Full Solutions
MATH307 December 2012

April 5, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

SOLUTION. Slopes of *p(x) = 0* at $x = 0$, $x = 2$ implies $p'(0) = p'(2) = 0$

$$p'(x) = 3a_1x^2 + 2a_2x + a_3$$

When $p'(0) = 0$, $3a_1(0)^2 + 2a_2(0) + a_3 = 0$
When $p'(2) = 0$, $3a_1(2)^2 + 2a_2(2) + a_3 = 0 \rightarrow 12a_1 + 4a_2 + a_3 = 0$
Hence, matrix A is

$$A = \begin{bmatrix} 12 & 4 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Question 1 (b)

SOLUTION. In part (a) we found that

$$A = \begin{bmatrix} 12 & 4 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The matrix A has 2 pivot columns and so by rank-nullity theorem

$$\dim(N(A)) = 4 - \dim(R(A)) = 4 - 2 = 2$$

and so there are two basis vectors for the nullspace. To find them we could row reduce A or notice that, at *x = 0*,

$$3a_1(0)^2 + 2a_2(0) + a_3 = 0$$

gives $a_3 = 0$. Using the value at *x=2*

$$12a_1 + 4a_2 + a_3 = 12a_1 + 4a_2 = 0,$$

gives $a_2 = -3a_1$. Since neither equation depends on a_4 then it is a free variable. Hence, the basis of *N(A)* is
$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} a_1 \\ -3a_1 \\ 0 \\ a_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \\ 0 \end{bmatrix} a_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} a_4$
Hence, $\vec{a_1} = \begin{bmatrix} 1 \\ -3 \\ 0 \\ 0 \end{bmatrix}, \vec{a_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Question 1 (c)

SOLUTION. $p(x)$ passes through $(0, 1), (1, 2)$ and $(2, 2)$
Mathematically, for *p(0)=1*,

$$a_1(0)^3 + a_2(0)^2 + a_3(0) + a_4 = a_4 = 1,$$

for $p(1)=2$,

$$a_1(1)^3 + a_2(1)^2 + a_3(1) + a_4 = a_1 + a_2 + a_3 + a_4 = 2,$$

and for $p(2)=2$,

$$a_1(2)^3 + a_2(2)^2 + a_3(2) + a_4 = 8a_1 + 4a_2 + 2a_3 + a_4 = 2.$$

The systems above can be written in a form of $B\vec{a} = \vec{b}$

$$\begin{bmatrix} 8 & 4 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

where

$$B = \begin{bmatrix} 8 & 4 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \vec{b} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

Question 1 (d)

SOLUTION. Since $\vec{a} = s_1\vec{a_1} + s_2\vec{a_2} = s_1 \begin{bmatrix} 1 \\ -3 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$

Using the equation from part (c), we have

$B\vec{a} = \vec{b}$

$$\begin{bmatrix} 8 & 4 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \vec{s} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

Therefore,

$$C = \begin{bmatrix} -4 & 1 \\ -2 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \vec{c} = \vec{b} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}.$$

Notice that for there to be a solution we need $\vec{c}$ in the range of C, $R(C)$. We know that $R(C)$ is the orthogonal complement to the nullspace of C^T, $N(C^T)$. Therefore, onsider the basis vector of $N(C^T)$

$$\begin{bmatrix} -4 & -2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vec{x} = 0 \rightarrow \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \vec{x} = 0$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} x_3$$

Since $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = 2 - 4 + 1 \neq 0$

In other words, $N(C^T)$ is not orthogonal to $\vec{c}$. This means that $\vec{c}$ cannot possibly be in the range of C and so there will be no solution to the problem.

Question 3 (b)

SOLUTION 1. If S is the null space of the matrix A from part (a), then the basis for S can be found as the basis of $N(A)$. Let's choose the simplest form $A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$.

