Full Solutions
MATH104 December 2011

April 4, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

Solution 1. The limit evaluates to the form 0/0,

$$\begin{aligned}\lim_{x\to 4}\frac{x^2-3x-4}{\sqrt{x}-2} &= \frac{4^2-3(4)-4}{\sqrt{4}-2}\\ &= \frac{16-12-4}{2-2}\\ &= \frac{0}{0}\end{aligned}$$

which is undefined. However, notice that in the denominator we can get rid of the square root by multiplying top and bottom by the conjugate

$$\lim_{x\to 4}\frac{x^2-3x-4}{\sqrt{x}-2}\cdot\frac{\sqrt{x}+2}{\sqrt{x}+2} = \lim_{x\to 4}\frac{(x^2-3x-4)(\sqrt{x}+2)}{x-4}.$$

Now notice that the quadratic polynomial on the top can factor,

$$x^2-3x-4 = (x-4)(x+1)$$

and therefore

$$\begin{aligned}&\lim_{x\to 4}\frac{(x^2-3x-4)(\sqrt{x}+2)}{x-4}\\ &= \lim_{x\to 4}\frac{(x-4)(x+1))(\sqrt{x}+2)}{x-4}\\ &= \lim_{x\to 4}(x+1)(\sqrt{x}+2)\\ &= 20.\end{aligned}$$

Therefore

$$\lim_{x\to 4}\frac{x^2-3x-4}{\sqrt{x}-2} = \frac{4^2-3(4)-4}{\sqrt{4}-2} = 20.$$

Solution 2. The limit evaluates to the form 0/0,

$$\begin{aligned}\lim_{x\to 4}\frac{x^2-3x-4}{\sqrt{x}-2} &= \frac{4^2-3(4)-4}{\sqrt{4}-2}\\ &= \frac{16-12-4}{2-2}\\ &= \frac{0}{0}\end{aligned}$$

which is undefined. 0/0 is a form for which we can apply l'Hospital's rule (if you know it):

$$\begin{aligned}
\lim_{x\to 4}\frac{x^2-3x-4}{\sqrt{x}-2} &= \lim_{x\to 4}\frac{\frac{d}{dx}\left(x^2-3x-4\right)}{\frac{d}{dx}\left(\sqrt{x}-2\right)}\\
&= \lim_{x\to 4}\frac{2x-3}{\frac{1}{2}x^{-1/2}}\\
&= \frac{5}{\frac{1}{2}4^{-1/2}}\\
&= \frac{5}{1/4}\\
&= 20
\end{aligned}$$

Solution 3. We can be a bit clever and factor $x^2-3x-4=(x-4)(x+1)$ then factor $x-4$ as a difference of squares where we think of x as the square of $\sqrt{x}$.
Hence

$$x-4=(\sqrt{x}-2)(\sqrt{x}+2).$$

Combining gives

$$\begin{aligned}
\lim_{x\to 4}\frac{x^2-3x-4}{\sqrt{x}-2} &= \lim_{x\to 4}\frac{(x-4)(x+1)}{\sqrt{x}-2}\\
&= \lim_{x\to 4}\frac{(\sqrt{x}-2)(\sqrt{x}+2)(x+1)}{\sqrt{x}-2}\\
&= \lim_{x\to 4}(\sqrt{x}+2)(x+1)\\
&= (\sqrt{4}+2)(4+1)\\
&= 20
\end{aligned}$$

Question 1 (b)

Solution. For $f(x)$ to be continuous at the point $x=1$, we need

$$2x^2-\ln(x)=x^3-4ax$$

at $x=1$. Substituting $x=1$ into this equation and solving for a yields

$$\begin{aligned}
2(1)^2-\ln(1) &= (1)^3-4a(1)\\
2-0 &= 1-4a\\
1 &= -4a\\
a &= -1/4
\end{aligned}$$

Therefore, a = -1/4.

Question 1 (c)

SOLUTION 1. One method of solving this problem is to use the quotient rule:

$$f'(x) = \frac{\left(\frac{d}{dx}\left[(x^2+3\sin^2 x)(e^{x^2})\right]\right)(x^4+7) - (x^2+3\sin^2 x)(e^{x^2})\left(\frac{d}{dx}(x^4+7)\right)}{(x^4+7)^2}$$

For the first derivative we need the product rule (and then the chain rule):

$$\begin{aligned}
\frac{d}{dx}\left[(x^2+3\sin^2 x)(e^{x^2})\right] &= \left(\frac{d}{dx}(x^2+3\sin^2 x)\right)e^{x^2} + (x^2+3\sin^2 x)\left(\frac{d}{dx}e^{x^2}\right) \\
&= (2x+6\sin x\cos x)e^{x^2} + (x^2+3\sin^2 x)(2xe^{x^2})
\end{aligned}$$

Then, by substitution

$$f'(x) = \frac{\left((2x+6\sin x\cos x)e^{x^2} + (x^2+3\sin^2 x)(2xe^{x^2})\right)(x^4+7) - (x^2+3\sin^2 x)e^{x^2}4x^3}{(x^4+7)^2}$$

Luckily, no further simplification is required for this question.

SOLUTION 2. Another method of solving this problem is to use the product rule. First re-write the denominator of the given function using exponent notation

$$\begin{aligned}
f(x) &= \frac{(x^2+3\sin^2 x)(e^{x^2})}{x^4+7} \\
&= (x^2+3\sin^2 x)(e^{x^2})(x^4+7)^{-1}
\end{aligned}$$

This is a product of three functions. The product rule can be generalized from its two-function version to:

$$\frac{d}{dx}(f\cdot g\cdot h) = \frac{df}{dx}\cdot g\cdot h + f\cdot\frac{dg}{dx}\cdot h + f\cdot g\cdot\frac{dh}{dx}$$

So taking the derivative, we obtain

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left((x^2+3\sin^2 x)(e^{x^2})(x^4+7)^{-1}\right) \\
&= \left(\frac{d}{dx}(x^2+3\sin^2 x)\right)(e^{x^2})(x^4+7)^{-1} \\
&\quad + (x^2+3\sin^2 x)\left(\frac{d}{dx}(e^{x^2})\right)(x^4+7)^{-1} \\
&\quad + (x^2+3\sin^2 x)(e^{x^2})\left(\frac{d}{dx}(x^4+7)^{-1}\right) \\
&= \left(2x+6\sin x\cos x\right)(e^{x^2})(x^4+7)^{-1} \\
&\quad + (x^2+3\sin^2 x)\left(2xe^{x^2}\right)(x^4+7)^{-1} \\
&\quad + (x^2+3\sin^2 x)(e^{x^2})\left(-4x^3(x^4+7)^{-2}\right)
\end{aligned}$$

No further simplification is required, for this question.

