Full Solutions
MATH110 December 2010

April 5, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

SOLUTION. The statement is false. Let's take polynomials that not only have a nonzero horizontal asymptote but that also always obtain their asymptote! We can choose constant functions for f and g (these are polynomials!) So suppose that $f(x) = g(x) = 1$. Then, the limit becomes

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1}{1} = 1$$

which is not *0* and does not diverge to $\pm\infty$.
Note: If, for some reason, you don't like constant polynomials there is a whole zoo of non-constant polynomials f and g with quotients approaching any horizontal asymptote you want, e.g.

$$\lim_{x\to\infty} \frac{17x+1}{x} = 17.$$

Question 1 (b)

SOLUTION. The most common reasons that a function f is non-differentiable at a point $x = a$ are as follows:
a) f has a sharp corner (called a cusp) at $x = a$, where the tangent lines from both sides of a converge to different values.
b) f has a vertical tangent line at $x = a$, that is, the slope of the tangent line at a is infinity.
Both of these conditions can occur for continuous functions. For example:
a) The function $f(x) = |x|$ is continuous, but has a sharp corner at $x =$ *0*.
b) The function $f(x) = \sqrt[3]{x}$ is continuous, but has a vertical tangent line at $x =$ *0*.
Thus it is clear that there are continuous functions that are not differentiable, so the statement Not all continuous functions are differentiable is true. To prove this, it is sufficient to give an example of a function that is continuous but not differentiable, like one of the functions shown above.

Question 1 (c)

SOLUTION. The statement is false. If we have two functions that only differ by a constant, when we take the derivative the constant will disappear, making the derivatives equal. However the original functions were not equal to start with.
A simple concrete example would be: suppose $f(x) = x^2 + 2$ and $g(x) = x^2$. Even though their derivatives are equal,

$$f'(x) = g'(x) = 2x$$

the original functions aren't equal.
More generally, if $f(x) = p(x)$ and $g(x) = p(x) + c$ where $p(x)$ is a differentiable function, then $f'(x) = g'(x) = p'(x)$ but $f(x) \neq g(x)$.

Question 2 (a)

SOLUTION. If we start by simplifying the expression of the function f we have:

$$f(x) = \frac{x^2x^3x^4x^5x^6}{x} = \frac{x^{20}}{x}$$

This function is not defined at 0 and for all the other value, we can divide and obtain that

$$f(x) = \frac{x^{20}}{x} = x^{19} \quad \text{if } x \neq 0$$

and so its derivative is

$$f'(x) = 19x^{18} \quad \text{if } x \neq 0$$

Question 2 (b)

Solution. The function $f(x)$ consists of one function dividing another. This means we should use the quotient rule to find the derivative.
Changing $\sqrt{x}$ to $x^{\frac{1}{2}}$, and applying the quotient rule we get:

$$\begin{aligned} f'(x) &= \frac{(x^{\frac{1}{2}})' \cos x - x^{\frac{1}{2}}(\cos x)'}{\cos^2 x} \\ &= \frac{\frac{1}{2}x^{-\frac{1}{2}} \cos x - x^{\frac{1}{2}}(-\sin x)}{\cos^2 x} \\ &= \frac{\frac{\cos x}{2\sqrt{x}} + \sqrt{x}\sin x}{\cos^2 x} \end{aligned}$$

This answer is sufficient, however, we can simplify this term by multiplying the numerator and the denominator by $2\sqrt{x}$ to get

$$= \frac{\cos x + 2x \sin x}{2\sqrt{x}\cos^2 x}$$

Question 2 (c)

Solution. The function we are trying to differentiate is the composition of four functions: three $\tan x$ functions and an x^2. Thus we will use the chain rule. This means taking the derivative of the outermost $\tan x$ function and then replacing x with the rest of the function; then multiplying by the deriviative of the next $\tan x$ and so on, until each part of the function has been differentiated.
(Recall that the derivative of $\tan x$ is $\sec^2 x$.)
$f'(x) = \sec^2(\tan(\tan(x^2))) \cdot \sec^2(\tan(x^2)) \cdot \sec^2(x^2) \cdot 2x$.

