}{4x^2 + y^2}.$$

and we wish to approximate f near (x,y) = (0,1). Hence the linear approximation we seek is:

$$\begin{aligned} L(x, y) &= \ln(1) + 0 \cdot (x - 0) + 2(y - 1) \\ &= 2y - 2 \end{aligned}$$

Using the linear approximation, we can estimate the value of f(0.1,1.2) as

$$f(0.1, 1.2) \approx L(0.1, 1.2) = 0.4.$$

Question 2 (b)

SOLUTION. Two planes are parallel if their normal vectors are parallel. Hence, we will evaluate the normal vectors for the given plane and the function f to determine the points where they are parallel.
By inspection of the equation of the plane 2x + 2y - z = 3, we can see that is normal vector is $\mathbf{n}_1 = [2, 2, -1]^T$. The normal vector to the function f at the point (x,y) is given by computing the gradient to F(x,y,z) = f(x,y) - z:

$$\nabla F(x, y, z) = [F_x(x, y, z), F_y(x, y, z), F_z(x, y, z)]^T = \left[\frac{8x}{4x^2 + y^2}, \frac{2y}{4x^2 + y^2}, -1\right]^T.$$

Hence, to determine points where the normal vectors are parallel, we need to solve the following two equations for (x,y):

$$\frac{8x}{4x^2 + y^2} = 2, \quad \frac{2y}{4x^2 + y^2} = 2$$

Taking the difference of the two equations gives us the condition y = 4x. Taking this result and plugging it into the first equation above gives:

$$\begin{aligned} \frac{8x}{4x^2 + y^2} = 2 \quad \rightarrow \quad 8x &= 8x^2 + 2y^2 \\ 8x &= 8x^2 + 32x^2 \\ x &= 5x^2 \end{aligned}$$

This equation has two solutions: x = 0 and x = 1/5. Notice that the x = 0 is invalid since the corresponding value of y is y = 4x = 0 and f is undefined at (0,0). Therefore, the only valid solution is x = 1/5 which corresponds to y = 4x = 4/5. The corresponding value of z is z = f(1/5,4/5) = ln(4/5).
Therefore, the tangent plane to f is parallel to 2x + 2y - z = 3 at the point P(a,b,c) = (1/5,4/5,ln(4/5)).

Question 3

SOLUTION. First we evaluate the first derivative of z with respect to t using the multivariable chain rule.

$$\frac{dz}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} \qquad = f_x \cdot 4t + f_y \cdot 3t^2$$

We then take the next derivative of z by taking the derivative with respect to t of the above result, using the multivariable chain rule again as well as the product rule.

$$\begin{aligned}\frac{d^2z}{dt^2} &= \frac{d}{dt}\left(f_x \cdot 4t + f_y \cdot 3t^2\right) \\ &= 4t(f_{xx} \cdot 4t + f_{xy} \cdot 3t^2) + 4f_x + 3t^2(f_{yx} \cdot 4t + f_{yy}3t^2) + 6t \cdot f_y\end{aligned}$$

Next, we recognize that if t = 1, then (x,y) = (2,1). Thus, we can use the information given in the question about the partial derivatives of f to evaluate d^2z/dt^2 at t = 1:

$$\begin{aligned}\frac{d^2z}{dt^2} &= 4t(f_{xx} \cdot 4t + f_{xy} \cdot 3t^2) + 4f_x + 3t^2(f_{yx} \cdot 4t + f_{yy}3t^2) + 6t \cdot f_y \\ \left.\frac{d^2z}{dt^2}\right|_{t=1} &= 4(2(4) + 3) + 4(5) + 3(4 - 4(3)) + 6(-2) \\ &= 4(11) + 20 + 3(-8) - 12 = 28\end{aligned}$$

Therefore, $\left.\frac{d^2z}{dt^2}\right|_{t=1} = 28$.

