Full Solutions
MATH104 December 2010

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Educational Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Exam Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Educational Resources.

Question 1 (a)

Easiness: 92/100

SOLUTION. The first thing we should try is direct substitution (it sometimes works). Unfortunately if we substitute in $x = 2$ we have $\frac{0}{0}$.
The next thing we try is to simplify this expression. Factoring is often a good idea:
$\lim_{x\to 2} \frac{x^2+x-6}{3x^2-5x-2} = \lim_{x\to 2} \frac{(x+3)(x-2)}{(3x+1)(x-2)} = \lim_{x\to 2} \frac{x+3}{3x+1}$
In the last equality we cancelled the common factor of (x-2). Note that, without the limit, $\frac{(x+3)(x-2)}{(3x+1)(x-2)}$ and $\frac{x+3}{3x+1}$ are not equal (because the function is not defined at $x = 2$). In taking the limit as $x \to 2$, however, we don't need to worry what happens to the function at the precise point $x = 2$.
To evaluate $\lim_{x\to 2} \frac{x+3}{3x+1}$ we can try direct substitution again. This time, when we directly substitute $x = 2$, we have 5/7. This is our answer.

Question 1 (b)

Easiness: 75/100

SOLUTION. Since the numerator, $g(x)$, approaches 6 as $x \to \infty$, the denominator must approach 2 in order for the fraction to approach 3. Hence,

$$\lim_{x\to\infty} f(x) = 3.$$

Question 1 (c)

Easiness: 70/100

SOLUTION. Using the discrete compound interest formula with 4 compounds per year, we know that

$$\$12000 = \$10000(1 + 0.12/4)^{4t},$$

where t is the time in years required for the \$10 000 investment to grow to \$12 000. We want to solve for t. Dividing both sides by \$10000,

$$1.2 = (1 + 0.03)^{4t}.$$

Taking the logarithm of both sides and simplifying,

$$\ln(1.2) = \ln\left(1.03^{4t}\right) = 4t\ln(1.03)$$

Hence, the investment will take

$$t = \frac{\ln(1.2)}{4\ln(1.03)}$$

years to grow to \$12000.
"A calculator would tell you that t is about 1.54 years."

Question 1 (d)

Easiness: 38/100

SOLUTION. As long as $N''(t)$ is negative, $N'(t)$ is decreasing. Differentiating $N(t)$,

$$N'(t) = 60t - 3t^2$$

Differentiating again,

$$N''(t) = 60 - 6t.$$

This is negative whenever $6t > 60$, i.e. $t > 10$. We can conclude that $N'(t)$ is decreasing for all $t > 10$. *(Furthermore, $N'(t)$ is continuous, we also know that it is increasing on the closed interval, $[10, \infty)$. On an exam, either answer, $(10, \infty)$ or $[10, \infty)$, would be acceptable.)*

Question 1 (e)

Easiness: 60/100

SOLUTION. First of all, both the numerator and denominator converge to zero as $x \to 3$, so we cannot solve the limit directly.
Instead, we need to make use of the fact that $f'(3) = 5$. Taking this as a hint, let's recall the definition of the derivative as the limit of a difference quotient:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

For us, $a = 3$ and $f'(3) = 5$, so we know the left hand side of the equation above. To find the limit of the quotient we are given, let us rewrite our quotient in a similar form as the difference quotient:

$$\begin{aligned}
\frac{x^2 - 3x}{f(3) - f(x)} &= x \frac{x - 3}{f(3) - f(x)} \\
&= x \left(\frac{f(3) - f(x)}{x - 3} \right)^{-1} \\
&= x \left(\frac{(-1)(f(x) - f(3))}{x - 3} \right)^{-1} \\
&= x(-1)^{-1} \left(\frac{f(x) - f(3)}{x - 3} \right)^{-1} \\
&= -x \left(\frac{f(x) - f(3)}{x - 3} \right)^{-1}
\end{aligned}$$

