Full Solutions
MATH103 April 2010

April 4, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

SOLUTION. Following the hint we check if the integrands above are continuous on the entire domain of integration.
i. $\int_{-1}^{1} \frac{\cos x}{\ln x}\, dx$. Since $\ln x$ is not even defined for negative numbers, the integrand $\frac{\cos x}{\ln x}$ is certainly not continuous on $(-1, 1)$.
ii. $\int_{-1}^{1} \frac{\cos x}{x}\, dx$. The integrand is not defined for x=*0*, so it is certainly not continuous on $(-1, 1)$.
iii. $\int_{-1}^{1} \frac{\cos x}{\sqrt{x}+1}\, dx$. Since $\sqrt{x}$ is not even defined for negative numbers, the integrand $\frac{\cos x}{\sqrt{x}+1}$ is certainly not continuous on $(-1, 1)$.
iv. $\int_{-1}^{1} \frac{\cos x}{x^2+1}\, dx$. Here the integrand is defined everywhere. As a composition of continuous functions, the integrand is continuous. Hence the Fundamental Theorem of Calculus applies here.
Final answer: iv.

Question 1 (b)

SOLUTION. We want to integrate

$$\int x^2 e^{2x} dx$$

with integration by parts. We have three possible products:
a) $1 \cdot x^2 e^{2x}$
b) $x \cdot xe^{2x}$
c) $x^2 \cdot e^{2x}$
The first product doesn't make sense to choose since the second term is neither easy to derivate nor to integrate.
The second product also doesn't make sense because the second term is not easy to integrate. Also choosing xe^{2x} as u and x as dv is not good since integrating x AND derivating xe^{2x} makes the equation more complicated.
So it must be the third product. It doesn't matter, if we choose e^{2x} as u or dv because the integral is the same (excapt for constants). If we choose x^2 as dv, we get $x^3/3$ as v, what makes the equation more complicated. So we take x^2 as u and get $du = 2x$ and e^{2x} dv and get $v = e^{2x}/2$. That simplified the equation. **So we take answer ii.**
Integration would work now like this:

$$\begin{aligned}
\int x^2 e^{2x} dx &= \frac{1}{2}x^2 e^{2x} - \int 2x \frac{1}{2} e^{2x} dx \\
&= \frac{1}{2}x^2 e^{2x} - \underbrace{\int xe^{2x} dx}_{\text{integrate by parts again}} \\
&= \frac{1}{2}x^2 e^{2x} - \frac{1}{2}xe^{2x} + \int \frac{1}{2} e^{2x} dx \\
&= \frac{1}{2}x^2 e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + c
\end{aligned}$$

Question 1 (c)

SOLUTION 1. Following the hints we can just plug in the argument of the functions and calculate

$$\begin{aligned}
\sin(x^3) &= x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots \\
\cos(x^{3/2}) &= 1 - \frac{x^3}{2!} + \frac{x^6}{4!} - \cdots \\
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \\
\cos(x^3) &= 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \cdots \\
e^{x^3} &= 1 + x^3 + \frac{x^6}{2!} + \cdots
\end{aligned}$$

Hence, the correct answer is *v.*

SOLUTION 2. A clever way of doing this problem is to consider the following:

- $S_3(0) = 1$, hence the candidate function needs to also satisfy this condition. Since $\sin(0) = 0 \neq 1$, choice *i.* can be eliminated.
- S_3 is increasing for small positive values of x, since, for small x, the term x^3 dominates. In particular, $S_3(x) > 1$ for small positive values of x. But since the cosine is decreasing for small positive values of x, and can also never be more than *1*, the choices *ii*, *iii*, and *iv* can all be eliminated.
- This only leaves *v.* Notice that this choice makes sense

  $$e^0 = 0,$$

  and e^{x^3} is increasing.

Question 1 (d)

SOLUTION. Let's first check if the summands of the offered series go to zero.

$$\begin{aligned}
\lim_{k\to\infty} k^{-2} = \lim_{k\to\infty} \frac{1}{k^2} &= 0. \\
\lim_{k\to\infty} \frac{1}{k} &= 0. \\
\lim_{k\to\infty} k &= \infty. \\
\lim_{k\to\infty} (1.01)^k &= \infty. \\
\lim_{k\to\infty} k^2 &= \infty.
\end{aligned}$$

Thus iii., iv., and v. can not converge. But the series i. and ii. are candidates for convergent series.
By a pairwise comparison of the summands of series i. and series ii. we see that $k^{-2} < 1/k$, and hence (Series i.) $<$ (Series ii.). We are told that only one series converges, so it can only be Series i. (If Series ii. would converge, and (Series i.) $<$ (Series ii.), then Series i. would also converge.)
Indeed this is true, by the p-series test with $p = 2$.
Thus, the correct answer is i.
Fun fact: $\sum_{k=1}^{\infty} k^{-2} = \frac{\pi^2}{6}$.

