Full Solutions
MATH102 December 2013

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Educational Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Exam Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Educational Resources.

Question A 01

SOLUTION. f has an **inflection point** wherever f'' *changes sign*. Thus we begin by looking for the zeroes of f'':

$$f(x) = \sin(x) + ax^2$$
$$f'(x) = \cos(x) + 2ax$$
$$f''(x) = -\sin(x) + 2a$$

Solving for a, we find that $a = \frac{1}{2}\sin(x)$. Since $-1 \le \sin(x) \le 1$, we have $-\frac{1}{2} \le a \le \frac{1}{2}$.
But by the same inequalities, $0 \le f''(x) \le 2$ when $a = \frac{1}{2}$. Similarly, $-2 \le f''(x) \le 0$ when $a = -\frac{1}{2}$. In these cases, f has no inflection points since f'' does not *change sign*.
Therefore $-\frac{1}{2} < a < \frac{1}{2}$. This is equivalent to answer (D).

Question A 03

SOLUTION. We start at the point $(0, 1)$, where the slope of the tangent line at this point is
$\frac{dy}{dx} = 2 - x^2 = 2 - [y(0)]^2 = 2 - 1 = 1$
By applying **Euler's method** with a step size of $\Delta t = 0.1$, our next point would be at the value $y(0.1) = y(0) + \left.\frac{dy}{dx}\right|_{x=0}(0.1 - 0) = 1 + 0.1 = 1.1$, which is answer (E).

Question A 04

SOLUTION. Noting that $f(x_1) = 0$, we solve for x_1 using the equation of the tangent line to f at $(x_0, f(x_0))$:

$$y - y_0 = m(x - x_0)$$
$$f(x_1) - f(x_0) = f'(x_0)(x_1 - x_0)$$
$$0 - f(x_0) = f'(x_0)(x_1 - x_0)$$
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

which is answer (E).

Question A 05

SOLUTION. Taking the second derivative of our function gives

$$y = \cos(x)$$
$$y' = -\sin(x)$$
$$y'' = -\cos(x)$$

To the immediate left of $x = \pi/2$, the second derivative is negative and to its immediate right it is positive. So we get an *overestimate* when $x < \pi/2$ and an *underestimate* when $x > \pi/2$. Thus the answer is (D).

Question A 08

SOLUTION. To solve this question, notice that the question is asking for **steady states** (i.e., where $y' = 0$). Solving this yields the values $y = 0, a$. Since the initial condition is $y = 2a$, $a > 0$, and since solutions never cross steady states, we see that $\lim_{t\to\infty} y(t) = a$ which is answer (C).

Question B 01

SOLUTION. Differentiate both sides of $\tan y = x$ with respect to x. (Don't forget to apply the chain rule!)

$$\begin{aligned}\frac{d}{dx}\tan y &= \frac{d}{dx}x\\ \sec^2 y\cdot\frac{dy}{dx} &= 1\\ \frac{dy}{dx} &= \frac{1}{\sec^2 y}.\end{aligned}$$

Now apply the trig identity $\sec^2 y = 1+\tan^2 y$ to get

$$\frac{dy}{dx} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

after noting that $\tan y = x \implies \tan^2 y = x^2$.

Question B 03

SOLUTION. Following the hint, we have

$$f'(x) = 5x^4 - 5x^3 = 5x^3(x-1)$$

The zeroes of f' are $x = 0, 1$.
On $(-\infty,0)$ (say at $x=-10$), $f'(x) > 0$ and so the function is increasing on this interval.
On $(0,1)$ (say at $x = 1/2$), $f'(x) < 0$ and so the function is decreasing on this interval.
On $(1,\infty)$ (say at $x = 10$), $f'(x) > 0$ and so the function is increasing on this interval.
Therefore we have a **local maximum** at $x = 0$ and a **local minimum** at$x = 1$.