We now recognize the second term as the difference quotient. Hence, applying the limit laws, we arrive at our final answer

$$\begin{aligned}
\lim_{x \to 3} \frac{x^2 - 3x}{f(3) - f(x)} &= \lim_{x \to 3} \left[-x \left(\frac{f(x) - f(3)}{x - 3} \right)^{-1} \right] \\
&= -\left(\lim_{x \to 3} x \right) \left(\lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} \right)^{-1} \\
&= -3 \left(f'(3) \right)^{-1} \\
&= -3 \left(5 \right)^{-1} \\
&= -\frac{3}{5}
\end{aligned}$$

Question 1 (f)

Easiness: 98/100

SOLUTION. By the chain rule,

$$\frac{dy}{dx} = \frac{1}{f(2x)} f'(2x) \cdot 2$$

When $x = 1$, this is equal to

$$\frac{1}{f(2)} f'(2) \cdot 2 = \frac{1}{3}(-5) \cdot 2 = -\frac{10}{3}.$$

Hence, the slope of the tangent line at $x = 1$ is equal to $-10/3$.

Question 1 (g)

SOLUTION. We want to find a solution to

$$f(x) = x$$

and so we begin by defining a function

$$g(x) = f(x) - x$$

so that we can think instead of finding roots to g(x). Recall that the Newton-Raphson Method for finding roots to a function g(x) gives a future approximation (x_{n+1}) based on a current approximation (x_n) as

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}.$$

Therefore, in our case

$$x_1 = x_0 - \frac{g(x_0)}{g'(x_0)} = 3 - \frac{g(3)}{g'(3)}$$

since $x_0 = 3$. The task now is to find what g(3) and g'(3) are. Since g(x)=f(x)-x then

$$g'(x) = f'(x) - 1$$

so we really only need to determine f(3) and f'(3). To do this we can use the tangent line which is also the linear approximation to f(x). Therefore we have that

$$f(x) \approx 5x - 7 = 5x - 15 + 8 = 8 + 5(x - 3) = f(3) + f'(3)(x - 3).$$

Therefore, we have that f(3)=8 and f'(3)=5. Similarly, we then have that,

$$g(3) = f(3) - 3 = 8 - 3 = 5$$
$$g'(3) = f'(3) - 1 = 5 - 1 = 4$$

and so we get that,

$$x_1 = 3 - \frac{g(3)}{g'(3)} = 3 - \frac{5}{4} = \frac{7}{4}.$$

Therefore the next approximation to the solution is $x_1 = 7/4$.

Question 1 (h)

Easiness: 98/100

SOLUTION. This requires us to find a point on the line and its slope.
When $x = 1$, $\ln x = 0$ so we know the point $(1, 0)$ is on the line.
If $y = \ln x$ then $\frac{dy}{dx} = \frac{1}{x}$ and $\frac{dy}{dx}|_{x=1} = 1$ so the slope of the line is 1.
The equation of the tangent line is then $y - 0 = 1(x - 1)$, by the point-slope formula, or $y = x - 1$.

Question 1 (i)

SOLUTION. We have that

$$f'(x) = \frac{2\ln x}{x}.$$

The second derivative is (using quotient rule),

$$f''(x) = \frac{\frac{2}{x}x - 2\ln x(1)}{x^2} = 2\frac{1 - \ln x}{x^2}.$$

In order to find intervals of concavity we must check where the second derivative is zero or does not exist. We see that the second derivative is zero whenever 1-lnx=0. This occurs if x=e. The second derivative doesn't exist if x=0 (the denominator vanishes) or if x<0 since then the logarithm cannot be evaluated. In fact we anticipate that x<0 is not even in the domain of the function. Therefore our intervals of interest are $(0, e), (e, \infty)$. We can make a table (or number line) as follows

	$(0, e)$	(e, ∞)
$1 - \ln x$	$+$	$-$
x^2	$+$	$+$
$f''(x)$	$+$	$-$
$f(x)$	Concave Up	Concave Down

Therefore we have that f(x) is concave up on $(0, e)$ and concave down on (e, ∞).

