Full Solutions
MATH215 December 2013

April 4, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

Question 1 (a)

Solution. With $y'' + 2y' + y = 0$, we first solve the characteristic equation $r^2 + 2r + 1 = (r+1)^2 = 0$ so $r = -1$.
As the root is repeated, we only get one solution of the form e^{-t}. The other solution will be te^{-t}.
The general solution is a linear combination of these two independent solutions, $y = Ae^{-t} + Bte^{-t}$.

Question 1 (b)

Solution. We first need to find the homogeneous solution. The characteristic polynomial $r^2 - 4r + 5$ has roots $r^{\pm} = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$ and hence the homogeneous solution is a linear combination of $e^{2x}\sin x$ and $e^{2x}\cos x$.
No such terms appear in the forcing function. The forcing function $x \sin x$ is a degree one polynomial times a sinusoidal function. We thus would guess a particular solution of form $x_p = (Ax+B)\sin x + (Cx+D)\cos x$.

Question 1 (c)

Solution. Picard's theorem (at least one version of it) states that initial value problems of the form $\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$ have a unique solution over some interval I containing x_0 provided f is continuous at (x_0, y_0).
If we rearrange the ODE problem $y\frac{dy}{dx} = x, \quad y(0) = 0$, we get $\frac{dy}{dx} = \frac{x}{y}, \quad y(0) = 0$. When $x = 0$, $y = 0$, but $\dfrac{x}{y}$ is not continuous at (0,0) because it is not defined. Therefore the theorem cannot be used and uniqueness is not guaranteed.

Question 1 (d)

Solution. The Laplace transform only considers positive values of t. Since the heaviside function is identically equal to 1 for $t > 0$. That is,

$$\mathcal{L}(1) = \int_0^\infty e^{-st}dt = \mathcal{L}(u(t)) = \int_0^\infty e^{-st}dt$$

Question 1 (e)

Solution. We let $\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ so that

$$\begin{aligned} x_1' &= y' = x_2 \\ x_2' &= y'' = -ty' - t^2 y = -tx_2 - t^2 x_1 \end{aligned}$$

We can now write the above in matrix notation $\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -t^2 & -t \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. This is a first-order linear system as requested.

Question 1 (f)

Solution. The Laplace transform of the 3rd derivative $x'''(t)$ is given by

$$\mathcal{L}(x'''(t)) = s^3X(s) - s^2x(0) - sx'(0) - x''(0)$$

Plugging this into the equation, and using that the Laplace transform is linear, we obtain

$$0 = \mathcal{L}(x'''(t) + x(t)) = s^3X(s) - s^2x(0) - sx'(0) - x''(0) + X(s)$$

Upon using $x(0) = 1, x'(0) = 2, x''(0) = 3$, we have $(s^3 + 1)X(s) = s^2 + 2s + 3$ so

$$X(s) = \frac{s^2 + 2s + 3}{s^3 + 1}$$

Question 2 (a)

SOLUTION. We begin with $y' + x^3y = 3x^3, \quad y(0) = 8$. We can use an integrating factor here, taking $\mu(x) = e^{\int x^3dx} = e^{\frac{x^4}{4}}$ where we chose the arbitrary constant to be 0. Multiplying both sides of the equation by $\mu(x)$ gives

$$e^{\frac{x^4}{4}}y' + e^{\frac{x^4}{4}}x^3 = 3x^3e^{\frac{x^4}{4}}$$

or

$$(e^{\frac{x^4}{4}}y)' = 3x^3e^{\frac{x^4}{4}}.$$

We can integrate both sides of the equation (doing a simple substitution on the right-hand side) giving us

$$\int \frac{d}{dx}(e^{\frac{x^4}{4}}y(x))dx = \int 3x^3e^{\frac{x^4}{4}}dx$$

$$e^{\frac{x^4}{4}}y = \underbrace{\int 3e^u du}_{u=x^4/4} = 3e^u + C = 3e^{\frac{x^4}{4}} + C$$

$$y = 3 + Ce^{\frac{-x^4}{4}}.$$

If $y(0) = 8$ then $8 = 3 + C$ so $C = 5$ and the final answer is
$y(x) = 3 + 5e^{\frac{-x^4}{4}}$.

