Full Solutions
MATH221 April 2010

April 4, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

SOLUTION. The rank of matrix A is found by determining the number of linearly independent columns or the number of linearly independent rows. They are equal.
1.) First, we have to reduce the matrix to row echelon form in order to see which columns contain pivot points(point with all zeros below it and occurs to the right and below any previous pivot points). These columns are the linearly independent columns.
2.) In this question, matrix U is given and is an echelon form of matrix A. Therefore, we don′t have to perform row reduction anymore.
3.)In order to see the pivot points clearly, we can switch columns 3 and 4 to get:

$$\text{U} = \begin{bmatrix} 1 & 2 & -2 & 3 & 4 \\ 0 & 1 & 3 & 2 & -2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3.) As you can tell from the edited matrix U, column 1, 2, and 3 contain the pivot points. This means that matrix A has a rank of 3.

Question 1 (b)

SOLUTION. 1.) To find the basis for the column space of A, we need to find the linearly independent columns of the row echelon form of the matrix. In this question, it is given as matrix U.
2.)In order to see the pivot points clearly, we can switch columns 3 and 4 to get:

$$\text{U} = \begin{bmatrix} 1 & 2 & -2 & 3 & 4 \\ 0 & 1 & 3 & 2 & -2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3.) Columns 1, 2, and 3 are linearly independent. Once we have determined where the linearly independent columns are, we take the linearly independent columns from the original matrix A corresponding to the linearly independent columns of matrix U. REMEMBER THAT I SWITCHED COLUMN 3 AND 4.
Therefore the basis for the column space of A are:
column 1 = $\begin{bmatrix} 1 & 2 & 1 & 1 \end{bmatrix}^T$
column 2 = $\begin{bmatrix} 2 & 5 & 4 & 3 \end{bmatrix}^T$
column 4 = $\begin{bmatrix} -2 & -1 & 5 & 1 \end{bmatrix}^T$

Question 1 (c)

SOLUTION. 1.) In this question, we are finding a basis for $[R(A^T)]$. To find the basis for the row space of A, we need to find the column space of the $[A^T]$ (transpose of matrix A). However, to make things easier for ourselves, this is just the linearly independent rows of the row echelon form of the matrix. In this question, it is given as matrix U.
2.)In order to see the pivot points clearly, we can switch columns 3 and 4 to get:

$$\text{U} = \begin{bmatrix} 1 & 2 & -2 & 3 & 4 \\ 0 & 1 & 3 & 2 & -2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3.) Rows 1, 2, and 3 are linearly independent. Unlike the column space, we take the rows from the echelon form as the basis, in this case from matrix U.
Remember we switched columns 3 and 4. Take the original form of U to find the basis of the row space.
Therefore, the basis for the row space of A are:
row 1 = $\begin{bmatrix} 1 & 2 & 3 & -2 & 4 \end{bmatrix}$
row 2 = $\begin{bmatrix} 0 & 1 & 2 & 3 & -2 \end{bmatrix}$

row 3 = $\begin{bmatrix} 0 & 0 & 0 & 1 & 2 \end{bmatrix}$

Question 1 (d)

SOLUTION. 1.) Let's use the row echelon form of matrix A, which is matrix U to compute the solution for Ax=0.
2.) Ax=0 means that the product of matrix A and the vector x=[x_1,x_2,x_3,x_4] has to equal to 0.
3.) In order to solve this, we must let the matrix U equal to 0 and solve for the most row reduced echelon matrix form. To see the pivot points clearly, we can switch columns 3 and 4 to get:

U = $\begin{bmatrix} 1 & 2 & -2 & 3 & 4 \\ 0 & 1 & 3 & 2 & -2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Then we follow these steps in order to put matrix U in the most RREF. 1.) R_1 = R_1 - 2R_2 2.) R_1 = R_1 + 8R_3 3.) R_2 = R_2 - 3R_3
The resulting RREF of matrix U is

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 24 \\ 0 & 1 & 0 & 2 & -8 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.) Now we solve for each x components to solve Ax=0
Remember that we switched columns 3 and 4, therefore, x_4(column 4) is now in x_3(column 3) and vice versa.
x_1=x_4-24x_5 x_2=-2x_4+8x_5 x_3=-2x_5 x_4= x_4 x_5=x_5

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_4 \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -24 \\ 8 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Question 1 (e)

SOLUTION. 1.) To find the basis for the null space of A, we need to find the reduced echelon form of matrix A. In this question, it is given as matrix U.
2.) To see the pivot points clearly, we can switch columns 3 and 4 to get:

