Full Solutions
MATH221 April 2009

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How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

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Question 2

SOLUTION 1. Before computing the determinant of A, we could first use row operations to put it into an easier form. Recall that the following relationships hold between row operations and the determinant:

1. If we add or subtract multiples of one row to another, the determinant is unchanged
2. If we swap rows, the determinant changes by (-1)
3. If we multiply a row by a constant, the determinant also multiplies by that constant

The first rule is the most powerful since we can manipulate the matrix without changing the determinant. If we want to use co-factor expansion to compute the determinant then it will be useful if we can place a lot of zeros in either a row or a column. If we subtract the first row from the third and fourth columns this will put zeros in the first column of each row. If we subtract x multiples of row 1 from row 2 then this will place a zero in column 1 or row 2. Let's perform these operations,

$$\begin{bmatrix} 1 & 1 & x & 1 \\ x & 1 & 1 & 1 \\ 1 & 1 & 1 & x \\ 1 & x & 1 & 1 \end{bmatrix} \underset{R3-R1, R4-R1}{\longrightarrow} \begin{bmatrix} 1 & 1 & x & 1 \\ 0 & 1-x & 1-x^2 & 1-x \\ 0 & 0 & 1-x & x-1 \\ 0 & x-1 & 1-x & 0 \end{bmatrix}.$$

Since we used rule 1 above of row operations, we didn't change the determinant at all! If we were to co-factor expand now, the first column is a good choice because only the entry in the first row will contribute anything. Therefore,

$$\det(A) = (1)\det\left(\begin{bmatrix} 1-x & 1-x^2 & 1-x \\ 0 & 1-x & x-1 \\ x-1 & 1-x & 0 \end{bmatrix}\right).$$

We could co-factor expand this final matrix as is but we can make it even easier if we add the first row to the third row since the the first column of the third row will be zero and the co-factor expansion will be easier to do,

$$\begin{bmatrix} 1-x & 1-x^2 & 1-x \\ 0 & 1-x & x-1 \\ x-1 & 1-x & 0 \end{bmatrix} \underset{R3+R1}{\longrightarrow} \begin{bmatrix} 1-x & 1-x^2 & 1-x \\ 0 & 1-x & x-1 \\ 0 & 2-x-x^2 & 1-x \end{bmatrix}$$

and, again, if we co-factor expand along the first column, only the entry in the first row contributes. Therefore,

$$\det(A) = (1)(1-x)\det\left(\begin{bmatrix} 1-x & -(1-x) \\ (x+2)(1-x) & 1-x \end{bmatrix}\right)$$

where we have factored the quadratic term to easily see the factor of $1-x$. Since there is a factor of $1-x$ in each row we can remove that from the matrix and by rule 3 above, this will change the determinant by $(1-x)^2$. Therefore,

$$\det(A) = (1)(1-x)^3\det\left(\begin{bmatrix} 1 & -1 \\ (x+2) & 1 \end{bmatrix}\right) = (1-x)^3(x+3).$$

If we expand this out we get $(1-x)^3(x+3) = 3-8x+6x^2-x^4$ but this simplification is an optional step.

SOLUTION 2. As a brute-force alternative we can compute the determinant directly, using co-factor expansion. Expanding on the first row gives

$$\det(A) = (1)\det\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & x \\ x & 1 & 1 \end{bmatrix}\right) - (1)\det\left(\begin{bmatrix} x & 1 & 1 \\ 1 & 1 & x \\ 1 & 1 & 1 \end{bmatrix}\right)$$
$$+ (x)\det\left(\begin{bmatrix} x & 1 & 1 \\ 1 & 1 & x \\ 1 & x & 1 \end{bmatrix}\right) - (1)\det\left(\begin{bmatrix} x & 1 & 1 \\ 1 & 1 & 1 \\ 1 & x & 1 \end{bmatrix}\right)$$

Performing one row swap in the first, third and fourth terms (which changes the sign of the determinant) yields

$$\det(A) = -(1)\det\left(\begin{bmatrix} x & 1 & 1 \\ 1 & 1 & x \\ 1 & 1 & 1 \end{bmatrix}\right) - (1)\det\left(\begin{bmatrix} x & 1 & 1 \\ 1 & 1 & x \\ 1 & 1 & 1 \end{bmatrix}\right)$$
$$- (x)\det\left(\begin{bmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{bmatrix}\right) + (1)\det\left(\begin{bmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & 1 \end{bmatrix}\right)$$