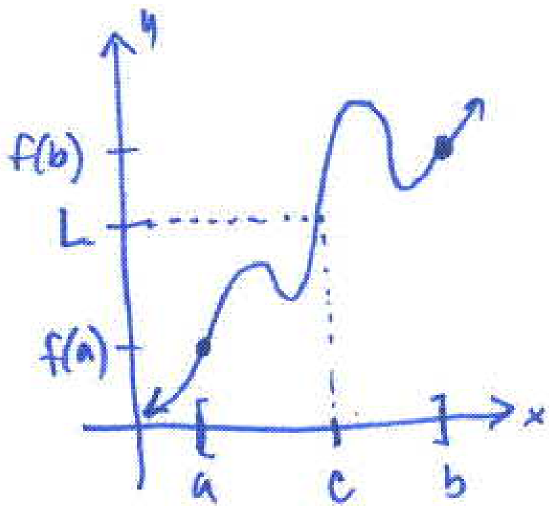
Question 3 (a)

Easiness: 5.0/5

Solution. Suppose the function f is continuous on the interval [a,b] and L is a number between $f(a)$ and $f(b)$. Then there is at least one number c in $[a, b]$ satisfying $f(c) = L$.

Question 3 (b)

Solution. The following picture is a sample illustration of the Intermediate Value Theorem.



Question 3 (c)

Easiness: 4.0/5

Solution. The simplest example of such a function would be the piecewise function:

$$f(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

The function f is clearly defined on $[-1, 1]$, but jumps from -1 to 1 at $x = 0$. Thus at no point in $[-1, 1]$ does $f(x) = 0$.

Two other variations:

$$f(x) = \begin{cases} 1/x & x \neq 0 \\ 1 & x = 0 \end{cases}$$

In this case, the function $1/x$ is never equal to zero; however, we must add the point $(0, 1)$ in the piecewise function to make it defined everywhere in our domain.

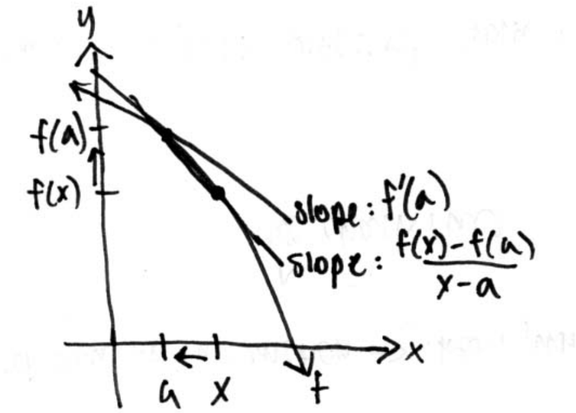
$$f(x) = \begin{cases} x & x \neq 0 \\ 1 & x = 0 \end{cases}$$

Here we have what would be a continuous function (the function x) - we've just removed the one point where it would equal zero and moved that point somewhere else, in this case, to $y = 1$.

You could come up with many other piecewise functions using these same ideas.

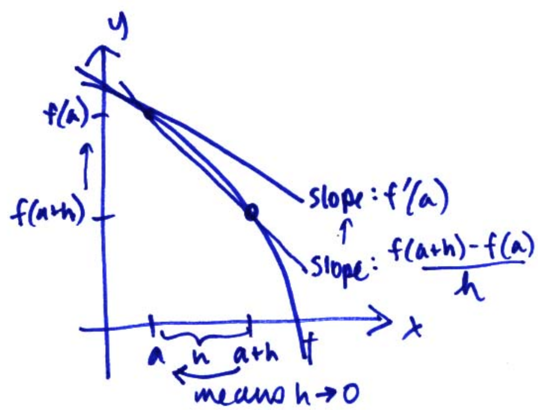
Question 4 (a)

Solution 1. A drawing illustrating the first definition given would look something like this:



See as $x \to a$, the slope of the line between $(a, f(a))$ and $(x, f(x))$ approaches the slope of the tangent line at $x = a$.