Question 1 (d)

SOLUTION. The following solution makes use of the product rule, rather than the quotient rule, for differentiation. Writing the given function as:

$$y(x) = (x+3)(2x+1)^{-1}$$

allows us to apply the product rule when taking the derivative:

$$\frac{dy}{dx} = y'(x) = \frac{1}{2x+1} - \frac{2x+6}{(2x+1)^2}$$

At $x = 0$, the derivative of y is equal to

$$\begin{aligned} y'(0) &= \frac{1}{2(0)+1} - \frac{2(0)+6}{(2(0)+1)^2} \\ &= 1-6 \\ &= -5 \end{aligned}$$

The general "point-slope form" for the equation of a straight line is

$$y - y_1 = m(x - x_1)$$

where (x_1, y_1) is a point on the line with slope m. We choose $x_1 = 0$ and hence $y_1 = y(0) = 3$. Plugging in these values yields

$$\begin{aligned} y - y(0) &= (-5)(x-0) \\ y - 3 &= -5x \\ y &= -5x + 3 \end{aligned}$$

Therefore, the required equation is y = -5x +3.

Question 1 (e)

SOLUTION. The absolute maximum of f may occur at the critical points (if it has any), or at the endpoints of its domain. We can solve this problem by finding the domain of f first, then by finding its critical points.

Determine the Domain of f

Because of the square root, the domain of f is defined where $2 - x^2$ is positive or zero. Therefore,

$$\begin{aligned} 0 &\leq 2 - x^2 \\ x^2 &\leq 2 \\ x &\leq \pm\sqrt{2} \end{aligned}$$

The domain of the given function is

$$[-\sqrt{2}, +\sqrt{2}]$$

Find the Critical Points of f

The critical points of f are located where its derivative is zero. Let's find the derivative of f:

$$\begin{aligned}
\frac{df}{dx} &= \frac{d}{dx}\left(x\sqrt{2-x^2}\right) \\
&= \left(\frac{d}{dx}(x)\right)\sqrt{2-x^2} + x\left(\frac{d}{dx}(\sqrt{2-x^2})\right) \\
&= \left(1\right)\sqrt{2-x^2} + x\left(\frac{1}{2\sqrt{2-x^2}}\frac{d}{dx}(2-x^2)\right) \\
&= \sqrt{2-x^2} + x\left(\frac{1}{2\sqrt{2-x^2}}(-2x)\right) \\
&= \sqrt{2-x^2} - \frac{2x^2}{2\sqrt{2-x^2}}
\end{aligned}$$

Setting this to zero and solving for x yields the critical points:

$$\begin{aligned}
0 &= \sqrt{2-x^2} - \frac{2x^2}{2\sqrt{2-x^2}} \\
\frac{2x^2}{2\sqrt{2-x^2}} &= \sqrt{2-x^2} \\
2x^2 &= 2(2-x^2) \\
2x^2 &= 4-2x^2 \\
4x^2 &= 4 \\
x^2 &= 1 \\
x &= \pm 1
\end{aligned}$$

Therefore, there are two critical points at -1 and +1.

Find Absolute Maximum on the Domain

The absolute maximum could occur at the two critical points, or at the endpoints of the domain.

$$\begin{aligned}
f(-\sqrt{2}) &= 0 \\
f(-1) &= -1 \\
f(+1) &= +1 \\
f(+\sqrt{2}) &= 0
\end{aligned}$$

The absolute maximum occurs at $x = +1$.

Question 1 (f)

SOLUTION. Taking the derivative of the given equation yields

$$\begin{aligned}
\frac{d}{dx}f'(x) &= \frac{d}{dx}\left(x\left[f(x)\right]^2 + x^2\right) \\
f''(x) &= \frac{d}{dx}\left(x\left[f(x)\right]^2\right) + \frac{d}{dx}\left(x^2\right) \\
&= \left[f(x)\right]^2 + x\left[2f(x)f'(x)\right] + 2x
\end{aligned}$$

When x=1, f is equal to 2, so

$$\begin{aligned} f''(1) &= \big[f(1)\big]^2 + (1)\big[2f(1)f'(1)\big] + 2(1) \\ &= \big[2\big]^2 + \big[2(2)f'(1)\big] + 2 \\ &= 6 + 4f'(1). \end{aligned}$$

We need the first derivative of f at $x = 1$. Using the expression we are given for the derivative of f,

$$\begin{aligned} f'(1) &= (1)\big[f(1)\big]^2 + (1)^2 \\ &= 2^2 + 1 \\ &= 5. \end{aligned}$$

Therefore,

$$\begin{aligned} f''(1) &= 6 + 4f'(1) \\ &= 6 + 4(5) \\ &= 26. \end{aligned}$$

Question 1 (g)

SOLUTION. The given function is increasing where its first derivative is positive. Taking the derivative yields

$$f'(x) = 2x - 1/x$$

We must now solve the inequality for x yields the interval where f is increasing. We will only be interested in positive values of x, as negative values of x are not in the domain of the given function (because of the natural logarithm). Therefore,

$$\begin{aligned} 0 &< 2x - 1/x, \quad x > 0 \\ 1/x &< 2x \\ 1 &< 2x^2 \\ \frac{1}{2} &< x^2 \end{aligned}$$