Question 1 (j)

SOLUTION. We have that

$$x(t)^2 + 2y(t)^2 = 9$$

and we want to find dy/dt when x=1, y=2 and dx/dt=3. Therefore, we implicitly differentiate and isolate for dy/dt,

$$
\begin{aligned}
2x\frac{\mathrm{d}x}{\mathrm{d}t} + 4y\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= -\frac{x}{2y}\frac{\mathrm{d}x}{\mathrm{d}t}.
\end{aligned}
$$

We then sub in x=1, y=2, and dx/dt=3 to get that

$$\left.\frac{\mathrm{d}y}{\mathrm{d}t}\right|_{x=1,y=2} = -\frac{1}{4}(3) = -\frac{3}{4}$$

and so we conclude that the y-coordinate is changing at -3/4 kilometers per minute at that moment.

Question 1 (k)

SOLUTION. On the closed interval $[-1, 8]$, the global maximum/minimum can occur at critical points or at the endpoints.
If $f(x) = x^{2/3}$, then $f'(x) = \frac{2}{3}x^{-1/3}$ which is never 0 but is undefined at $x = 0$. The only critical point on the interval is at $x = 0$.
Now we evaluate f at the critical point and at the endpoints

$$f(0) = 0,$$

$f(-1) = (-1)^{2/3} = 1$, and $f(8) = 8^{2/3} = 4$.
In comparing these we find the global maximum is 4 occurring at $x = 8$ and the global minimum is 0 occurring at $x = 0$.

Question 1 (l)

SOLUTION. Using the chain and product rules, we compute $f'(x) = 2a\cos(2x) + \cos(x) - x\sin(x) = 0$ at a critical point.
If $x = \pi$ is a critical point then $2a\cos(2\pi) + \cos(\pi) - \pi\sin(\pi) = 2a - 1 - 0 = 0$ so $a = 1/2$.

Question 1 (m)

SOLUTION. In general if

$$p(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + ... + c_n(x-a)^n$$

is the nth degree Taylor polynomial to $f(x)$ centred at $x = a$ then the coefficient is

$$c_m = \frac{f^{(m)}(a)}{m!}.$$

If

$$f(x) = e^{2x+1}$$

then

$$f'(x) = 2e^{2x+1}$$
$$f''(x) = 4e^{2x+1}$$
$$f'''(x) = 8e^{2x+1}.$$

Since we are centering about 0,

$$c_3 = \frac{f'''(0)}{3!} = \frac{8e}{6}.$$

Question 1 (n)

SOLUTION. We use the linear approximation of $\ln(x)$ at $x = 1$ to estimate $\ln(0.8)$. (Approximating at $x = 1$ is a logical choice, as 1 is close to 0.8, and we know that $\ln(1) = 0$.)
We know $\frac{d}{dx}\ln(x) = \frac{1}{x}$, which is equal to 1 when $x = 1$. Substituting this into the linear approximation formula, we get,

$$\ln(0.8) \approx \ln(1) + 1(0.8 - 1) = 0 - 0.2 = -0.2$$

Question 2 (a)

SOLUTION. To find the slope of the tangent line, we consider y as a function of x near the point (1,2) and use implicit differentiation to find $\frac{dy}{dx}$:

$$3x^2 + y^2 + 2xy\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

Isolating $\frac{dy}{dx}$,

$$\left(2xy + 3y^2\right)\frac{dy}{dx} = -3x^2 - y^2.$$

Hence,

$$\frac{dy}{dx} = \frac{-3x^2 - y^2}{2xy + 3y^2}.$$

The slope of the tangent line to the curve at $(x, y) = (1, 2)$ is then

$$\frac{dy}{dx} = \frac{-3(1)^2 - 2^2}{2(1)(2) + 3(2)^2} = \frac{-7}{16}.$$

Since the tangent line passes through the point $(1, 2)$ and has slope $-7/16$, its equation is

$$y - 2 = \frac{-7}{16}(x - 1).$$

Question 2 (b)

Solution. First note that the difference between the y-coordinates of this point and (1,2) is $31/16 - 2 = -1/16$, which is fairly small. It is probably small enough to use the tangent line from part (a) to find a reasonable approximation for the x-coordinate of this point. (We are not asked to estimate the error in this approximation.)
The x-coordinate of the point on the tangent line with y-coordinate 31/16 can be found by substituting y=31/16 into the equation for the tangent line and solving for x.