Solution 2. A drawing illustrating the first definition given would look something like this:



As $h \to 0$, the slope of the line between $(a, f(a))$ and $(a + h, f(a + h))$ approaches the slope of the tangent line at $x = a$.

Question 4 (b)

Solution 1. We are trying to find $f'(a)$ where $f(x) = 2x + 1$ and $a = 0$. Using the first definition of the derivative, we need to know $f(0)$ which we calculate here:
$f(0) = 2(0) + 1 = 1$
Plugging this, and $f(x)$ into the definition of the derivative, we get:

$$
\begin{aligned}
f'(0) &= \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} \\
&= \lim_{x\to 0} \frac{2x+1-1}{x} \\
&= \lim_{x\to 0} \frac{2x}{x} \\
&= \lim_{x\to 0} 2 \\
&= 2
\end{aligned}
$$

SOLUTION 2. We are trying to find $f'(a)$ where $f(x) = 2x+1$ and $a = 0$.
Using the second definition of the derivative, we will need to know $f(a+h)$ and $f(a)$. Since $a = 0$, they are as follows:
$f(0+h) = 2(0+h) + 1 = 2h + 1$
$f(0) = 2(0) + 1 = 1$
Plugging these into our definition we get:

$$
\begin{aligned}
f'(0) &= \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} \\
&= \lim_{h\to 0} \frac{2h+1-1}{h} \\
&= \lim_{h\to 0} \frac{2h}{h} \\
&= \lim_{h\to 0} 2 \\
&= 2
\end{aligned}
$$

Question 5 (a)

Easiness: 2.5/5

SOLUTION. The two curves will intersect at $x = 3$ if their y-values are the same at $x = 3$. To find the y-values, we just plug $x = 3$ into each curve.
The first curve gives:

$$
\begin{aligned}
y &= \frac{1}{6}\sqrt{2(3)^2+7} \\
&= \frac{1}{6}\sqrt{25} \\
&= \frac{5}{6}
\end{aligned}
$$

The second curve gives:

$$\begin{aligned}
y &= -2(3)^2 + 7(3) - \frac{13}{6} \\
&= -18 + 21 - \frac{13}{6} \\
&= 3 - \frac{13}{6} \\
&= \frac{18}{6} - \frac{13}{6} \\
&= \frac{5}{6}
\end{aligned}$$

These two y-values are equal, so the curves do intersect at $x = 3$.

Question 5 (b)

SOLUTION. First we need to calculate the slope of the tangent line to each curve at $x = 3$. We do this by taking the derivative and then plugging in $x = 3$.
For the first curve we use the chain rule to find the derivative:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{6} \cdot \frac{1}{2}(2x^2 + 7)^{-\frac{1}{2}} \cdot 4x \\
&= \frac{4x}{12\sqrt{2x^2 + 7}} \\
&= \frac{x}{3\sqrt{2x^2 + 7}}
\end{aligned}$$

Plugging in $x = 3$ to find the slope we get:
$\frac{3}{3\sqrt{2(3)^2+7}} = \frac{1}{\sqrt{25}} = \frac{1}{5}$.
For the second curve we simply use the power rule to calculate the derivative:
$\frac{dy}{dx} = -4x + 7$
Plugging in $x = 3$ gives a slope of:
$-4(3) + 7 = -5$.
We know that two slopes are perpendicular if they are negative reciprocals of each other, i.e. if one slope is $\frac{a}{b}$ then the other slope must be $-\frac{b}{a}$. For this problem, this relationship is true: the first slope, $\frac{1}{5}$ is the negative reciprocal of -5. Thus the tangent lines to these two curves are perpendicular at $x = 3$.