Question 4 (a)

SOLUTION. For $T(x, y, z) = 5e^{-2x^2-y^2-3z^2}$, the gradient is

$$\begin{aligned}\nabla T(x, y, z) &= 5e^{-2x^2-y^2-3z^2}\langle -4x, -2y, -6z\rangle \\ &= 10e^{-2x^2-y^2-3z^2}\langle -2x, -y, -3z\rangle\end{aligned}$$

At the point $P(1, 2, -1)$, we have $\nabla T(1, 2, -1) = 10e^{-9}\langle -2, -2, 3\rangle$. Each component has units of C/m.
The vector from $P(1, 2, -1)$ to $Q(1, 1, 0)$ is $\langle 0, -1, 1\rangle$
A unit vector in this direction is

$$\widehat{\mathbf{u}} = \frac{\langle 0, -1, 1\rangle}{|\langle 0, -1, 1\rangle|} = \frac{1}{\sqrt{2}}\langle 0, -1, 1\rangle$$

Each component is dimensionless because the units of the vector $\mathbf{u}$ and its magnitude are the same.
The rate of change of T at P in direction $\widehat{\mathbf{u}}$ is

$$D_{\widehat{\mathbf{u}}}T(1, 2, -1) = \nabla T(1, 2, -1) \cdot \widehat{\mathbf{u}} = 10e^{-9}\left(\frac{5}{\sqrt{2}}\right) = 25\sqrt{2}e^{-9}.$$

Its units are C/m.

Question 4 (b)

SOLUTION. For any given unit vector $\widehat{\mathbf{u}}$, the rate of change of temperature at P in direction $\widehat{\mathbf{u}}$ is the scalar

$$\begin{aligned}D_{\widehat{\mathbf{u}}}T(1, 2, -1) &= \nabla T(1, 2, -1) \cdot \widehat{\mathbf{u}} \\ &= |\nabla T(1, 2, -1)|\ |\widehat{\mathbf{u}}| \cos(\theta) \\ &= |\nabla T(1, 2, -1)| \cos(\theta)\end{aligned}$$

where θ is the angle between the given vectors. (Recall that $|\widehat{\mathbf{u}}| = 1$.) For optimum *decrease*, we should make the directional derivative as small as possible. Since our only degree of freedom is the angle above and we know that

$$-1 \le \cos\theta \le 1$$

we should arrange $\cos\theta = -1$ by choosing $\theta{=}\pi$, i.e., by making $\widehat{\mathbf{u}}$ parallel to $-\nabla T(1, 2, -1)$:

$$\widehat{\mathbf{u}} = \frac{-\nabla T(1,2,-1)}{|-\nabla T(1,2,-1)|} = \frac{\langle 2, 2, -3\rangle}{\sqrt{17}}$$

This is the direction of most rapid temperature decrease.

Question 4 (c)

SOLUTION. When T is decreasing most rapidly from its value at P, the actual rate of change (a scalar) is based on the direction chosen in part (b):

$$\begin{aligned} D_{\widehat{\mathbf{u}}}T(1,2,-1) &= \nabla T(1,2,-1) \cdot \frac{-\nabla T(1,2,-1)}{|-\nabla T(1,2,-1)|} \\ &= \frac{-\left|\nabla T(1,2,-1)\right|^2}{|\nabla T(1,2,-1)|} \\ &= -\left|\nabla T(1,2,-1)\right| \\ &= -10e^{-9}\sqrt{17}. \end{aligned}$$

This has units of C/m.
In plain English, the words can absorb the minus sign: "The maximum rate of decrease of temperature at point P is

$$10e^{-9}\sqrt{17}\mathrm{C/m}$$

obtained in the direction $\langle 2, 2, -3\rangle / \sqrt{17}$."

Question 5 (a)

SOLUTION. We wish to maximize the value of $f(x, y, z) = z$ subject to the constraints

$$x + y + z = 2, \quad x^2 + y^2 + z^2 = 2.$$

So we use Lagrange multipliers. The objective function, Λ, is

$$\Lambda(x, y, z, \mu, \nu) = f(x, y, z) + \mu(x + y + z - 2) + \nu(x^2 + y^2 + z^2 - 2)$$

Computing the partial derivatives of Λ with respect to all the variables x, y, z and the Lagrange multipliers μ, ν and setting them equal to zero gives:

$$
\begin{aligned}
(1) \quad & \frac{\partial \Lambda}{\partial x} = \mu + 2\nu x = 0 \\
(2) \quad & \frac{\partial \Lambda}{\partial y} = \mu + 2\nu y = 0 \\
(3) \quad & \frac{\partial \Lambda}{\partial z} = 1 + \mu + 2\nu z = 0 \\
(4) \quad & \frac{\partial \Lambda}{\partial \mu} = x + y + z - 2 = 0 \\
(5) \quad & \frac{\partial \Lambda}{\partial \nu} = x^2 + y^2 + z^2 - 2 = 0
\end{aligned}
$$