U = $\begin{bmatrix} 1 & 2 & -2 & 3 & 4 \\ 0 & 1 & 3 & 2 & -2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

3.) Because we switched columns 3 and 4, we can see that Columns 1, 2, and 3 are linearly independent. Therefore the free variables are contained in columns 4 and 5 :
column 4 = $\begin{bmatrix} 3 & 2 & 0 & 0 \end{bmatrix}^T$
column 5 = $\begin{bmatrix} 4 & -2 & 2 & 0 \end{bmatrix}^T$
4.) Then we follow these steps in order to put matrix U in the most RREF. 1.) R_1 = R_1 - 2R_2 2.) R_1 = R_1 + 8R_3 3.) R_2 = R_2 - 3R_3
The resulting RREF of matrix U is

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 24 \\ 0 & 1 & 0 & 2 & -8 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

5.) Now we solve for each x components to solve Ax=0

Remember that we switched columns 3 and 4, therefore, x_4(column 4) is now in x_3(column 3) and vice versa.
x_1=x_4-24x_5 x_2=-2x_4+8x_5 x_3=-2x_5 x_4= x_4 x_5=x_5

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_4 \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -24 \\ 8 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

6.) Therefore, a basis for the null space of A are $\begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and

$$\begin{bmatrix} -24 \\ 8 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Question 1 (f)

SOLUTION. 1.) To express a_5 as a linear combination of a_1, a_2 and a_4, we take n multiples of each vector that will add up to a_5.
2.) One way to do this is by putting the vectors in a system of equations and then putting this system of equations into an augmented matrix:
a_1 + 2a_2 - 2a_4 = 4a_5 0+ a_2 + 3a_4 = -2a_5 0 + 0 + a_4 = 2a_5

$$\begin{bmatrix} 1 & 2 & -2 & 4 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

3.) Reduce to row echelon form through these steps: 1.) R_2 = R_2 -3R_3 2.) R_1 = R_1 + 2R_3 3.) R_1 = R_1 - 2R_2
and you will get

$$\begin{bmatrix} 1 & 0 & 0 & 24 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

4.) This shows the linear combination:
a_1 = 24 a_2 = -8 a_4 = 2
a_5 = 24a_1 - 8a_2 + 2a_4
PROOF

$$\begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix}$$

$$= 24 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - 8 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 24 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 16 \\ 8 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -4 \\ 6 \\ 2 \\ 0 \end{bmatrix}$$

Question 1 (g)

SOLUTION. 1.) The dimension of the null space of A^T is dim(N(A^T)) = n(rows) - r(A)(rank of matrix).
2.) The RREF of matrix A is:

$$rref(A)\begin{bmatrix}1 & 0 & 0 & 0 & 24\\0 & 1 & 0 & 0 & 8\\0 & 0 & 1 & 0 & 2\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$

Therefore, the rank is 3.
3.) A^T is the transpose of matrix A

$$\begin{bmatrix}1 & 2 & 1 & 1\\2 & 5 & 4 & 3\\3 & 8 & 7 & 5\\-2 & -1 & 5 & 1\\4 & 6 & 2 & 2\end{bmatrix}$$

rows from original matrix A = 4 r(A) = 3
3.) dim(N(A^T)) = 4 - 3 = 1

Question 1 (h)

SOLUTION. 1.) Transform the system of equations into an augmented matrix. x,y,z,&w all represent one column each and the number of system of equations represent the rows of the matrix. The answers to each equation will be on the right-most of the matrix.

2.) $$\begin{vmatrix}1 & 2 & 3 & -2 & 4\\2 & 5 & 8 & -1 & 6\\1 & 4 & 7 & 5 & 2\\1 & 3 & 5 & 1 & 2\end{vmatrix}$$

3.) Let′s reduce it to row echelon form by following these steps: 1. R_2 = -2R_1 + R_2 2. R_3 = -1R_1 + R_3 3. R_4 = -1R_1 + R_4 4. R_3 = -2R_2 + R_3 5. R_4 = -1R_2 + R_4 6. R_2 = -3R_3 + R_2 7. R_1 = 2R_3 + R_1 8. R_1 = -2R_2 + R_1

and we will end up with $$\begin{vmatrix}1 & 0 & -1 & 0 & 24\\0 & 1 & 2 & 0 & -8\\0 & 0 & 0 & 1 & 2\\0 & 0 & 0 & 0 & 0\end{vmatrix}$$

The right most values in column 5 are the answers.
4.) Therefore, the system of equations correspond to:
x_1= x_3 + 24 x_2 = -2x_3 - 8 x_3= x_3 x_4= 2
All solutions are represented by

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$$

$$= x_3\begin{bmatrix}1\\-2\\1\\0\end{bmatrix} + \begin{bmatrix}24\\-8\\0\\2\end{bmatrix}$$

Question 2 (a)

SOLUTION. To show that the set is orthogonal, we show that the dot product of each pair of vectors in the set is zero.