Question B 05

SOLUTION 1. Recall that the definition of the derivative of a function f at x is

$$f'(x) := \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Following hint 1, we see that if $x = 3$ and $f(x) = \sqrt{x}$, then

$$\begin{aligned}f'(3) &= \lim_{h\to 0}\frac{\sqrt{3+h}-\sqrt{3}}{h} = L\\ L &= \frac{1}{2\sqrt{3}}\end{aligned}$$

SOLUTION 2. The slower solution uses hint 2 and multiplies the numerator and the denominator by the conjugate of the numerator

$$
\begin{aligned}
L &= \lim_{h\to 0} \frac{\sqrt{3+h}-\sqrt{3}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{3+h}-\sqrt{3}}{h} \cdot \frac{\sqrt{3+h}+\sqrt{3}}{\sqrt{3+h}+\sqrt{3}} \\
&= \lim_{h\to 0} \frac{3+h-3}{h(\sqrt{3+h}+\sqrt{3})} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{3+h}+\sqrt{3})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{3+h}+\sqrt{3}} \\
&= \frac{1}{\sqrt{3}+\sqrt{3}} \\
&= \frac{1}{2\sqrt{3}}
\end{aligned}
$$

Question B 06

SOLUTION. Polynomials are continuous, so the only point where we are concerned with the continuity of f is $x = 1$. It remains to find a value a such that $\lim_{x\to 1^-} f(x) = f(1) = \lim_{x\to 1^+} f(x)$.

- By the definition of $f(x)$ we have $f(1) = a - 1$.
- The right-sided limit is $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} a - x = a - 1$.
- The left-sided limit is $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 = 1$.

Hence we need to find a value a such that $a - 1 = 1$. It follows that the value a that makes $f(x)$ continuous on $(-\infty, \infty)$ is $a = 2$.

Question B 07

Easiness: 100/100

SOLUTION. The solution of the differential equation $\frac{dy}{dt} = ky$ is

$$y(t) = y(0) \cdot e^{kt}$$

where in this case y is the number of viral particles, t is the time in days, and k is some constant.
We are given that $y(0) = 1000$, $k = 0.05$. Thus we have $y(t) = 1000e^{0.05t}$.
Solving for t when $y(t) = 350000$ yields:

$$
\begin{aligned}
350000 &= 1000e^{0.05t} \\
350 &= e^{0.05t} \\
\ln 350 &= 0.05t \\
t &= \frac{\ln 350}{0.05} \text{ days}
\end{aligned}
$$

Question B 08

Solution. (i)

$$\lim_{x\to 0^+} e^{1/x} = \lim_{h\to\infty} e^h = \infty$$

(ii)

$$\lim_{x\to\infty} e^{1/x} = \lim_{h\to 0} e^h = e^0 = 1$$

(iii)

$$\lim_{x\to 0^+} \frac{1}{ax + e^{1/x}} = \lim_{h\to\infty} \frac{1}{\frac{a}{h} + e^h} = 0$$

(iv)

$$\lim_{x\to\infty} \frac{1}{ax + e^{1/x}} = \lim_{h\to 0^+} \frac{1}{\frac{a}{h} + e^h} = 0$$

Question C 01

Solution. The volume of a cone is given by

$$V = \frac{1}{3}\pi r^2 h$$

By the physical application of the problem, we can assume that the base radius of the cone changes as the height increases. First, we should express the radius of the cone as a function of its height.
By similar triangles, and the fact that $r = 1$ when $h = 10$, we find that the following relation holds:

$$\frac{r}{h} = \frac{1}{10} \qquad \Longleftrightarrow \qquad r = \frac{h}{10}$$

Thus the volume of the shell can be expressed entirely in terms of h:

$$\begin{aligned} V &= \frac{1}{3}\pi\left(\frac{h}{10}\right)^2 h \\ &= \frac{\pi}{300}h^3 \end{aligned}$$

The rate of change in the volume of the cone can be computed by differentiating V with respect to t, giving us

$$\frac{dV}{dt} = \frac{\pi}{100}h^2 \cdot \frac{dh}{dt}$$

Since we are told that the height of the cone changes at a constant rate of 0.1 cm/year, the rate of change of in the cone's volume when $h = 10$ cm and $r = 1$ cm is given by

