Full Solutions
MATH105 April 2014

April 16, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

SOLUTION. The planes are orthogonal if the scalar product of their normal vector is zero.

$$n_Q = (-1, 5, -3)$$
$$n_P = (3, -\frac{9}{5}, -4)$$

$$n_Q \cdot n_P = (-1) \cdot 3 + 5 \cdot (-\frac{9}{5}) + (-3) \cdot (-4) = 0.$$

Hence the two planes are orthogonal.

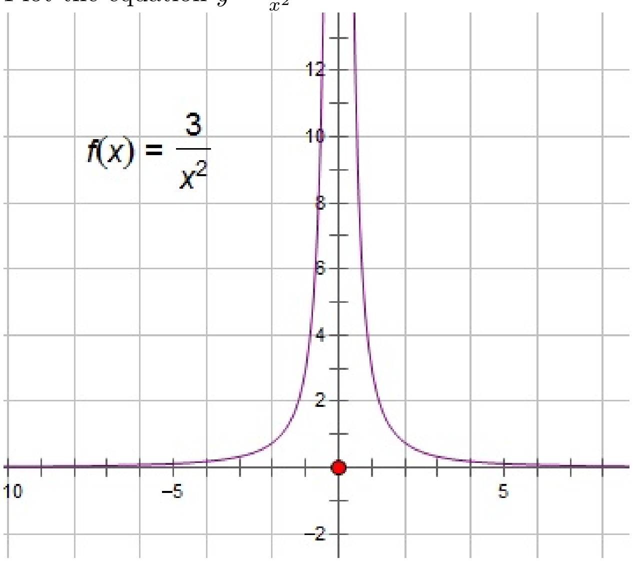
Question 1 (b)

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

Setting $V = \pi$,

$$\frac{\pi x^2 y}{3} = \pi$$
$$x^2 y = 3$$
$$y = \frac{3}{x^2}$$

Plot the equation $y = \frac{3}{x^2}$



Question 1 (c)

SOLUTION. First, take the derivative with respect to y using the chain rule and treating x as a constant. Then use take the derivative with respect to x using the product rule and the chain rule:

$$
\begin{aligned}
\frac{\partial^2}{\partial x \partial y} f(x,y) &= \frac{\partial^2}{\partial x \partial y} \sin(xy) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\sin(xy)\right) \\
&= \frac{\partial}{\partial x}\left(\cos(xy)x\right) \\
&= -\sin(xy)xy + \cos(xy)
\end{aligned}
$$

Question 1 (d)

SOLUTION. Set $a = 5$, $b = 15$ and $n = 50$.

$$
\begin{aligned}
\Delta x = \frac{b-a}{n} = \frac{15-5}{50} = \frac{1}{5}; \\
x_i = a + i\Delta x = 5 + \frac{i}{5}; \\
\frac{x_{i-1}+x_i}{2} = \frac{\left(5+\frac{i-1}{5}\right)+\left(5+\frac{i}{5}\right)}{2} = 5 + \frac{2i-1}{10}.
\end{aligned}
$$

Hence, the Riemann sum is:
$R = \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right) \cdot \Delta x = \sum_{i=1}^{50}\left(5+\frac{2i-1}{10}\right)^8 \cdot \frac{1}{5}$.

Question 1 (e)

SOLUTION. Split the interval of integration to $[1, 3]$ and $[3, 5]$:

$$
\begin{aligned}
\int_1^5 f(x)dx &= \int_1^3 f(x)dx + \int_3^5 f(x)dx \\
&= \int_1^3 3dx + \int_3^5 xdx \\
&= 3x\Big|_1^3 + \frac{x^2}{2}\Big|_3^5 \\
&= (9-3) + \left(\frac{25}{2} - \frac{9}{2}\right) \\
&= 14.
\end{aligned}
$$

Question 1 (f)