Question 2 (b)

SOLUTION. As the ODE is separable, we can re-express with the differentials separated:

$$(x^2 + 1)dx = (t^2 + 1)dt$$

which can be integrated to yield

$$\int (x^2+1)dx = \frac{x^3}{3} + x = \int (t^2+1)dt = \frac{t^3}{3} + t + C$$

We merge the two integration constants to C. From our initial condition that $x(0) = -1$ we obtain

$$\frac{(-1)^3}{3} - 1 = \frac{0^3}{3} + 0 + C$$

and hence C = *-4/3*. Left in implicit form the solution reads

$$\frac{x^3}{3} + x = \frac{t^3}{3} + t - \frac{4}{3}$$

Question 3 (a)

SOLUTION. We have a homogeneous ODE $x'' + 2x' + 5x = 0$ with *x(0) = 0, x'(0) = 1*.

The roots of the corresponding characteristic polynomial $\lambda^2+2\lambda+5 = 0$ are $\lambda = \frac{-2\pm\sqrt{2^2-4(1)(5)}}{2} = \frac{-2\pm\sqrt{-16}}{2} = \frac{-2\pm4i}{2} = -1 \pm 2i$.

The roots are complex, and we have a general solution of

$$x(t) = C_1 e^{-t}\sin(2t) + C_2 e^{-t}\cos(2t)$$

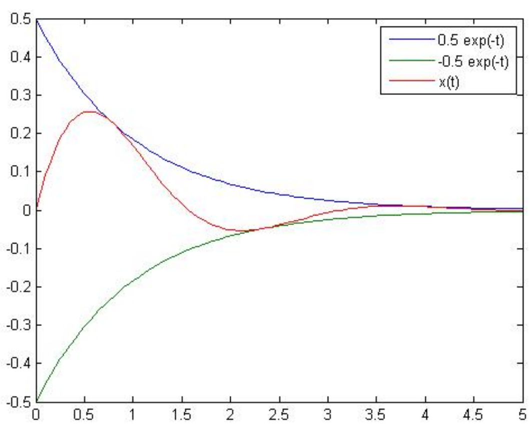
We can use the initial conditions to find the constants. $x(0) = 0 + C_2 = 0$ implies $C_2 = 0$. Next, use the fact that the second term vanishes to calculate the first derivative of the solution.

$$x'(t) = -C_1 e^{-t}\sin(2t) + 2C_1 e^{-t}\cos(2t) = 2C_1$$

Since *x'(0) = 1* this implies $C_1 = 1/2$. The final solution is therefore

$$x(t) = \frac{1}{2}e^{-t}\sin(2t).$$

A plot of the solution is given below. Note that the solution is bounded between $x(t) = \pm\frac{1}{2}e^{-t}$ so there is an envelope. The solution must fit between the upper and lower envelope functions. At $t = 0$, $x = 0$, and the solution is increasing at $t = 0$ (the derivative is positive).



Question 3 (b)

SOLUTION. We seek a particular solution since the homogeneous solution will decay to 0 as $t \to \infty$. As $\sin(2t) = \text{Im}(e^{2it})$, we will look for a particular solution of form $x(t) = \text{Im}(\phi(t))$ where $\phi = Ae^{2it}$ satisfies the ODE in complex form:

$$\phi'' + 2\phi' + 5\phi = e^{2it}$$

so that

$$-4Ae^{2it} + 4iAe^{2it} + 5Ae^{2it} = e^{2it}$$

which upon cancelling the e^{2it} terms and simplification becomes

$$A(1 + 4i) = 1$$

Solving for A we obtain

$$A = \frac{1}{1+4i} = \frac{1-4i}{(1+4i)(1-4i)} = \frac{1-4i}{17}.$$

Now that we have A, we can find x by using A and Euler's identity:

$$\begin{aligned} x(t) &= \text{Im}\left(\frac{1-4i}{17}e^{2it}\right) \\ &= \frac{1}{17}\text{Im}((1-4i)(\cos(2t) + i\sin(2t))) \\ &= \frac{1}{17}\text{Im}(\cos(2t) + 4\sin(2t) + i(\sin(2t) - 4\cos(2t))) \\ &= \frac{1}{17}\sin(2t) - \frac{4}{17}\cos(2t) \end{aligned}$$

The steady solution is $x(t) = \frac{1}{17}\sin(2t) - \frac{4}{17}\cos(2t)$.

