Full Solutions
MATH103 April 2009

April 4, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

SOLUTION. As we see in the formula for the limit of the Riemann sum

$$\lim_{N\to\infty}\sum_{k=1}^{N}\left(\frac{b-a}{N}\right)\cdot f\left(a+\frac{k(b-a)}{N}\right)=\int_a^b f(x)dx$$

the factor that is devided by N, must be $b-a$. So, we know that $b-a=3$.
Now we search for the variable

$$x_k = a+\frac{k(b-a)}{N}=a+\frac{3k}{N}.$$

The only term, that matches this expression is

$$-2+\frac{3k}{N}$$

and thus a must be equal to *-2*. With *3=b-a* $=$ *b+2* we find that *b=1*.
The last thing is to find the function *f*. From the formula of the limit of the Riemann sum we know that

$$f\left(a+\frac{k(b-a)}{N}\right)=\left[\left(a+\frac{k(b-a)}{N}\right)^3-1\right],$$

$$f\left(-2+\frac{3k}{N}\right)=\left(-2+\frac{3k}{N}\right)^3-1,$$

$$f(x)=x^3-1.$$

So the answer is **(b)**,

$$\lim_{N\to\infty}\sum_{k=1}^{N}\left(\frac{3}{N}\right)\cdot\left[\left(-2\frac{k3}{N}\right)^3-1\right]=\int_{-2}^{1}(x^3-1)dx.$$

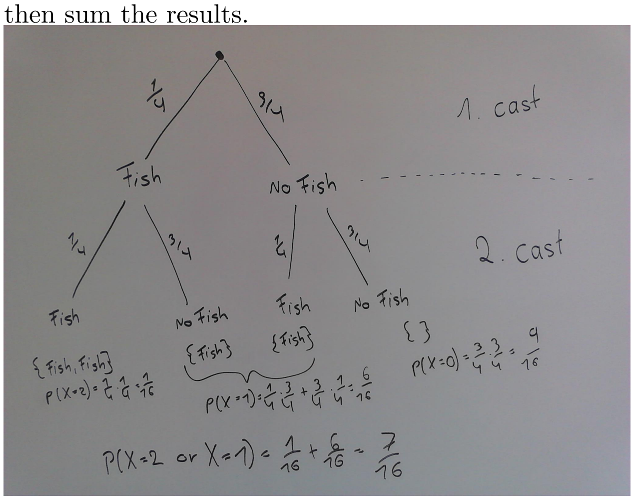
Question 1 (b)

SOLUTION 1. This is a Bernoulli process with probability of success p = *1/4*, and n = *2* repetitions. Let X be the random variable that counts the number of successful casts. Then

$$\begin{aligned}\text{Prob(at least 1 success)} &= \text{Prob}(X\geq 1)\\ &= \text{Prob}(X=1 \text{ or } X=2)\\ &= \text{Prob}(X=1)+\text{Prob}(X=2)\\ &= C(2,1)\left(\frac{1}{4}\right)^1\left(1-\frac{1}{4}\right)^{2-1}\\ &\quad + C(2,2)\left(\frac{1}{4}\right)^2\left(1-\frac{1}{4}\right)^{2-2}\\ &= \frac{7}{16}.\end{aligned}$$

Hence, answer **(b)** is correct.

SOLUTION 2. We can also use a tree diagram to solve this. Recall that we multiply along branches, and then sum the results.



Question 1 (c)

SOLUTION. We need to check for which of these differential equations $\frac{dy}{dt} = f(y)$ it holds that $f(3) = 0$.
For the first one we see that *2(3) - 6 = 0*. So, y=3 is a steady state.
For the second equation we have 3^2+9=18 $\neq 0$. So, y=*3* is *no* steady state.
The third equation $\frac{1}{3-3}$ is not even defined for y=3.
For the forth equation we have $cos(\pi\ 3) \leq 0$, and y=*3* is *no* steady state.
Last we have the differential equation $\frac{dy}{dt} = 3$ which is never equal to *0* and has no steady state.
So, the answer is (a).