Question 1 (e)

SOLUTION. Let's check first the if the integrand goes to 0, as a necessary condition for convergence.

i. $\lim_{x\to\infty} e^{3x} = \infty$

ii. $\lim_{x\to\infty} x^{-1} = 0$✓

iii. $\lim_{x\to\infty} x^{-1/2} = 0$✓

iv. $\lim_{x\to\infty} x^{-2} = 0$✓

v. $\lim_{x\to\infty} \frac{x^2}{1000} = \infty$

This only leaves candidates *ii.*, *iii.*, and *iv.*.

Since $x^{1/2} < x < x^2$ we have $x^{-1/2} > x^{-1} > x^{-2}$ and thus

$$\int_1^\infty x^{-1/2}\,dx > \int_1^\infty x^{-1}\,dx > \int_1^\infty x^{-2}\,dx$$

Since we are told that only one of the five integrals converges it must be *iv.*. (E.g., if the integral of *ii.* was finite, and the integral in *iv.* is less than the integral in *ii.*, then *iv.* would also necessarily be finite.)
And indeed, *iv.* does converge: By the integral test,

$$\int_1^\infty x^{-2}\,dx$$

converges if

$$\sum_{n=1}^{\infty} n^{-2}$$

converges. This sum does converge by the p-series test. **Hence the correct answer is *iv.*.**
As a fun fact,

$$\sum_{n=1}^{\infty} n^{-2} = \frac{\pi^2}{6}$$

Question 1 (f)

Solution. Using the hint, we first integrate *f(x)* and make sure it integrates to 1. Keep in mind that $f(x) = 0$ if $x \notin [-1, 1]$:

$$\begin{aligned}1 &= \int_{-\infty}^{\infty} f(x)\,dx \\ &= \int_{-1}^{1} f(x)\,dx \\ &= \int_{-1}^{1} (Ax + Bx^2)\,dx \\ &= A\frac{x^2}{2} + B\frac{x^3}{3}\Big|_{-1}^{1} \\ &= B\frac{2}{3}.\end{aligned}$$

Note that it is no surprise that A cancels here, because the term Ax is odd an thus cancels when integrating from *-1* to *1*.
The calculation above tells us that $\boldsymbol{B = 3/2}$.
The constant A needs to ensure that $f(x) \geq 0$, for all x. For very small (in absolute value) negative numbers x, the linear term Ax dominates the quadratic term $3/2\ x^2$, so we suspect that $A = 0$ is the only option to keep $f(x)$ positive. But let's do the math:
To show that $f(x)$ is positive it suffices to show that $f(x)$ is positive at the global minimum of $f(x)$. By the extreme value theorem, this global minimum will occur at either a critical point, or at the endpoints of the interval. We can find a critical point (convince yourself it is a local minimum) by setting $f'(x)$ to zero:

$$f'(x) = A + 2(3/2)x = A + 3x = 0.$$

Hence x = $-A/3$ is the local minimum of $f(x)$. At this point the value of f is

$$f(-A/3) = A(-A/3) + 3/2(-A/3)^2 = -A^2/3 + A^2/6 = -A^2/6 \leq 0$$

for any A. Since we need all points, including this critical point, to satisfy $f(x) \geq 0$ then we demand **A=0**. With this choice of A, notice that $f(-1) = f(1) = 3/2 > 0$ so we have that the local minimum is the global minimum and therefore that $f(x) \geq 0$ for all x.
Final answer: iii.

Question 1 (g)

SOLUTION. The mean value is
$(1)p(1) + (3)p(3) + (5)p(5) = \frac{1}{2} + \frac{3}{4} + \frac{5}{4} = \frac{10}{4} = \frac{5}{2}$
and so the correct answer is (i)

Question 1 (h)

SOLUTION. Saying that $y(t)$ does not change any more is the same as saying that $y(t)$ is constant. A function is constant if its derivative $\frac{dy}{dt}$ is equal to 0. So we look for values of y that satisfy $0 = 1 - 4y^2$. This equation only holds for $y = \pm 1/2$.
Hence the the right answer is **v**.
(Note: The answer y=$-1/2$ is not a choice here because of the initial condition $y(0)=0$. The solution of the differential equation starts at $y=0$ and then increases to $y=1/2$. With a different initial condition the other equilibrium y=$-1/2$ would be approached.)

