Full Solutions
MATH221 December 2011

December 4, 2014

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Educational Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Exam Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Educational Resources.

Question 1 (a)

Easiness: 90/100

SOLUTION. We can write this system in matrix form:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 2 & 3 & 5+t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

To know if the system is consistent, we want to look at the row reduced form of the augmented matrix. First, perform the operations $L_2 \to L_2$-L_1 and $L_3 \to L_3$-$2L_1$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 3 & 5+t & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & -1 & -1+t & 0 \end{bmatrix}$$

Then, perform $L_2 \to$ (-1)L_2

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & -1+t & 0 \end{bmatrix}$$

Perform $L_3 \to L_3$+L_2

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & t & -1 \end{bmatrix}$$

If we read the last row of this matrix, we obtain the equation

$$tx_3 = -1$$

The system is inconsistent if $t = 0$ since that would give us the equation 0 = -1. Hence the system is consistent for all non-zero values of t.

Question 1 (b)

Easiness: 58/100

SOLUTION. In the first part of the question, we calculated the row reduced form of the augmented matrix of the system and obtained:

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & t & -1 \end{bmatrix}$$

For which we saw that unless $t = 0$, there is always a unique solution (since the matrix has full rank in that case).
The third row of the matrix translates into

$$tx_3 = -1$$

Which is equivalent to

$$x_3 = -\frac{1}{t}$$

(recall that we suppose here that t is NOT equal to 0.)
The second row gives

$$x_2 + x_3 = -1$$

and so

$$x_2 - \frac{1}{t} = -1$$

which gives

$$x_2 = -1 + \frac{1}{t} = \frac{1-t}{t}$$

And finally, the first row gives

$$x_1 + 2x_2 + 3x_3 = 1$$

Substituting for x_2 and x_3 we obtain

$$x_1 + \frac{2(1-t)}{t} - \frac{3}{t} = 1$$

And hence

$$x_1 = \frac{t - 2(1-t) + 3}{t} = \frac{3t+1}{t}$$

And we obtain that the system has a unique solution (when t is not zero) which depends on t and the solution is

$$\begin{bmatrix} x_1 \\ \\ x_2 \\ \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{3t+1}{t} \\ \\ \frac{1-t}{t} \\ \\ -\frac{1}{t} \end{bmatrix}$$

Question 2 (a) — **Easiness: 100/100**

SOLUTION. We know that a matrix is invertible if and only if its determinant is not 0.

$$A = \begin{bmatrix} 3 & 2 & t+3 \\ 3 & 4 & t+1 \\ 2 & 2 & t+1 \end{bmatrix}$$

Then,

$$\begin{aligned}
\det(A) &= \begin{vmatrix} 3 & 2 & t+3 \\ 3 & 4 & t+1 \\ 2 & 2 & t+1 \end{vmatrix} \\
&= 3 \cdot 4 \cdot (t+1) + 2 \cdot (t+1) \cdot 2 + (t+3) \cdot 3 \cdot 2 \\
&\quad - (t+3) \cdot 4 \cdot 2 - (t+1) \cdot 2 \cdot 3 - (t+1) \cdot 2 \cdot 3 \\
&= 12(t+1) + 4(t+1) + 6(t+3) \\
&\quad - 8(t+3) - 6(t+1) - 6(t+1) \\
&= 4(t+1) - 2(t+3) \\
&= 2t - 2
\end{aligned}$$

Hence the determinant of the matrix A is zero if and only if $t = 1$. And so, the matrix is invertible for all values of t different than 1.