$u1^T$*u2 = $\begin{bmatrix}0 & 2 & 1 & 2\end{bmatrix}$ * $\begin{bmatrix}1\\-2\\0\\2\end{bmatrix}$ = 0 * 1 + 2 * -2 + 1 * 0 + 2 * 2 = -4 + 4 = 0

$u1^T$*u3 = $\begin{bmatrix}0 & 2 & 1 & 2\end{bmatrix}$ * $\begin{bmatrix}2\\0\\2\\-1\end{bmatrix}$ = 0 * 2 + 2 * 0 + 1 * 2 + 2 * -1 = 2 - 2 = 0

$u2^T$*u3 = $\begin{bmatrix}1 & -2 & 0 & 2\end{bmatrix}$ * $\begin{bmatrix}2\\0\\2\\-1\end{bmatrix}$ = 1 * 2 + -2 * 0 + 0 * 2 + 2 * -1 = 2 - 2 = 0

This is true, so the set is orthogonal.

Question 2 (b)

SOLUTION. To do this, we can set up u1, u2, u3 and v as a matrix equation and solve for the coefficients a1, a2, a3 that produce v.

M = $\begin{bmatrix}u1 & u2 & u3\end{bmatrix} = \begin{bmatrix}0 & 1 & 2\\2 & -2 & 0\\1 & 0 & 2\\2 & 2 & -1\end{bmatrix}$

M * a = v

Solving this equation, a = $\begin{bmatrix}-1 & 1 & 1\end{bmatrix}$

So, v = u2 + u3 - u1

Question 2 (c)

SOLUTION. To find a vector orthogonal to all of u_1, u_2, u_3, we must find a vector whose dot product with each of u_1, u_2, u_3 is 0. Similar to part (b), we can arrange u_1, u_2, u_3 in a matrix.

$M = \begin{bmatrix}u1 & u2 & u3\end{bmatrix} = \begin{bmatrix}0 & 1 & 2\\2 & -2 & 0\\1 & 0 & 2\\2 & 2 & -1\end{bmatrix}$

To find the vector we are looking for, we can solve for the null space of the transpose of M. This will satisfy the constraint that the dot product of the vector u_4 and each of u_1, u_2, u_3 is 0.

So we solve

$$M^T u_4 = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

This gives us

$$u_4 = \begin{bmatrix}-2/3\\-1/3\\2/3\\0\end{bmatrix}$$

Question 3 (a)

SOLUTION. The characteristic polynomial of A is just the determinant of $A - I\lambda$

$$A-\lambda I=\begin{bmatrix}2-\lambda & -1 & 0\\ -3 & -\lambda & -2\\ 3 & 1 & 3-\lambda\end{bmatrix}$$
$$det(A-I\lambda)=(2-\lambda)[-\lambda(3-\lambda)+2]+[-3(3-\lambda)+6]=(2-\lambda)(-3\lambda+\lambda^2+2)+3\lambda-3=-\lambda^3+5\lambda^2-5\lambda+1$$

Question 3 (b)

SOLUTION. No content found.

Question 4

SOLUTION. In order to multiply matrices, we must first ensure that the number of columns in the first matrix matches the number of rows in the second matrix. This is because we multiply the number in the first column in the first matrix by the number in the first row in the second matrix, then the number in the second column and second row, and so on (i.e. multiply the first row by the first column). The number in the first row and column of the matrix product of these two matrices will be the sum of the numbers multiplied column by row. The first row will then be multiplied by the second column, and this will make up the number in the second column, first row in the matrix product. This pattern will continue until there are no more columns in the second matrix. Then, we move on to the second row in the first matrix and do the same thing, with the sum of the products now filling out the second row in the product matrix. We will continue this pattern until there are no more rows in the first matrix.
Since we have three matrices, we will start by multiplying the first two, then multiplying this matrix product by the third matrix to get our answer. Note that the number of columns in matrix one is equal to the number of rows in matrix two, which is three.
Multiplying matrix one by two, we get

$$\begin{pmatrix}3 & 1 & 2\\ 2 & 0 & 0\end{pmatrix}\begin{pmatrix}3 & 1 & 0\\ 3 & 0 & 2\\ 5 & 0 & 0\end{pmatrix}=\begin{pmatrix}22 & 3 & 2\\ 6 & 2 & 0\end{pmatrix}$$

Now, multiply this matrix by matrix three. Again note that the number of columns in the new matrix is equal to the number of rows in matrix three. $\begin{pmatrix}22 & 3 & 2\\ 6 & 2 & 0\end{pmatrix}\begin{pmatrix}2 & 0\\ -1 & 0\\ 0 & 3\end{pmatrix}=\begin{pmatrix}41 & 6\\ 10 & 0\end{pmatrix}$
This result is the solution we are looking for.