Question 5 (c)

SOLUTION. To find a solution for b, c, we need to satisfy two conditions: (1) $\sin x$ and $x^2 + bx + c$ must intersect at $x = 0$ and (2) the tangent line of $\sin x$ at $x = 0$ must be perpendicular to the tangent line of $x^2 + bx + c$ at $x = 0$. For this problem it doesn't matter which condition you test first; in this case, we will start with the second condition and then proceed to the first condition.
We begin by finding the slope of the tangent line to each curve at $x = 0$. For $\sin x$ the derivative is $\cos x$ and so the slope at $x = 0$ is $\cos(0) = 1$. For $x^2 + bx + c$ the derivative is $2x + b$ and so the slope at $x = 0$ is $2(0) + b = b$. To satisfy condition (2), these two slopes must be perpendicular, i.e. negative reciprocals of each other. Since the tangent slope of $\sin x$ is 1, this means we need the tangent slope of $x^2 + bx + c$ to equal to -1. However, we know the tangent slope of $x^2 + bx + c$ is b, so this means $b = -1$.
Having found b we now turn to the first condition of the definition to find c. The first condition requires that $\sin x$ and $x^2 + bx + c$ intersect at $x = 0$. This means that $\sin(0)$ must equal $(0)^2 + (-1)(0) + c$. In equations, this is:

$$\begin{aligned} \sin(0) &= (0)^2 + -1(0) + c \\ 0 &= c \end{aligned}$$

So $c = 0$.
Thus the curve $y = x^2 - x$ satisfies both of our conditions and intersects orthogonally the curve $y = \sin x$ at $x = 0$.

Question 6 (a)

Solution. Given a relation or rule that takes input values (usually denoted x) and produces output values (usually denoted y), the relation is a function if each x-value is associated with only one y-value. In other words, the graph of the relation must pass the "vertical line test" - if you can draw a vertical line that intersects the graph at two points, the relation is not a function.
In this case, a single x-value in the domain will be associated with two y-values from the upper and lower half of the circle. This is the same as saying that a circle fails the vertical line test, as a vertical line drawn through any point in the domain will intersect the graph in two places.

Question 6 (b)

Solution. Our differentiation rules at this point only address functions that are in the form $y = f(x)$. Thus our first step is to transform the equation of a circle into this form by solving for y.

$$\begin{aligned} x^2 + y^2 &= 25 \\ y^2 &= 25 - x^2 \\ y &= \pm\sqrt{25 - x^2} \end{aligned}$$

Because our point of concern is on the lower half of the circle, we will use the negative square root.
We now proceed to find the slope of the tangent line at (3, -4). First we take the derivative using the chain rule.

$$\begin{aligned} f(x) &= -\sqrt{25 - x^2} \\ f'(x) &= -\frac{1}{2}(25 - x^2)^{-\frac{1}{2}}(-2x) \\ &= \frac{x}{\sqrt{25 - x^2}} \end{aligned}$$

To find the slope, we plug $x = 3$ into the derivative to get:
$f'(3) = \dfrac{3}{\sqrt{25 - 3^2}} = \dfrac{3}{4}$
We now have a slope, 3/4, and a point, $(3, -4)$. Plugging these into our equation of a line we get:
$y + 4 = \frac{3}{4}(x - 3)$

Question 6 (c)

Solution. If two lines are perpendicular, their slopes are negative reciprocals of each other, that is, if one line has slope a/b, the other will have slope $-b/a$. Therefore, what we're really concerned about here is the slope of each line mentioned in the problem statement. We already have the slope of the line from part (b), namely 3/4, so it remains to find the slope of the line through (3, -4) and the origin. This hardly requires any calculus - we just calculate the slope using the two points (0,0) and (3, -4).

$\frac{y_2-y_1}{x_2-x_1} = \frac{-4-0}{3-0} = \frac{-4}{3}$
In this case, we have slope $-\frac{4}{3}$ and the line from part (b) has slope $\frac{3}{4}$. These two numbers are negative reciprocals so the two lines are perpendicular.