Question 2

SOLUTION. To find the limits of the integral that calculates the area, we need to find the intersection points of the two functions f and g and so we set the functions equal.

$$f(x) \ = g(x)$$

That means

$$x^3 - 4x = -x^2 + 2x$$

and leads to

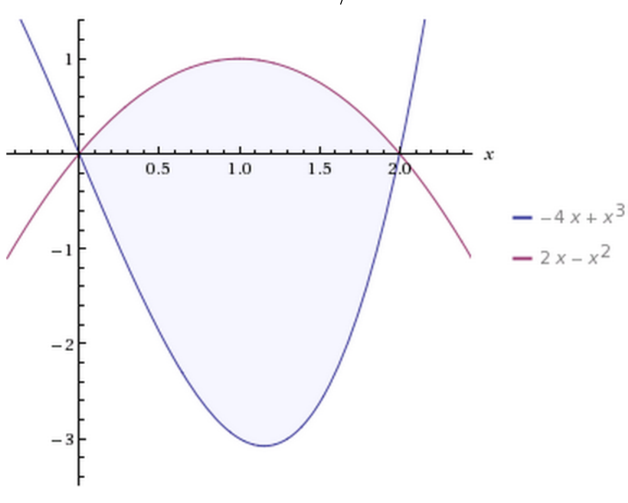
$$x(x+3)(x-2)=0$$

So, we find that the functions intersect three times at $x=-3,\ x=0,\ x=2$. However, we have a constraint that $x \geq 0$ and so we will only be concerned with the interval [0,2]. We need to integrate the function $f-g$ or $g-f$ (depending on which is the upper and which is the lower curve).
Now we check which function is the upper function. One way to do this is to simply plug in any point in the interval and compare the function values. On [0,2] we choose *x = 1* and get *f(1) = -3*, *g(1)=1*. Hence $g > f$ on [0,2].
Therefore, the area A is given by

$$A = \int_0^2 (g(x) - f(x))dx.$$

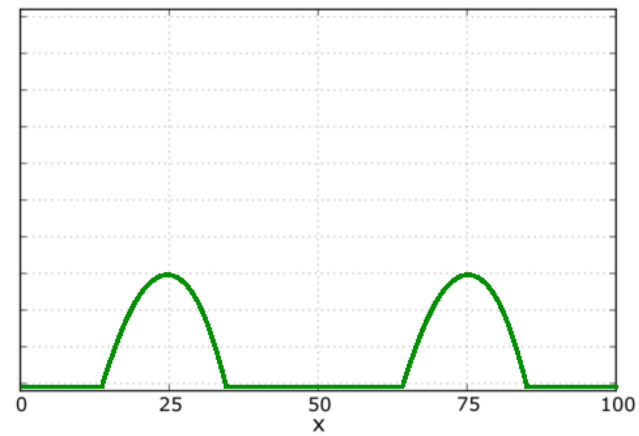
$$\begin{aligned} A &= \int_0^2 (g(x) - f(x))dx \\ &= \int_0^2 (-x^2 + 2x - x^3 + 4x)dx \\ &= \left[-\frac{x^3}{3} + 6\frac{x^2}{2} - \frac{x^4}{4} \right]_0^2 \\ &= -\frac{2^3}{3} + 6\frac{2^2}{2} - \frac{2^4}{4} - 0 \\ &= \frac{16}{3} \end{aligned}$$

Therefore the area of A is 16/3.



Question 3 (a)

SOLUTION. The derivative *p(x)* is zero whenever *F(x)* is flat, and *p(x)* is positive whenever *F(x)* is increasing.



Question 3 (b)

SOLUTION. The spill most likely occurred around the x=25km and/or at the x=75km margin. This is where the cumulative probability $F(x)$ changes the most, which means that this is where the probability density function $p(x)$ has its maximal values.