$$\frac{31}{16} - 2 = -\frac{7}{16}(x-1)$$

Solving for x,

$$x = 1 - \frac{16}{7}\left(\frac{31}{16} - 2\right) = 1 - \frac{16}{7}\left(-\frac{1}{16}\right) = 1 + \frac{1}{7} = \frac{8}{7}.$$

Making a linear approximation, we conclude that the point on the *curve* with y-coordinate 31/16 is approximately the same as the point on the *tangent line* with y-coordinate 31/16.
Hence, the x-coordinate of the point on the curve with y-coordinate 31/16 is approximately 8/7.

Question 3

Easiness: 98/100

Solution. In order to find dq/dp, we differentiate both sides of the supply equation with respect to p.

$$3p^2 + 1 = (6q^2 + 2q)\frac{dq}{dp}$$

Solving for dq/dp,

$$\frac{dq}{dp} = \frac{3p^2+1}{6q^2+2q}$$

Hence, the elasticity of supply is

$$-\frac{p}{q}\frac{dq}{dp} = -\frac{p}{q}\frac{3p^2+1}{6q^2+2q}.$$

Evaluating this when $p = 3$ and $q = 2$, we obtain

$$-\frac{3}{2}\frac{3(3)^2+1}{6(2)^2+2(2)} = -\frac{3}{2}\frac{28}{28} = -\frac{3}{2}.$$

Question 4

Easiness: 96/100

Solution. Let $A(q)$ denote the average cost per kilogram of producing q kilograms of this chemical,

$$A(q) = \frac{C(q)}{q} = 3q^{1/3} + 50 + \frac{10000}{q}.$$

The domain of interest is $(0, \infty)$ because the company cannot produce negative kilograms of their product. In order to find the absolute minimum of the average cost function, we begin by looking for its critical points. The derivative of A is

$$A'(q) = q^{-2/3} - \frac{10000}{q^2} = \frac{1}{q^2}(q^{4/3} - 10000)$$

This exists everywhere in our domain and is equal to zero when $q^{4/3} - 10000 = 0$, that is, when

$$q^{\frac{4}{3}} = 10000 = 10^4.$$

Since $q \geq 0$, we can take the positive fourth root of both sides to obtain

$$q^{\frac{1}{3}} = 10.$$

Hence,

$$q = 10^3 = 1000$$

which is our final answer.
Note:
It is easy to check that this is indeed an absolute minimum by noting that $A'(q) < 0$ when $q < 1000$ and $A'(q) > 0$ when $q > 1000$. Since $A(q)$ is decreasing when $q < 1000$ and increasing when $q > 1000$, $A(q)$ must have an absolute minimum at $q = 1000$.
Hence, the company should produce 1000 kg of chemical in order to minimize its average cost.

Question 5

Solution. The quantity we want to minimize (the "objective function"), is the profit, P. The profit is equal to the revenue, pq, minus the cost, $C(q)$.

$$P = pq - C(q).$$

Our constraint is the demand equation, which gives us a relationship between p and q:

$$p = 400 - 50q.$$

We now use the constraint to express the objective function P in terms of one variable only. Since C is already expressed in terms of q, it will be easiest to express P as a function of q rather than p.

$$P(q) = (400 - 50q)q - C(q) = 400q - 50q^2 - C(q)$$

The domain of interest is $[0, 8]$, because we cannot produce a negative quantity of jackets, and in order to sell more than 8 jackets per month, we'd actually have to start paying people to take them (according to the demand equation). In order to find the absolute maximum value of $P(q)$ in this interval, we start by looking for its critical points.

$$P'(q) = 400 - 100q - C'(q) = 400 - 100q - \frac{800}{q+5}$$

This exists everywhere inside the domain. In order to search for its zeros, it is helpful to take out a factor of $\frac{100}{q+5}$ (which is never equal to zero), so that we are left with a polynomial in q. (Hopefully we'll be able to find the roots of that polynomial.)