Therefore, we have

$$x > +\frac{1}{\sqrt{2}}$$

The negative solution was ruled out because of the requirement that x be a positive number.
The interval on which f is increasing is

$$\left(\frac{1}{\sqrt{2}}, \infty\right)$$

Question 1 (h)

SOLUTION. Let the value of the stamp collection at time t be $v(t)$, and the time at which the collection was purchased be time $t = 0$. Then

$$v(0) = 10$$

million dollars, and

$$v(t) = 10e^{0.12t}.$$

If we wait until the collection has tripled in value, then we would be waiting until $v(t) = 30$ million dollars. So, we solve the following equation for t:

$$\begin{aligned} 30 &= 10e^{0.12t} \\ 3 &= e^{0.12t} \\ \ln(3) &= 0.12t \\ t &= \frac{\ln(3)}{0.12} \end{aligned}$$

Therefore, we'll be waiting

$$t = \frac{\ln(3)}{0.12}$$

years.
Note: This is a little more than 9 years. Also, using the rules of logarithms, we could also express the solution as

$$t = \ln(3^{25/3})$$

Whether this simplification would be necessary is up to your instructor.

Question 1 (i)

SOLUTION. To determine the equation of the tangent line we will need the slope of the line at the given point. To find the slope at the given point, we may use implicit differentiation and find an expression for y'.

$$\begin{aligned} \frac{d}{dx}\left(xy^2 + 2xy\right) &= \frac{d}{dx}(8) \\ y^2 + 2xyy' + 2y + 2xy' &= 0 \end{aligned}$$

At the point (1,2), this equation becomes

$$\begin{aligned} (2)^2 + 2(1)(2)y' + 2(2) + 2(1)y' &= 0 \\ 4 + 4y' + 4 + 2y' &= 0 \\ 6y' &= -8 \\ y' &= -4/3 \end{aligned}$$

Using the "point-slope" equation of a straight line with slope $m =$ y' that passes through the point (x_0,y_0)

$$y - y_0 = m(x - x_0)$$

we have

$$y - (2) = (-\frac{4}{3})\big(x - (1)\big)$$
$$y = -\frac{4}{3}x + \frac{10}{3}$$

which is the equation of the tangent line at (1,2).

Question 1 (j)

SOLUTION. We can consider statements I, II, III individually to which of them are true.

Statement I: f + g is Concave Up Everywhere

Let

$$H = f + g$$

Statement I is true if the second derivative of H is positive for all real numbers x. The second derivative of H is

$$H'' = f'' + g''.$$

However,

$$f''(x)$$

and

$$g''(x)$$

must be positive for all real numbers (because f' and g' are continuous and increasing for all x), so

$$H''(x)$$

must be positive for all x. Statement I is true.

Statement II: f - g is Concave Up Everywhere

If we let

$$H = f - g$$

then

$$H'' = f'' - g''.$$

However,

$$f''(x)$$

could be less than

$$g''(x)$$

for all real numbers, so

$$H''(x)$$

may not be positive for all x. Statement II is false.

Statement III: fg is Concave Up Everywhere

If we let

$$H = f \cdot g$$

then

$$H'' = f''g + 2f'g' + fg''.$$

We know that

$$f''(x), g''(x)$$

are both positive for all real numbers, but we don't necessarily have that

$$f'(x), g'(x), f(x), g(x)$$

are necessarily positive. So it is possible that $f''g + fg''$ is negative and outweighs the positive term $2f'g'$. For example, choose

$$f(x) = x^2 - 1, \; g(x) = x^2$$

Then

$$f'(x) = g'(x) = 2x$$

is increasing but

$$f''g + 2f'g' + 2fg'' = (2)(x^2) + 2(2x)(2x) + (x^2 - 1)(2) = 12x^2 - 2$$

which is negative for small values of x (e.g. when $x = 0$).
Statement III is false.

The Answer

The answer is (A) I only.

Question 1 (k)

SOLUTION. The correct answer is (d). Let's examine each possibility.
(a) Since $g(x)$ is continuous, we can use the extreme value theorem which says exactly what (a) is telling us - there is a maximum value on the interval $[0, 1]$
(b) This is saying that if two values are equal then their function values are equal. Since a function can only take one output value for each input value, this is a true statement.
(c) Since $g(0) = 1 > 1/2 > 0 = g(1)$ and $g(x)$ is continuous, then the intermediate value theorem tells us exactly what this statement says, there is a value c in $[0, 1]$ such that $g(c) = 1/2$
(d) This statement is false. The difference between (c) and this problem is that the number $3/2$ does not lie between $g(0)$ and $g(1)$. A counter example would be a straight line joining $g(0)$ and $g(1)$. This can be described by $y = -x + 1$.
(e) This is true as this is the definition of continuity and $g(x)$ is given to be continuous.

Question 1 (l)

SOLUTION. If f is differentiable at a, then f must also be continuous at a. Hence, the limit

$$\lim_{x \to a} f(x)$$

exists and is equal to $f(a)$. The correct answer is therefore **B**.

Question 1 (m)

SOLUTION. We wish to minimize the quantity $S = m + n$ subject to $mn = 50$. Using our constraint we rewrite S as

$$S = m + \frac{50}{m}.$$

To determine the minimum of this quantity S, we first find the critical points of S:

$$\frac{dS}{dm} = 1 - \frac{50}{m^2} = 0 \quad \to \quad m^2 = 50.$$

Dropping the negative solution, we find

$$m = \sqrt{50}.$$

The corresponding value for n is determined from the constraint $mn = 50$, and is therefore equal to

$$n = 50/\sqrt{50} = \sqrt{50}.$$

Is this critical point a local minimum? Apply the second derivative test:

$$\frac{d^2S}{dm^2} = \frac{100}{m^3} > 0$$

at the critical point. Therefore, this critical point is a minimum of S and the two values m, n that have product equal to 50 and the smallest sum are

$$m = \sqrt{50},\ \ n = \sqrt{50}.$$

Question 1 (n)

SOLUTION. The linear approximation, $L(x)$, to a function, $f(x)$, at the point a, is given by

$$L(x) = f(a) + f'(a)(x - a).$$

From the information given we can write the linear approximation to the function $f(x) = x^{1/3}$ at $a = 8$ as follows:

$$\begin{aligned} L(x) &= 8^{1/3} + \frac{1}{3}8^{-2/3}(x-8) \\ &= 2 + \frac{1}{12}(x-8). \end{aligned}$$

Thus, the linear approximation gives us the approximate value for $(7.5)^{1/3}$:

$$\begin{aligned} (7.5)^{1/3} \approx L(7.5) &= 2 + \frac{1}{12}(7.5-8) \\ &= 2 - \frac{1}{24} \\ &= \frac{47}{24}. \end{aligned}$$

(This answer is already expressed in lowest terms.)