Question 2 (b)

SOLUTION. When $t = 0$, the matrix A is

$$A = \begin{bmatrix} 3 & 2 & 3 \\ 3 & 4 & 1 \\ 2 & 2 & 1 \end{bmatrix}$$

One way to obtain the inverse of a matrix is to row-reduce the augmented matrix [A|I]. Let's proceed with this method.

$$[A|I] = \left[\begin{array}{ccc|ccc} 3 & 2 & 3 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

Perform $L_2 \to L_2$-L_1 and $L_3 \to L_3$-$(2/3)L_1$

$$\sim \left[\begin{array}{ccc|ccc} 3 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & -2 & -1 & 1 & 0 \\ 0 & 2/3 & -1 & -2/3 & 0 & 1 \end{array}\right]$$

Perform $L_3 \to 3L_3$

$$\sim \left[\begin{array}{ccc|ccc} 3 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & -2 & -1 & 1 & 0 \\ 0 & 2 & -3 & -2 & 0 & 3 \end{array}\right]$$

Perform $L_3 \to L_3$-L_2

$$\sim \left[\begin{array}{ccc|ccc} 3 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & -2 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 3 \end{array}\right]$$

Perform $L_3 \to$ (-1)L_3

$$\sim \left[\begin{array}{ccc|ccc} 3 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & -2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & -3 \end{array}\right]$$

Perform $L_1 \to L_1$-L_2 and $L_2 \to L_2$+2L_3

$$\sim \left[\begin{array}{ccc|ccc} 3 & 0 & 5 & 2 & -1 & 0 \\ 0 & 2 & 0 & 1 & 3 & -6 \\ 0 & 0 & 1 & 1 & 1 & -3 \end{array}\right]$$

Perform $L_1 \to L_1$-5L_3

$$\sim \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & -3 & -6 & 15 \\ 0 & 2 & 0 & 1 & 3 & -6 \\ 0 & 0 & 1 & 1 & 1 & -3 \end{array}\right]$$

Perform $L_1 \to (1/3)L_1$ and $L_2 \to (1/2)L_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -2 & 5 \\ 0 & 1 & 0 & 1/2 & 3/2 & -3 \\ 0 & 0 & 1 & 1 & 1 & -3 \end{array}\right]$$

And we can conclude that

$$A^{-1} = \begin{bmatrix} -1 & -2 & 5 \\ 1/2 & 3/2 & -3 \\ 1 & 1 & -3 \end{bmatrix}$$

Question 3 (a)

SOLUTION. Eigenvalues are values of the variable λ such that the system

$$A - \lambda I$$

has a non-trivial nullspace or equivalently, when it has determinant 0.
We compute this determinant:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -6-\lambda & 8 \\ -4 & 6-\lambda \end{vmatrix} \\ &= (-6-\lambda)(6-\lambda) - 8(-4) \\ &= -36 + \lambda^2 + 32 \\ &= \lambda^2 - 4 \end{aligned}$$

And so we conclude that this determinant is zero if and only if λ is 2 or -2. (Note: this equation is called the characteristic polynomial of the linear system).
For each eigenvalue, an eigenvector is simply a non-trivial solution (non-trivial means not zero since that's an obvious solution that is always present).
For $\lambda = 2$ we obtain the system

$$
\begin{aligned}
A - 2I &= \begin{bmatrix} -8 & 8 \\ -4 & 4 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}
\end{aligned}
$$

And so the general solution is a vector of the form $[t,\ t\ ]$, which means that the eigenspace for the eigenvalue 2 is spanned by the vector [1, 1].
For λ = -2 we obtain the system

$$
\begin{aligned}
A + 2I &= \begin{bmatrix} -4 & 8 \\ -4 & 8 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}
\end{aligned}
$$

And so the general solution is a vector of the form $[2t,\ t\ ]$, which means that the eigenspace for the eigenvalue -2 is spanned by the vector [2, 1].

Question 3 (b)

SOLUTION 1. First, let's compute the matrix B

$$
\begin{aligned}
B &= A^2 + 3A + 2I \\
&= \begin{bmatrix} -6 & 8 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} -6 & 8 \\ -4 & 6 \end{bmatrix} + 3 \begin{bmatrix} -6 & 8 \\ -4 & 6 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} -18 & 24 \\ -12 & 18 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \\
&= \begin{bmatrix} -12 & 24 \\ -12 & 24 \end{bmatrix}
\end{aligned}
$$

And clearly this matrix has determinant zero.