Question 4 (a)

SOLUTION. We will use the fact that $\mathcal{L}(u(t-a)f(t-a)) = e^{-as}F(s)$.
We recognize that

$$g(t) = u(t-1)e^{t-1} = u(t-1)f(t-1)$$

where $f(t) = e^t$. Thus,

$$\mathcal{L}(g(t)) = e^{-s}\mathcal{L}(e^t) = \frac{e^{-s}}{s-1}$$

Question 4 (b)

SOLUTION. We take the Laplace transform of the equation:

$$\mathcal{L}(y'' - 2y' + y) = \mathcal{L}(g(t))$$

$$\underbrace{s^2Y(s) - sy(0) - y'(0)}_{\mathcal{L}(y'')} - 2\underbrace{(sY(s) - y(0))}_{\mathcal{L}(y')} + \underbrace{Y(s)}_{\mathcal{L}(y)} = \underbrace{\frac{e^{-s}}{s-1}}_{\mathcal{L}(g(t))}$$

$$s^2Y(s) - 0 - 1 - 2(sY(s) - 0) + Y(s) = (s^2 - 2s + 1)Y(s) - 1 = \frac{e^{-s}}{s-1}$$

Since $s^2 - 2s + 1 = (s-1)^2$ we arrive at our final answer

$$Y(s) = \frac{e^{-s}}{(s-1)^3} + \frac{1}{(s-1)^2}$$

Question 4 (c)

SOLUTION. We have $Y(s) = \frac{e^{-s}}{(s-1)^3} + \frac{1}{(s-1)^2}$ and we need to find $y(t) = \mathcal{L}^{-1}(Y(s))$.
The most important thing here is finding a function $f(t)$ such that $\mathcal{L}(f(t)) = 1/(s-a)^n$. The key identity to realize is that $\mathcal{L}(t^n f(t)) = (-1)^n \frac{d^n}{ds^n} F(s)$.
With this knowledge, since $\mathcal{L}(e^t) = 1/(s-1)$, we have $\mathcal{L}(te^t) = (-1)(-1/(s-1)^2) = 1/(s-1)^2$ and $\mathcal{L}(t^2e^t) = 2/(s-1)^3$ (note the factor of 2!). Then

$$Y(s) = e^{-s}\mathcal{L}(\frac{1}{2}t^2e^t) + \mathcal{L}(te^t).$$

We use one more identity that $\mathcal{L}(u(t-a)f(t-a)) = e^{-as}F(s)$ which allows us to invert the first term (using a=1) on the right-hand side. We find

$$y(t) = u(t-1)\frac{1}{2}(t-1)^2e^{t-1} + te^t.$$

Question 5 (a)

SOLUTION. We begin by finding the eigenvalues and eigenvectors.
The characteristic polynomial of $\begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}$ is $\lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2)$ which has roots $\lambda = 3, -2$. The find the eigenvector with eigenvalue λ, we seek the nullspace of $A - \lambda I$.
With $\lambda = 3$, we have:

$$\begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix} \vec{v} = \vec{0}.$$

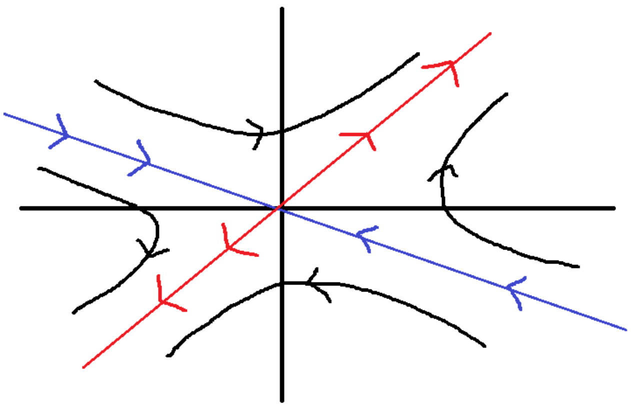
From the equation implied by the first row (the second is redundant), $-3v_1 + 3v_2 = 0$ so we can take $\vec{v} = \langle 1, 1 \rangle$.
With $\lambda = -2$, we have:

$$\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \vec{v} = \vec{0}.$$

From the equation implied by the first row (the second is the same), $2v_1 + 3v_2 = 0$ so we can take $\vec{v} = \langle 3, -2\rangle$ as an eigenvector.

The general solution is $\vec{x}(t) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + C_2 \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-2t}$.