Question 2 (a)

SOLUTION. To determine where *f'(x)=0*, we look at the numerator of *f'(x)* and note that *f'(x)=0* when the numerator is zero, i.e.

$$(x^2 - 1)(x^2 - 6) = 0,$$

which implies x^2-1=0 or x^2-6=0. Solving each equation and noting that both positive and negative solutions are valid, we obtain

$$f'(x) = 0 \text{ if and only if } x = \pm 1 \text{ or } x = \pm\sqrt{6}.$$

To determine where *f'(x)* does not exist, we set the denominator to 0:

$$x^2 - 3 = 0.$$

So, *f'(x)* does not exist when $x = \pm\sqrt{3}$.

"Remark: Note that the denominator and numerator are never simultaneously 0. If they were, say at x=a, we would have to take the limit of f'(x) for $x \to a$ to determine if *f'(x)* did not exist or was zero (or another finite number)."

Question 2 (b)

SOLUTION. To determine where *f"(x)=0*, we look at the numerator of *f"(x)* and note that *f"(x)=0* when the numerator is zero, i.e.

$$2x(x^2 + 9) = 0.$$

Since x^2+9=0 has no real solutions, the only value of x that satisfies the above equation is when x=0.
To determine where *f"(x)* does not exist, we set the denominator to 0, i.e.

$$x^2 - 3 = 0.$$

So, *f"(x)* does not exist when $x = \pm\sqrt{3}$.
"Remark: Note that the denominator and numerator are never simultaneously 0. If they were, say at x=a, we would have to take the limit of *f"(x)* as $x \to a$ to determine if *f"(x)* did not exist or was zero (or another finite number)."

Question 2 (c)

SOLUTION 1. Factoring the derivative yields

$$f'(x) = \frac{(x-1)(x+1)(x-\sqrt{6})(x+\sqrt{6})}{(x-\sqrt{3})^2(x+\sqrt{3})^2}$$

Now the following sign table is straight forward:

	$(-\infty;-\sqrt{6})$	$(-\sqrt{6};-\sqrt{3})$	$(-\sqrt{3};-1)$	$(-1;1)$	$(1;\sqrt{3})$	$(\sqrt{3};\sqrt{6})$	$(\sqrt{6};\infty)$
$x-1$	$-$	$-$	$-$	$-$	$+$	$+$	$+$
$x+1$	$-$	$-$	$-$	$+$	$+$	$+$	$+$
$x-\sqrt{6}$	$-$	$-$	$-$	$-$	$-$	$-$	$+$
$x+\sqrt{6}$	$-$	$+$	$+$	$+$	$+$	$+$	$+$
$(x-\sqrt{3})^2$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$(x+\sqrt{3})^2$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$f'(x)$	$+$	$-$	$-$	$+$	$-$	$-$	$+$
$f(x)$	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$

And so we can conclude that the function is increasing on:

$$(-\infty, -\sqrt{6}) \cup (-1, 1) \cup (\sqrt{6}, \infty)$$

and decreasing on:

$$(-\sqrt{6}, -\sqrt{3}) \cup (-\sqrt{3}, -1) \cup (1, \sqrt{3}) \cup (\sqrt{3}, \sqrt{6})$$

Solution 2. **If you don't like a sign table:**
From Question 2 (a), we know that the critical points of $f(x)$, when $f'(x)$ or does not exist, are at $x = \pm 1$, $\pm$, and $\pm$.

- Testing a value less than -, such as -3, we get f'(-3) = 2/3 > 0.
- Testing a point between - and -, such as -2, yields f'(-2) = -2 < 0.
- Testing a point between - and -1, such as -1.5, we get f'(-1.5) = -25/3 < 0.
- Finally, we test a point between -1 and 1, such as 0, and get f'(0) = 2/3 > 0.

Normally, we would have to test points between the positive critical points as well, but since our function only has x^2 terms in it (*i.e. f ' is even*), we know that our negative test points will have identical values to their corresponding positive test points.
(For example, testing -2 and 2 produce identical values of -2, so $f(x) < 0$ between - and - as well as between and .)
Therefore: $f(x)$ is

- increasing on the intervals $(-\infty, -\sqrt{6}) \cup (-1, 1) \cup (\sqrt{6}, \infty)$

and

- decreasing on the intervals $(-\sqrt{6}, -\sqrt{3}) \cup (-\sqrt{3}, -1) \cup (1, \sqrt{3}) \cup (\sqrt{3}, \sqrt{6})$.

Note that the critical points are not included in the intervals of increase and decrease.

Question 2 (d)

Solution. In order to find the concavity of the function, we first try to find points where the second derivative is zero or undefined. These points will provide endpoints of intervals for concavity.
From part (b) we know that $f''(x)$ is zero exactly when $x = 0$, and that $f''(x)$ is undefined for $x = \pm$
We can now either make a sign table or pick test points between those values to determine whether we are concave up (second derivative positive) or concave down (second derivative negative).
For convenience, recall that

$$f''(x) = \frac{2x(x^2+9)}{(x^2-3)^3} = \frac{2x(x^2+9)}{(x+\sqrt{3})^3(x-\sqrt{3})^3}$$

	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, 0)$	$(0, \sqrt{3})$	$(\sqrt{3}, \infty)$
$2x$	$-$	$-$	$+$	$+$
x^2+9	$+$	$+$	$+$	$+$
$(x+\sqrt{3})^3$	$-$	$+$	$+$	$+$
$(x-\sqrt{3})^3$	$-$	$-$	$-$	$+$
$f''(x)$	$-$	$+$	$-$	$+$
Concavity	Concave Down	Concave Up	Concave Down	Concave Up

Therefore $f(x)$ is concave up on $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$ and concave down on $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$.