Question 1 (d)

SOLUTION. In total we have 100 students. So we need to find the number of points such that 50 students have more and 50 students have less points.
We line up the marks of the students:

$$\underbrace{\underbrace{20,20,20,20,20}_{5\text{times}},\underbrace{30,30,\dots,30}_{15\text{ times}},\underbrace{40,\dots,40}_{10\text{ times}},\underbrace{50,\dots,50}_{15\text{ times}},\underbrace{60,\dots,60}_{5\text{ times}}}_{50\text{ students}},\underbrace{70,\dots,70}_{15\text{ times}},\dots$$

Unfortunately we end up between 60 and 70. So, we need to think closer. The column at the figure between 20 and 30 marks means, that the student got an amount of marks in the interval $[20, 30]$.
So, the students in the column between 60 and 70 got an amount of marks between 60 and 70. The students in the column between 70 and 80 got an amount of marks between 70 and 80. So, the 70 lies exaclty in the middle.
Hence the mean is about 70 and answer **c)** is correct.

Question 1 (e)

SOLUTION. We find, that we need to solve the differential equation

$$
\begin{aligned}
x''(t) &= \cos(t)\\
x'(0) &= 0\\
x(0) &= 1
\end{aligned}
$$

We integrate $x''(t) = \cos(t)$ and get $x'(t) = \sin(t) + c_1$. With $0 = x'(0) = \sin(0) + c_1 = c_1$ we find $c_1 = 0$. We continue with integrating $x'(t) = \sin(t)$, which leads to $x(t) = -\cos(t) + c_2$. With $x(0) = 1$ we find $1 = x(0) = -\cos(0) + c_2 = -1 + c_2$ and get $c_2 = 2$.
The velocity beeing $x'(t) = \sin(t)$ reaches -1 at $t = \frac{3}{2}\pi$, and we get
$x\left(\frac{3}{2}\pi\right) = -\cos\left(\frac{3}{2}\pi\right) + 2 = -0 + 2 = 2$.
So, the answer is (d).

Question 1 (f)

Solution. By using the formula for the bernoulli experiment
$P(X = k) = \binom{N}{k} p^k(1-p)^{N-k}$
for $p = 0.7$, $N = 10$ and $k = 8$ and get
$P(X = 8) = \binom{10}{8} 0.7^8 \cdot (0.3)^2 = 45 \cdot 0.7^8 \cdot 0.3^2$
So, the answer is (a).

Question 2

Solution. We just plug in the functions
$\frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)^2 = \left(\sum_{k=0}^{\infty} x^k\right)^2 = \sum_{k=0}^{\infty} x^k \cdot \sum_{m=0}^{\infty} x^m$
Next we start to multiply the two infinite sums term by term, and sort the result by powers of x.

$$
\begin{aligned}
\sum_{k=0}^{\infty} x^k \cdot \sum_{m=0}^{\infty} x^m &= \left(1 + x + x^2 + x^3 + \dots\right)\left(1 + x + x^2 + x^3 + \dots\right)\\
&= (1 \cdot 1) + (x \cdot 1 + 1 \cdot x) + (1 \cdot x^2 + x \cdot x + x^2 \cdot 1)\\
&\quad + (1 \cdot x^3 + x \cdot x^2 + x^2 \cdot x + x^3 \cdot 1) + \dots\\
&= 1 + 2x + 3x^2 + 4x^3 + \dots
\end{aligned}
$$

Now we can see the pattern: There are always $k + 1$ combinations that give the term x_k. Hence the Taylor series is given by $\frac{1}{(1-x)^2} = \frac{1}{(1-x)^2} = \sum_{k=0}^{\infty}(k+1)x^k$.

Question 3 (a)

Solution. To solve the differential equation

$$\frac{dx}{dt} = \frac{x^2 - 2}{2x}$$

with separation of variables we rewrite the equation as

$$\frac{2x dx}{x^2 - 2} = dt$$

and integrate both sides

$$\int \frac{2x}{x^2-2}dx = \int 1dt.$$

The integral on the right hand side simply gives t+c. The integrand on the left hand side is of the form $\frac{f'(x)}{f(x)}$, with *f(x)* = x^2-*2*, which has anti-derivative $\ln|f(x)| = \ln|x^2-2|$. If you don't see this easily you can simply use the substitution u = x^2-*2*. Either way, we obtain

$$\ln(|x^2-2|) = t + c$$

for a constant c.
To solve this equation for x we first take the exponential on both sides

$$|x^2-2| = e^{t+c} = e^c\, e^t = de^t,$$

where d is another constant, which has to be positive. Finally, to remove the absolute value we write

$$\pm(x^2-2) = |x^2-2| = de^t,$$

i.e.

$$x^2 - 2 = \pm de^t = \tilde{c}e^t$$

for another constant $\tilde{c}$ that has no restriction on its sign (i.e. can be positive or negative). Now we can finally solve for x and obtain

$$x(t) = \pm\sqrt{\tilde{c}e^t + 2}.$$

Question 3 (b)