Question 7

Solution. It is clear that this function is continuous and differentiable everywhere, except maybe at $x = 1$.
Let's start with checking continuity. For $f(x)$ to be continuous at $x = 1$ the left and right limits of the function must be the same. Hence we set

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$$
$$\lim_{x\to 1} -x^2 = \lim_{x\to 1} ax^3 + bx - 2$$

Because these functions are polynomials we can simply substitute the x-value to get:

$$-(1)^2 = a(1)^3 + b(1) - 2$$
$$-1 = a + b - 2$$

This gives us one equation in terms of a and b. We can't proceed any further with this equation right now, so we will turn to the second condition that the function be differentiable. This means that the derivatives from the right and left of $x = 1$ must be equal. In other words:

$$(-x^2)' = (ax^3 + bx - 2)'$$

at $x = 1$

$$-2x = 3ax^2 + b$$

at $x = 1$
Plugging in $x = 1$ this yields

$$-2(1) = 3a(1)^2 + b$$

$$-2 = 3a + b$$

which gives another equation in terms of a and b.
We know both equations must hold if $f(x)$ is to be continuous and differentiable, so we can solve them as a system of equations to find a and b. Solving the first equation for b we get:

$$b = 1 - a$$

Substituting this into the second equation we get:

$$-2 = 3a + 1 - a$$
$$-3 = 2a$$
$$\frac{-3}{2} = a$$

Plugging this back into $b = 1 - a$ we get that $b = \frac{5}{2}$.
(Note that the system of equations can be solved multiple ways - using a different substitution or elimination would also work.)
So $a = -\frac{3}{2}$ and $b = \frac{5}{2}$ are values that satisfy the conditions that $f(x)$ be continuous and differentiable.

Question 8 (a)

Solution. Let $f(x) = \sin x$ and $g(x) = \cos x$. Find $f\left(\frac{\pi}{3}\right)$, $g\left(\frac{\pi}{3}\right)$, $f'\left(\frac{\pi}{3}\right)$, and $g'\left(\frac{\pi}{3}\right)$
First, let's take derivatives
$f'(x) = \cos x$
and
$g'(x) = -\sin x$.
Plugging in all the values we need, we see that we need to evaluate
$f(\frac{\pi}{3}) = \sin(\frac{\pi}{3})$
$g(\frac{\pi}{3}) = \cos(\frac{\pi}{3})$
$f'(\frac{\pi}{3}) = \cos(\frac{\pi}{3})$
$g'(\frac{\pi}{3}) = -\sin(\frac{\pi}{3})$
So it suffices to evaluate $\sin(\frac{\pi}{3})$ and $\cos(\frac{\pi}{3})$
Drawing a triangle with angles 30 degrees, 60 degrees and 90 degrees (since $\frac{\pi}{3}$ is 60 degrees), we see that the side lengths are 1, $\sqrt{3}$ and 2. Since the shortest side is opposite the shortest angle, the middle angle is opposite the middle side and the longest angle is opposite to the longest side, we have that
$\sin(\frac{\pi}{3}) = \dfrac{O}{H} = \dfrac{\sqrt{3}}{2}$
and
$\cos(\frac{\pi}{3}) = \dfrac{A}{H} = \dfrac{1}{2}$. Hence, we have
$f(\frac{\pi}{3}) = \sin(\frac{\pi}{3}) = \dfrac{\sqrt{3}}{2}$
$g(\frac{\pi}{3}) = \cos(\frac{\pi}{3}) = \dfrac{1}{2}$
$f'(\frac{\pi}{3}) = \cos(\frac{\pi}{3}) = \dfrac{1}{2}$
$g'(\frac{\pi}{3}) = -\sin(\frac{\pi}{3}) = -\dfrac{\sqrt{3}}{2}$

Question 8 (b)