SOLUTION 2. **Note: this is a more fancy solution, no worries if you don't like it as much.**
The matrix B is a polynomial in the matrix A that we can factor as

$$
\begin{aligned}
B &= A^2 + 3A + 2I \\
&= (A + I)(A + 2I)
\end{aligned}
$$

And since the determinant is multiplicative, we have that

$$
\begin{aligned}
\det(B) &= \det((A + I)(A + 2I)) \\
&= \det(A + I)\det(A + 2I)
\end{aligned}
$$

Now in part (a) we saw that -2 is an eigenvalue and hence the determinant of $A + 2I$ is zero and thus so is the determinant of the matrix B.

Question 4 (a)

SOLUTION. A matrix is diagonalizable if you can find a basis of eigenvectors, here that means we look for three linearly independent eigenvectors.
To find eigenvectors, we first need to find eigenvalues. For this, we compute the characteristic polynomial:

$$\begin{aligned}\det(A-\lambda I) &= \begin{vmatrix} 2-\lambda & 0 & 0 \\ 1 & -\lambda & -1 \\ -1 & 2 & 3-\lambda \end{vmatrix} \\ &= (2-\lambda)((-\lambda)(3-\lambda)+2) \\ &= (2-\lambda)(\lambda^2-3\lambda+2) \\ &= (2-\lambda)(\lambda-2)(\lambda-1) \\ &= -(\lambda-2)^2(\lambda-1)\end{aligned}$$

And so A has the eigenvalues 1 and 2. We now employ two facts about eigenvectors:

1. The eigenvectors of distinct eigenvalues are linearly independent.
2. The number of linearly independent eigenvector for a given eigenvalue (= dimension of the eigenspace = geometric multiplicity) is between 1 and the algebraic multiplicity of that eigenvalue.

Recall that the algebraic multiplicity corresponds to the exponent in the characteristic polynomial:

$$\det(A-\lambda I) = -(\lambda-2)^{2}(\lambda-1)^{1}$$

means that the algebraic multiplicity of the eigenvalue 2 is two, and of the eigenvalue 1 is one.
With this we conclude that

1. There is exactly one eigenvector corresponding to the eigenvalue 1, and it is linearly independent from any eigenvector corresponding to the eigenvalue 2.
2. There is either one or two eigenvectors corresponding to the eigenvalue 2. We can find out how many by calculating the dimension of the eigenspace.
3. If the dimension of the eigenspace corresponding to the eigenvector 2 is two, then there are three linearly independent eigenvectors and A is diagonalizable. If the dimension is only one, then A has only two linearly independent eigenvectors and is not diagonalizable.

Hence, we only need to find the dimension of that eigenspace. Since that eigenspace is the nullspace of the matrix A - $2I$, we simply compute the rank of that matrix:

$$\begin{aligned}A-2I &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & -2 & -1 \\ -1 & 2 & 1 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\end{aligned}$$

And since the rank is 1 this eigenspace will be of dimension 2. Therefore matrix A is diagonalizable.

Question 4 (b)

SOLUTION. **Step 1.** If v is an eigenvector for a matrix A for some eigenvalue λ, then we can write

$$Av = \lambda v$$

Step 2. Since we assume that the matrix A is invertible, we can multiply on the above equation by that inverse on the left on each side and obtain

$$A^{-1}Av = A^{-1}\lambda v$$

Which is equivalent to

$$v = \lambda A^{-1}v$$

Step 3. We would now like to divide by λ, but before we should check that this is allowed, i.e. that λ is not zero. Since A in invertible, $Aw = 0 = 0(w)$ implies $w = 0$. Thus $\lambda = 0$ is **not** an eigenvalue for A and we can safely bring λ to the LHS in the equation above. This yields

$$\frac{1}{\lambda}v = A^{-1}v$$

which would mean that v is an eigenvector for the eigenvalue $1/\lambda$ of the inverse of the matrix A. So we can conclude that the statement is **true**.