Solution. Recall the solution from part (a):

$$x(t) = \pm\sqrt{\tilde{c}e^t + 2}.$$

To satisfy the initial condition $x(0) = 2$, we calculate

$$2 = x(0) = \pm\sqrt{\tilde{c}e^0 + 2}$$

We see, that this can only be fullfilled with the plus sign.

$$x(0) = \sqrt{\tilde{c}+2} = 2,$$

$$\tilde{c} = 2.$$

So, the solution of the differential equation with initial condition is

$$x(t) = \sqrt{2e^t + 2}.$$

Question 4 (a)

SOLUTION. For calculating this integral we use substitution.

$$\int_{-1}^{0} \frac{1}{\sqrt{1-x}} dx = \int_{-1}^{0} (1-x)^{-\frac{1}{2}} dx.$$

We define u=1-x, du=-dx. The new limits of the integral are u(-1)=1-(-1)=2 and u(0)=1-0=1. Hence

$$\int_{2}^{1} u^{-\frac{1}{2}}(-du) = \int_{1}^{2} u^{-\frac{1}{2}} du = \left[2u^{\frac{1}{2}}\right]_1^2 = 2(\sqrt{2}-1).$$

Question 4 (b)

SOLUTION. For calculating this integral we use integration by parts two times. We take $dv = x\,dx$ and $u = (\ln(x))^2$. Then $v = \dfrac{x^2}{2}$ and $du = 2\ln(x)\frac{1}{x}\,dx$ (don't forget the derivative $1/x$ of *ln(x)*).

$$\begin{aligned}
\int x(\ln(x))^2 dx &= \left[\frac{x^2}{2}(\ln(x))^2\right] - \int \frac{x^2}{2} 2\ln(x)\frac{1}{x} dx \\
&= \frac{x^2}{2}(\ln(x))^2 - \int \underbrace{x}_{dv} \underbrace{\ln(x)}_{u}\, dx \\
&= \frac{x^2}{2}(\ln(x))^2 - \frac{x^2}{2}\ln(x) + \int \frac{x^2}{2}\frac{1}{x} dx \\
&= \frac{x^2}{2}(\ln(x))^2 - \frac{x^2}{2}\ln(x) + \int \frac{x}{2} dx \\
&= \frac{x^2}{2}(\ln(x))^2 - \frac{x^2}{2}\ln(x) + \frac{x^2}{4} + c
\end{aligned}$$

for a constant c.

Question 4 (c)

SOLUTION. Before we can calculate the integral, we use partial fraction.

$$\begin{aligned}
\frac{x+7}{(x-3)(x+2)} &= \frac{A}{x-3} + \frac{B}{x+2} \\
&= \frac{A(x+2) + B(x-3)}{(x-3)(x+2)} = \frac{x\overbrace{(A+B)}^{=1} + \overbrace{2A-3B}^{=7}}{(x-3)(x+2)}
\end{aligned}$$

So, we get two equations to solve for A and B

$$\begin{aligned} A + B &= 1, \\ 2A - 3B &= 7 \end{aligned}$$

and we get $A = 2,\ B = -1$.
So we find that

$$\begin{aligned} \int \frac{x+7}{(x-3)(x+2)} dx &= \int \left(\frac{2}{x-3} - \frac{1}{x+2} \right) dx \\ &= 2\ln(x-3) - \ln(x+2) + C \end{aligned}$$

Question 5 (a)

Solution. We take the disk method to calculate

$$\begin{aligned} \int_0^{12} \left(\sqrt{14-x}\right)^2 \pi dx &= \pi \int_0^{12} (14-x) dx \\ &= \pi \left(14x - \frac{x^2}{2} \right)\Bigg|_0^{12} \\ &= 96\pi. \end{aligned}$$

Question 5 (b)

Solution. To find the amount A of Vitamin A in the carrot, we need to multiply every infinitly small disk with unit cm^3 with the passend concentration of Vitamin A and then integrate over all disks:

$$\begin{aligned} A &= \int_0^{12} \frac{1}{12} e^{-\frac{x}{12}} \left(\sqrt{14-x}\right)^2 \pi dx \\ &= \frac{\pi}{12} \int_0^{12} e^{-\frac{x}{12}} (14-x) dx \\ &= \frac{\pi}{12} \left[\int_0^{12} 14 e^{-\frac{x}{12}} dx - \int_0^{12} x e^{-\frac{x}{12}} dx \right] \end{aligned}$$