Solution. To find the equation of a tangent line we need both the slope and a point on the line.
To find the slope, we simply take the derivative of the function using the product rule and plug in the value $x = \frac{\pi}{3}$
$\frac{dy}{dx} = 4((\sin x)' \cos x + \sin x(\cos x)') = 4(\cos^2 x - \sin^2 x)$
We know from part (a) that $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ and $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$. So the value of the derivative at $x = \frac{\pi}{3}$ is:
$$4\left(\left(\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2\right) = 4\left(\frac{1}{4} - \frac{3}{4}\right) = -\frac{4}{2} = -2.$$
So the slope of the tangent line at $x = \frac{\pi}{3}$ is -2.
In order to find a point on the line, we will use the point on the graph of the original function at $x = \frac{\pi}{3}$. We already have the x-coordinate, so we only need to find the y-coordinate by plugging the x-value into the original function. This gives:
$y = 4\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right) = 4 \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2} = \sqrt{3}$
Now that we have the slope -2 and the point $\left(\frac{\pi}{3}, \sqrt{3}\right)$, we use the point-slope formula to get the equation of the tangent line.
$y - \sqrt{3} = -2\left(x - \frac{\pi}{3}\right)$

Note that this would be a sufficient answer, however, if simplified to $y = mx + b$ form, the equation would be:
$y = -2x + \frac{2\pi}{3} + \sqrt{3}$

Question 9 (a)

SOLUTION. The domain of a function is the set of all values where the function is defined. The function

$$f(x) = \frac{x+1}{x-1}$$

is defined everywhere except where its denominator is equal to zero. Setting the denominator equal to zero and solving...

$$x - 1 = 0$$

gives that the denominator is zero when $x = 1$. Thus $f(x)$ is defined everywhere except $x = 1$. In interval notation this is written $(-\infty, 1) \cup (1, \infty)$, or, equivalently

$$(-\infty, \infty) \backslash \{1\}$$

Question 9 (b)

SOLUTION. To find the horizontal asymptotes, we evaluate the following limit:
$\lim_{x\to\pm\infty} \frac{x+1}{x-1}$
To evaluate this limit, we need only consider the leading terms - the x in both the numerator and denominator. If we think of factoring out the x from top and bottom and then canceling, we get:
$\lim_{x\to\pm\infty} \frac{x(1+\frac{1}{x})}{x(1-\frac{1}{x})} = \lim_{x\to\pm\infty} \frac{1+\frac{1}{x}}{1-\frac{1}{x}}$
The two $\frac{1}{x}$ terms go to zero as x goes to plus or minus infinity, so the limit is as follows:
$\lim_{x\to\pm\infty} \frac{1+\frac{1}{x}}{1-\frac{1}{x}} = \frac{1}{1} = 1$
So $f(x)$ has a horizontal asymptote, given by the equation $y = 1$.
Now to find the vertical asymptotes, we need to find all values $x = a$ such that $\lim_{x\to a} f(x)$ diverges to $\pm\infty$. The only candidate for a vertical asymptote for this function is the point where the function is undefined, $x = 1$, found in part (a).
Indeed, the limit
$\lim_{x\to 1} \frac{x+1}{x-1}$
goes to infinity. How does this happen? First consider that when the denominator of a fraction is small, the entire fraction is large (see below for an example). In our limit, as the x-values get closer and closer to 1, the number in the denominator is getting smaller and smaller, closer and closer to zero. Using the previous fact about fractions, as the denominator becomes smaller, the entire fraction grows, meaning that the entire function is going to positive or negative infinity. So $f(x)$ has a vertical asymptote at $x = 1$.
Example: The denominator of the fraction
$\frac{1}{.01}$
is small, but when it is simplified as follows:
$\frac{1}{.01} = \frac{1}{\frac{1}{100}} = 100$
we can see that entire function is large - in fact it's a large integer, not even a fraction anymore.

Question 9 (c)

SOLUTION. The x-intercepts of $f(x)$ are all points of the form $(a, 0)$, that is, all points where $y = 0$. To find these x-values, we simply set the function $f(x) = 0$ and solve for x.

$$\begin{aligned} \frac{x+1}{x-1} &= 0 \\ x+1 &= 0 \\ x &= -1 \end{aligned}$$

So $f(x)$ has an x-intercept at $(-1, 0)$.
The y-intercept of $f(x)$ is the point of the form $(0, b)$, that is, where $x = 0$. To find the y-intercept therefore, we simply plug $x = 0$ into the function to find the y-value.
$f(0) = \frac{0+1}{0-1} = \frac{1}{-1} = -1$
So the y-intercept is $(0, -1)$.