We see that we choose x_2 and x_3 to be free variables when we set $x_1 = -x_2 - x_3$. That is, the kernel $N(A)$ (and with it the subspace S) can be expressed as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

In other words, a basis for S is given by the two vectors

$$b_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad b_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

SOLUTION 2. The dimension of S is 2, because if we fix two of x_1, x_2, x_3 we can always choose the remaining such that the equation $x_1 + x_2 + x_3 = 0$ is satisfied. So let us choose $x_2 = r$ and $x_3 = s$. Then $x_1 =$ $-x_2-x_3=-r-s$. That is, S can be expressed as all vectors of the form $\begin{bmatrix} -r-s \\ r \\ s \end{bmatrix} = r \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. In other words, a basis for S is given by the vectors $b_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad b_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Question 3 (c)

SOLUTION. Since S is the set of all possible linear combinations of its basis vectors, and $R(B)$ is the set of all possible linear combinations of its columns, we can simply populate the columns of B with (non-trivial linear combinations of) the basis vectors of S. For example $B = \begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $B = \begin{bmatrix} -1 & -1 & -\pi & -2 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & \pi & 1 \end{bmatrix}$

Question 4 (a)

SOLUTION. The inner product is defined as

$$\langle f, g \rangle = \int_0^1 f(t)\overline{g(t)}dt$$

This is the continuous equivalent of the inner product for vectors, where we have the sum of the multiplied components of the vectors.

Question 4 (b)

SOLUTION. To compute the coefficient, start with

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e_n(t)$$

where $f(t) = t^2 - t$ and $e_n(t) = e^{2\pi i n t}$. Also, we have $L=1-0=1$. Take the inner product of $f(t)$ with e_m. The only term in the infinite sum that remains is the one with $n = m$, and in this case $\langle e_n, e_n \rangle = 1$. Thus

$$\langle f, e_m \rangle = \sum_{n=-\infty}^{\infty} c_n \langle e_n, e_m \rangle = c_m$$

and we get the formula

$$c_m = \langle f, e_m \rangle = \int_0^1 f(t)e^{-2\pi imt}\,dt = \int_0^1 f(t)\overline{g(t)}\,dt$$

Since $\overline{g(t)} = e^{-2\pi imt}$ we have $g(t) = e^{2\pi imt}$, and so we overall find that

$$c_n = \langle t^2 - t, e^{2\pi int} \rangle$$

Question 4 (c)

SOLUTION. From part (b) we have the formula $c_n = \int_0^1 f(t)e^{-2\pi int}\,dt$. Plugging in $n = 0$ we calculate

$$\begin{aligned} c_0 &= \int_0^1 f(t)e^{-2\pi i(0)t}\,dt \\ &= \int_0^1 (t^2 - t)(1)dt \\ &= \frac{1}{3} - \frac{1}{2} = -\frac{1}{6} \end{aligned}$$

Question 4 (d)

SOLUTION. Consider the Fourier series

$$t^2 - t = \sum_{n=-\infty}^{\infty} c_n e^{2\pi int}$$

for $0 \leq t \leq 1$. Parseval's formula says

$$\int_0^1 |f(t)^2|dt = \langle f, f \rangle = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Plugging $f(t) = t^2 - t$ in the left hand side we calculate

$$\langle f, f \rangle = \int_0^1 (t^4 - 2t^3 + t^2)dt = \frac{1}{5} - \frac{1}{2} + \frac{1}{3} = \frac{1}{30}$$

Plugging in the given and calculated values for c_n in the right hand side we compute

$$\begin{aligned}\sum_{n=-\infty}^{\infty}|c_n|^2 &= \sum_{n=-\infty}^{-1}|c_n|^2+|c_0|^2+\sum_{n=1}^{\infty}|c_n|^2\\ &= \sum_{n=-\infty}^{-1}\frac{1}{4\pi^4n^4}+\left|-\frac{1}{6}\right|+\sum_{n=1}^{\infty}\frac{1}{4\pi^4n^4}\\ &= \frac{1}{36}+2\sum_{n=1}^{\infty}\frac{1}{4\pi^4n^4}\\ &= \frac{1}{36}+\frac{1}{2\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}\end{aligned}$$

Combining the left hand side and the right hand side of Parseval's formula we can find

$$\frac{1}{30}-\frac{1}{36}=\frac{1}{180}=\frac{1}{2\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}$$

Therefore, the infinite sum is

$$\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$$

Question 4 (e)

