Full Solutions
MATH102 December 2011

April 4, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a) i

SOLUTION. Given the data we have, the best way to approximate the derivative is by using secant lines! For an example, let's first estimate $f'(1.25)$. For this, lets use the secant line from x=1 and x=1.5. This gives

$f'(1.25) \approx \dfrac{f(1.5)-f(1)}{1.5-1} = \dfrac{1.06-1.12}{0.5} = \dfrac{-0.06}{0.5} = -0.12.$

Proceeding as above, we can fill in the table as follows:

:{| style="text-align:center; padding: 10px; border-collapse: collapse;" |style="padding: 5px; border-style: solid; border-width: 1px"|*x* |style="padding: 5px; border-style: solid; border-width: 1px"|*1.25* |style="padding: 5px; border-style: solid; border-width: 1px"|*1.75* |style="padding: 5px; border-style: solid; border-width: 1px"|*2.25* |style="padding: 5px; border-style: solid; border-width: 1px"|*2.75* |- |style="padding: 5px; border-style: solid; border-width: 1px"|*f' (x)* |style="padding: 5px; border-style: solid; border-width: 1px"| -0.12 |style="padding: 5px; border-style: solid; border-width: 1px"| -0.08 |style="padding: 5px; border-style: solid; border-width: 1px"| -0.04 |style="padding: 5px; border-style: solid; border-width: 1px"| 0.06 |}

For the second derivative, we use the same logic above except we do it with the derivative as our function. This gives

$f''(2) \approx \dfrac{f'(2.25)-f'(1.75)}{2.25-1.75} = \dfrac{-0.04-(-0.08)}{0.5} = \dfrac{0.04}{0.5} = 0.08$

and this completes the question.

Question 1 (a) ii

SOLUTION. This is a quotient of functions and so let $f(x) = x$ and $g(x) = \ln(\frac{1}{x})$. Then, by the quotient rule, we have

$$\frac{d}{dx}\frac{x}{\ln(\frac{1}{x})} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Now, $f'(x) = 1$ and for $g'(x)$, we use the chain rule to see that

$g'(x) = \dfrac{1}{\frac{1}{x}}(-x^{-2}) = -\dfrac{1}{x}$

Note, we could also have noticed that

$g(x) = \ln(\frac{1}{x}) = -\ln(x)$ and hence $g'(x) = -\dfrac{1}{x}$.

Thus, combining all this information, we have

$$\begin{aligned}\frac{d}{dx}\frac{x}{\ln(\frac{1}{x})} &= \frac{(1)(\ln(\frac{1}{x})) - x\left(\frac{-1}{x}\right)}{(\ln(\frac{1}{x}))^2}\\ &= \frac{\ln(\frac{1}{x})+1}{(\ln(\frac{1}{x}))^2}\\ &= \frac{-\ln(x)+1}{(-\ln(x))^2}\\ &= \frac{1-\ln(x)}{(\ln(x))^2}\end{aligned}$$

and any of the last three answers would be accepted.

Question 1 (b) i

SOLUTION 1. To do this problem, we rearrange the equation to isolate y and then take the derivative of y with respect to x:

$$e^{2x} + y(1-x) = 1$$
$$\rightarrow \quad y = \frac{1-e^{2x}}{1-x}$$

Using the quotient rule, we evaluate dy/dx and we are done:

$$\frac{dy}{dx} = \frac{(-2e^{2x})(1-x) - (-1)(1-e^{2x})}{(1-x)^2} = \frac{-3e^{2x} + 2xe^{2x} + 1}{(x-1)^2}.$$

Hence, at $x = 0$ we obtain

$$\frac{dy}{dx}(0) = \frac{-3e^0 + 0 + 1}{(-1)^2} = -2$$

Solution 2. We can also use implicit differentiation. By differentiating with respect to x we obtain:

$$2e^{2x} + \frac{dy}{dx}(1-x) + y(-1) = 0$$

(Watch out for the product rule on the term y(1-x))
The difference is that we need to know the value of y when $x = 0$, but if we plug that in the original equation we obtain

$$e^0 + y(1-0) = 1$$

and so we have that $y = 0$. We use this in what we obtained by differentiating implicitly:

$$2e^0 + \frac{dy}{dx} + 0 = 0$$

and so we have that

$$\frac{dy}{dx}(0) = -2$$

Question 1 (b) ii

Solution. The tangent line must have the same slope as the function $f(x)$ at $x = 1/2$. Solving for the slope m, we get

$$m = f'(1/2) = -\pi \sin\left(\frac{\pi}{2}\right) = -\pi.$$

The tangent line must also touch the function at $x = 1/2$. We use this to solve for the y-intercept:

$$
\begin{aligned}
y &= mx + b \\
\cos\left(\frac{\pi}{2}\right) &= -\pi\left(\frac{1}{2}\right) + b \\
0 &= -\frac{\pi}{2} + b \\
b &= \frac{\pi}{2}
\end{aligned}
$$