Question 2 (e)

Solution. **Local Maxima and Minima**
To find local maxima and minima, we consider critical points which are points where the derivative is zero or does not exist. In part (c), we got that the critical points were

$$x = \pm 1, \pm\sqrt{3}, \pm\sqrt{6}.$$

In that same part we got that the function was

- increasing on $(-\infty, -\sqrt{6}) \cup (-1, 1) \cup (\sqrt{6}, \infty)$
- decreasing on $(-\sqrt{6}, -\sqrt{3}) \cup (-\sqrt{3}, -1) \cup (1, \sqrt{3}) \cup (\sqrt{3}, \sqrt{6})$.

1. We can immediately exclude - and as potential maximum or minimum because the function is not even defined there.
2. We notice that to the left of *x=-* the function is increasing and then on the right it is decreasing. Therefore, we have a *local maximum* at *x=-*.
3. To the left of *x=-1*, the function is decreasing and then to the right, it is increasing. Therefore *x=-1* is a *local minimum*.
4. To the left of *x=1*, the function is increasing and to the right it is decreasing. Therefore *x=1* is a *local maximum*.
5. To the left of *x=* the function is decreasing and to the right it is increasing. Therefore *x=* is a *local minimum*.

Therefore we conclude that we have two local maxima at $x = -\sqrt{6}, 1$ and two local minima at $x = -1, \sqrt{6}$.
Inflection Points
To find inflection points, we consider where the second derivative is zero or does not exist. In part (d) we got that the second derivative vanished or failed to exist at

$$x = 0, \pm\sqrt{3}.$$

In that same part we got that the function was

- concave down on $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$ and
- concave up on $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$.

1. Like with the max and min test, we immediately exclude $x = \pm\sqrt{3}$ because these are vertical asymptotes. While the concavity may change on either side of these points, they are not technically defined as inflection points.
2. We see that the only point left to test is *x=0*. We notice that the function is concave up to the left of *x=0* and concave down to the right. Therefore, we conclude that since the concavity changes across this point, *x=0*, is an *inflection point*.

Exact coordinates
We are asked to get the coordinates and we sub the x values for our maxima, minima, and inflection point into the function to find the y values. Therefore we conclude that

- $\left(-\sqrt{6}, 1 - \frac{4\sqrt{6}}{3}\right)$ is a local maxima,
- $\left(-1, \frac{1}{2}\right)$ is a local minima,
- $\left(1, \frac{3}{2}\right)$ is a local maxima,
- $\left(\sqrt{6}, 1 + \frac{4\sqrt{6}}{3}\right)$ is a local minima, and
- $(0, 1)$ is an inflection point.

Question 2 (f)

Solution. **Vertical Asymptotes**
We know that vertical asymptotes occur when the denominator is zero. Here our denominator is

$$x^2 - 3$$

which is zero if $x=\pm\sqrt{3}$. Therefore we have vertical asymptotes at $x=-\sqrt{3}$ and $x=\sqrt{3}$.
To see how the function looks near the asymptotes we could take some limits. However, from part (d) we have that the function is decreasing to the left of $x=-\sqrt{3}$ as well as to the right. Therefore we conclude that

$$\lim_{x \to -\sqrt{3}^-} f(x) = -\infty$$

while

$$\lim_{x \to -\sqrt{3}^+} f(x) = +\infty$$

We have the exact same conclusion about the other vertical asymptote at $x=\sqrt{3}$ since it has the same decreasing properties as $x=\sqrt{3}$.
Horizontal or Slant Asymptotes
We check for horizontal or slant asymptotes. Since the degree of the numerator is larger than the degree of the denominator then we do not have any horizontal asymptotes. However, the degree of the numerator is exactly one larger than the degree of the denominator and so we do expect slant asymptotes. To figure out the slant asymptote we perform polynomial long division

$$\begin{array}{rrrrrrrrrrr} & & & & x & + & 1 & & & & \\ \cline{5-11} x^2 & + & 0x & - & 3) \; x^3 & + & x^2 & - & 2x & - & 3 \\ & & & & x^3 & + & 0x^2 & - & 3x & & \\ \cline{5-11} & & & & & & x^2 & + & x & - & 3 \\ & & & & & & x^2 & + & 0x & - & 3 \\ \cline{5-11} & & & & & & & & x & & \end{array}$$

Therefore we write $x^3 + x^2 - 2x - 3 = (x^2 - 3)(x + 1) + x$. So we get that

$$\begin{aligned} \frac{x^3 + x^2 - 2x - 3}{x^2 - 3} &= \frac{(x^2 - 3)(x + 1) + x}{x^2 - 3} \\ &= x + 1 + \frac{x}{x^2 - 3} \end{aligned}$$

Notice that for the last term,

$$\lim_{x \to \infty} \frac{x}{x^2 - 3} = 0.$$

Therefore as x gets big we see that

$$\frac{x^3 + x^2 - 2x - 3}{x^2 - 3}$$

looks like $x+1$. If we take x going to $-\infty$ we see that the limit of the last term still vanishes and it still looks like $x+1$. Therefore, we conclude that $x+1$ is a slant asymptote to the function.
Summary
Finally then we conclude that the equations for the vertical asymptotes are