The origin is a saddle point because the eigenvalues are real and of opposite sign. Along the $\langle 1, 1\rangle$ direction, the solution grows exponentially. Along the $\langle 3, -2\rangle$ direction, the solution decays exponentially. At all other points, it will approach the line spanned by $\langle 1, 1\rangle$ as $t \to \infty$. This is displayed in the figure.



Question 5 (b)

SOLUTION 1. We need a particular solution. To obtain it, we will use the method of undetermined coefficients.

If $\vec{x}_p = \begin{bmatrix} ate^{3t} + be^{3t} \\ cte^{3t} + de^{3t} \end{bmatrix}$ then $\vec{x_p}' = \begin{bmatrix} (a+3b)e^{3t} + 3ate^{3t} \\ (c+3d)e^{3t} + 3cte^{3t} \end{bmatrix}$ and

$$Ax_p + \begin{bmatrix} 25e^{3t} \\ 0 \end{bmatrix} = \begin{bmatrix} 3cte^{3t} + (3d+25)e^{3t} \\ (2a+c)te^{3t} + (2b+d)e^{3t} \end{bmatrix}$$

Therefore if $\vec{x_p}' = A\vec{x_p} + \begin{bmatrix} 25e^{3t} \\ 0 \end{bmatrix}$ we get

$$\begin{bmatrix} (a+3b)e^{3t} + 3ate^{3t} \\ (c+3d)e^{3t} + 3cte^{3t} \end{bmatrix} = \begin{bmatrix} 3cte^{3t} + (3d+25)e^{3t} \\ (2a+c)te^{3t} + (2b+d)e^{3t} \end{bmatrix}.$$

In comparing the coefficients of e^{3t} from row 1, the coefficients of te^{3t} from row 1, the coefficients of e^{3t} from row 2, and the coefficients of te^{3t} from row 2, the following equations must hold:

$$\begin{aligned} a + 3b &= 3d + 25 \\ 3a &= 3c \\ c + 3d &= 2b + d \\ 3c &= 2a + c \end{aligned}$$

We can turn this into a matrix equation and row-reduce:

$$\begin{bmatrix}1 & 3 & 0 & -3\\ 1 & 0 & -1 & 0\\ 0 & 2 & -1 & -2\\ 1 & 0 & -1 & 0\end{bmatrix}\begin{bmatrix}a\\ b\\ c\\ d\end{bmatrix}=\begin{bmatrix}25\\ 0\\ 0\\ 0\end{bmatrix}$$

Subtracting row 2 from row 4 and row 1 from row 2 yields:

$$\begin{bmatrix}1 & 1 & 1 & -1\\ 0 & -3 & -1 & 3\\ 0 & 2 & -1 & -2\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}a\\ b\\ c\\ d\end{bmatrix}=\begin{bmatrix}25\\ -25\\ 0\\ 0\end{bmatrix}$$

We next multiply the third row by 3 and then subtract the second row twice:

$$\begin{bmatrix}1 & 1 & 1 & -1\\ 0 & -3 & -1 & 3\\ 0 & 0 & -5 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}a\\ b\\ c\\ d\end{bmatrix}=\begin{bmatrix}25\\ -25\\ -50\\ 0\end{bmatrix}$$

Next, divide the third row by -5 and then add the result to the second row, and subtract it from the first row:

$$\begin{bmatrix}1 & 1 & 0 & -1\\ 0 & -3 & 0 & 3\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}a\\ b\\ c\\ d\end{bmatrix}=\begin{bmatrix}15\\ -15\\ 10\\ 0\end{bmatrix}$$

In our last step we divide the second row by -3 and then subtract the result from the first row:

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}a\\ b\\ c\\ d\end{bmatrix}=\begin{bmatrix}10\\ 5\\ 10\\ 0\end{bmatrix}$$

We see that d is a free parameter, and since we only need one particular we choose *d=0* for simplicity. Then we read off *a = 10, b - d = 5, c = 10.* Putting this together we find a particular solution $\vec{x_p}=\begin{bmatrix}10te^{3t}+5e^{3t}\\ 10te^{3t}\end{bmatrix}$.

The general solution is the homogeneous solution (which we found in part (a)) plus a particular solution: $x(t)=C_1\begin{bmatrix}1\\ 1\end{bmatrix}e^{3t}+C_2\begin{bmatrix}3\\ -2\end{bmatrix}e^{-2t}+\begin{bmatrix}10te^{3t}+5e^{3t}\\ 10te^{3t}\end{bmatrix}$.