Therefore, the tangent line to $f(x) = \cos(\pi x)$ at $x = 1/2$ is:

$$y = -\pi x + \frac{\pi}{2}$$

Question 1 (b) iii

Solution. To determine the equation of the tangent line to $y = f^{-1}(x)$ at the point $x = 1$ we need two things: (1) the slope of $f^{-1}(x)$ at $x = 1$ and (2) the value of a point that the tangent line crosses through. (1) Recall this important property of inverse functions:

$$y = f^{-1}(x) \quad \Leftrightarrow \quad x = f(y)$$

From this rule, we can use implicit differentiation to get the derivative for f^{-1} in terms of f, f'.

$$
\begin{aligned}
\frac{d}{dx}(x) &= \frac{d}{dx}(f(y)) \\
1 &= f'(y)\frac{dy}{dx} \quad \rightarrow \quad \frac{dy}{dx} = \frac{1}{f'(f^{-1}(x))}
\end{aligned}
$$

Thus, the slope of the inverse function at the point $x = 1$ is given by m where

$$m = \frac{1}{f'(f^{-1}(1))}.$$

We know that the tangent line must touch the inverse function at the point $x = 1$. Using the rules of inverse functions, since $f(0) = 1$, we know that $f^{-1}(1) = 0$, and thus

$$m = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{2}.$$

(2) We get by the above point as well

$$f^{-1}(1) = 0 \rightarrow (x_1, y_1) = (1, 0).$$

Using these two things in the standard equation of a line, we get an equation for the tangent line

$$
\begin{aligned}
(y - y_1) &= m(x - x_1) \\
(y - 0) &= \frac{1}{2}(x - 1).
\end{aligned}
$$

Rearranging the equation to slope-intercept form (as the question asks), we get our final answer:

$$y = \frac{x}{2} - \frac{1}{2}.$$

Question 1 (c)

SOLUTION. (Remember z is a root of $f(x) = x^3 - x + 1$, if $f(z) = 0$.)
By Newton's method, we can improve on our guess for a root to the equation, by computing the following:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

where x_1 is our improved guess and $f(x) = x^3 - x + 1$. Performing the computation gives:

$$x_1 = (-1) - \frac{f(-1)}{f'(-1)} = -1 - \frac{(-1)^3 - (-1) + 1}{3(-1)^2 - 1} = -\frac{3}{2}.$$

Therefore, our improved estimate for the root of $f(x) = x^3 - x + 1$ is:

$$x_1 = -\frac{3}{2}$$

Question 2 (a) i

SOLUTION.

$$A = \lim_{x\to\infty} \frac{2x^4 - 3x^3 + 5}{2x^3 - 5x^4}$$

To evaluate this limit, first we divide both the numerator and denominator by the largest power of x that appears in this fraction. In this case, x^4 is the largest power.

$$A = \lim_{x\to\infty} \frac{2 - 3/x + 5/x^4}{2/x - 5}.$$

Now all that's left to finish the problem, is calculating the limits separately:

$$\begin{aligned} A &= \lim_{x\to\infty} \frac{2 - 3/x + 5/x^4}{2/x - 5} \\ &= \frac{\lim_{x\to\infty} 2 - 3/x + 5/x^4}{\lim_{x\to\infty} 2/x - 5} && = \frac{2}{-5} \end{aligned}$$

Therefore,

$$A = -\frac{2}{5}.$$

Question 2 (a) ii

SOLUTION. Notice that we cannot simply plug in $x = 3$ as we will get $B = 0/0$, an indeterminate form. To evaluate this limit, factor the numerator and denominator of the fraction as much as possible:

$$B = \lim_{x\to 3} \frac{x^2 - 2x - 3}{x^2 - 3x} = \lim_{x\to 3} \frac{(x-3)(x+1)}{(x-3)x}.$$

Notice that there is a term that can be cancelled out from the numerator and denominator and rewrite the limit:

$$B = \lim_{x\to 3} \frac{x+1}{x}.$$

Using the properties of limits, we can now evaluate this limit by plugging in the value $x = 3$

$$B = \frac{\lim_{x\to 3} x + 1}{\lim_{x\to 3} x} = \frac{4}{3} \quad \to \quad B = \frac{4}{3}.$$

Question 2 (a) iii

SOLUTION. To evaluate this limit, we notice that:

$$\lim_{x\to\infty} e^{-x} = 0.$$

So we can rewrite C, letting the variable $z = e^{-x}$.

$$C = \lim_{x\to\infty} \frac{\sin(e^{-x})}{e^{-x}} = \lim_{z\to 0} \frac{\sin(z)}{z} = 1.$$