To begin with, we observe that ν can not be zero, because if it was, then equations (1) and (2) would force $\mu = 0$, while equation (3) would force $\mu = -1$; a contradiction. Hence, we combine equations (1) and (2) to see that the maximum of f occurs on the plane $y = x$:

$$\begin{cases} \mu + 2\nu x = 0 \\ \mu + 2\nu y = 0 \end{cases} \quad \rightarrow \quad y = x$$

Taking this result and using (4), we find

$$2x + z - 2 = 0 \quad \rightarrow \quad x = \frac{2-z}{2}$$

Taking these two results and using (5) we have an equation for z:

$$x^2 + y^2 + z^2 - 2 = \left(\frac{2-z}{2}\right)^2 + \left(\frac{2-z}{2}\right)^2 + z^2 - 2 = 0$$

Multiplying this equation by 4 and expanding all the quadratic terms we get:

$$8 - 8z + 6z^2 - 8 = 6z^2 - 8z = 0$$

Solving this equation for z we get

$$z = 0, \ \frac{4}{3}.$$

Taking the larger of the two results we obtain the maximum of $f(x, y, z) = z$ subject to the given constraints which is $z = 4/3$.

Question 5 (b)

SOLUTION. From our answer in part (a), we found that there were two critical points of the objective function,

$$\Lambda(x, y, z) = f(x, y, z) + \mu(x + y + z - 2) + \nu(x^2 + y^2 + z^2 - 2),$$

on C. Those critical points corresponded to the critical points of the function $f(x,y,z) = z$ restricted to the curve C. The values of z corresponding to those critical points were $z = 0$ and $z = 4/3$. In contrast to part (a) where we looked for the larger of the two, we are now interested in the smaller value $z = 0$.

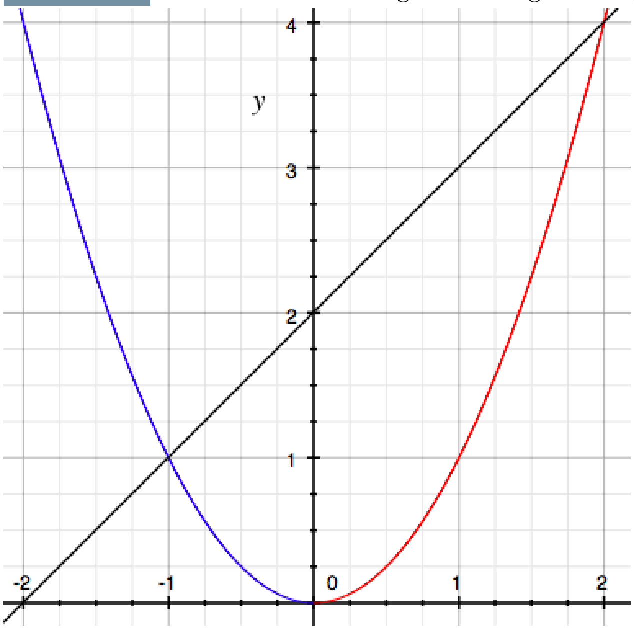
To find the corresponding x and y value, recall the condition $x = y$ from part (a). Plugging this into the equation for the line, we obtain

$$x + x + 0 = 2 \quad \rightarrow \quad x = y = 1$$

Therefore, the lowest point on C is (1,1,0).

Question 6 (a)

SOLUTION. If we examine the region of integration by drawing a picture...