Question 4 (a)

SOLUTION. Let $u = \sqrt{x}$ so $du = \frac{dx}{2\sqrt{x}}$.
Cross multiplying yields $2\sqrt{x}du = dx$.
Since $u = \sqrt{x}$, the above is equal to $2udu = dx$.
Note further that $u(4) = \sqrt{4} = 2$ and $u(9) = \sqrt{9} = 3$
Substituting gives
$\int_4^9 e^{\sqrt{x}}dx = \int_2^3 2ue^u du = 2\int_2^3 ue^u du$
We integrate by parts. Let $v = u$ and $dw = e^u du$ so

$$\begin{aligned} v &= u \qquad & w &= e^u \\ dv &= du \qquad & dw &= e^u du \end{aligned}$$

Hence

$$\begin{aligned} \int_4^9 e^{\sqrt{x}}dx &= 2\int_2^3 ue^u du \\ &= 2(ue^u \mid_2^3 - \int_2^3 e^u du) \\ &= 2(3e^3 - 2e^2 - e^u \mid_2^3) \\ &= 2(3e^3 - 2e^2 - e^3 + e^2) \\ &= 2(2e^3 - e^2) \\ &= 4e^3 - 2e^2 \end{aligned}$$

completing the integral.

Question 4 (b)

SOLUTION. Let $u = \ln(x)$ so that $du = \frac{1}{x}dx$.
Notice that $u(e) = \ln(e) = 1$ and $u(e^2) = \ln(e^2) = 2\ln(e) = 2$. Hence
$\int_e^{e^2} \frac{1}{x\ln(x)}\, dx = \int_1^2 \frac{1}{u}\, du = \ln(u) \mid_1^2 = \ln(2) - \ln(1) = \ln(2)$
completing the question.

Question 4 (c)

SOLUTION. We proceed by partial fractions. Let A and B be constants such that

$$\frac{x}{(2-x)(1+2x)} = \frac{A}{2-x} + \frac{B}{1+2x} = \frac{A(1+2x)+B(2-x)}{(2-x)(1+2x)}$$

Since the denominators are the same, the numerators must be equal and hence

$$x = A(1+2x) + B(2-x)$$

Now, since this equality holds for all values of x, we can plug in specific values for x to get the coefficients. At $x = 2$, we have

$$2 = A(1+2(2)) + B(2-(2)) = 5A$$

and so $A = \frac{2}{5}$. At $x = -\frac{1}{2}$, we have

$$-\tfrac{1}{2} = A(1+2(-\tfrac{1}{2})) + B(2-(-\tfrac{1}{2})) = \tfrac{5}{2}B$$

and so $B = \frac{-1}{5}$. Hence, we have

$$\begin{aligned}
&\int_0^1 \frac{x}{(2-x)(1+2x)}\, dx \\
&= \int_0^1 \frac{\frac{2}{5}}{2-x}\, dx + \int_0^1 \frac{\frac{-1}{5}}{1+2x}\, dx \\
&= \left(\frac{2}{5}\ln|2-x|(-1) - \frac{1}{5}\ln|1+2x|\frac{1}{2}\right)\Bigg|_0^1 \\
&= \left(\frac{-2}{5}\ln|2-x| - \frac{1}{10}\ln|1+2x|\right)\Bigg|_0^1 \\
&= \frac{-2}{5}\ln|2-1| - \frac{1}{10}\ln|1+2(1)| - \frac{-2}{5}\ln|2-0| - \frac{1}{10}\ln|1+2(0)| \\
&= -\frac{1}{10}\ln|3| - \frac{-2}{5}\ln|2| \\
&= -\frac{\ln(3)}{10} + \frac{2\ln(2)}{5}
\end{aligned}$$

Question 5 (a)

SOLUTION. Let $u = x$ and $dv = \cos x dx$. This gives $du = dx$ and $v = \sin x$. Hence, using integration by parts, we have

$$
\begin{aligned}
\int_0^1 x\cos x\,dx &= x\sin x\,|_0^1 - \int_0^1 \sin x\,dx \\
&= (1)\sin(1) - (0)\sin(0) - (-\cos(x))|_0^1 \\
&= \sin(1) - (-\cos(1) + \cos(0)) \\
&= \sin(1) + \cos(1) - 1
\end{aligned}
$$

completing the problem.