Therefore, our final answer is

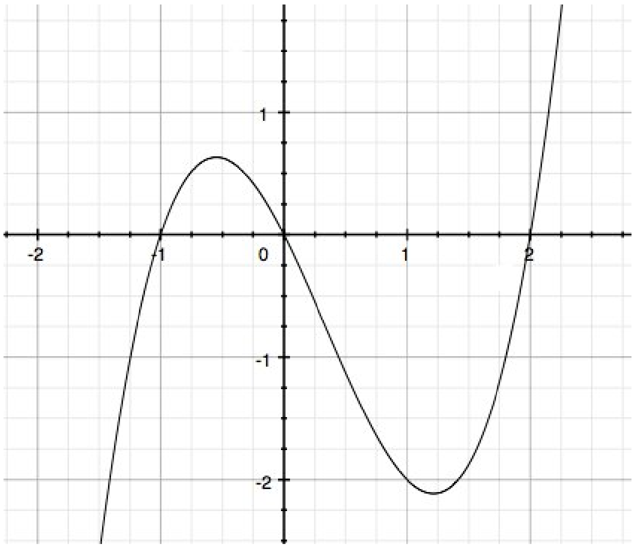
$$C = 1$$

Question 2 (b) i

SOLUTION. Let's follow the hint and sketch a graph of the polynomial on the right hand side. Note that

$$\frac{dy}{dt} = y^3 - y^2 - 2y = y(y^2 - y - 2) = y(y-2)(y+1).$$

The roots are at 0, 2 and -1.



We are given that y(0)=1, which is in between 0 and 2. In this region, y and $y+1$ are both positive and $y-2$ is negative. Therefore, $\frac{dy}{dt} < 0$, which means that y is decreasing (throughout the interval (0,2)). As y approaches zero, we see that $\frac{dy}{dt}$ also approaches zero. (Note that y could never cross zero, since $\frac{dy}{dt} = 0$ at the point where $y = 0$. In other words, if y ever hits 0, it stays there!)
Therefore,

$$\lim_{t\to\infty} y(t) = 0.$$

Question 2 (b) ii

SOLUTION. To solve this, we compare the left and right-hand sides of the equation,

$$\frac{dy}{dx} = 2xy.$$

1. $y = xe^{2x}$.

$$\begin{cases} \frac{dy}{dx} &= e^{2x} + 2xe^{2x} \\ 2xy &= 2x^2e^{2x} \end{cases} \quad \rightarrow \quad \frac{dy}{dx} \neq 2xy$$

Therefore, (1.) is not the answer.
2. $y = \dfrac{1}{2x}$.

$$\begin{cases} \frac{dy}{dx} &= -\frac{1}{2x^2} \\ 2xy &= 1 \end{cases} \quad \rightarrow \quad \frac{dy}{dx} \neq 2xy$$

Therefore, (2.) is not the answer.
3. $y = -e^{-x}$.

$$\begin{cases} \frac{dy}{dx} &= e^{-x} \\ 2xy &= -2xe^{-x} \end{cases} \quad \rightarrow \quad \frac{dy}{dx} \neq 2xy$$

Therefore, (3.) is not the answer.
4. $y = -3e^{x^2}$.

$$\begin{cases} \frac{dy}{dx} &= -6xe^{x^2} \\ 2xy &= -6xe^{x^2} \end{cases} \quad \rightarrow \quad \frac{dy}{dx} = 2xy$$

Therefore, (4.) IS the answer.
5. $y = 3e^{-x^2}$.

$$\begin{cases} \frac{dy}{dx} &= -6xe^{-x^2} \\ 2xy &= 6xe^{-x^2} \end{cases}$$

Hence, $\frac{dy}{dx} \neq 2xy$ and therefore, (5.) is not the answer.
Finally, the in case 4 we have $\frac{dy}{dx} = 2xy$.

Question 2 (b) iii

SOLUTION. By Euler's method, we can determine an approximation for the values $y_n = y(n\Delta t)$ by the formula,

$$\begin{aligned} y_{n+1} &= y_n + \Delta t\, f(y_n), \quad f(y) = 3y - y^2 \\ &= y_n + \Delta t\,(3y_n - y_n^2) \end{aligned}$$

Thus, to determine y_1 we evaluate the following:

$$\begin{aligned} y_1 &= y_0 + \Delta t\,(3y_0 - y_0^2) \\ &= 1 + (1/2)(3 - 1^2) \\ &= 2. \end{aligned}$$

Applying Euler's method for the following two steps, we get y_2...

$$\begin{aligned} y_2 &= y_1 + \Delta t\,(3y_1 - y_1^2) \\ &= 2 + (1/2)(6 - 2^2) \\ &= 3. \end{aligned}$$

and y_3...

$$\begin{aligned} y_3 &= y_2 + \Delta t\,(3y_2 - y_2^2) \\ &= 3 + (1/2)(9 - 3^2) \\ &= 3. \end{aligned}$$

Therefore,

$$y_1 = 2,\ y_2 = 3,\ y_3 = 3$$

Question 2 (c)