Question 5 (a)

SOLUTION. To find the standard matrix of this linear transformation, we simply consider the image of the three base vectors.
The vector [1, 0, 0] is sent to the vector [-1, 0, 0] and the vectors [0, 1, 0] and [0, 0, 1] simply remain where they are (since they belong to the yz-plane itself). Hence the standard matrix of this linear transformation is

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Question 5 (b)

SOLUTION. Since

$$b_3 = 7b_2 - 4b_1$$

we have that

$$\begin{aligned} T(b_3) &= T(7b_2 - 4b_1) \\ &= 7T(b_2) - 4T(b_1) \\ &= 7\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 4\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 10 \end{bmatrix} \end{aligned}$$

To find the matrix of T with respect to the standard basis, we need to compute the image of each of these standard vectors.

$$\begin{aligned} T(e_1) &= T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \\ &= T(b_2) \\ &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} T(e_2) &= T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\ &= T(b_2 - b_1) \\ &= T(b_2) - T(b_1) \\ &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} & = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}$$

And so the matrix of T in the standard basis is

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

"Note that you can check that your result makes sense by verifying this matrix on the vector b_3, indeed we have that"

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \end{bmatrix}$$

Question 6 (a)

SOLUTION. First we must determine if the vectors are orthogonal. We know that two vectors are orthogonal if their dot product is zero.
Consider $v_1 \cdot v_2$. This gives:

$$(1/2)(1/2) + (1/2)(1/2) + (1/2)(-1/2) + (1/2)(-1/2) = 0$$

All other dot products are calculated the same way and all vanish. Hence, v_1, v_2, v_3, v_4 is an orthogonal set.
Second, we must determine if the vectors are linearly independent. We will do so by placing them in a matrix and using Gaussian elimination.

$$\begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}$$

We multiply the whole matrix by 2 to get

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Subtracting the first row from the following three rows, we get

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & 0 & -2 \\ 0 & -2 & -2 & 0 \end{bmatrix}$$

We rearrange the rows so that the second row becomes the last row; each of the last three rows is also multiplied by -1/2.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

We subtract row 2 from row 1 and row 2 from row 3 to get

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

We subtract row 3 from row 1 and row 3 from row 4 to get

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Multiplying the last row by -1/2 gives

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Which finally simplifies to

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Because the matrix reduces to an identity matrix the columns must be linearly independent. Thus v_1, v_2, v_3, v_4 are a basis for $\mathbb{R}^4$.

Question 6 (b)

SOLUTION. We create the augmented matrix $[v_1 v_2 v_3 v_4 b]$ and use Gaussian elimination.

$$\begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & -1/2 & -1/2 & 2 \\ 1/2 & -1/2 & 1/2 & -1/2 & 3 \\ 1/2 & -1/2 & -1/2 & 1/2 & 4 \end{bmatrix}$$

We multiply the whole matrix by 2 to get

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & -1 & -1 & 4 \\ 1 & -1 & 1 & -1 & 6 \\ 1 & -1 & -1 & 1 & 8 \end{bmatrix}$$

Subtracting the first row from the following three rows, we get

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & -2 & -2 & 2 \\ 0 & -2 & 0 & -2 & 4 \\ 0 & -2 & -2 & 0 & 6 \end{bmatrix}$$

We rearrange the rows so that the second row becomes the last row; each of the last three rows is also multiplied by -1/2.

$$\left[\begin{array}{ccccc} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

We subtract row 2 from row 1 and row 2 from row 3 to get

$$\left[\begin{array}{ccccc} 1 & 0 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

We subtract row 3 from row 1 and row 3 from row 4 to get

$$\left[\begin{array}{ccccc} 1 & 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$$

Multiplying row 4 by -1/2, we get

$$\left[\begin{array}{ccccc} 1 & 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Which finally simplifies to

$$\left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

So our scalars will be $x_1 = 5, x_2 = -2, x_3 = -1$, and $x_4 = 0$.
In other words $b = 5v_1 - 2v_2 - v_3$.