SOLUTION. Using the hint given in the question, we have

$$\begin{aligned}\int_0^1\cos(2\pi t)(t^2-t)dt &= \int_0^1\frac{1}{2}(e^{2\pi it}+e^{-2\pi it})(t^2-t)dt\\ &= \int_0^1\frac{1}{2}e^{2\pi it}(t^2-t)dt+\int_0^1\frac{1}{2}e^{-2\pi it}(t^2-t)dt\\ &= \frac{1}{2}\int_0^1(t^2-t)e^{2\pi it}dt+\frac{1}{2}\int_0^1(t^2-t)e^{-2\pi it}dt\\ &= \frac{1}{2}\langle t^2-t,e^{-2\pi it}\rangle+\frac{1}{2}\langle t^2-t,e^{2\pi it}\rangle\\ &= \frac{1}{2}c_{-1}+\frac{1}{2}c_1\end{aligned}$$

From part (d) we know that $c_n=\frac{1}{2\pi^2n^2}$, which we plug into our equation above to calculate

$$\begin{aligned}\int_0^1\cos(2\pi t)(t^2-t)dt &= \frac{1}{2}\left(\frac{1}{2\pi^2(-1)^2}\right)+\frac{1}{2}\left(\frac{1}{2\pi^2(1)^2}\right)\\ &= \frac{1}{4\pi^2}+\frac{1}{4\pi^2}=\frac{1}{2\pi^2}\end{aligned}$$

Question 1 (e)

SOLUTION. First, define the matrix C and the vector $\boldsymbol{c}$

```
C=[-4, -2; -2, 1; 0, 1];
c=[2; 2; 1];
```

Then solve the least squares equation $C\boldsymbol{s} = \boldsymbol{c}$ for $\boldsymbol{s}$

```
s=(C'*C)\(C'*c);  % or s=inv(C'*C)*C'*c;
```

Finally, from $\boldsymbol{s}$ we retrieve the coefficient vector $\boldsymbol{a}$

```
a_1 = [1; -3; 0; 0];
a_2 = [0; 0; 0; 1];
a = s(1)*a_1 + s(2)*a_2;
```

This determines the function *p(x)*

```
p = @(x) a(1)*x^3 + a(2)*x^2 + a(3)*x + a(4);
```

All that is left now is to plot the points *(0, 1)*, *(1, 2)* and *(2,2)* as well as the polynomial *p(x)* on the interval *[0, 2]*:

```
hold on
plot(0, 1, 'o')
plot(1, 2, 'o')
plot(2, 2, 'o')
fplot(p, [0, 2])  % or eg plot(linspace(0, 2), p(linspace(0, 2)))
```

Question 2 (a)

SOLUTION. With N = 4, we will consider f(x) at 5 evenly spaced values of x between 0 to 1, where $x_n = \frac{n}{N}$, $0 < n < N$.

This gives us 4 segments joining the points.

For $f(x_n)$ where $0 < n < N$, we can approximate $f < math >$''(x_{n}) = \frac{(f(x_{n+1}) - f(x_{n})) - (f(x_{n}) - f(x_{n-1}))}{(\Delta x)^{2}} for $0 < n < N$

This is equivalent to

$$\frac{1}{(\Delta x)^2}\begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ f(x_3) \\ f(x_4) \end{bmatrix}$$

Since we know that $f(0) = 1$ and $f'(1) = 1$, we can add a row to this matrix to include these boundary conditions to get L

$$L = \frac{1}{(\Delta x)^2}\begin{bmatrix} (\Delta x)^2 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & -\Delta x & \Delta x \end{bmatrix}$$

To find Q, we will consider the second half of the ODE

$$xf(x).$$

We have $f''(x_n)$ for $0 < n < N$, so we will find $x(f(x_n)$ for these points. We can ignore n = 0 and n = N because here, we are considering their boundary conditions instead. Since in the equation, Q is multiplied by $(\Delta x)^2$, we divide by $(\Delta x)^2$.

$$Q = \frac{1}{(\Delta x)^2}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \Delta x & 0 & 0 & 0 \\ 0 & 0 & 2\Delta x & 0 & 0 \\ 0 & 0 & 0 & 3\Delta x & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \frac{1}{(\Delta x)}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We will combine L and Q to determine b

$(L + (\Delta x)^2 Q)\vec{F} = \vec{b}$

We have arranged rows 1 to 3 of the matrices to correspond to the left hand side of the ODE when plugged into this equation, so the corresponding rows of $\vec{b}$ will have entries equal to 1. Row 0 and 4 will have entries equal to their corresponding boundary conditions, so both are equal to one as well.