SOLUTION. To draw a qualitatively accurate sketch of the derivative of the function (i.e

$$f'(x)$$

), we need to observe some qualitative features of the slope:
(1) We notice that the slope of the function is zero (i.e: function has a horizontal tangent line) at $x = -1$ and $x = 1$. (i.e

$$f'(x) = 0$$

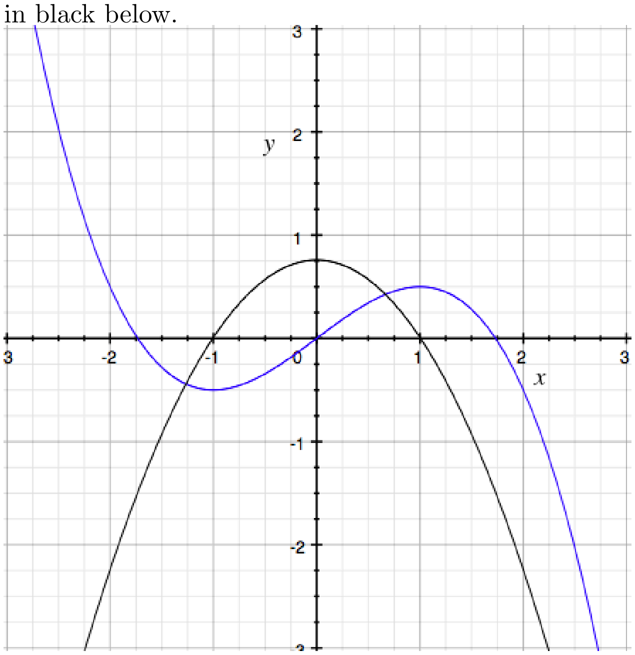
at $x = -1$ and $x = 1$)
(2) We notice the function has positive slope on $(-1, 1)$ (i.e

$$f'(x) > 0$$

on $(-1, 1)$)
(3) We notice the function has negative slope on $(-3, -1) \cup (1, 3)$ (i.e

$$f'(x) < 0$$

on $(-3, -1) \cup (1, 3)$).
So we need to sketch a curve for $f'(x)$ that the properties described in points (1)-(3). Such a curve is plotted in black below.



Question 3

SOLUTION 1. One way to determine the point where the distance between the point P and the ellipse are closest is to determine the equation of the normal line to the ellipse that passes through the point P.
We can use implicit differentiation to determine the slope of the tangent line at a point (x,y) on the ellipse:

$$\frac{2x}{9} + 2y\frac{dy}{dx} = 0 \quad \rightarrow \quad \frac{dy}{dx} = m = -\frac{x}{9y}$$

Remember that the slope of the normal line to a function at a point x is just the negative reciprocal of the slope at the same point. i.e: if m is slope of the tangent line (as above) and M is the slope of the normal line, both at a point x, then:

$$M = -\frac{1}{m} = \frac{9y}{x}$$

Therefore the equation for the line that is normal to the ellipse at point (a,b) that passes through the point P on the ellipse is given by

$$(y - 0) = \frac{9b}{a}(x - 4/9)$$

Since this normal line will also pass through the point where the normal line is being taken, we can solve for (a,b) by solving the following:

$$\begin{aligned}(b - 0) &= \frac{9b}{a}(a - 4/9) \\ a &= 9(a - 4/9) \\ 8a = 4 \quad &\rightarrow \quad a = \frac{1}{2}\end{aligned}$$

To figure out the corresponding values of b, we use the fact that the point (a,b) must lie on the ellipse:

$$\frac{a^2}{9} + b^2 = 1 \quad \rightarrow \quad b = \pm\frac{\sqrt{35}}{6}$$

Notice that we have two solutions! Therefore, there are two points on the ellipse that are closest to the point P. The two points are:

$$(a, b) = \left(\frac{1}{2}, \frac{\sqrt{35}}{6}\right), \left(\frac{1}{2}, -\frac{\sqrt{35}}{6}\right)$$

Solution 2. This can also be treated as an optimization problem. So, we need to determine the quantity that we want to optimize. We want to determine the point on the given ellipse, (x, y), that is closest to the point $P = (x_0, y_0) = (4/9, 0)$.
Thus, we want to minimize the distance between the two points which we will denote by, L. We obtain an expression for L using the formula for distance between two points:

$$\begin{aligned}L &= \sqrt{(x - x_0)^2 + (y - y_0)^2} \\ &= \sqrt{\left(x - \frac{4}{9}\right)^2 + y^2}\end{aligned}$$

Our constraint in this problem is that the point (x, y) must lie on the ellipse,

$$\frac{x^2}{9} + y^2 = 1 \quad \rightarrow \quad y^2 = 1 - \frac{x^2}{9}$$

Replacing y^2 in our expression for L with what is given by our constraint, we obtain

$$L(x) = \sqrt{\left(x - \frac{4}{9}\right)^2 + 1 - \frac{x^2}{9}}.$$

To minimize L, we need to determine the critical points of L. So we compute L' and set it to zero.

$$L'(x) = \frac{2(x - 4/9) - 2x/9}{2\sqrt{(x - 4/9)^2 + 1 - x^2/9}} = \frac{8x - 4}{9\sqrt{(x - 4/9)^2 + 1 - x^2/9}} = 0.$$

The solution to the above equation is x = 1/2. To figure out the corresponding value(s) of y, we substitute the x value into the equation of the ellipse:

$$y^2 = 1 - \frac{1}{36} \quad \rightarrow \quad y = \pm\frac{\sqrt{35}}{6}.$$

So there are two critical points on the ellipse:

$$(x, y) = \left(\frac{1}{2}, \frac{\sqrt{35}}{6}\right), \left(\frac{1}{2}, -\frac{\sqrt{35}}{6}\right)$$