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

Since $\left(\frac{1}{2}(f'(x))^2\right)' = f'(x)f''(x)$, we have:

$$\begin{aligned}
\int_1^2 f'(x)f''(x)dx &= \frac{1}{2}(f'(x))^2\Big|_1^2 \\
&= \frac{1}{2}\left(f'(2)\right)^2 - \frac{1}{2}\left(f'(1)\right)^2 \\
&= \frac{9}{2} - 2 \\
&= \frac{5}{2}
\end{aligned}$$

Question 1 (g)

SOLUTION. Recall the formula for integration by parts

$$\int u dv = uv - \int v du.$$

Let $u = \cos^{-1}(y)$ and $dv = dy$. Then $du = \frac{-1}{\sqrt{1-y^2}}dy$ and $v = y$.

$$\begin{aligned}
\int \cos^{-1}(y)dy &= y\cos^{-1}(y) - \int y(\cos^{-1}(y))'dy \\
&= y\cos^{-1}(y) + \int \frac{y}{\sqrt{1-y^2}}dy
\end{aligned}$$

Now, recognizing y as the derivative of $1 - y^2$ (to within a factor of -1/2), we can do a substitution with $x = 1 - y^2$ and $dx = -2ydy$.

$$\begin{aligned}
\int \cos^{-1}(y)dy &= y\cos^{-1}(y) - \frac{1}{2}\int \frac{1}{\sqrt{x}}dx \\
&= y\cos^{-1}(y) - \sqrt{x} + C \\
&= y\cos^{-1}(y) - \sqrt{1-y^2} + C
\end{aligned}$$

where we computed $\int \frac{1}{\sqrt{x}}dx = \int x^{-1/2}dx = \frac{1}{-1/2+1}x^{-1/2+1} + C = 2\sqrt{x} + C$ by the reverse power rule.

Question 1 (h)

SOLUTION. Following the remark in the hint, we begin by factoring out $\cos x$, the derivative of $\sin x$ so that we can later do a substitution. In doing this substitution, we will need to express the integrand in terms of powers of $\sin x$ with a single factor of $\cos x$.

$$\begin{aligned}
\int \cos^3(x)\sin^4(x)dx &= \int \cos(x)\cos^2(x)\sin^4(x)dx \\
&= \int \cos(x)(1-\sin^2(x))\sin^4(x)dx \\
&= \int (\sin^4(x) - \sin^6(x))\cos(x)dx
\end{aligned}$$

With $u = \sin(x)$, $du = \cos(x)dx$ giving

$$\begin{aligned}\int (u^4 - u^6)du &= \frac{u^5}{5} - \frac{u^7}{7} + C \\ &= \frac{\sin^5(x)}{5} - \frac{\sin^7(x)}{7} + C.\end{aligned}$$

Question 1 (i)

Solution. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

Let $t = \frac{x+1}{2}$. Then $dt = \frac{dx}{2}$.

$$\begin{aligned}\int \frac{dx}{\sqrt{3-2x-x^2}} &= \int \frac{dx}{\sqrt{4-(x+1)^2}} \\ &= \int \frac{dx}{2\sqrt{1-\left(\frac{x+1}{2}\right)^2}} \\ &= \int \frac{dt}{\sqrt{1-t^2}} \\ &= \sin^{-1} t + C \\ &= \sin^{-1}\left(\frac{x+1}{2}\right) + C.\end{aligned}$$

Question 1 (j)

Solution. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

Factor the denominator:
$x^2 - x - 6 = (x-3)(x+2)$
Use partial fraction:

$$\begin{aligned}\frac{x-13}{x^2-x-6} &= \frac{A}{x-3} + \frac{B}{x+2} \\ &= \frac{Ax+2A+Bx-3B}{(x-3)(x+2)} \\ &= \frac{(A+B)x+(2A-3B)}{(x-3)(x+2)}\end{aligned}$$