Question 5 (b)

SOLUTION. Via the hint we have that
$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$
multiplying by x gives
$x\cos(x) = x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + ... = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n)!}$
and then integrating gives
$\int x\cos(x)\,dx = \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{144} - \frac{x^8}{5760} + ... = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+2}}{(2n+2)(2n)!}$
Taking into account the end points, we have
$\int_0^1 x\cos(x)\,dx = \frac{1^2}{2} - \frac{1^4}{8} + \frac{1^6}{144} - \frac{1^8}{5760} + ... = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+2)(2n)!}$
and simplifying and using only the first three terms, we have
$\int_0^1 x\cos(x)\,dx \approx \frac{1}{2} - \frac{1}{8} + \frac{1}{144} = \frac{55}{144}$
completing the question

Question 6 (a)

SOLUTION. Using separation of variables we can write

$$\frac{dh}{dt} = -kh^{1/3}$$

as

$$h^{-1/3}\,dh = -k\,dt, \quad \text{which we integrate to get} \quad \int h^{-1/3}\,dh = \int -k\,dt.$$

Both integrals are solved straight forward to yield

$$\frac{3}{2}h^{2/3} = -kT + C_1$$

which we solve for h to obtain

$$h(t) = \left(C_2 - \frac{2}{3}kt\right)^{3/2},$$

where C_2 is another constant.
All that's left to do now is to use the initial condition $h(0) = h_0$ to solve for C_2:

$$h_0 = h(0) = \left(C_2 - \frac{2}{3}k\cdot 0\right)^{3/2}, \quad \text{which yields} \quad C_2 = h_0^{2/3}.$$

Therefore the height $h(t)$ at any time t is given by

$$h(t) = \left(h_0^{2/3} - \frac{2}{3}kt\right)^{3/2}.$$

To double-check your answer you can take the derivative of h(t) and check that the given differential equation is indeed satisfied:

$$\begin{aligned}\frac{dh}{dt} &= \frac{3}{2}\left(h_0^{2/3} - \frac{2}{3}kt\right)^{1/2} \cdot \left(-\frac{2}{3}k\right) \\ &= -k\underbrace{\left(h_0^{2/3} - \frac{2}{3}kt\right)^{1/2}}_{=h^{1/3}}\end{aligned}$$

Question 6 (b)

SOLUTION. To find the time t_{empty} when the height $h(t_{empty})$ is zero we calculate

$$0 = h(t_{\text{empty}}) = \left(h_0^{2/3} - \frac{2}{3}kt_{\text{empty}}\right)^{3/2}.$$

Solving for t_{empty} yields

$$t_{\text{empty}} = \frac{3}{2k}h_0^{2/3}.$$

Note: The differential equation is no longer valid after this time, because a negative height of fluid in a tank does not make sense.

Question 7 (a)

SOLUTION 1. To get from the cumulative function to the probability density function, you need to take the derivate, i.e.

$$p(t) = F'(t) = \frac{1}{12}e^{-t/12}.$$

Now we apply the definition of the expected value to the random variable X that denotes the waiting time:

$$\overline{X} = \int_0^\infty tp(t)\,dt.$$

This is an improper integral, so we treat it with caution. To solve it, we apply integration by parts with u=t and $dv = e^{-t/12}dt$ and obtain

$$
\begin{aligned}
\overline{X} &= \int_0^\infty tp(t)\,dt \\
&= \lim_{T\to\infty}\left(\frac{1}{12}\int_0^T te^{-t/12}\,dt\right) \\
&= \lim_{T\to\infty}\left(\frac{1}{12}t(-12e^{-t/12})\bigg|_0^T + \frac{1}{12}\int_0^T 12e^{-t/12}\,dt\right) \\
&= \lim_{T\to\infty}\left(-Te^{-T/12} + 0 - 12e^{-t/12}\Big|_0^T\right) \\
&= \lim_{T\to\infty}\left(-12e^{-T/12} + 12e^0\right) \\
&= 12;
\end{aligned}
$$

SOLUTION 2. From class and the course notes you may remember that waiting time typically has a probability density function and cumulative function of the form

$$p(t) = ke^{-kt}, \quad \text{and} \quad F(t) = 1 - e^{-kt},$$

respectively. In this case, the expected waiting time is given by *1/k*. Here k = *1/12* so that the expected value is *12*.

Question 7 (b)

SOLUTION. Again, let X be the continuous random variable that denotes the waiting time. Since $F'(t) = p(t)$ we can use the Fundamental theorem of calculus to find the probability that the cell will divide between *3* and *6* hours:

$$
\begin{aligned}
p(3 \le X \le 6) &= \int_3^6 p(t)\,dt \\
&= F(6) - F(3) = (1 - e^{-6/12}) - (1 - e^{-3/12}) \\
&= e^{-1/4} - e^{-1/2} \\
&(\approx 17.2\%)
\end{aligned}
$$

Question 7 (c)

SOLUTION. The median t_{med} is defined as

$$F(t_{\text{med}}) = \frac{1}{2}$$

Since *F(t)* is given we can easily calculate that

$$\frac{1}{2} = F(t_{\text{med}}) = 1 - e^{-t_{\text{med}}/12}, \quad \text{i.e.} \quad e^{-t_{\text{med}}/12} = \frac{1}{2}.$$

After cross multiplying we get $e^{t_{\text{med}}/12} = 2$. Taking the logarithm *ln(x)* on both sides yields the final answer

$$t_{\text{med}} = 12\ln(2).$$

Note: From the class notes you may remember that the median waiting time is given by ln(2)/k. Here k=1/12, see part (a), so our calculated answer agrees with the answer we expected.