$$\begin{aligned}
P'(q) &= \frac{100}{q+5}\left((4-q)(q+5) - 8\right) \\
&= \frac{100}{q+5}\left(4q + 20 - q^2 - 5q - 8\right) \\
&= -\frac{100}{q+5}\left(q^2 + q - 12\right) \\
&= -\frac{100}{q+5}(q+4)(q-3)
\end{aligned}$$

This is equal to zero when q is equal to 3 or -4, but only 3 lies in the domain of interest. Although it is not necessary, it is easy to check that the profit is actually maximized when $q = 3$ because when $q < 3$, $P'(q) > 0$, and when $q > 3$, $P'(q) < 0$. (Hence, the profit function is increasing when $q < 3$ and decreasing when $q > 3$.) Therefore, the profit is maximized when 3 jackets are sold per month.
Also, notice that taking the derivative of profit and setting it to zero, is equivalent to finding the point at which the marginal profit is zero or when the marginal revenue is equal to the marginal cost

Question 6

SOLUTION. **Step 1: Find objective function**
We have that the stock price is

$$P(t) = 0.25 + 30t\mathrm{e}^{-t/5}$$

and are asked to find when the stock is increasing most rapidly. Notice we are **not** asked to find when the price of a stock is a maximum. This makes sense since if you're looking to buy the stock, you want it when its value is increasing the fastest. If you buy the stock at its maximum price then your stock will only lose value as time moves forward. Therefore the objective function is to maximize the rate of change, R(t), of the stock price or $P'(t)$. We get,

$$R(t) = P'(t) = 30\mathrm{e}^{-t/5} - 6t\mathrm{e}^{-t/5} = (30 - 6t)\mathrm{e}^{-t/5}.$$

Step 2: Find critical points of the objective function R(t)
We start by finding critical points of R(t). First take the derivative of R(t) and set it to zero.

$$R'(t) = -6\mathrm{e}^{-t/5} - \frac{1}{5}(30 - 6t)\mathrm{e}^{-t/5} = \left(\frac{6}{5}t - 12\right)\mathrm{e}^{-t/5} = 0.$$

Since the exponential can never be zero we must have that,

$$\begin{aligned}
\left(\frac{6}{5}t - 12\right) &= 0 \\
t &= 10.
\end{aligned}$$

Thus t=10 is the only critical point of R(t).

Step 3: Is the critical point a maximum or a minimum?

We have still not answered the question, we must still maximize R. We have that t is in $[0, \infty)$ and so we do not have a full closed interval to use the extreme value theorem. However, since we only have one critical point we know that if R(10) is a local maximum then it is also an absolute maximum and if R(10) is a local minimum then it is also an absolute minimum.

We will use the first derivative test. Notice, if we pick something slightly less than 10, then R'(t)<0 so R(t) is decreasing. Similarly if we pick something slightly bigger than 10 then R'(t)>0 so R(t) is increasing. Therefore, by the first derivative test we conclude that t=10 is a local **minimum**. Since it is the only critical point, it is also an absolute minimum.

Step 4: Look at the boundaries of the interval of t to find the absolute maximum

How do we then find the absolute maximum? Even though we do not have a closed interval for R(t), we can see what happens for large time, i.e. we can compute

$$\lim_{t\to\infty} R(t).$$

Doing so involves taking an infinite limit of $te^{-t/5}$. The exponential decays much faster than the linear term grows so this term goes to zero. Therefore we have that,

$$\begin{aligned}&\lim_{t\to\infty} R(t)\\ &= \lim_{t\to\infty} 30e^{-t/5} - 6\lim_{t\to\infty} 6te^{-t/5}\\ &= 0.\end{aligned}$$

Therefore we see that for large time, R(t) goes to zero. We can plug in the left endpoint,

$$R(0) = (30 - 6(0))e^{-0/5} = 30.$$

Step 5: Conclusion

Therefore, we see that R starts at 30, decreases to the critical point at t=10 (note that R(10) $<$ 0) and then increases towards 0 for t large. Therefore, R is the biggest when t=0, or right when the rumour starts spreading. Therefore, the price of the stock is increasing most rapidly when t=0.

Good Luck for your exams!