Comparing the coefficients we get:

$$\begin{aligned}A+B &= 1 \\ 2A-3B &= -13\end{aligned}$$

Solve and get:

$$\begin{aligned} A &= -2 \\ B &= 3 \end{aligned}$$

Hence:

$$\begin{aligned} \int \frac{x-13}{x^2-x-6}dx &= \int \frac{-2}{x-3} + \frac{3}{x+2}dx \\ &= -2\ln|x-3| + 3\ln|x+2| + C \end{aligned}$$

Question 1 (k)

Solution. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

$$\begin{aligned} \mathbb{E}(X) &= \int_{-\infty}^{\infty} xf(x)dx \\ &= \int_{-\infty}^{1} xf(x)dx + \int_{1}^{\infty} xf(x)dx \\ &= \int_{-\infty}^{1} 0dx + \int_{1}^{\infty} x \cdot \frac{3}{2}x^{-\frac{5}{2}}dx \\ &= 0 + \int_{1}^{\infty} \frac{3}{2}x^{-\frac{3}{2}}dx \\ &= \lim_{b\to\infty} \int_{1}^{b} \frac{3}{2}x^{-\frac{3}{2}}dx \\ &= \lim_{b\to\infty} -3x^{-\frac{1}{2}}\Big|_1^b \\ &= \lim_{b\to\infty} -3b^{-\frac{1}{2}} - (-3(1)^{-\frac{1}{2}}) \\ &= 3. \end{aligned}$$

Question 1 (l)

Solution. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

Since $\displaystyle\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$, we have:
$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$.
Since $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$, we have:
$\sum_{n=1}^{\infty} \left(-\frac{2}{5}\right)^{n-1} = \sum_{n=0}^{\infty} \left(-\frac{2}{5}\right)^n = \frac{1}{1-(-\frac{2}{5})} = \frac{5}{7}$.
Hence
$\sum_{n=1}^{\infty} \left[\left(\frac{1}{3}\right)^n + \left(-\frac{2}{5}\right)^{n-1}\right] = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n + \sum_{n=1}^{\infty} \left(-\frac{2}{5}\right)^{n-1} = \frac{1}{2} + \frac{5}{7} = \frac{17}{14}$.

Question 1 (m)

Solution 1. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

$\int f(x)dx = 1$ when $f(x)$ is a probability density function. Hence, we have:

$$\begin{aligned} 1 &= \int_{-1}^{1} (1 + k|x|)dx \\ &= 2\int_{0}^{1} (1 + kx)dx \\ &= 2\left(x + \frac{kx^2}{2}\right)\Big|_0^1 \\ &= 2\left(1 + \frac{k}{2}\right) \\ &= 2 + k. \end{aligned}$$

Note that in the second equality we use $\int_{-1}^{1} f(x)dx = 2\int_{0}^{1} f(x)dx$ since $f(x)$ is an even function; i. e. $f(x) = f(-x)$.
Finally we get $1 = 2 + k$. Solve it to get $k = -1$.

Solution 2. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

$\int f(x)dx = 1$ when $f(x)$ is a probability density function. Hence, we have:

$$\begin{aligned} 1 &= \int_{-1}^{1} (1 + k|x|)dx \\ &= \int_{-1}^{0} (1 + k|x|)dx + \int_{0}^{1} (1 + k|x|)dx \\ &= \int_{-1}^{0} (1 - kx)dx + \int_{0}^{1} (1 + kx)dx \\ &= \left(x - \frac{kx^2}{2}\right)\Big|_{-1}^{0} + \left(x + \frac{kx^2}{2}\right)\Big|_0^1 \\ &= -\left(-1 - \frac{k}{2}\right) + \left(1 + \frac{k}{2}\right) \\ &= 2 + k. \end{aligned}$$

Solving $1 = 2 + k$ gives us $k = -1$.

Question 1 (n)

Solution. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

Let $t = xy$ and $G(t) = \int_1^t h(s)ds$. Then $G'(t) = h(t)$. By chain rule, we get:
$f_x(x, y) = G'(t)\frac{dt}{dx} = G'(xy)y = h(xy)y$.
Hence, $f_x(2, 5) = h(2 \cdot 5) \cdot 5 = h(10) \cdot 5 = 2 \cdot 5 = 10$.