$\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Δx is the spacing between the points. Since the points go from $0 < x < 1$, are evenly spaced and we have 5 of them, $\Delta x = \frac{1}{4} = 0.25$

We can plug $\Delta x = 0.25$ into our expressions for Q and L above to get

$L = 16 \begin{bmatrix} \frac{1}{16} & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & \frac{-1}{4} & \frac{1}{4} \end{bmatrix}$

$\mathrm{Q} = 4 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Question 2 (b)

SOLUTION. Set dx, L, Q and b in Matlab to be equal to values above:

dx = 1/N

L = 1/dx^2*[[dx^2 0 0 0]; diag(ones(3),-1) + diag(-2*ones(3)) + diag(ones(3),1); [0 0 -dx dx]]

for m = 1:N

qVector(m) = m-1

end

qVector(N+1) = 0

Q = 1/dx*diag(qVector)

f = (L + dx^2*Q)

Find index of value corresponding that has x closest to 1/2:

Find points corresponding to x values on either side or equal to 1/2 and approximate f(1/2) as point on line joining both points:

Get f(1/2):

xi = floor((1/2)/dx)

xf = ceil((1/2)/dx)

m = (f(xf) - f(xi))/dx b = f(x0) - m(x0) ans = m(1/2) + b

Question 3 (a)

SOLUTION. The definition of S contains a dot product which reveals the vector orthogonal to S:

$$\begin{aligned} S &= \left\{ [x_1, x_2, x_3]^T : x_1 + x_2 + x_3 = 0 \right\} \\ &= \left\{ [x_1, x_2, x_3]^T : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0 \right\} \end{aligned}$$

Since the vector $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is orthogonal to S, and the nullspace of A (which is S) is orthogonal to the row space of A, we can choose any matrix A whose row space is spanned by $\begin{bmatrix}1\\1\\1\end{bmatrix}$. Such as $A = \begin{bmatrix}1 & 1 & 1\end{bmatrix}$ or $A = \begin{bmatrix}1 & 1 & 1\\2 & 2 & 2\\0 & 0 & 0\\\pi & \pi & \pi\end{bmatrix}$.

Question 3 (d)

SOLUTION. To begin with, we define the matrix B using the result from part (c):

```
B = [-1 -1; 1 0; 0 1];
```

Then, use the formula for the projection matrix to define P:

```
P = B*inv(B'*B)*B';
```

This solves (i). Note that all possible choices of B would have resulted in the same projection matrix P. Finally, the vector x in S closest to $[0, 1, 0]^T$ is the projection of that vector onto S, that is:

```
x = P*[0; 1; 0]
```

Question 3 (e)

SOLUTION. First, we show that Q is a projection, that is $Q^2 = Q$:

$$Q^2 = (I - P)(I - P) = I - P - P + P^2 = I - P - P + P = I - P = Q$$

since $P^2 = P$ because P is a projection.
Next we show that $N(Q) = R(P)$. So let x be in the nullspace of Q. Then $0 = Qx = (I\text{-}P)x = x\text{-}Px$ and hence $Px = x$ which implies in particular that x is in the range of P. Next, let x be in the range of P, and choose y such that $x = Py$. Multiplying both sides with P yields $Px = P^2y = Py = x$, which we can rewrite as $(I\text{-}P)x = 0$ and thus x is in the nullspace of Q. Therefore $N(Q) = R(P)$. But P was chosen such that $R(P) = S$ and hence