We let the former integral as is for now and use integration by parts for the latter. Let $u = x$, $du = dx$, $dv = e^{-x/12} dx$ and v = -12$e^{-x/12}$. Then

$$
\begin{aligned}
A &= \frac{\pi}{12}\left[\int_0^{12} 14e^{-\frac{x}{12}}dx - \int_0^{12} xe^{-\frac{x}{12}}dx\right]\\
&= \frac{\pi}{12}\left[\int_0^{12} 14e^{-\frac{x}{12}}dx - \left(x(-12)e^{-\frac{x}{12}}\Big|_0^{12} - \int_0^{12}(-12)e^{-\frac{x}{12}}\,dx\right)\right]\\
&= \frac{\pi}{12}\left[14\int_0^{12} e^{-\frac{x}{12}}\,dx - 12\int_0^{12} e^{-\frac{x}{12}}\,dx + (12)(12)e^{-1} - 0\right]\\
&= \frac{\pi}{12}\left[2\int_0^{12} e^{-\frac{x}{12}}\,dx + (12)(12)e^{-1}\right]\\
&= \frac{\pi}{12}\left[2(-12)e^{-\frac{x}{12}}\Big|_0^{12} + (12)(12)e^{-1}\right]\\
&= \frac{\pi}{12}\left[-2(12)(e^{-1}-1) + (12)(12)e^{-1}\right]\\
&= \pi\left[-2e^{-1} + 2 + 12e^{-1}\right]\\
&= \pi\left[\frac{10}{e} + 2\right]\\
&= \frac{2\pi(5+e)}{e}.
\end{aligned}
$$

Question 5 (c)

SOLUTION. We are searching for a $x_m \in [0,12]$ such that
$\int_0^{x_m}\left(\sqrt{14-x}\right)^2\pi dx = \frac{1}{2}\int_0^{12}\left(\sqrt{14-x}\right)^2\pi dx = \frac{1}{2}96\pi = 48\pi$.
We calculate

$$
\begin{aligned}
\int_0^{x_m}\left(\sqrt{14-x}\right)^2\pi dx &= \pi\int_0^{x_m}(14-x)dx\\
&= \pi\left(14x - \frac{x^2}{2}\right)\Big|_0^{x_m}\\
&= \pi\left(14x_m - \frac{x_m^2}{2}\right) = 48\pi
\end{aligned}
$$

We cancel the π and continue
$0 = x_m^2 - 28x_m + 96 = (x_m - 24)(x_m - 4)$

$$
x_{m_1} = 4, x_{m_2} = 24
$$

Since $24 \notin [0,12]$ we conclude $x_m = 4$.
You need to cut 4cm from $x = 0$.

Question 6 (a)

SOLUTION. Since we look for the maximum of s, we set its derivative to zero.

$$
0 = \frac{d}{dr}50\sin\left(\frac{\pi r}{10}\right) = 50\frac{\pi}{10}\cos\left(\frac{\pi r}{10}\right),
$$

$$\cos\left(\frac{\pi r}{10}\right) = 0,$$

Due to the restriction on r the only possible zero is when

$$\frac{\pi r}{10} = \frac{\pi}{2}$$

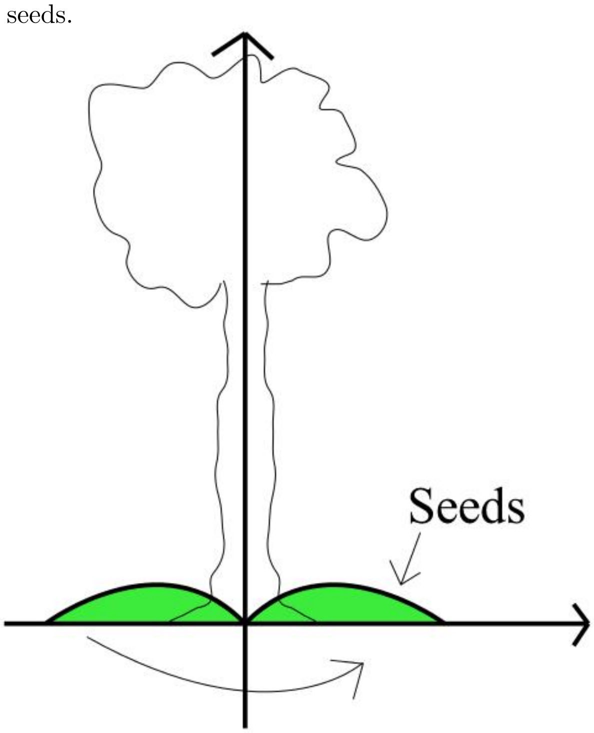
which is attained at

$$r = 5$$

Note that $\sin(\pi(5)/10) = \sin(\pi/2) = 1$ gives indeed the maximum value for sine and hence for s.