Question 2 (a)

SOLUTION. THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!

Since $\dfrac{\sqrt[3]{k^4+1}}{\sqrt{k^5+9}} \approx \dfrac{\sqrt[3]{k^4}}{\sqrt{k^5}} = \dfrac{k^{4/3}}{k^{5/2}} = \dfrac{1}{k^{5/2-4/3}} = \dfrac{1}{k^{7/6}}$, we want to compare $\dfrac{\sqrt[3]{k^4+1}}{\sqrt{k^5+9}}$ with $\dfrac{1}{k^{7/6}}$.

Let $a_k = \dfrac{1}{k^{7/6}} > 0$ and $b_k = \dfrac{\sqrt[3]{k^4+1}}{\sqrt{k^5+9}} > 0$. Recall $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^\alpha}$ converges if and only if $\alpha > 1$. Hence $\displaystyle\sum_{k=1}^{\infty} a_k$ converges.

On the other hand, by comparison test, if $\displaystyle\lim_{k\to\infty} \frac{b_k}{a_k} = 1$ then $\displaystyle\sum_{k=1}^{\infty} a_k$ and $\displaystyle\sum_{k=1}^{\infty} b_k$ converge or diverge at the same time. Hence, if we can prove $\displaystyle\lim_{k\to\infty} \frac{b_k}{a_k} = 1$, then we know that $\displaystyle\sum_{k=1}^{\infty} b_k$ converges.

$$\begin{aligned}
\lim_{k\to\infty} \frac{b_k}{a_k} &= \lim_{k\to\infty} k^{7/6} \frac{\sqrt[3]{k^4+1}}{\sqrt{k^5+9}} = \lim_{k\to\infty} k^{7/6} \frac{\sqrt[3]{(1+k^{-4})\cdot k^4}}{\sqrt{(1+9k^{-5})\cdot k^5}} \\
&= \lim_{k\to\infty} k^{7/6} \frac{\sqrt[3]{(1+k^{-4})}}{\sqrt{(1+9k^{-5})}} \frac{k^{4/3}}{k^{5/2}} = \lim_{k\to\infty} k^{7/6} \frac{\sqrt[3]{(1+k^{-4})}}{\sqrt{(1+9k^{-5})}} k^{4/3-5/2} \\
&= \lim_{k\to\infty} k^{7/6} \frac{\sqrt[3]{(1+k^{-4})}}{\sqrt{(1+9k^{-5})}} k^{-7/6} = \lim_{k\to\infty} \frac{\sqrt[3]{(1+k^{-4})}}{\sqrt{(1+9k^{-5})}} \\
&= \lim_{k\to\infty} \frac{\sqrt[3]{(1+k^{-4})}}{\sqrt{(1+9k^{-5})}} = \frac{1}{1} = 1.
\end{aligned}$$

Question 2 (b)

SOLUTION. THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!

$$\begin{aligned}
\lim_{k\to\infty} \sqrt[k]{\frac{1}{10^{k+1}(k+1)!}} &= \lim_{k\to\infty} \frac{1}{10\sqrt[k]{10\cdot(k+1)!}} = \lim_{k\to\infty} \frac{1}{10\sqrt[k]{k!\cdot 10(k+1)}} \\
&= \frac{1}{10} \lim_{k\to\infty} \frac{1}{\sqrt[k]{k!}} \cdot \lim_{k\to\infty} \frac{1}{\sqrt[k]{10(k+1)}} = \frac{1}{10}\cdot\frac{1}{\infty}\cdot 1 = 0.
\end{aligned}$$

Note that we use two well-known facts above:

1. $\lim_{n\to\infty} \sqrt[n]{n!} = \infty$
2. $\lim_{n\to\infty} \sqrt[n]{an+b} = 1$ (for any constants $a > 0$ and $b \in \mathbb{R}$).