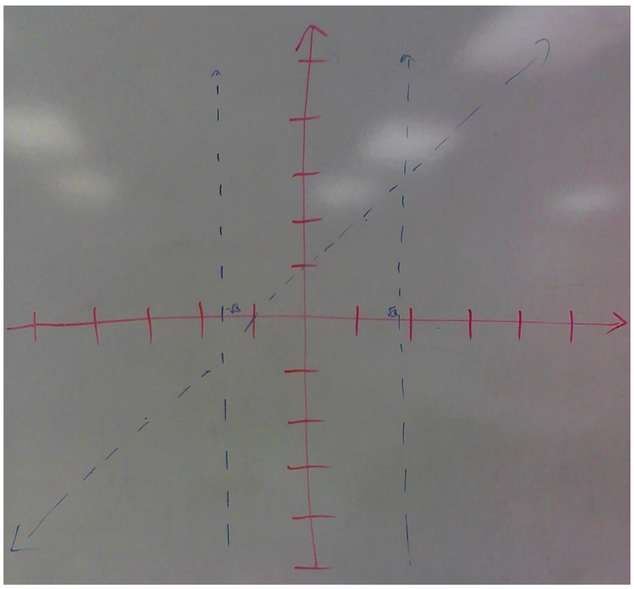
$$x = -\sqrt{3}$$
$$x = \sqrt{3}$$

while the equation of the slant asymptote is

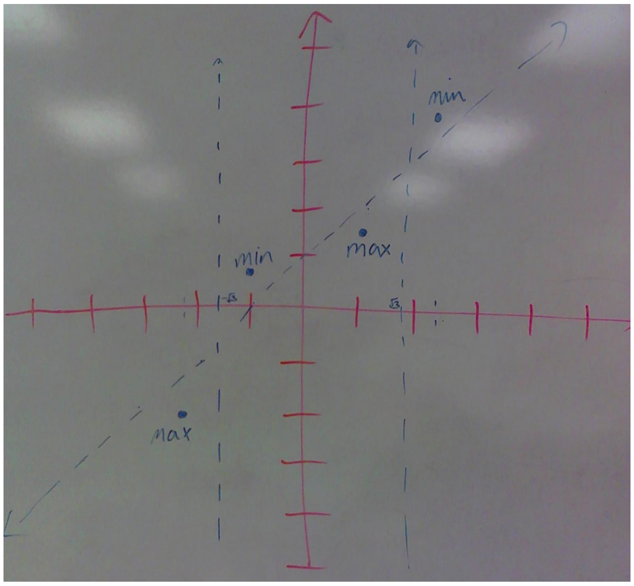
$$y = x + 1.$$

Question 2 (g)

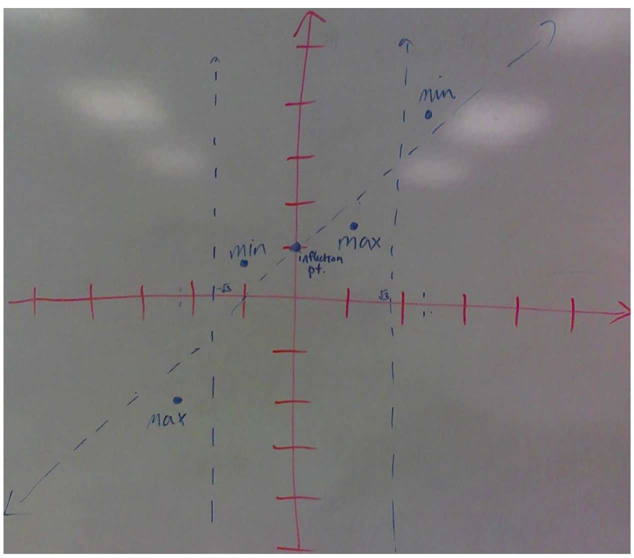
SOLUTION. Following the order given in Hint 2, we first draw the asymptotes.



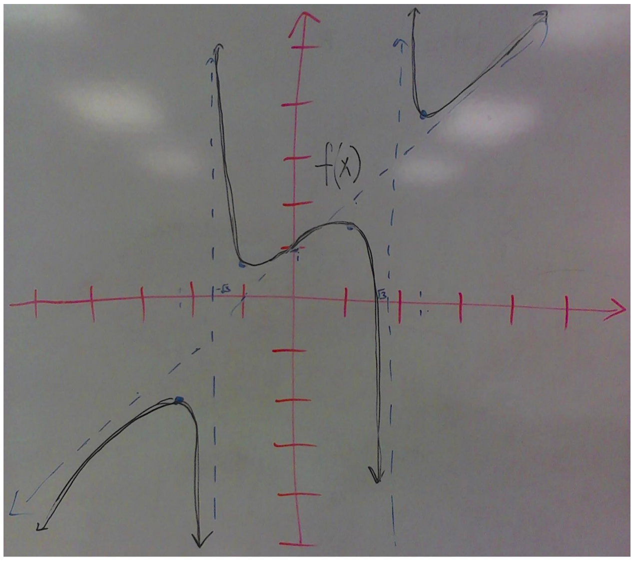
We then draw the maxima and minima.



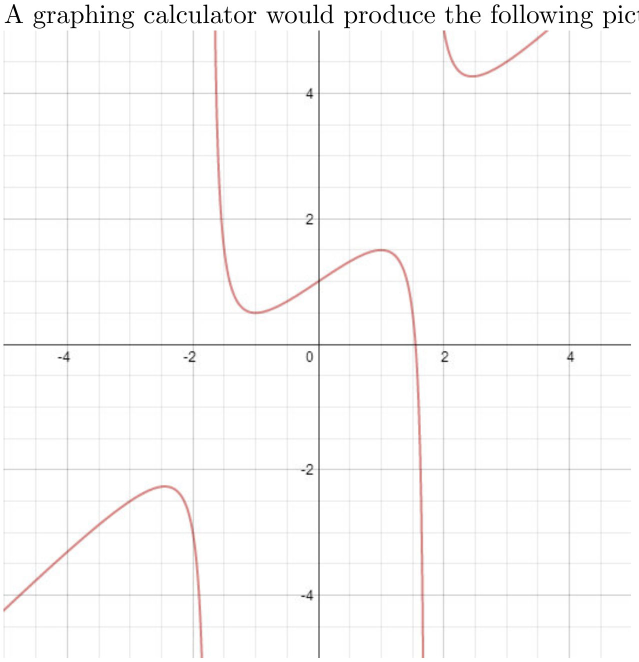
We draw the inflection point.