... we see that if we integrate with respect to y first, that we can specify the lower bound of integration as $y = x^2$ and the upper bound as $y = x + 2$. Looking at where the lines meet, we can see that the bounds of integration for x is x = -1 to x = 2. (To find these intersection points solve the equation $x^2 = x + 2$.)
From these observations, we can write I as follows

$$I = \int_{-1}^{2} \int_{x^2}^{x+2} f(x,y)\, \mathrm{d}y \mathrm{d}x$$

Question 6 (b)

SOLUTION. Using our answer from part (a), we can write I with the given f as

$$I = \int_{-1}^{2} \int_{x^2}^{x+2} \frac{\exp(x)}{2-x}\, dydx.$$

We can evaluate the integral as follows:

$$
\begin{aligned}
I &= \int_{-1}^{2}\int_{x^2}^{x+2} \frac{\exp(x)}{2-x}\, dydx \\
&= \int_{-1}^{2} \frac{\exp(x)}{2-x}\left(x+2-x^2\right)\, dx \\
&= \int_{-1}^{2} \frac{\exp(x)}{2-x}(-x+2)(x+1)\, dx \\
&= \int_{-1}^{2} \exp(x)(x+1)\, dx.
\end{aligned}
$$

Using integration by parts, with *u=x+1* and dv = *exp(x)*,

$$
\begin{aligned}
I &= \int_{-1}^{2} \exp(x)(x+1)\, dx \\
&= \exp(x)(x+1)\Big|_{-1}^{2} - \int_{-1}^{2} \exp(x)\, dx \\
&= 3\exp(2) - \left(\exp(x)\Big|_{-1}^{2}\right) = 2\exp(2) + \exp(-1)
\end{aligned}
$$

Therefore,

$$\int_{-1}^{2}\int_{x^2}^{x+2} \frac{\exp(x)}{2-x}\, dydx = 2e^2 + e^{-1}$$

Question 7

SOLUTION. Since D is the unit disc, we know that it is a circle of radius 1 and so it has area π, i.e.

$$A(D) = \pi$$

The centre of D is the origin and so *(a,b)* = *(0,0)*. To evaluate the integral,

$$\iint_D \sqrt{(x-0)^2 + (y-0)^2}\, dx\, dy = \iint_D \sqrt{x^2+y^2}\, dx\, dy$$

we use polar coordinates:

$$\iint_D \sqrt{x^2+y^2}\, dx\, dy = \int_0^{2\pi}\int_0^1 r^2\, dr\, d\theta$$

where we have used the fact that $dx\, dy = r\, dr\, d\theta$ (i.e. the Jacobian of the transformation from Cartesian to Polar coordinates) and $\sqrt{x^2+y^2} = r$. Evaluating the integral gives:

$$\int_0^{2\pi}\int_0^1 r^2\, dr\, d\theta = 2\pi \int_0^1 r^2\, dr = \frac{2\pi}{3}.$$

Hence, the average distance from any point within D and the centre of D (the origin) is $\frac{1}{A(D)}\iint_D \sqrt{x^2+y^2}\, dx\, dy = \frac{2}{3}$.

Question 8

SOLUTION. We find that the integration boundaries are $z \in [0, y^2]$, $y \in [0, 1-x]$, $x \in [0, 1]$
Hence, the integral to calculate is

$$\begin{aligned}
\int_0^1 \int_0^{1-x} \int_0^{y^2} z \, \mathrm{d}z\mathrm{d}y\mathrm{x} &= \int_0^1 \int_0^{1-x} \left[\frac{z^2}{2}\right]_0^{y^2} \mathrm{d}y\mathrm{d}x \\
&= \int_0^1 \int_0^{1-x} \frac{y^4}{2} \, \mathrm{d}y\mathrm{d}x \\
&= \int_0^1 \left[\frac{y^5}{10}\right]_0^{1-x} \mathrm{d}x \\
&= \int_0^1 \frac{(1-x)^5}{10} \, \mathrm{d}x \\
&= \left[-\frac{(1-x)^6}{60}\right]_0^1 \\
&= \frac{1}{60}
\end{aligned}$$

Question 9

SOLUTION. The region of integration is bounded from above by $z = x^2+y^2$ and from below by $x^2+y^2+z^2 = 6$.
The intersection of the paraboloid and the sphere is found by substituting the first equation into the second, and solving for z

$$x^2 + y^2 + z^2 = z + z^2 = 6,$$

hence $z_{1,2} = 2, -3$. Since we see on the sketch from Hint 1, that the two surfaces intersect at a positive z value, the intersection is at $z = 2$.
We switch to cylindrical coordinates and express the z-variable in dependence of the radius $r = \sqrt{x^2+y^2}$.