SOLUTION 2. This problem can also be done using variation of parameters.

Given the homogeneous solution $C_1\begin{bmatrix}1\\ 1\end{bmatrix}e^{3t}+C_2\begin{bmatrix}3\\ -2\end{bmatrix}e^{-2t}$, we form a fundamental matrix $\mathcal{X}=\begin{bmatrix}e^{3t} & 3e^{-2t}\\ e^{3t} & -2e^{-2t}\end{bmatrix}$ using independent solutions of the homogeneous problem as columns. Observe that $\mathcal{X}'(t)=A\mathcal{X}(t)$.

For the nonhomogeneous equation, we will try to find a solution of the form $\vec{x}(t)=\mathcal{X}(t)\vec{g}(t)$ where $\vec{x}'(t)=A\vec{x}+\begin{bmatrix}25e^{3t}\\ 0\end{bmatrix}$. With this solution form, the equation reads:

$$\vec{x}'(t)=\mathcal{X}'(t)\vec{g}(t)+\mathcal{X}(t)\vec{g}'(t)=A\vec{x}(t)+\begin{bmatrix}25e^{3t}\\ 0\end{bmatrix}$$

$$\vec{x}'(t) = A\underbrace{\mathcal{X}\vec{g}(t)}_{=\vec{x}(t)} + \mathcal{X}(t)\vec{g}'(t) = A\vec{x}(t) + \begin{bmatrix} 25e^{3t} \\ 0 \end{bmatrix}$$

so that

$$\mathcal{X}\vec{g}' = \begin{bmatrix} 25e^{3t} \\ 0 \end{bmatrix}.$$

We can now work on solving for $\vec{g}$ (note the primed in $\vec{g}'$).

$$\vec{g}' = \mathcal{X}^{-1}\begin{bmatrix} 25e^{3t} \\ 0 \end{bmatrix} = \frac{1}{e^{3t}(-2e^{-2t}) - e^{3t}(3e^{-2t})}\begin{bmatrix} 2e^{-2t} & -3e^{-2t} \\ -e^{3t} & e^{3t} \end{bmatrix} = \frac{e^{-t}}{5}\begin{bmatrix} 2e^{-2t} & 3e^{-2t} \\ e^{3t} & -e^{3t} \end{bmatrix}\begin{bmatrix} 25e^{3t} \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 5e^{5t} \end{bmatrix}.$$

Integrating gives

$$\vec{g} = \begin{bmatrix} 10t + K_1 \\ e^{5t} + K_2 \end{bmatrix}$$

for arbitrary K_1, K_2.

We finally have a general solution $\vec{x}(t) = \mathcal{X}(t)\vec{g}(t) = \begin{bmatrix} 10te^{3t} + 3e^{3t} + K_1e^{3t} + 3K_2e^{-2t} \\ 10te^{3t} - 2e^{3t} + K_1e^{3t} - 2K_2e^{-2t} \end{bmatrix} = \begin{bmatrix} 10te^{3t} + 3e^{3t} \\ 10te^{3t} - 2e^{3t} \end{bmatrix} + K_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{3t} + K_2\begin{bmatrix} 3 \\ -2 \end{bmatrix}e^{-2t}$.

This is in agreement with the first solution. Observe that based on the numbers in that solution, a=c=10, and b-d=5.

Question 6 (a)

Solution. __NOTOC__ If $x' = 0$ then either $x = 0$ or $x = 3 - y$.

- Using $x = 0$ in $y' = 0$ we get $y = 0, 1$. This gives two critical points: (0,0) and (0,1).
- Using $x = 3 - y$ in $y' = 0$ we get $y = 0, 2$. In this case, *x=3* when *y=0*, and *x=1* when *y=2*. This gives two more critical points (1,2) and (3,0).

Considering the full system, we have

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3x - x^2 - xy \\ y + xy - y^2 \end{pmatrix}$$

Labelling $f(x,y) = 3x - x^2 - xy$ and $g(x,y) = y + xy - y^2$, the Jacobian of the system at (x,y) is

$$J = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 3 - 2x - y & -x \\ y & 1 + x - 2y \end{pmatrix}$$

The Jacobian evaluated at the critical points gives us information about the local stability of the critical points.

At *(x,y) = (0,0)*

$$J(0,0) = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

which has eigenvalues of +1 and +3. Thus (0,0) is an unstable node (both eigenvalues are real and positive).