The distance between each of these points and P is the same:

$$L = \sqrt{\left(\frac{1}{2} - \frac{4}{9}\right)^2 + \left(\frac{\sqrt{35}}{6}\right)^2} = \sqrt{\left(\frac{1}{18^2}\right) + \left(\frac{35}{36}\right)} \approx 0.988$$

We also need to consider the distance between P and the left and right-most points of the ellipse, which occur at (-3,0) and (3,0). Computing value of L for each of these points gives L approximately equal to 3.444 and 2.556, respectively. The distances between P and each of the two endpoints are both larger than the distance between the first two points that we found. Therefore, the (x,y) coordinates of the points that minimize the distance between the ellipse and P are:

$$(x, y) = \left(\frac{1}{2}, \frac{\sqrt{35}}{6}\right), \left(\frac{1}{2}, -\frac{\sqrt{35}}{6}\right)$$

Question 4 (i)

SOLUTION. When considering limits at infinity, only that term with the highest degree plays a role. So we can conclude instantly that the limit at either positive or negative infinity will be positive infinity.
To illustrate why this is true, consider the full computation below. We have that

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{1}{4}x^4 - 2x^3 + \frac{9}{2}x^2 \\
&= \lim_{x\to\infty} \frac{1}{4}x^4(1 - 2\frac{1}{x} + \frac{9}{2x^2}) \\
&= \lim_{x\to\infty} \frac{1}{4}x^4 \\
&= +\infty
\end{aligned}$$

Question 4 (ii)

SOLUTION. Taking the derivative gives

$$f'(x) = x^3 - 6x^2 + 9x = x(x^2 - 6x + 9) = x(x-3)^2.$$

Which allows to write a sign table for the derivative.

		0		3	
x	$-$	0	$+$		$+$
$(x-3)^2$	$+$		$+$	0	$+$
$f'(x)$	$-$	0	$+$	0	$+$
$f(x)$	$\searrow$		$\nearrow$		$\nearrow$

And so we see that the function is decreasing up to $x = 0$, where it has a global minimum and then is increasing.

Question 4 (iii)

SOLUTION. The previous part gave us that

$$f'(x) = x^3 - 6x^2 + 9x = x(x^2 - 6x + 9) = x(x-3)^2.$$

Let's compute the second derivative of the function:

$$f''(x) = 3x^2 - 12x + 9$$

and then factor it

$$\begin{aligned} f''(x) &= 3x^2 - 12x + 9 \\ &= 3(x^2 - 4x + 3) \\ &= 3(x-1)(x-3) \end{aligned}$$

This is plenty enough to get the sign table of the second derivative.

		1		3	
$3(x-1)$	$-$	0	$+$		$+$
$(x-3)$	$-$		$-$	0	$+$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\cup$		$\cap$		$\cup$

And so we see that there are two inflection points, one at $x = 1$ and one at $x = 3$.

Question 4 (iv)

SOLUTION. In part *ii)* of this question we obtained the sign table of the derivative:

		0		3	
x	$-$	0	$+$		$+$
$(x-3)^2$	$+$		$+$	0	$+$
$f'(x)$	$-$	0	$+$	0	$+$
$f(x)$	$\searrow$		$\nearrow$		$\nearrow$

And we see there that the function is decreasing on the interval $(-\infty,0)$ and increasing on the intervals (0,3) and $(3,+\infty)$.

Question 4 (v)

SOLUTION. In part *iii)* of this question we obtained the sign table of the second derivative:

		1		3	
$3(x-1)$	$-$	0	$+$		$+$
$(x-3)$	$-$		$-$	0	$+$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\cup$		$\cap$		$\cup$

And so we see that function is concave up on the intervals $(-\infty, 1)$ and $(3, +\infty)$ while it is concave down on the interval (1,3).

Question 4 (vi)

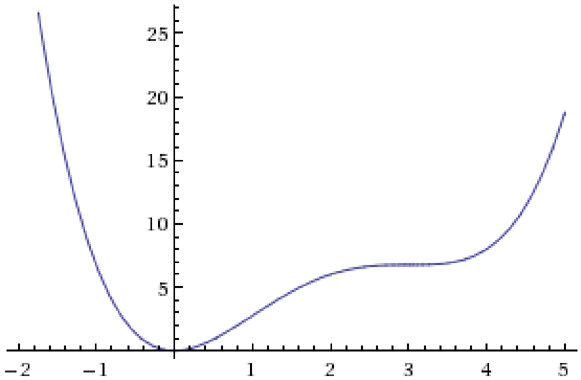
SOLUTION. This function is a polynomial of degree 4, so the curve sketching shouldn't offer too many difficulties. We have collected the following information:

- The function has no horizontal asymptotes (from part i))
- The function has a global minimum at $x = 0$ (from part ii))
- The function has two inflection points at $x = 1$ and $x = 3$ (from part iii))
- The function is decreasing up to its minimum and then increasing, except at $x = 3$ where it has an inflexion point (from part iv))
- The function is concave up all the way up to $x = 1$, then concave down until $x = 3$ and then concave up again (from part v))