Question 9 (d)

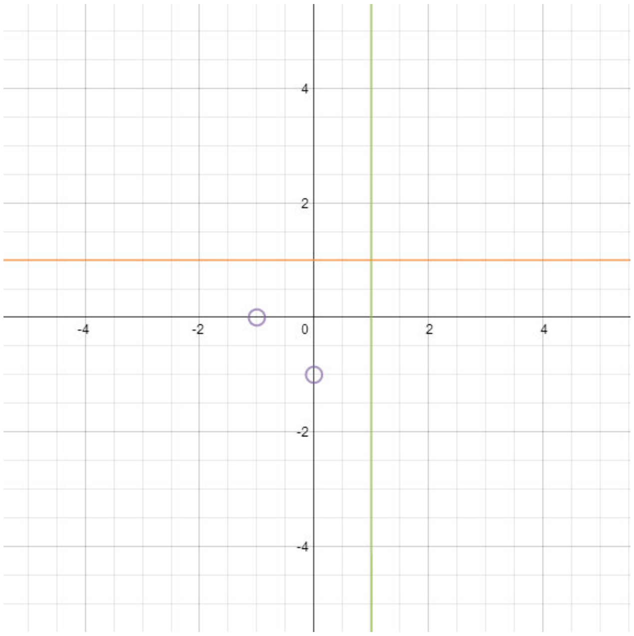
SOLUTION. We start by calculating the derivative of $f(x)$, using the quotient rule:
$f'(x) = \frac{(x-1)-(x+1)}{(x-1)^2} = \frac{-2}{(x-1)^2}$.
The critical points of the function are the x-values where the derivative is either undefined or equal to zero. The derivative here is undefined when the denominator is equal to zero. Setting the denominator equal to zero...
$(x-1)^2 = 0$
...and solving for x, we get that $f'(x)$ is undefined when $x = 1$.
To find the remaining critical points, we set the derivative equal to zero and try solving for x.
$\frac{-2}{(x-1)^2} = 0$
As it turns out, there are no x-values that will satisfy this equation. Thus our only critical point is at $x = 1$.

Question 9 (e)

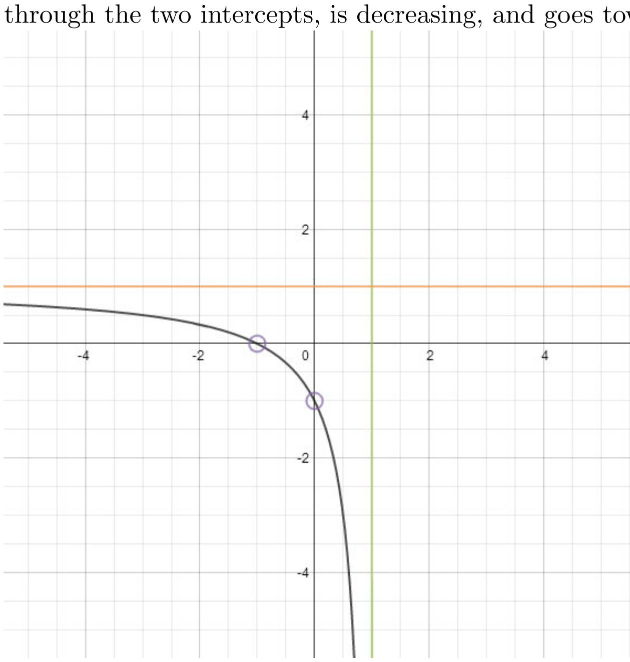
SOLUTION. We know that where a function is increasing or decreasing depends on the sign of its derivative. Therefore we consider the derivative we found in part (d).
$f'(x) = \frac{-2}{(x-1)^2}$
Note that the sign of the numerator is always negative because the numerator is a constant. Thus the only place the sign of the derivative can change is in the denominator. However, because the x-value in the denominator is squared, the denominator will always be positive. Since the numerator is always negative and the denominator always positive, the entire derivative will be negative for all x for which it is defined. This means that the function $f(x)$ is decreasing, on the interval $(\infty, 1) \cup (1, \infty)$.