At (x,y) = (0,1)

$$J(0,1) = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}$$

which has eigenvalues of +2 and -1 (being lower triangular, the eigenvalues are the diagonal entries). As the eigenvalues are real and of opposite sign, (0,1) is a saddle point.

At (x,y) = (3,0)

$$J(3,0) = \begin{pmatrix} -3 & -3 \\ 0 & 4 \end{pmatrix}$$

having eigenvalues of -3 and +4 (the matrix is upper triangular and its eigenvalues are the diagonal entries). We thus have (3,0) is a saddle point.

At (x,y) = (1,2)

$$J(1,2) = \begin{pmatrix} -1 & -1 \\ 2 & -2 \end{pmatrix}.$$

The eigenvalues satisfy the characteristic polynomial $\lambda^2 + 3\lambda + 4 = 0$ so $\lambda^{\pm} = \frac{-3\pm\sqrt{9-16}}{2} = -3/2 \pm \sqrt{7}i/2$. These eigenvalues have a nonzero imaginary part giving rise to a spiral/ellipse. As there real components are negative, we have a stable spiral at (1,2).

Question 6 (b)

SOLUTION. __NOTOC__ We will need to analyze the linearized system at each critical point.

At (x,y) = (0,0)

$$J = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

which has eigenvectors $(1,0)^T$ and $(0,1)^T$ with corresponding eigenvalues 3 and 1 respectively. The local solution structure centred at (0,0) is

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{3t} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{t}$$

Along both eigenvectors, the trajectories move away, with the $(1,0)^T$ direction being dominant, since it is multiplied by the larger term, e^{3t}.

At (x,y) = (0,1)

$$J = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}.$$

To find the eigenvector corresponding to $\lambda = 2$, we seek the solution to $(J - 2I)\vec{v} = \vec{0}$ so that

$$\begin{pmatrix} 0 & 0 \\ 1 & -3 \end{pmatrix} \vec{v} = \vec{0}$$

Or $v_1 - 3v_2 = 0$, so we take $\vec{v} = (3,1)^T$.
To find the eigenvector corresponding to $\lambda = -1$, we seek the solution to $(J - (-1)I)\vec{v} = \vec{0}$ so that

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \vec{v} = \vec{0}$$

Or $v_1 = 0$, $v_2 = \textit{anything}$, so we take $\vec{v} = (0,1)^T$.
The local solution structure centred at (0,1) is

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}$$

The solution grows exponentially along the $(3,1)^T$ direction and decays exponentially along the $(0,1)^T$ direction. Starting anywhere other than along the $(0,1)^T$ vector, the solution will approach the $(3,1)^T$ vector.

At *(x,y) = (3,0)*

$$J = \begin{pmatrix} -3 & -3 \\ 0 & 4 \end{pmatrix}.$$

To find the eigenvector corresponding to $\lambda = -3$, we seek the solution to $(J - (-3)I)\vec{v} = \vec{0}$ so that

$$\begin{pmatrix} 0 & -3 \\ 0 & 7 \end{pmatrix} \vec{v} = \vec{0}$$

Or $v_2 = 0$, $v_1 = \textit{anything}$, so we take $\vec{v} = (1,0)^T$.
To find the eigenvector corresponding to $\lambda = 4$, we seek the solution to $(J - 4I)\vec{v} = \vec{0}$ so that

$$\begin{pmatrix} -7 & -3 \\ 0 & 0 \end{pmatrix} \vec{v} = \vec{0}$$

Or $-7v_1 - 3v_2 = 0$, so we take $\vec{v} = (-3,7)^T$.
The local solution structure centred at (3,0) is

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-3t} + \begin{pmatrix} -3 \\ 7 \end{pmatrix} e^{4t}$$

The solution grows exponentially along the $(-3,7)^T$ direction and decays exponentially along the $(1,0)^T$ direction. Starting anywhere other than along the $(1,0)^T$ vector, the solution will approach the $(-3,7)^T$ vector.