Finally, we use information about where the function is increasing and decreasing, as well as its concavity, to connect these points and finish the graph.



A graphing calculator would produce the following picture:



Question 3

SOLUTION. Let's call the dimension of the length and width of the base x (they are the same since it is a **square** base. Let's call the height of the box y. We want to find the dimensions of a box (x and y) that minimizes the **cost** of building it but such that it has a prescribed volume (16m^3). The volume requirement is known as our **constraint** and it allows us to find a relationship between our two variables so that we have only one independent variable. The volume is

$$V = x^2y = 16$$

and so we conclude that

$$y = \frac{16}{x^2}.$$

We now need to consider the cost. The total cost will be the sum of building the 4 sides (C_S), the base (C_B), and the top (C_T),

$$C = 4C_S + C_B + C_T.$$

We have that the cost per square metre of the sides is \$20 but for now let's call this number m. We then have that

$$C_S = mA_S$$

where A_S is the area of the sides. We know the top costs half as much per square metre and so,

$$C_T = \frac{m}{2}A_T.$$

Similarly the base costs twice as much per square metre,

$$C_B = 2mA_B.$$

where A_B and A_T are the base and top areas respectively. In order to continue, we need to write each surface area in terms of our independent variable x. The surface area of both the base and the top is

$$A_B = A_T = x^2.$$

The surface area of each of the sides is

$$A_S = xy = x\frac{16}{x^2} = \frac{16}{x}$$

where we have used our relationship for y in terms of x. We can now write the cost

$$\begin{aligned} C &= 4C_S + C_B + C_T \\ &= 4m\frac{16}{x} + 2mx^2 + \frac{m}{2}x^2 \\ &= m\left(\frac{64}{x} + \frac{5}{2}x^2\right). \end{aligned}$$

We seek to find critical points of this function and therefore we take its derivative and set it to zero,

$$\frac{\mathrm{d}C}{\mathrm{d}x} = m\left(-\frac{64}{x^2} + 5x\right) = 0$$

and we see that the only critical point is

$$x^3 = \frac{64}{5}.$$

Before continuing, it is worth noting that the cost per square metre of the sides (m=\$20) was irrelevant information. In order to test if our critical point is indeed a minimum we will use the second derivative test (since there is no closed interval we cannot test the endpoints). The second derivative is,

$$\frac{\mathrm{d}^2C}{\mathrm{d}x^2} = m\left(\frac{128}{x^3}+5\right).$$

If we sub in our critical point,

$$\frac{\mathrm{d}^2C}{\mathrm{d}x^2} = m\left(128\frac{5}{64}+5\right) = m(10+5) = 15m = 300$$

where here we have used the fact that m=20. Since the second derivative is positive, the objective function, C is concave up and thus we have a minimum at our critical point. Therefore the dimensions that minimize the cost of producing the box with volume $16\mathrm{m}^3$ is

$$x = \left(\frac{64}{5}\right)^{1/3} \qquad y = \frac{16}{x^2} = 5^{2/3}.$$

Question 4

SOLUTION. We begin by assembling all the information we know about the problem:

1. The volume of water in the conical tank, which we will denote as V_1 is given by the formula $V_1 = 1/3\pi r^2 h_1$ where h_1 is the height of water in the tank.
2. The volume of water in the cylindrical tank, which we will denote as V_2 is given by the formula $V_2 = \pi r^2 h_2$ where h_2 is the height of water in the tank.
3. Due to similar triangles, we know that the ratio of radius/height within the conical tank is always constant, thus for the conical tank $r/h = 4/5$.
4. The rate of change of the volume of water in the first tank is dV_1/dt. Because the water is pouring from the first tank into the second one, the change in the volume of water in the second tank, or dV_2/dt will be the same as $-dV_1/dt$.
5. The height of the water in the conical tank is dropping by .5 m/min, which is given by the derivative $dh_1/dt = -0.5$.
6. We are looking for the change of the water's height in the second tank, which is given by dh_2/dt.

We note that for this problem, we are mostly concerned with change in height and change in volume, not change in radius. Therefore, we will rewrite our two volume formulas in terms of h as follows:
First, we use fact #3 and solve for r in terms of h, or $r = \frac{4}{5}h$. We can then rewrite the formula given in #1 as a function of h_1.

$$V_1(h_1) = \frac{1}{3}\pi\left(\frac{4}{5}\right)^2 (h_1)^3.$$

For the volume formula given in #2, the radius doesn't change (because the tank has straight sides) and equals *4*. So we have a volume formula as a function of h_2.

$$V_2(h_2) = \pi(4)^2 h_2.$$

We now have two volume formulas in terms of h. Because the question is asking about change over time, we will implicitly differentiate with respect to t. This gives:

$$\frac{dV_1}{dt}(h_1) = \pi \left(\frac{4}{5}\right)^2 (h_1)^2 \frac{dh_1}{dt}$$

for the first tank and

$$\frac{dV_2}{dt}(h_2) = \pi (4)^2 \frac{dh_2}{dt}$$

for the second tank.
If we consider the first formula, we realize from #5 that we know $dh_1/dt = -0.5$ and furthermore, the statement of the question tells us that $h = 3$. We plug these into the first formula to get

$$\begin{aligned} \tfrac{dV_1}{dt}(3) &= \pi \left(\tfrac{4}{5}\right)^2 (3)^2(-0.5) \\ &= -\tfrac{72}{25}\pi \end{aligned}$$

Now we remember from #4 that $dV_2/dt = -dV_1/d_2$. So $dV_2/dt = (72/25)\pi$. Plugging that into our second volume formula we get

$$\frac{72}{25}\pi = \pi 16 \frac{dh_2}{dt}$$

Finally, we know from #6 that we're trying to find dh_2/dt. Thus we simply solve the previous equation for dh_2/dt to get:

$$\frac{dh_2}{dt} = \frac{9}{50}$$

Which is our answer.