To get a better idea of what to sketch, we quickly compute the second coordinates of the three points of interests:

- Global minimum at (0,0)
- Inflection points at (1, 11/4) and (3, 27/4)

This allows us to sketch the graph of this function. As we saw above, this function is always positive, reaches a global minimum at the point (0,0) and has two inflection points. This should give you a picture that looks like:

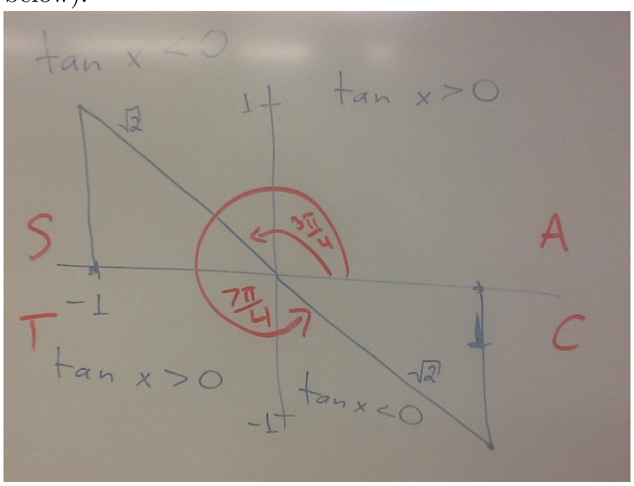


Question 5

SOLUTION 1. To determine the absolute max and min of the function f(x) on the interval we need to determine the critical points of the function and evaluate the function f(x) at those points, as well as the endpoints of the interval. The absolute max(min) of f(x) on the interval $[0, 2\pi]$ is given by the greatest(least) value of the function at those points.
To determine the critical points, we evaluate solve $f'(x) = 0$ for x.

$$0 = f'(x) = \cos(x) + \sin(x).$$

and thus when $\cos(x) \neq 0$
$-1 = \dfrac{\sin(x)}{\cos(x)} = \tan(x)$
If $\cos(x)$ is 0, then we know that $x = \frac{\pi}{2}$ or $x = \frac{3\pi}{2}$. At these points, $\sin(x) \neq 0$ and thus $0 \neq \cos(x) + \sin(x)$. Hence we may suppose that $\cos(x) \neq 0$.
We know that $\tan(x)$ is negative in the second and fourth quadrants. We also know that a right triangle with side lengths $1 : 1 : \sqrt{2}$ has angles $\dfrac{\pi}{4}$, $\dfrac{\pi}{4}$ and $\dfrac{\pi}{2}$. Thus, we have that $x = \dfrac{3\pi}{4}$ and $x = \dfrac{7\pi}{4}$ (see the diagram below).



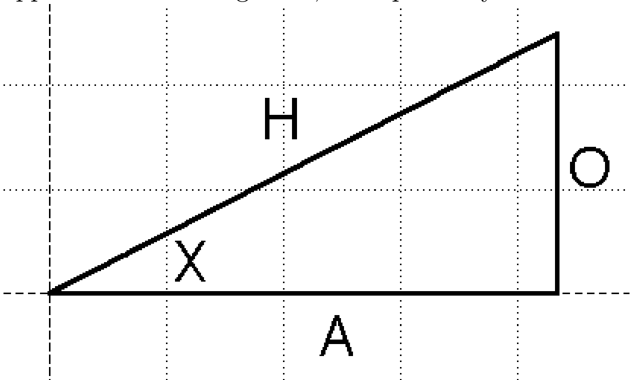
Thus, we plug in all these values to see that
$f(0) = \sin(0) - \cos(0) = -1$
$f(\frac{3\pi}{4}) = \sin(\frac{3\pi}{4}) - \cos(\frac{3\pi}{4}) = \dfrac{1}{\sqrt{2}} - \dfrac{-1}{\sqrt{2}} = \dfrac{2}{\sqrt{2}} = \sqrt{2}$
$f(\frac{7\pi}{4}) = \sin(\frac{7\pi}{4}) - \cos(\frac{7\pi}{4}) = \dfrac{-1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}} = \dfrac{-2}{\sqrt{2}} = -\sqrt{2}$

$f(2\pi) = \sin(2\pi) - \cos(2\pi) = 0 - 1 = -1$
and thus, the maximum occurs at $\frac{3\pi}{4}$ and its value is $\sqrt{2}$ and the minimum occurs at $\frac{7\pi}{4}$ and its value is $-\sqrt{2}$

SOLUTION 2. Proceed as in solution one to see that

$$f'(x) = \cos(x) + \sin(x) = 0 \quad \to \quad \cos(x) = -\sin(x).$$

Solving this equation may appear difficult but if we consider some basic trigonometry, it should be no trouble. Consider this drawing of a right triangle with hypotenuse of length H, and an angle x with adjacent and opposite sides of lengths A, O respectively.