Question 6 (b)

SOLUTION. The total number of seeds is the sum of all m^2 around the tree times the passend concentration of seeds.



Because the area around the tree is a ring, the field of seeds looks like a the function rotated around the tree (y-axis).

To calculate this integral, we use the shell method $\int 2\pi x h(x) dx$

For integration we use integration by parts.

$$\begin{aligned}\int_0^{10} 2\pi \underbrace{r}_{=u} 50 \underbrace{\sin\left(\frac{\pi r}{10}\right) dr}_{dv} &= 100\pi \left[-r\cos\left(\frac{\pi r}{10}\right)\frac{10}{\pi}\Big|_0^{10} + \frac{10}{\pi}\int_0^{10} \cos\left(\frac{\pi r}{10}\right) dr \right] \\ &= 100\left[-10 \cdot 10\cos(\pi) + 0 + 10\sin\left(\frac{\pi r}{10}\right)\frac{10}{\pi}\Big|_0^{10} \right] \\ &= 10000.\end{aligned}$$

There are 10000 seeds around the tree.

Question 7 (a)

SOLUTION. To find the expected lenght, we multiply the length x with the probability density function *p(x)* and integrate over all possible lengths. In other words, the mean, or expected value, *E[X]* is given by

$$E[X] = \int_a^b xp(x)\,dx.$$

In this problem $a = 1$, $b = e$ and *p(x)* $= 1/x$, thus

$$E[X] = \int_1^e x \cdot \frac{1}{x} dx = \int_1^e dx = x|_1^e = e - 1.$$

Question 7 (b)

SOLUTION. $P\{X \leq 2\} = \int_1^2 p(x)dx = \int_1^2 \frac{1}{x} dx = [\ln(x)]_1^2 = \ln(2)$.

Question 7 (c)

SOLUTION. At first we calculate the Variance $Var[X]$.

$$\begin{aligned}
Var[X] &= E[X^2] - (E[X])^2 \\
&= \int_1^e x^2 \cdot \frac{1}{x} dx - (e-1)^2 \\
&= \int_1^e x dx - (e-1)^2 \\
&= \left.\frac{x^2}{2}\right|_1^e - (e-1)^2 \\
&= \frac{e^2}{2} - \frac{1}{2} - e^2 + 2e - 1 \\
&= -\frac{e^2}{2} + 2e - \frac{3}{2} \\
&= -\frac{1}{2}(e-3)(e-1) \\
&= \frac{1}{2}(3-e)(e-1) \quad (\approx 0.242)
\end{aligned}$$

Then

$$\sigma = \sqrt{\frac{1}{2}(3-e)(e-1)} \approx 0.492$$

Question 8 (a)

Solution. Since the growth rate is the derivative of the height for any time $t \geq 0$, we need to integrate the function $g(t)$ to find the height depending on t.
$h(t) = \int g(t)dt = \int ae^{\frac{t}{2}}dt = 2ae^{\frac{t}{2}} + c$
for a constant $c \in \mathbb{R}$.
We find that constant by using the initial condition.
$0 = h(0) = 2a + c,$

$$c = -2a.$$

So, the function of height is
$h(t) = 2a(e^{\frac{t}{2}} - 1)$.

Question 8 (b)

Solution. By knowing that $h(1) = 1$, we calculate
$1 = h(1) = 2a(e^{\frac{1}{2}} - 1)$

$$a = \frac{1}{2(e^{\frac{1}{2}} - 1)}.$$

So, the function of height is
$h(t) = \frac{e^{\frac{t}{2}}-1}{e^{\frac{1}{2}}-1}$.

Question 8 (c)

Solution. The change of height between day 4 and 5 is

$$\begin{aligned} h(5) - h(4) &= \frac{e^{\frac{5}{2}} - 1}{e^{\frac{1}{2}} - 1} - \frac{e^{\frac{4}{2}} - 1}{e^{\frac{1}{2}} - 1} \\ &= \frac{e^{2.5} - e^2}{e^{\frac{1}{2}} - 1} = e^2 \frac{e^{\frac{1}{2}} - 1}{e^{\frac{1}{2}} - 1} = e^2 \end{aligned}$$

Good Luck for your exams!