Question 5

SOLUTION. The answer is **false**.
Let A be the 2 by 2 identity matrix. Then
$\det(-A)=\det(-I)=\det\left(\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\right)=(-1)(-1)-(0)(0)=1$
where as
$\det(A)=\det(I)=\det\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)=(1)(1)-(0)(0)=1\neq-\det(-A)$.

Question 6

SOLUTION. The answer to this statement is false. I will prove this by doing one example.
We will start by letting the matrix A be:

$$A = \begin{bmatrix} 1 & 3 & 2 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 2 & 0 & 2 & 4 \end{bmatrix}$$

and vectors b, c be

$$b = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

$$, c = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \\ 4 \end{bmatrix} .$$

We can also define x to be:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Note that there are only 4 x values because there are only 4 rows in the matrix A.
We can solve the equation $Ax = b$ by doing Gaussian Elimination on the augmented matrix $[A|b]$:

$$\begin{bmatrix} 1 & 3 & 2 & 1 & | & 1 \\ 1 & 0 & 2 & 1 & | & 0 \\ 1 & 0 & 1 & 2 & | & 1 \\ 1 & 1 & 1 & 1 & | & 2 \\ 2 & 0 & 2 & 4 & | & 2 \end{bmatrix}$$

After performing Gaussian elimination, we get the matrix in its reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & | & 4 \\ 0 & 1 & 0 & 0 & | & 1/3 \\ 0 & 0 & 1 & 0 & | & -5/3 \\ 0 & 0 & 0 & 1 & | & -2/3 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

and from this we can see that $Ax = b$ does indeed have a unique solution being

$$x = \begin{bmatrix} 4 \\ 1/3 \\ -5/3 \\ -2/3 \end{bmatrix}$$

We have satisfied the first argument that $Ax = b$ has a unique solution. Now, does c? We will again perform Gaussian Elimination, this time on the augmented matrix $[A|c]$, to see if $Ax = c$ also has a unique solution.

$$\begin{bmatrix} 1 & 3 & 2 & 1 & | & 0 \\ 1 & 0 & 2 & 1 & | & 1 \\ 1 & 0 & 1 & 2 & | & 3 \\ 1 & 1 & 1 & 1 & | & 0 \\ 2 & 0 & 2 & 4 & | & 4 \end{bmatrix}$$

After performing only a few steps of Gaussian Elimination, we come to the augmented matrix:

$$\begin{bmatrix} 1 & 3 & 2 & 1 & | & 0 \\ 0 & 1 & 0 & 0 & | & -1/3 \\ 0 & 0 & 1 & -1 & | & -2 \\ 0 & 0 & 0 & -1 & | & -7/3 \\ 0 & 0 & 0 & 0 & | & 14 \end{bmatrix}$$

This matrix is telling us that $0 * x_1 + 0 * x_2 + 0 * x_3 + 0 * x_4 = 14$, but this is impossible! Therefore, there are no solutions to $Ax = c$, and we reach the conclusion that the answer is false.

Question 7 (a)

SOLUTION. Let $\vec{y} = c\vec{v}$ $A\vec{y} = A(c\vec{v}) = c(A\vec{v}) = c(\lambda\vec{v}) = \lambda(c\vec{v}) = \lambda\vec{y}$ Therefore $c\vec{v}$ is also an eigenvector with the same eigenvalue λ

Question 7 (b)

SOLUTION.

$$\begin{aligned}(A + 7I)v = Av + 7v = \lambda v + 7v &= (\lambda + 7)v \\ (A + 7I)v &= (\lambda + 7)v \\ Eigenvalue\ is\ &\lambda + 7\end{aligned}$$

Question 7 (c)

SOLUTION.

$$\begin{aligned}Av &= \lambda v \\ Left\ multiplying\ both\ sides\ by\ &A^{-1} \\ A^{-1}Av &= A^{-1}\lambda v \\ Dividing\ both\ sides\ by\ &\lambda, \\ \frac{1}{\lambda}Iv &= A^{-1}v \\ A^{-1}v &= (\frac{1}{\lambda})v \\ So\ we\ have\ eigenvalue\ &\frac{1}{\lambda}\end{aligned}$$

Question 8 (a)

SOLUTION. No content found.