- The largest value of r in the region of integration is where the sphere and the paraboloid intersect, namely at $r = \sqrt{z} = \sqrt{2}$. The integration limits for the radius are $0 \le r \le \sqrt{2}$.
- The variable z is bounded from below by $z = r^2$ and from above by $r^2 + z^2 = 6$, that is, $z = \sqrt{6-r^2}$. So the integration boundaries are $r^2 \le z \le \sqrt{6-r^2}$.
- The angle ϕ does not depend on z or r and completes the whole circle $0 \le \phi \le 2\pi$.
- The integrand function in cylindrical coordinates is $x^2 + y^2 = r^2$

Hence, the integral to calculate is

$$\begin{aligned}
\int_0^{\sqrt{2}} \int_{r^2}^{\sqrt{6-r^2}} \int_0^{2\pi} r^2 r \, \mathrm{d}\phi\mathrm{d}z\mathrm{d}r &= \int_0^{\sqrt{2}} \int_{r^2}^{\sqrt{6-r^2}} 2\pi r^3 \, \mathrm{d}z\mathrm{d}r \\
&= \int_0^{\sqrt{2}} 2\pi r^3 (\sqrt{6-r^2} - r^2) \, \mathrm{d}r \\
&= \int_0^{\sqrt{2}} 2\pi r^3 (6-r^2)^{\frac{1}{2}} \, \mathrm{d}r - \int_0^{\sqrt{2}} 2\pi r^5 \, \mathrm{d}r
\end{aligned}$$

Now we perform integration by parts in the first integral, where we consider $u = 2\pi r^2$ and $dv = r(6-r^2)^{\frac{1}{2}}$. Then $du = 4\pi r$ and $v = -\frac{1}{3}(6-r^2)^{\frac{3}{2}}$. Overall we obtain

$$\begin{aligned}
&= \left[-2\pi r^2 \frac{1}{3}(6-r^2)^{\frac{3}{2}}\right]_0^{\sqrt{2}} + \int_0^{\sqrt{2}} 4\pi r \frac{1}{3}(6-r^2)^{\frac{3}{2}}\,\mathrm{d}r - \left[2\pi \frac{r^6}{6}\right]_0^{\sqrt{2}} \\
&= -2\pi 2\frac{1}{3}(6-2)^{\frac{3}{2}} + 0 - \left[4\pi \frac{1}{15}(6-r^2)^{\frac{5}{2}}\right]_0^{\sqrt{2}} - 2\pi\sqrt{2}^6 \frac{1}{6} + 0 \\
&= -\frac{32}{3}\pi - \frac{4}{15}\pi 32 + \frac{4}{15}\pi 36\sqrt{6} - \frac{8}{3}\pi \\
&= \frac{\pi}{15}(144\sqrt{6} - 328) \approx 5.18
\end{aligned}$$

Question 10

SOLUTION. The combination $x^2 + y^2 + z^2$ suggests a change to spherical coordinates. The domain is compatible with this: in spherical coordinates, the whole space is covered when:

- θ runs from 0 to 2π,
- ϕ runs from 0 to π, and
- ρ runs from 0 to $+\infty$.

In spherical coordinates, the differential volume element is:

$$dV = \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi.$$

And the function to integrate becomes:

$$(1 + (x^2 + y^2 + z^2)^3)^{-1} = (1 + (\rho^2)^3)^{-1} = \frac{1}{1+\rho^6}.$$

Thus the integral of interest becomes:

$$I = \int_{\rho=0}^{\infty} \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} \frac{\rho^2 \sin(\phi)}{1+\rho^6} d\theta\, d\phi\, d\rho = \int_{\rho=0}^{\infty} \frac{\rho^2}{1+\rho^6}\, d\rho \underbrace{\int_{\phi=0}^{\pi} \sin(\phi)\, d\phi}_{=2} \underbrace{\int_{\theta=0}^{2\pi} d\theta}_{=2\pi}$$

Hence

$$I = 4\pi \int_{\rho=0}^{\infty} \frac{\rho^2}{1+\rho^6}\, d\rho$$

Substitute:

- $u = \rho^3$
- $du = 3\rho^2\, d\rho$

And you obtain:

$$I = \frac{4\pi}{3} \int_{u=0}^{\infty} \frac{du}{1+u^2} = \frac{4\pi}{3} \tan^{-1}(u)\Big|_{u=0}^{\infty} = \frac{2\pi^2}{3}.$$

Good Luck for your exams!