Hence, the radius of convergence is $\frac{1}{0} = \infty$.

Question 2 (c)

SOLUTION. THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!

Using $\dfrac{1}{1-y} = \displaystyle\sum_{n=0}^{\infty} y^n$, we have

$\frac{3}{x+1} = \frac{3}{1-(-x)} = 3 \cdot \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty} 3(-1)^n x^n$;
$-\frac{1}{2x-1} = \frac{1}{1-2x} = \sum_{n=0}^{\infty}(2x)^n = \sum_{n=0}^{\infty} 2^n x^n$;
Therefore,

$$\frac{3}{x+1} - \frac{1}{2x-1} = \sum_{n=0}^{\infty} 3(-1)^n x^n + \sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty}(3(-1)^n + 2^n)x^n.$$

Hence, $b_n = 3(-1)^n + 2^n$.

Question 3 (a)

Solution. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

The object function is $(x+1)^2 + (y-2)^2$ and the constraint is $x^2 + y^2 = 125$, i. e. $x^2 + y^2 - 125 = 0$. Let $f(x, y, \lambda) = (x+1)^2 + (y-2)^2 + \lambda(x^2+y^2-125)$. By the method Lagrange multipliers, set $f_x = 0, f_y = 0, f_\lambda = 0$:
$f_x(x, y, \lambda) = 2(x+1) + 2\lambda \cdot x = 0$; (1)
$f_y(x, y, \lambda) = 2(y-2) + 2\lambda \cdot y = 0$; (2)
$f_\lambda(x, y, \lambda) = x^2 + y^2 - 125 = 0$; (3)
From (1), we get $x = \frac{-1}{\lambda+1}$ and from (2), we get $y = \frac{2}{\lambda+1}$. Plug them into (3) gives us $\lambda + 1 = \pm\frac{1}{5}$.
When $\lambda + 1 = \frac{1}{5}$, $x = -5$, $y = 10$, $(x+1)^2 + (y-2)^2 = 80$;
When $\lambda + 1 = -\frac{1}{5}$, $x = 5$, $y = -10$, $(x+1)^2 + (y-2)^2 = 180$.
Hence Max=180, Min=80.

Question 3 (b)

Solution. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

We want to find where $\sqrt{(x-(-1))^2 + (y-2)^2}$ attains its minimum. Equivalently, we need to find where $(x-(-1))^2 + (y-2)^2$ is the smallest. Note that $(x-(-1))^2 + (y-2)^2 = (x+1)^2 + (y-2)^2$ is the same as the last question. Thus we know that the maximum is at $(x, y) = (5, -10)$, and minimum is at $(-5, 10)$.

Question 4 (a)

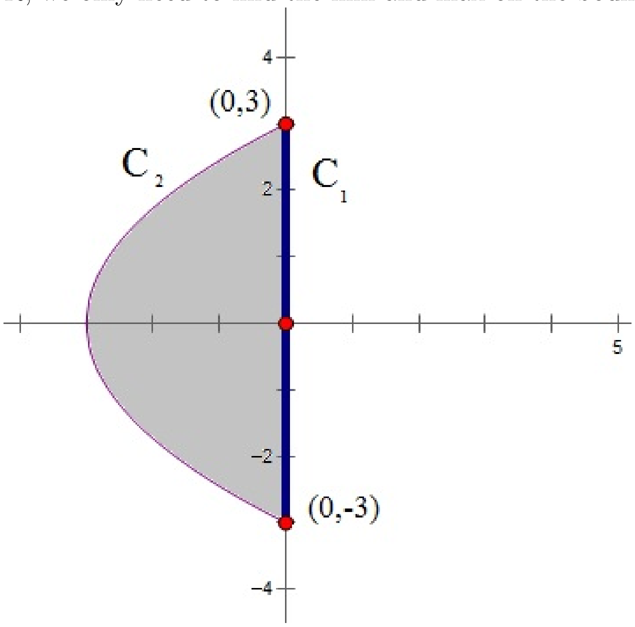
Solution. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