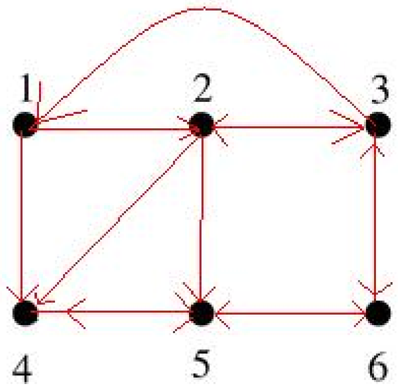
$$N(Q) = R(P) = S$$

Interchanging the roles of P and Q, after all $Q = I\text{-}P$ implies that $P = I\text{-}Q$, we obtain that $R(Q) = N(P)$. We claim that $N(P) = S^T$. This is quick since $Px = 0$ is equivalent to saying that x has no component in S which is equivalent to saying that x is orthogonal to S. From part (a) we remember that the vector $[1, 1, 1]^T$ spans the orthogonal complement of S and hence

$$R(Q) = S^T = \text{span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$$

Question 5 (a)

SOLUTION. Proceeding as in the hint, we see that graph should look like



Question 5 (b)

SOLUTION. No content found.

Question 5 (c)

SOLUTION. No content found.

Question 5 (d)

SOLUTION. No content found.

Question 6 (a)

SOLUTION. Since all matrices are real, **hermitian**, **real symmetric** and **symmetric** are equivalent, and so are **unitary** and **orthogonal**.

***U* and *V* are: hermitian ◯, real symmetric ◯, unitary ⊗, orthogonal ⊗** We easily see that U is symmetric but V is not. Hence, together, they are neither hermitian nor real symmetric. Since the columns of U and V are orthonormal, both matrices are unitary and orthogonal. This can be seen for U since

$$\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix}\cdot\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix}=1$$
$$\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix}\cdot\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\\0\end{bmatrix}=0$$
$$\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix}\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}=0$$
$$\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\\0\end{bmatrix}\cdot\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\\0\end{bmatrix}=1$$
$$\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\\0\end{bmatrix}\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}=0$$
$$\begin{bmatrix}0\\0\\1\end{bmatrix}\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}=1$$

The corresponding vector multiplications give the same result for V.

Σ is: hermitian $\bigotimes$, real symmetric $\bigotimes$, unitary $\bigcirc$, orthogonal $\bigcirc$ Since Σ is diagonal it is cleary symmetric. However, the first and the third column vectors don't have length 1 and hence Σ is not unitary or orthogonal.

A^*A and AA^* are: hermitian $\bigotimes$, real symmetric $\bigotimes$, unitary $\bigcirc$, orthogonal $\bigcirc$ Following the rules of transposition of matrix products we quickly check that $(A^*A)^* = A^*((A^*)^*) = A^*A$ and correspondingly $(AA^*)^* = ((A^*)^*)A^* = AA^*$ so that A^*A and AA^* are indeed unitary and hence also real symmetric. Next, observe that

$$A^*A = (U\Sigma V^*)^*(U\Sigma V^*) = V\Sigma^2V^{-1}$$

since, from the above, U and V are unitary. The equation above however describes a diagonalization of A^*A, and hence the eigenvalues of A^*A are found on the diagonal of Σ^2, namely $4, 1, \frac{1}{4}$. However, all eigenvalues of a unitary matrix must have an absolute value of 1. Hence A^*A is not unitary and not orthogonal. The equivalent argument also shows that AA^* is not unitary and not orthogonal.

A is: hermitian $\bigcirc$, real symmetric $\bigcirc$, unitary $\bigcirc$, orthogonal $\bigcirc$ Since the largest singular value corresponds to the matrix norm of A it follows that $\|A\| = 2$, while unitary and orthogonal matrices have norm 1. To check if A is symmetric we start by checking the entries $a^{2,1}$ and $a^{1,2}$ for equality.

$$
\begin{aligned}
A &= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 2/\sqrt{6} & -1/\sqrt{6} & -1/\sqrt{6} \end{bmatrix} \\
&= \begin{bmatrix} 2/\sqrt{2} & 1/\sqrt{2} & 0 \\ 2/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 2/\sqrt{6} & -1/\sqrt{6} & -1/\sqrt{6} \end{bmatrix} \\
&= \begin{bmatrix} * & \frac{2}{\sqrt{6}} + \frac{1}{2} & * \\ \frac{2}{\sqrt{6}} & * & * \\ * & * & * \end{bmatrix}
\end{aligned}
$$

We see that $a^{2,1} \neq a^{1,2}$ and hence A is not hermitian or real symmetric.