Question 9 (f)

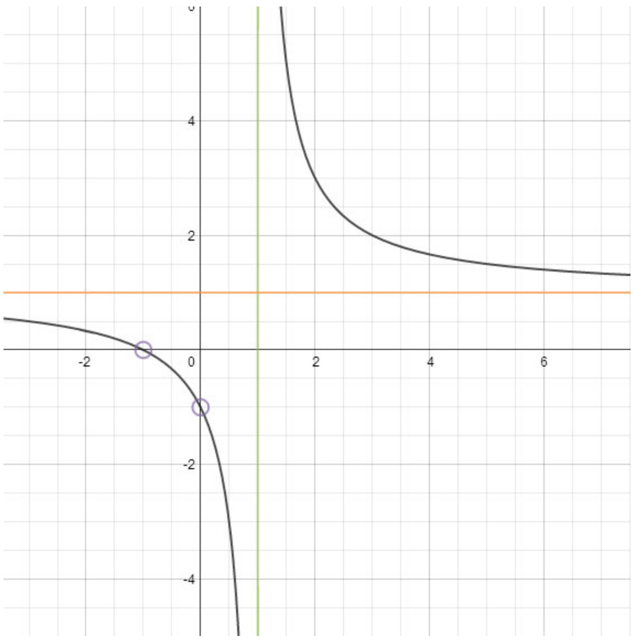
SOLUTION. First we draw the asymptotes and x- and y-intercepts. Using parts (b) and (c) of the question, we know there are two asymptotes: x = 1 and y = 1, and two intercepts: (0, -1) and (-1, 0).



From this picture, it is pretty clear to know where the left half of the function should go - it must pass through the two intercepts, is decreasing, and goes towards the horizontal and vertical asymptotes.



The right half of the function is a little trickier, but not by much. It must either go in the upper-right or lower-right quadrant formed by the two asymptotes. However, in the lower quadrant, the function would have to be increasing. Thus it must be in the upper right quadrant, decreasing from infinity and going towards 1.



Question 10

SOLUTION. As was noted in the hint, the curve has horizontal tangent lines at values where its derivative is equal to zero. Thus we first take the derivative of the curve, using the quotient rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{(\cos x - 1)' \cdot x - (\cos x - 1)\cdot (x)'}{x^2}\\ &= \frac{-\sin x \cdot x - (\cos x - 1)\cdot 1}{x^2}\\ &= \frac{-x\sin x - \cos x + 1}{x^2}\end{aligned}$$

We set this equal to zero and simplify to get:

$$\begin{aligned}\frac{-x\sin x - \cos x + 1}{x^2} &= 0\\ -x\sin x - \cos x + 1 &= 0\\ -x\sin x - \cos x &= -1\end{aligned}$$

So we are trying to find x-values that satisfy this final equation; if such an x-value exists, y has a horizontal tangent at that point.

From here, there is no specific method or surefire trick to find the answer - the only thing required is an understanding of the behavior of $\sin x$ and $\cos x$, especially considering their values as points on the unit circle.

The most useful observation here is that on the unit circle, when $\cos x$ is zero, $\sin x$ is ± 1 and vice versa. This is one step towards a solution, because if we can find a place where $\sin x = 0$ and $\cos x = 1$, we will have found a combination of $x \sin x$ and $\cos x$ that will equal -1, which is what we want.

In fact, when $x = 0$ we know that $\sin x = 0$ and $\cos x = 1$. On the right hand side of the equation above this gives -1, just like we want. Thus $x = 0$ is one solution to our equation; our curve will have a horizontal tangent line here.

Because $\sin x$ and $\cos x$ are periodic, we know that this solution will work anytime we are at the same angle on the unit circle, i.e. the angles $2\pi, 4\pi, 6\pi, \ldots$ and so on. So there are an infinite number of solutions,

$0, 2\pi, 4\pi, \ldots$ and so the curve has an infinite number of horizontal tangent lines.

Good Luck for your exams!