Question 5 (a)

SOLUTION. We can calculate the slope of the line by using two points on the line, say $(0, 8)$ and $(8, 0)$. These two points give us a slope of -1.
Because the line is tangent to the graph at $(p = 2, q = 6)$ and its slope is -1, we know that $\frac{dq}{dp} = -1$ at $(2, 6)$. Thus, by using the elasticity formula above, we get $\epsilon = \frac{p}{q}\frac{dp}{dq} = \frac{2}{6}(-1) = -1/3$.

Question 5 (b)

SOLUTION. As $\epsilon = -1/3 > -1$, the product is price inelastic and to increase revenue, price should be increased.

Question 5 (c)

SOLUTION. The elasticity approximates the percentage change in demand over the percentage increase in price.
If x denotes the percent change in price, at p=2\$, for a 5% increase in demand then we need to find x so that $\epsilon = 5/x$. Thus, using the elasticity calculated in the previous question,

$$\begin{aligned} x &= 5/\epsilon \\ &= 5/(-1/3) \\ &= 5(-3) \\ &= -15\% \end{aligned}.$$

This is a negative number, as we'd expect (to increase demand, the price must decrease).
The price should be decreased by about 15% to yield a 5% increase in demand.

Question 5 (d)

SOLUTION. To maximize revenue, we need the elasticity $\epsilon = \frac{p}{q}\frac{dq}{dp} = -1$, which is equivalent to stating $\frac{dp}{dq} = -\frac{q}{p}$.
Consider the line joining the origin to the point $(p, q = f(p))$ on the demand vs price curve. As the hint suggests, the slope of such a line is q/p, which is the negative of what we want $\frac{dp}{dq}$ to be at the price where revenue is maximized.
Thus, to maximize the revenue, we can take a series of lines joining the origin to the point $(p_0, f(p_0))$ for different values of p_0 and determine the line such that if it is reflected across the line $p = p_0$ (so its slope is negated), it is the tangent line to the graph $q = f(p)$ at the point (p_0, q_0). The slope of the tangent line is $\frac{dq}{dp}|_{p=p_0,q=q_0}$.
This has a nice geometric interpretation: the line segment joining the origin to the point (p_0, q_0) has same length as the line segment of the tangent line joining (p_0, q_0) to the $p-$axis. In fact, an isosceles triangle is formed.
Note: it is tempting to say that the revenue is maximized at p_0 if the line joining the origin to (p_0, q_0) meets the curve $q = f(p)$ at a right angle (recall two curves are orthogonal if the product of their slopes at their intersection point is -1). But the slope of the line is q/p and not p/q, so this doesn't hold. We need $\frac{p}{q}\frac{dq}{dp} = -1$ and not $\frac{q}{p}\frac{dq}{dp} = -1$.

Question 6 (a)

SOLUTION. Beginning with definition of the linear approximation of a function $f(x)$ at a point a, we find that the linear approximation to $\ln(x)$ at $a = 1$ is given by:

$$\begin{aligned} L(x) &= f(a) + f'(a)(x-a) \\ &= \ln(1) + \frac{1}{1}(x-1) \\ &= x - 1 \end{aligned}$$

We used that $f'(x) = 1/x$. Thus the approximation for $\ln(0.9)$ is:

$$\ln(0.9) \approx L(0.9) = -0.1$$

Question 6 (b)

SOLUTION. Here we are working with $x = 0.9$ and $a = 1$. We find that the error takes the following value

$$|\text{Error}| \leq \frac{M}{2}(0.9-1)^2 = \frac{M}{200}$$

where M is the maximum value of the absolute value of the second derivative on the interval [0.9, 1]. In other words, we would like to estimate how large the second derivative can be on this interval.
Given that

$$|f''(x)| = \left|\frac{-1}{x^2}\right| = \frac{1}{x^2}$$

and since this is a decreasing function on the interval [0.9, 1] we can deduce that the maximum value it takes on the interval is at $x = 0.9$ and hence

$$M = \frac{1}{0.9^2} = \frac{100}{81}$$

This allows us to find the upper bound on the error:

$$|\text{Error}| \leq \frac{\frac{100}{81}}{200} = \frac{1}{162}$$

Question 6 (c)

SOLUTION. We find the second derivative of f(x)=ln(x),

$$\begin{aligned} f'(x) &= \frac{1}{x} \\ f''(x) &= -\frac{1}{x^2}. \end{aligned}$$

By looking at the sign of the second derivative of the function $\ln(x)$, we see that the function is concave down for all x. Thus, the linear approximation is too large. (One can see this visually by drawing a tangent line to the $\ln(x)$ function at any point on the curve).
Since, we know the linear approximation overestimates the value $\ln(0.9)$, we know that the true value of $\ln(0.9)$ must lie within the interval:
$\left[-0.1 - \frac{1}{162}, -0.1\right]$,
where we have used our upper bound on the error from part (b) to help define the lower bound on our interval.

Question 6 (d)

SOLUTION. Beginning with definition of the quadratic approximation of a function $f(x)$ at a point a, we find that the quadratic approximation to $\ln(x)$ at $a = 1$ is given by:

$$\begin{aligned} Q(x) &= f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 \\ &= \ln(a) + \frac{1}{a}(x-1) + \frac{1}{2}\left(\frac{-1}{a^2}\right)(x-1)^2 \\ &= \ln(1) + \frac{1}{1}(x-1) + \frac{1}{2}\left(\frac{-1}{1^2}\right)(x-1)^2 \\ &= 0 + (x-1) - \frac{1}{2}(x-1)^2 \end{aligned}$$

Thus the quadratic approximation for $\ln(0.9)$ is:

$$\ln(0.9) \approx Q(0.9) = -0.1 - \frac{0.01}{2} = -0.105$$

Question 6 (e)

Solution. We expect that the error term will be larger than or equal to the magnitude of the second-order term in the quadratic approximation. The reason for this is because the two terms have a similar form, but in the error, the M term replaces the $f''(a)$ term in the quadratic correction, where M is the maximum value of $|f''(x)|$ on the interval $[0.9, 1]$. Indeed the error term was greater than the magnitude of the quadratic correction term in the quadratic approximation: $(1/162 = 0.0061... > 0.005)$

Good Luck for your exams!