Compute the derivatives:
$T_x(x, y) = 2x - 2y + 6 \qquad T_y(x, y) = \frac{1}{3}y^2 - 2x - 6$;
$T_{xx}(x, y) = 2, \quad T_{xy}(x, y) = -2, \quad T_{yy}(x, y) = \frac{2}{3}y$.
Set $T_x(x, y) = 0$ and $T_y(x, y) = 0$ to find critical points:
$2x - 2y + 6 = 0$
$\frac{1}{3}y^2 - 2x - 6 = 0$.
From the first equality we get $x = y - 3$. Plugging this into the second equality, we get $\frac{1}{3}y^2 - 2y = 0$. Solving it gives $y = 0, 6$, $(x, y) = (-3, 0)$, and $(x, y) = (3, 6)$. Hence, critical points are $(-3, 0)$ and $(3, 6)$.
For $(-3, 0)$, $T_{xx} = 2$, $T_{yy} = \frac{2}{3}y = 0$ and $T_{xy} = -2$, so we have a saddle point. (Recall that if $T_{xx}T_{yy} = 0$ and $T_{xy} \neq 0$ then it is a saddle point.)
For $(3, 6)$, $T_{xx} = 2$, $T_{yy} = \frac{2}{3}y = 4$, and $T_{xx} \cdot T_{yy} - T_{xy}^2 = 2 \cdot \frac{2}{3}y - (-2)^2 = 2 \cdot 4 - (-2)^2 = 4 > 0$, so it is a local minumum. (Recall that if $T_{xx} > 0, T_{yy} > 0$ and $T_{xx}T_{yy} - T_{xy}^2 > 0$ then it is a local minimum.)

Question 4 (b)

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

We just need to check the local maximum, the local minimum and the points on the boundary. From question $4a$ above, we know that it has only one local minimum $(3,6)$ and no local maximum. Since $(3,6)$ is not on R, we only need to find the min and max on the boundary.



On C_1:
$C_1 = \{x = 0, y \in [-3,3]\}$,

$$T(x,y) = \frac{1}{9}y^3 - 6y \qquad T_y = \frac{1}{3}y^2 - 6 < 0$$

Hence T decreases and max $= T(0,-3) = 15$, min $= T(0,3) = -15$;
On C_2:
$C_2 = \{x = \frac{1}{3}y^2 - 3, y \in [-3,3]\}$,
$T(x,y) = \frac{1}{9}y^3 + \left(\frac{1}{3}y^2 - 3\right)^2 - 2\left(\frac{1}{3}y^2 - 3\right)y + 6\left(\frac{1}{3}y^2 - 3\right) - 6y = \frac{y^4}{9} - \frac{5}{9}y^3 - 9.$
$T_y = \frac{4}{9}y^3 - \frac{15}{9}y^2 = \frac{y^2}{9}(4y - 15) < 0.$
Hence T decreases and max $= T(0,-3) = 15$, min $= T(0,3) = -15$.
Therefore, max $= 15$ and min $= -15$.

Question 5 (a)

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

We first solve for the differential equation:

$$
\begin{aligned}
\frac{dB}{dt} &= aB - m \\
dB &= (aB - m)dt \\
\frac{dB}{aB - m} &= dt \quad \text{(separating the variables)} \\
\int_0^t \frac{dB}{aB - m} &= \int_0^t dt \quad \text{(integrate both sides from 0 to } t\text{)} \\
\frac{1}{a}\ln(aB(t) - m) - \frac{1}{a}\ln(aB(0) - m) &= t \\
\frac{1}{a}\ln(aB(t) - m) &= t + \frac{1}{a}\ln(aB(0) - m) \\
\ln(aB(t) - m) &= at + \ln(aB(0) - m) \\
\ln(aB(t) - m) &= \ln(e^{at}) + \ln(aB(0) - m) \\
\ln(aB(t) - m) &= \ln(e^{at} \cdot (aB(0) - m)) \\
aB(t) - m &= e^{at} \cdot (aB(0) - m) \\
B(t) &= \frac{1}{a}((aB(0) - m)e^{at} + m).
\end{aligned}
$$