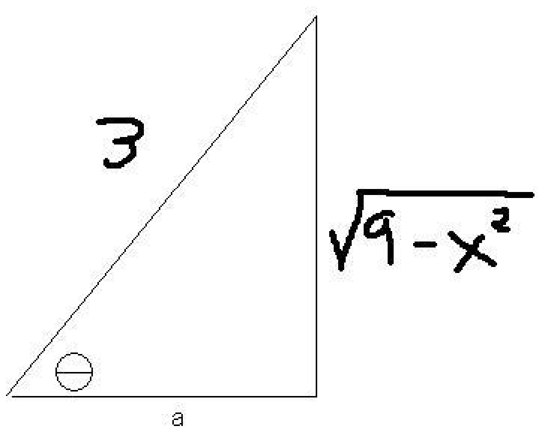
$$
\begin{aligned}
\frac{dV}{dt} &= \frac{\pi}{100}(10\text{cm})^2 \cdot 0.1 \frac{\text{cm}}{\text{year}} \\
&= \frac{\pi}{10} \frac{\text{cm}^3}{\text{year}}
\end{aligned}
$$

Question C 02 (b)

SOLUTION. The term with α tends to make the value of F decrease since it has a negative sign in front of it. Because of this, and the fact that this term depends on both the population of fish and the number of fishermen, one should conclude that this term models the interaction between fishermen and the fish. (i.e. the interaction of fisherman and fish leads to less fish because they get caught by the fishermen).
The constant α represents the rate at which a single fisherman catches fish.

Question A 02

SOLUTION. The correct answer is (B).
Let $\theta = \arcsin\left(\frac{\sqrt{9-x^2}}{3}\right)$. Then $\sin(\theta) = \frac{\sqrt{9-x^2}}{3}$. This gives the following triangle.



To solve for the last side, simply use the Pythagorean theorem which gives that the adjacent side equals $a^2 = 3^2 - (\sqrt{9-x^2})^2 = 9 - 9 + x^2 = x^2$ and so $a = x$ (we only need to consider the positive root since a has to be positive). Hence, taking the cos in our triangle above, we see that $\cos(\theta) = \frac{a}{3} = \frac{x}{3}$ and this is exactly the value we wanted:
$\cos\left(\arcsin\left(\frac{\sqrt{9-x^2}}{3}\right)\right) = \frac{x}{3}$

Question A 06

SOLUTION. No content found.

Question A 07

SOLUTION. The extrema of sine and cosine functions must occur at values that are 2π multiples of $0, \pi/2, \pi, 3\pi/2$. Plugging in 2000 shows that answers (B) and (C) do not satisfy this condition.

Plugging in 2000 into (A),(D) and (E) gives us the values
$60 + 120\sin(\pi/2) = 180$
$60 + 60\sin(0) = 60$
$60 + 60\cos(0) = 120$
respectively. Thus, the answer is (E).

Question B 02

SOLUTION. No content found.

Question B 04

SOLUTION. No content found.

Question C 02 (a)

SOLUTION. The constant I represents the rate at which the company adds fish to the lake.

Question C 02 (c)

SOLUTION. No content found.

Question C 02 (d)

SOLUTION. No content found.

Question C 03

SOLUTION. We begin by finding all the derivatives and the zeroes. For the original function, we have
$0 = x^2e^{-x}$
and since $e^{-x} > 0$ always, only $x = 0$ is a root.
Next we take the derivative and see that
$f'(x) = 2xe^{-x} - x^2e^{-x} = xe^{-x}(2 - x)$
As before, finding the roots gives us that $x = 0, 2$. Lastly taking the second derivative, we have that
$f''(x) = 2(e^{-x} - xe^{-x}) - (2xe^{-x} - x^2e^{-x}) = e^{-x}(2 - 2x - 2x + x^2) =$
The zeroes of the second derivative are given by $x^2 - 4x + 2 = 0$.
Using the quadratic formula gives us the roots
$x = \frac{-(-4)\pm\sqrt{(-4)^2-4(1)(2)}}{2} = \frac{4\pm\sqrt{8}}{2} = \frac{4\pm2\sqrt{2}}{2} = 2 \pm \sqrt{2}$
Next we make sign charts using the first and second derivative to determine critical points and inflection points respectively.

Question C 04

SOLUTION. No content found.

Good Luck for your exams!