The following relations hold:

$$\cos(x) = \frac{A}{H}, \quad \sin(x) = \frac{O}{H}.$$

To determine the angles, x, that satisfy the relation $\cos(x) = -\sin(x)$, we substitute the above relations and get

$$\frac{A}{H} = -\frac{O}{H} \quad \to \quad A = -O.$$

Therefore, angles with adjacent side length equal to the opposite side length, but with one extending in the negative direction and one in the positive direction are the solutions to our above equation. This implies that the magnitudes of the sides are equal and must occur in the second and fourth quadrants. A right triangle with two sides equal must have angles $\frac{\pi}{4}$, $\frac{\pi}{4}$ and $\frac{\pi}{2}$. Hence, the angles of the triangles we require are (in the second and fourth quadrants respectively - see the previous solution for the picture)

$$x = \frac{3\pi}{4}, \frac{7\pi}{4}$$

and thus, they are our critical points. Evaluating $f(x)$ at the critical points and the endpoints we get the following:

$$f(0) = 0 - 1 = -1, \quad f\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}} - \frac{-1}{\sqrt{2}} = \sqrt{2}$$
$$f\left(\frac{7\pi}{4}\right) = \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}, \quad f\left(2\pi\right) = 0 - 1 = -1$$

Therefore, we can see that this function has a unique absolute maximum and unique absolute minimum:

$$\text{Abs max} = f\left(\frac{3\pi}{4}\right) = \sqrt{2}, \quad \text{Abs min} = f\left(\frac{7\pi}{4}\right) = -\sqrt{2}$$

Question 6

SOLUTION. First, we take derivatives and see that
$f'(x) = \frac{1}{x^3}(3x^2) = \frac{3}{x}$
$g'(x) = 3(\ln(x))^2\frac{1}{x} = \frac{3(\ln(x))^2}{x}$
Setting the derivatives equal to each other gives
$\frac{3}{x} = \frac{3(\ln(x))^2}{x}$
which implies that
$(\ln(x))^2 = 1$
Hence, we have that $\ln(x) = 1$ or $\ln(x) = -1$. The first gives the solution $x = e$ and the second gives $x = e^{-1}$ both seen by exponentiating both sides. Both of these values are in the domain of the original function and so this completes the proof.

Question 7 (i)

SOLUTION. The question is asking us to find the rate of change of the variable x. (i.e: We want to solve for dx/dt). Using implicit differentiation and the equation $y = x^2$ we will derive a value for these quantities as follows:

$$\frac{d}{dt}y = \frac{d}{dt}x^2$$
$$\frac{dy}{dt} = 2x\frac{dx}{dt} \quad \rightarrow \quad \frac{dx}{dt} = \frac{1}{2x}\frac{dy}{dt}$$

From the information about the rate of change of y at the point P, we make the appropriate substitutions in the equation to determine dx/dt:

$$\frac{dx}{dt} = \frac{1}{2(2)}8 = 2 \quad \rightarrow \quad \frac{dx}{dt} = 2$$

Question 7 (ii)

SOLUTION. As discussed in the hint,

$$D(t)^2 = x(t)^2 + y(t)^2$$

Differentiating yields

$$2D(t)\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

From the previous part, we have that

$$\left.\frac{dx}{dt}\right|_{x=2} = 2$$

and we are given that

$$\left.\frac{dy}{dt}\right|_{x=2} = 8$$

and so plugging in the point $(x(t), y(t)) = (2, 4)$, and $D(t) = \sqrt{x(t)^2 + y(t)^2}$ (D must be positive) we have

$$\begin{aligned}
2\sqrt{x(t)^2 + y(t)^2}\frac{dD}{dt} &= 2D(t)\frac{dD}{dt} = 2x(t)\frac{dx}{dt} + 2y(t)\frac{dy}{dt} \\
2\sqrt{(2)^2 + (4)^2}\frac{dD}{dt} &= 2(2)(2) + 2(4)(8) \\
2\sqrt{20}\frac{dD}{dt} &= 72 \\
\frac{dD}{dt} &= \frac{36}{\sqrt{20}} = \frac{18}{\sqrt{5}} = \frac{18}{5}\sqrt{5}
\end{aligned}$$

completing the question. All answers in the last line are equivalently correct.

Question 7 (iii)

Solution. Notice that

$\tan(\theta) = \frac{y(t)}{x(t)}$

Taking derivatives with respect to time gives

$\sec^2(\theta)\frac{d\theta}{dt} = \frac{\frac{dy}{dt} \cdot x(t) - \frac{dx}{dt} \cdot y(t)}{x(t)^2}$

From the previous parts, we have that

$\left.\frac{dx}{dt}\right|_{x=2} = 2$

and we are given that

$\left.\frac{dy}{dt}\right|_{x=2} = 8$

and so plugging in the point $(x(t), y(t)) = (2, 4)$ into the triangle, we see that

$\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{1}{\frac{2}{\sqrt{2^2+4^2}}} = \frac{\sqrt{20}}{2} = \sqrt{5}$

Using all this information, we plug in the point $(2, 4)$ and see that

$$\begin{aligned}
\sec^2(\theta)\frac{d\theta}{dt} &= \frac{\frac{dy}{dt} \cdot x(t) - \frac{dx}{dt} \cdot y(t)}{x(t)^2} \\
(\sqrt{5})^2\frac{d\theta}{dt} &= \frac{(8)(2) - (2)(4)}{(2)^2} \\
5\frac{d\theta}{dt} &= 2 \\
\frac{d\theta}{dt} &= \frac{2}{5}
\end{aligned}$$

and this completes the question.