Question 8 (b)

SOLUTION. No content found.

Question 9

SOLUTION. We can make a matrix out of the vectors that span W $A = \begin{bmatrix} -1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$ $rref(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ therefore the columns of A are linearly independent and we can compute the projection matrix P using the equation
$P = A(A'A)^{-1}A'$

$(A'A) = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 4 & 14 \end{bmatrix}$

$(A'A)^{-1} = \frac{1}{3*14-4*4} \begin{bmatrix} 14 & -4 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 7/13 & -2/13 \\ -2/13 & 3/26 \end{bmatrix}$

$P = \begin{bmatrix} -1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 7/13 & -2/13 \\ -2/13 & 3/26 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -9/13 & 7/26 \\ 3/13 & 1/13 \\ 1/13 & 5/26 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

$P = \begin{bmatrix} 25/26 & -2/13 & 3/26 \\ -2/13 & 5/13 & 6/13 \\ 3/26 & 6/13 & 17/26 \end{bmatrix}$ The vector in W closest to the vector v can be found by taking the projection of $\underline{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $P \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 12/13 \\ 9/13 \\ 16/13 \end{bmatrix}$

Question 10

SOLUTION. Notice that $A = SDS^{-1}$ which is the diagonalization formula where $S = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ $A^{10} = SD^{10}S^{-1} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}^{10} \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 2^{10} \\ 5 & 2^{11} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 6-5(2)^{10} & -3+3(2)^{10} \\ 10-5(2)^{11} & -5+3(2)^{11} \end{bmatrix}$

Question 11

SOLUTION. To compute the determinant, we must perform Gaussian elimination on this matrix until it is in upper triangular form, at which point the determinant of the matrix will be the product of the main diagonal.

To get this matrix into upper triangular form, take row 2 and subtract it from row 1, then take row 3 and subtract it from row 1, and finally take row 4 and subtract it from row 1. This produces the matrix:

$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Note this matrix is in upper triangular form since there are all $0'$s below the main diagonal.

The determinant is the product of the diagonal, which is just $1*1*3*2 = 6$

Therefore, the determinant of the matrix $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 4 & 3 \\ 1 & 1 & 1 & 3 \end{bmatrix}$ is 6.

Question 12 (a)

SOLUTION. We can set up a matrix where each column represents the number of people who move from a given province to another. This matrix can be set up as follows:

$M = \begin{bmatrix} \text{Alberta} \to \text{Alberta} & \text{BC} \to \text{Alberta} & \text{Manitoba} \to \text{Alberta} \\ \text{Alberta} \to \text{BC} & \text{BC} \to \text{BC} & \text{Manitoba} \to \text{BC} \\ \text{Alberta} \to \text{Manitoba} & \text{BC} \to \text{Manitoba} & \text{Manitoba} \to \text{Manitoba} \end{bmatrix}$

Using the numbers provided, we end up with:

$M = \begin{bmatrix} 0.9 & 0.07 & 0.06 \\ 0.05 & 0.91 & 0.07 \\ 0.05 & 0.02 & 0.87 \end{bmatrix}$

Hitting a column vector $p = \begin{bmatrix} \text{population(Alberta)} \\ \text{population(BC)} \\ \text{population(Manitoba)} \end{bmatrix}$ with M will give us the new populations of each province.

Question 12 (b)

SOLUTION. Given that we are in steady state, and with the transition matrix $M = \begin{bmatrix} 0.9 & 0.07 & 0.06 \\ 0.05 & 0.91 & 0.07 \\ 0.05 & 0.02 & 0.87 \end{bmatrix}$, we have that $Ms = s$.
Reworking this a bit, we get

$$Ms = sMs - s = 0(M - 1)s = 0(M - I)s = 0$$

Solving for the null space of $(M - I)$, we get our steady state vector $s = \begin{bmatrix} 0.6700 \\ 0.6505 \\ 0.3578 \end{bmatrix}$
Given that the population of BC is 1 million, we need to multiply by $1,000,000/0.6505$ to get our final population vector. This turns out to be:
$s * 1,000,000/0.6505 = \begin{bmatrix} 1,030,000 \\ 1,000,000 \\ 550,000 \end{bmatrix}$
So the population of Manitoba is 550,000, and the population of Alberta is 1,030,000.

Good Luck for your exams!