Question 8

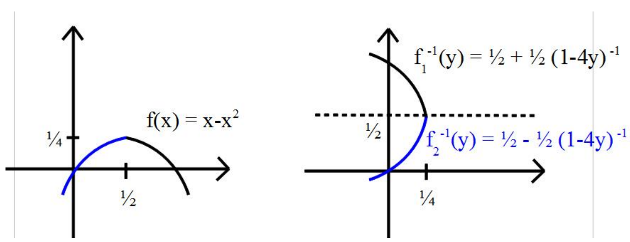
SOLUTION 1. We want to use the disk method and so we need to take the inverse of the function $f(x) = x(1-x)$.

$$f(x) = y = x - x^2$$

$$x^2 - x + y = 0$$

$$x_{1/2} = \frac{1 \pm \sqrt{1-4y}}{2}$$

We get two possible inverse functions $f_1^{-1}(y) = \frac{1}{2} + \frac{1}{2}\sqrt{1-4y}$ and $f_2^{-1}(y) = \frac{1}{2} - \frac{1}{2}\sqrt{1-4y}$.



On the picture we see that for our purpose we need the function f_2^{-1}. To find the boundaries for the integral we calculate

$$f(x=0) = 0, \qquad f\left(x = \frac{1}{2}\right) = \frac{1}{4}.$$

Now we use the formula for the disk method $V = \int \pi r(z)^2 dz$. Here V is the volume and r is the function of the radius. We need the function f_2^{-1} for the radius.

$$
\begin{aligned}
V &= \int_0^{\frac{1}{4}} \pi \left(\frac{1}{2} - \frac{\sqrt{1-4y}}{2}\right)^2 dy \\
&= \pi \int_0^{\frac{1}{4}} \left(\frac{1}{4} - \frac{1}{2}\sqrt{1-4y} + \frac{1-4y}{4}\right) dy \\
&= \pi \int_0^{\frac{1}{4}} \left(\frac{1}{2} - \frac{1}{2}\sqrt{1-4y} - y\right) dy \\
&= \frac{\pi}{2} \int_0^{\frac{1}{4}} \left(1 - (1-4y)^{\frac{1}{2}} - 2y\right) dy \\
&= \frac{\pi}{2} \left[y - \frac{2}{3}(1-4y)^{\frac{3}{2}} \left(-\frac{1}{4}\right) - y^2\right]_0^{\frac{1}{4}} \\
&= \frac{\pi}{2} \left(\frac{1}{4} + \frac{1}{6}\left(1 - 4(1/4)\right)^{\frac{3}{2}} - (1/4)^2 - 0 - \frac{1}{6}(1-4(0))^{\frac{3}{2}} + 0^2\right) \\
&= \frac{\pi}{2} \left(\frac{1}{4} - \frac{1}{16} - \frac{1}{6}\right) \\
&= \frac{\pi}{96}
\end{aligned}
$$

SOLUTION 2. For this problem it is much easier to use the shell method.
For using the shell method we need to know the height h of the trumpet for every $x \in [0, \frac{1}{2}]$. This is

$$h(x) = f(1/2) - f(x) = \frac{1}{4} - x(1-x).$$

Now we can easily integrate

$$
\begin{aligned}
V &= \int_0^{\frac{1}{2}} 2\pi x h(x) dx \\
&= 2\pi \int_0^{\frac{1}{2}} x\left(\frac{1}{4} - x + x^2\right) dx \\
&= 2\pi \left[\frac{1}{4}\frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4}\right]_0^{\frac{1}{2}} \\
&= 2\pi \left[\frac{1}{4}\frac{1}{8} - \frac{1}{8}\frac{1}{3} + \frac{1}{16}\frac{1}{4} - 0\right] \\
&= \frac{\pi}{96}
\end{aligned}
$$

Good Luck for your exams!