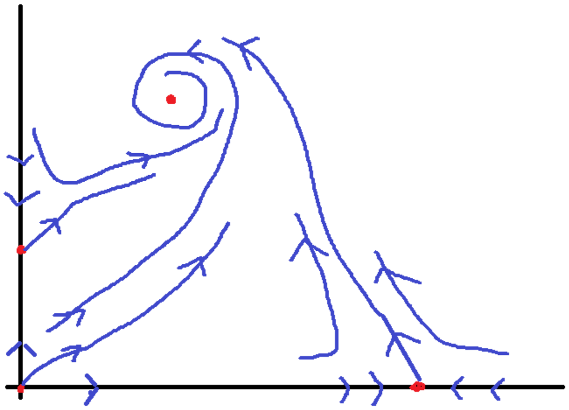
At *(x,y) = (1,2)*

$$J = \begin{pmatrix} -1 & -1 \\ 2 & -2 \end{pmatrix},$$

which as we saw in part (a) has complex eigenvalues with negative real part. By plugging in a point, say $(1,0)^T$ into the local linear system we can determine the direction of the spiral.

$$\begin{pmatrix} -1 & -1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

which points up and to the left at (1,0). Therefore the spiral is counterclockwise.
This is all put together in the very rough global phase portrait.



Question 6 (c)

SOLUTION. The only stable equilibrium solution is (1,2), which is a stable spiral. Saddle points and sources are unstable. While little can be done rigorously, our prediction is that with $x(0) = y(0) = 1$ that $(x(t), y(t)) \to (1, 2)$ as $t \to \infty$.

Question 7 (a)

SOLUTION. Setting $dy/dt = 0$, we have $-y(y-1)^2 = 0$ so $y = 0, 1$.

Looking at the sign of dy/dt, we have $\begin{cases} y' > 0, & y < 0 \\ y' < 0, & 0 < y < 1 \\ y' < 0, & y > 1 \end{cases}$.

Thus, if we start below $y = 0$, $y(t)$ increases; if we start above $y = 0$ but below $y = 1$, $y(t)$ decreases. Either way, we approach $y = 0$ so $y = 0$ is a stable equilibrium solution.
If we start below $y = 1$ but above $y = 0$, $y(t)$ decreases and moves away from $y = 1$ ($y' < 0$) and if we start above $y = 1$, $y(t)$ decreases and approaches $y = 1$ ($y' < 0$). However, because the solution moves away from $y = 1$ locally on at least one side, $y = 1$ is an unstable equilibrium solution.
All possible values for $\lim_{t\to\infty} y(t)$ include 0 and 1. If $y(0) < 0$, $y \to 0$ as $t \to \infty$. If $0 < y(0) < 1$, $y \to 0$ as $t \to \infty$. Finally, if $y(0) > 1$ then $y \to 1$ as $t \to \infty$.

Question 7 (b)

SOLUTION. We have $dy/dt = -y(y-1)^2$ so that if $y(t_0) = y_0$ then

$$y(t_0 + h) \approx y(t_0) + h\frac{dy}{dt}|_{t=t_0} = y(t_0) + h[-y_0(y_0 - 1)^2]$$

We denote $y_0 = 3/2$ and $t_0 = 0$ so that

$$y(1) \approx y_1 = y_0 + h[-y_0(y_0 - 1)^2] = 3/2 + 1\left[\frac{-3}{2}\left(\frac{1}{2}\right)^2\right] = 3/2 - 3/8 = 9/8.$$

Continuing in this process,

$$y(2) \approx y_2 = y_1 + h[-y_1(y_1-1)^2] = \frac{9}{8} - \frac{9}{8}\left(\frac{1}{8}\right)^2 = \frac{9}{8} - \frac{9}{8^3} = \frac{9\times 8^2 - 9}{8^3} = \frac{567}{512}.$$

Notice how this approximated solution slowly approaches the equilibrium value y=1.

Question 7 (c)

SOLUTION. Implementing the same notation as in part (b), we have

$$y(2) \approx y_1 = y_0 + h[-y_0(y_0-1)^2] = 3/2 + 2(-3/8) = 3/4.$$

We observe that $0 < y_1 < 1$ and that $\frac{dy}{dt}|_{y=3/4} < 0$ so that

$$y_2 = y_1 + h\frac{dy}{dt}|_{y=3/4} < y_1.$$

Given the information we know from part (a) that solution curves passing through a point $y_0 < 1$ approach $y = 0$ we anticipate that if these approximations were to carry on for a long time, the approximations would (likely) approach $y = 0$.
This disagrees with the prediction in part (a) in that if $y(0) = 3/2$, the exact solution curve should approach $y = 1$ as $t \to \infty$. The step size of $h = 2$ is too large a step to take to accurately resolve the exact solution of the initial value problem.

Good Luck for your exams!