Question 7

SOLUTION. We are not going to multiply A with itself a hundred times, at least not directly. Instead, we are going to diagonalize A first, because the 100th power of a diagonal matrix is much easier to calculate. Indeed, if D is the diagonal matrix corresponding to A (i.e. is a matrix with the eigenvalues of A on the diagonal), and if T is the transformation matrix of eigenvectors of A, then

$$\begin{aligned} A^n &= (TDT^{-1})^n \\ &= \underbrace{TDT^{-1}TDT^{-1} \cdots TDT^{-1}}_{n \text{ times}} \\ &= TD^nT^{-1} \end{aligned}$$

So let's find D, T and T^{-1}. We need the eigenvalues and eigenvectors of A, so we calculate the roots of the characteristic polynomial $\det(A - \lambda I)$:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1/2 - \lambda & 0 & 1/3 \\ 0 & 1/2 - \lambda & 1/2 \\ 1/2 & 1/2 & 1/6 - \lambda \end{vmatrix} \\ &= (\frac{1}{2} - \lambda)(\frac{1}{2} - \lambda)(\frac{1}{6} - \lambda) + 0 + 0 - \frac{1}{3}(\frac{1}{2} - \lambda)\frac{1}{2} - 0 - (\frac{1}{2} - \lambda)\frac{1}{2}\frac{1}{2} \\ &= (\frac{1}{2} - \lambda)(\frac{1}{12} - \frac{1}{2}\lambda - \frac{1}{6}\lambda + \lambda^2 - \frac{1}{6} - \frac{1}{4}) \\ &= (\frac{1}{2} - \lambda)(\lambda^2 - \frac{2}{3}\lambda - \frac{1}{3}) \\ &= (\frac{1}{2} - \lambda)(\lambda + \frac{1}{3})(\lambda - 1) \end{aligned}$$

And so the eigenvalues are 1/2, -1/3 and 1.
We compute an eigenvector for each eigenvalue using row reduction on the system $A - \lambda I$.
For $\lambda = 1$ we have the system

$$\begin{aligned} A - I &= \begin{bmatrix} -1/2 & 0 & 1/3 \\ 0 & -1/2 & 1/2 \\ 1/2 & 1/2 & -5/6 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & -2/3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

And so this eigenspace is spanned by the eigenvector [2, 3, 3].
For $\lambda = 1/2$ we have the system

$$\begin{aligned} A - (1/2)I &= \begin{bmatrix} 0 & 0 & 1/3 \\ 0 & 0 & 1/2 \\ 1/2 & 1/2 & -1/3 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

And so this eigenspace is spanned by the eigenvector [1, -1, 0].
For λ = -1/3 we have the system

$$\begin{aligned} A + (1/3)I &= \begin{bmatrix} 5/6 & 0 & 1/3 \\ 0 & 5/6 & 1/2 \\ 1/2 & 1/2 & 1/2 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 1 & 1 \\ 5 & 0 & 2 \\ 0 & 5 & 3 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & 3 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

And so this eigenspace is spanned by the eigenvector [2, 3, -5].
Using this basis of eigenvectors, we have that

$$A = T \cdot D \cdot T^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & -1/3 \end{bmatrix} \quad T = \begin{bmatrix} 2 & 1 & 2 \\ 3 & -1 & 3 \\ 3 & 0 & -5 \end{bmatrix}$$

We need now to compute the inverse of the matrix T. We obtain

$$T^{-1} = \begin{bmatrix} 1/8 & 1/8 & 1/8 \\ 3/5 & -2/5 & 0 \\ 3/40 & 3/40 & -1/8 \end{bmatrix}$$