Question 6 (b)

SOLUTION. From the eigenvalue decomposition

$$A^T A = V \Sigma^2 V^{-1}$$

we see that the eigenvalues of $A^T A$ are the squares of the diagonal entries in Σ, that is

$$\lambda_1 = 4, \quad \lambda_2 = 1, \quad \lambda_3 = \frac{1}{4}$$

Further, the corresponding eigenvectors of $A^T A$ are the column vectors of V, namely

$$\vec{v_1} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \quad \vec{v_2} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad \vec{v_3} = \begin{bmatrix} \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix}$$

Question 6 (c)

SOLUTION. To begin with, recall that unitary matrices don't change the norm of a vector, in particular $\|Ux\| = \|x\|$ and $\|V^*x\| = \|x\|$ for all vectors x. Using the definition of the matrix norm of A we find that

$$
\begin{aligned}
\|A\| &= \max_{\|x\|=1} \|Ax\| \\
&= \max_{\|x\|=1} \|U(\Sigma V^* x)\| \\
&= \max_{\|x\|=1} \|\Sigma V^* x\| \\
&= \max_{\|y\|=1} \|\Sigma y\| \\
&= \|\Sigma\|
\end{aligned}
$$

where $y = V^*x$ satisfies $\|y\| = \|V^*x\| = \|x\| = 1$. In other words, the matrices A and Σ have the same norm. The norm of the diagonal matrix Σ is simply the largest absolute value of the its diagonal entries. Therefore $2 = \|\Sigma\| = \|A\|$.

Question 6 (d)

SOLUTION. Taking the inverse we find

$$A^{-1} = (U\Sigma V^*)^{-1} = (V^*)^{-1}\Sigma^{-1}U^{-1}$$

Since U and V are real and unitary they satisfy $U^{-1} = U^* = U^T$ and $(V^*)^{-1} = V$. Hence

$$A^{-1} = V\Sigma^{-1}U^T$$

So we are left with having to calculate the inverse of Σ. Luckily, this is easy for a diagonal matrix, simply inverse the diagonal entries:

$$\Sigma^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Put together, the SVD of A^{-1} is:

$$A^{-1} = V\Sigma^{-1}U^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Question 6 (e)

SOLUTION. From part (c), $A^{-1} = V\Sigma^{-1}U^T$, and using the same logic as part (c) we find that $\|A^{-1}\| = \|\Sigma^{-1}\| = 2$. The condition number is then

$$\|A^{-1}\| \cdot \|A\| = 2 \cdot 2 = 4.$$

Question 6 (f)

SOLUTION. Let's start by calculating the nullspace of $\hat{A}$, that is, we are looking for vectors x such that

$$0 = \hat{A}x = U \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^*x = U\hat{\Sigma}V^*x$$

Since U is unitary, $Uy = 0$ implies $y = 0$. Hence, for $\hat{A}x = 0$ is only possible if and only if $\hat{\Sigma}V^*x = 0$. Let's abbreviate $V^*x = z$. Then, in order to find the nullspace of $\hat{A}$ we are looking for vectors z such that $\hat{\Sigma}z = 0$. This equation can quickly be solved

$$z = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = te_3,$$

for any real value of t. Since $z = V^*x$ we solve for x and find

$$x = (V^*)^{-1}z = Vz = tVe_3 = t\begin{bmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$$

Therefore, $N(\hat{A}) = t\begin{bmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$
Next, observe that

$$\hat{A} - A = U\hat{\Sigma}V^* - U\Sigma V^* = U(\hat{\Sigma} - \Sigma)V^*$$

Again following the logic of part (c) it holds that

$$\|\hat{A}\| = \|\hat{\Sigma} - \Sigma\| = \left\| \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1/2 \end{bmatrix} \right\| = 1/2$$

Good Luck for your exams!