Question 8 (i)

SOLUTION. Let $d(t)$ be the distance between your car and the moose, where $t = 0$ represents the time when your car begins backing up. The moose began charging one second before t=0, i.e. at t = -1, and so, when the car begins backing up the moose is 20m-8m/s*(1s) = 12m away.
Let the origin be the position of the car at time $t = 0$. Consider the moose. If the moose is 12m away at $t = 0$ and is moving towards the car at a speed of 8 m/s, we can define the position of the moose as $m(t)$ where

$$m(t) = 12 - 8t.$$

If we set $c(t)$ be the position of the car at time t. The car is accelerating away from the moose at 2 m/s^2, then

$$c''(t) = -2 \quad \rightarrow \quad c'(t) = -2t + c_1$$

Since the car is not moving at $t = 0$, we have that $c'(0) = c_1 = 0$. Thus $c'(t) = -2t$. From this, we obtain that $c(t) = -t^2 + c_2$. Since we set the position of the car at $t = 0$ to be $c(0) = 0$, then $c(t) = -t^2$.
The distance between the car and the moose is thus (note that the distance is always a positive number)

$$d(t) = |c(t) - m(t)| = |-t^2 + 8t - 12|$$

Question 8 (ii)

SOLUTION. From our answer in part i), the distance between the moose and the car is given by d(t), where

$$d(t) = |-t^2 + 8t - 12|.$$

To solve for the time when the moose hits the car we solve the equation d(T) = 0 to get T.

$$\begin{aligned} d(T) = |-T^2 + 8T - 12| &= 0 \\ |-(T-6)(T-2)| &= 0 \quad \rightarrow \quad T = 2, 6 \end{aligned}$$

Since the moose will not hit the car twice, we choose the smaller of the two times, and say the moose will hit the car at

$$T = 2.$$

Question 8 (iii)

SOLUTION. We consider the same problem as in part **(i)**, but now we have a car with acceleration a. We will set up the equation for the distance between the moose and car and try to determine values of a such that there is no time where the distance between the car and the moose is equal to 0.
As before, the position of the moose as a function of time, $m(t)$, is given by:

$$m(t) = 12 - 8t.$$

The position of the car however is now given by

$$c(t) = -\frac{a}{2}t^2.$$

Hence the distance between the moose and the car is given by

$$d(t) = |c(t) - m(t)| = \left| -\frac{a}{2}t^2 + 8t - 12 \right|.$$

What we want to do now is to choose a value for the acceleration of the car, a, such that the equation d(T) = 0 has no (real) solution for T. (i.e: Choose a so that there is no time where the distance between car and moose is 0).

$$d(T) = \left| -\frac{a}{2}T^2 + 8T - 12 \right| = 0 \quad \rightarrow \quad T = \frac{-8 \pm \sqrt{64 - 24a}}{-a}$$

To ensure that there are no real solutions, we must look for values of a such that *64 - 24a* is less than zero.

$$64 - 24a < 0 \quad \rightarrow \quad a > \frac{8}{3}$$

Therefore, we need a car that has acceleration greater than 8/3 to ensure that the moose will not hit our car next time this exact situation occurs.

Question 9

SOLUTION. First, using Heron's formula, we have that the semiperimeter is $s = 1$ and so
$A^2 = (1-a)(1-a)(1-b)$.
Next, the perimeter is given by $2 = 2a + b$ and so $b = 2 - 2a$. Substituting this into the area formula, we have
$A^2 = (1-a)^2(1-(2-2a)) = (1-a)^2(2a-1)$
Taking the derivative implicitly with respect to a via the chain rule and product rule, we have

$$\begin{aligned}
2AA' &= (2(1-a)(-1))(2a-1) + (1-a)^2(2) \\
&= -2(1-a)(2a-1) + 2(1-a)^2 \\
&= 2(1-a)(-(2a-1) + (1-a)) \\
&= -2(1-a)(3a-2)
\end{aligned}$$

Setting the derivative to 0 and optimizing gives a=1 and $a = 2/3$. Now, looking at the Perimeter restraint $2 = 2a + b$
We see that the maximum either occurs at the end points or when a=2/3.
To find the endpoints we consider degenerating the triangle into a straight line. We get a vertical straight line when $b = 0$. In this case $2 = 2a + b$ yields $a = 1$. Similarly, we get a horizontal straight line when b is as large as possible, which is when $b = 2a$. In this case $2 = 2a + b$ yields $2 = 2a + 2a$ and hence $a = 1/2$.
At the endpoints, the area of the triangle is 0 and so at the maximum will be at $a = 2/3$. Thus,
$A(2/3) = \sqrt{(1-(2/3))^2(2(2/3)-1)} = \sqrt{(1/9)(1/3)} = \dfrac{1}{3\sqrt{3}}$
completing the proof.

Good Luck for your exams!