And so

$$
\begin{aligned}
A^n &= (T \cdot D \cdot T^{-1})^n \\
&= T \cdot D^n \cdot T^{-1} \\
&= \begin{bmatrix} 2 & 1 & 2 \\ 3 & -1 & 3 \\ 3 & 0 & -5 \end{bmatrix} \cdot \begin{bmatrix} 1^n & 0 & 0 \\ 0 & (1/2)^n & 0 \\ 0 & 0 & (-1/3)^n \end{bmatrix} \cdot \begin{bmatrix} 1/8 & 1/8 & 1/8 \\ 3/5 & -2/5 & 0 \\ 3/40 & 3/40 & -1/8 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 2^{-n} & 2(-1/3)^n \\ 3 & -2^{-n} & (-1)^n 3^{1-n} \\ 3 & 0 & -5(-1/3)^n \end{bmatrix} \cdot \begin{bmatrix} 1/8 & 1/8 & 1/8 \\ 3/5 & -2/5 & 0 \\ 3/40 & 3/40 & -1/8 \end{bmatrix} \\
&= \begin{bmatrix} 1/4 + (3/5)(1/2)^n + (3/20)(-1/3)^n & 1/4 - (2/5)(1/2)^n + (3/20)(-1/3)^n & 1/4 - (1/4)(-1/3)^n \\ 3/8 - (3/5)(1/2)^n + (9/40)(-1/3)^n & 3/8 + (2/5)(1/2)^n + (9/40)(-1/3)^n & 3/8 - (3/8)(-1/3)^n \\ 3/8 - (3/8)(-1/3)^n & 3/8 - (3/8)(-1/3)^n & 3/8 + (5/8)(-1/3)^n \end{bmatrix}
\end{aligned}
$$

So we can compute x_{100}

$$
\begin{aligned}
x_{100} &= A^{100} x_0 \\
&= \begin{bmatrix} 1/4 + (3/5)2^{-100} + (3/20)3^{-100} & 1/4 - (2/5)2^{-100} + (3/20)3^{-100} & 1/4 - (1/4)3^{-100} \\ 3/8 - (3/5)2^{-100} + (9/40)3^{-100} & 3/8 + (2/5)2^{-100} + (9/40)3^{-100} & 3/8 - (3/8)3^{-100} \\ 3/8 - (3/8)3^{-100} & 3/8 - (3/8)3^{-100} & 3/8 + (5/8)3^{-100} \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 6 \end{bmatrix} \\
&= \begin{bmatrix} 4 + (1/2)^{100} \\ 6 - (1/2)^{100} \\ 6 \end{bmatrix}
\end{aligned}
$$

As n becomes very large, x_n approaches

$$
\lim_{n \to \infty} x_n = \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix}
$$

Question 8 (a)

SOLUTION. The basis of the column space of A are the columns of A corresponding to the linearly independent columns of the row-reduced version of A. We will find the row reduced matrix for A by performing Gaussian elimination as follows:

$$
\begin{bmatrix} 1 & 0 & -1 & 2 & 4 \\ 0 & 1 & -2 & 0 & -1 \\ -1 & -2 & 5 & 1 & 4 \\ 1 & 0 & -1 & 1 & 2 \end{bmatrix}
$$

Add row 1 to row 3; subtract row 1 from row 4.

$$
\begin{bmatrix} 1 & 0 & -1 & 2 & 4 \\ 0 & 1 & -2 & 0 & -1 \\ 0 & -2 & 4 & 3 & 8 \\ 0 & 0 & 0 & -1 & -2 \end{bmatrix}.
$$

Add twice row 2 to row 3.

$$
\begin{bmatrix} 1 & 0 & -1 & 2 & 4 \\ 0 & 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & -1 & -2 \end{bmatrix}.
$$

Multiply row 3 by 1/3.
$$\begin{bmatrix} 1 & 0 & -1 & 2 & 4 \\ 0 & 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 & -2 \end{bmatrix}.$$
Subtract twice row 3 from row 1; add row 3 to row 4
$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$
The row reduced version of A has columns 1, 2, and 4 as linearly independent vectors. Therefore, the basis of the column space will be the first, second and fourth column of the **original** matrix A. Hence a basis for the column space is:
$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Question 8 (b)

SOLUTION. By the rank-nullity theorem, we know that $\dim(\mathbb{R}^5) = \dim(\text{column space}) + \dim(\text{kernel})$. Because A is a linear transformation from $\mathbb{R}^5$, and the dimension of the column space of A is 3, we conclude that the dimension of the kernel is 2.

Good Luck for your exams!