Substitute the information we know

$$a = \frac{1}{50},$$

$B(0) = 30000$

$$
\begin{aligned}
B(t) &= \frac{1}{a}((aB(0) - m)e^{at} + m) \\
&= 50((600 - m)e^{0.02t} + m).
\end{aligned}
$$

Question 5 (b)

Solution 1. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

From $B(t) = 50((600 - m)e^{0.02t} + m)$, we can see that if $600 - m = 0$, then $B(t)$ is a constant. Hence, $m = 600$.

Solution 2. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

If $B(t)$ is a constant then $B'(t)$ must be 0. Therefore

$$
\begin{aligned}
0 &= B'(0) \\
&= aB(0) - m \\
&= 0.02 \cdot 30000 - m \\
&= 600 - m
\end{aligned}
$$

Hence, $m = 600$.

Question 6 (a)

SOLUTION. THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!

Since $e^x = 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!}$, let $x = \frac{1}{\pi}$ and we get

$e^{\frac{1}{\pi}} = 1 + \sum_{k=1}^{\infty} \frac{(\frac{1}{\pi})^k}{k!} = 1 + \sum_{k=1}^{\infty} \frac{1}{\pi^k k!}$

Hence

$\sum_{k=1}^{\infty} \frac{1}{\pi^k k!} = e^{\frac{1}{\pi}} - 1$

Question 6 (b)

SOLUTION. THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!

Since $\sum_{n=1}^{\infty} \frac{na_n - 2n + 1}{n+1}$ converges, we have $\lim_{n\to\infty} \frac{na_n - 2n + 1}{n+1} = 0$. Hence:

$$\begin{aligned}
0 &= \lim_{n\to\infty} \frac{na_n - 2n + 1}{n+1} \\
&= \lim_{n\to\infty} \left((a_n - 2) \cdot \frac{n}{n+1} + \frac{1}{n+1} \right) \\
&= \lim_{n\to\infty} (a_n - 2) \cdot \frac{n}{n+1} + \lim_{n\to\infty} \frac{1}{n+1} \\
&= \lim_{n\to\infty} (a_n - 2) \cdot \frac{n}{n+1} + 0 \\
&= \lim_{n\to\infty} (a_n - 2) \cdot \lim_{n\to\infty} \frac{n}{n+1} \\
&= \lim_{n\to\infty} (a_n - 2) \cdot 1 \\
&= \lim_{n\to\infty} (a_n - 2)
\end{aligned}$$

Therefore $\lim_{n\to\infty} (a_n - 2) = 0$. Or rather, $\lim_{n\to\infty} a_n = 2$.

On the other hand, $\ln\left(\frac{a_n}{a_{n+1}}\right) = \ln a_n - \ln a_{n+1}$. Hence,

$$\begin{aligned}
-\ln a_1 + \sum_{n=1}^{k} \ln\left(\frac{a_n}{a_{n+1}}\right) &= -\ln a_1 + \sum_{n=1}^{k} (\ln a_n - \ln a_{n+1}) \\
&= -\ln a_1 + (\ln a_1 - \ln a_2) + (\ln a_2 - \ln a_3) + \cdots + (\ln a_k - \ln a_{k+1}) \\
&= \ln a_{k+1}
\end{aligned}$$

Hence,

$$\begin{aligned}
-\ln a_1 + \sum_{n=1}^{\infty} \ln\left(\frac{a_n}{a_{n+1}}\right) &= \lim_{k\to\infty} \left(-\ln a_1 + \sum_{n=1}^{k} \ln\left(\frac{a_n}{a_{n+1}}\right) \right) \\
&= \lim_{k\to\infty} \ln a_{k+1} \\
&= \ln 2.
\end{aligned}$$

Good Luck for your exams!