Now the first two terms are the same so we combine them to get

$$\det(A) = -(2)\det\left(\begin{bmatrix} x & 1 & 1 \\ 1 & 1 & x \\ 1 & 1 & 1 \end{bmatrix}\right) - (x)\det\left(\begin{bmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{bmatrix}\right)$$
$$+ (1)\det\left(\begin{bmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & 1 \end{bmatrix}\right)$$

To compute the remaining 3×3 determinants we can use a variety of techniques. One such technique is to co-factor expand again. For example for the first matrix, if we expand along the first row,

$$\det\left(\begin{bmatrix} x & 1 & 1 \\ 1 & 1 & x \\ 1 & 1 & 1 \end{bmatrix}\right) = (x)(1-x) - (1)(1-x) + (1)(0) = -x^2 + 2x - 1.$$

Similarly for the other two matrices we have

$$\det\left(\begin{bmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{bmatrix}\right) = (x)(x^2-1) - (1)(x-1) + (1)(1-x) = x^3 - 3x + 2$$

and

$$\det\left(\begin{bmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) = x(x-1) - (1)(0) + (1)(1-x) = x^2 - 2x + 1$$

Combining everything, we have

$$\begin{aligned} \det(A) &= -2(-x^2 + 2x - 1) - (x)(x^3 - 3x + 2) + (x^2 - 2x + 1) \\ &= -x^4 + 6x^2 - 8x + 3 \end{aligned}$$

Question 4

SOLUTION. Recall that the orthogonal projection onto W are all of the vectors that are orthogonal to the normal of W. Therefore, the orthogonal projection onto W will place objects onto W itself. If we can find an orthonormal basis for this plane $\{v_1, v_2\}$ then the orthogonal projection onto W (the linear map T), will be given by

$$T(v) = \langle v, v_1\rangle v_1 + \langle v, v_2\rangle v_2.$$

for $v \in \mathbb{R}^3$ where $\langle\cdot\rangle$ is the standard dot product or inner product. We first note that in fact w_1 and w_2 are orthogonal (check!), and therefore $\{w_1, w_2\}$ is an orthogonal basis for W.

We can form the orthonormal basis by transforming $\{w_1, w_2\}$ into unit vectors by setting $v_1 = \frac{w_1}{\|w_1\|} = \frac{1}{5}\begin{pmatrix}3\\0\\4\end{pmatrix}$ and $v_2 = \frac{w_2}{\|w_2\|} = \frac{1}{5\sqrt{2}}\begin{pmatrix}4\\5\\-3\end{pmatrix}$.

Now the *standard matrix of* T is the matrix of T relative to the standard basis $e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$, i.e. it is the matrix generated by transforming each of the unit vectors where the first column of T is the transformation of e_1, the second column the transformation of e_2, and the third column the transformation of e_3.

Let us now determine these coefficients for each unit vector. For the first unit vector,

$T(e_1) = \langle e_1, v_1\rangle v_1 + \langle e_1, v_2\rangle v_2 = \frac{3}{5}v_1 + \frac{4}{5\sqrt{2}}v_2 = \frac{3}{25}\begin{pmatrix}3\\0\\4\end{pmatrix} + \frac{4}{50}\begin{pmatrix}4\\5\\-3\end{pmatrix} = \frac{17}{25}e_1 + \frac{2}{5}e_2 + \frac{6}{25}e_3,$

so the first column of the standard matrix of T is $\begin{bmatrix}17/25\\2/5\\6/25\end{bmatrix}$.

Continuing for the second unit vector,

$T(e_2) = \langle e_2, v_1\rangle v_1 + \langle e_2, v_2\rangle v_2 = \frac{1}{\sqrt{2}}v_2 = \frac{1}{10}\begin{pmatrix}4\\5\\-3\end{pmatrix} = \frac{2}{5}e_1 + \frac{1}{2}e_2 + (-\frac{3}{10})e_3,$

so the second column is $\begin{bmatrix}2/5\\1/2\\-3/10\end{bmatrix}$.

Finally for the third unit vector,

$T(e_3) = \langle e_3, v_1\rangle v_1 + \langle e_3, v_2\rangle v_2 = \frac{4}{5}v_1 - \frac{3}{5\sqrt{2}}v_2 = \frac{4}{25}\begin{pmatrix}3\\0\\4\end{pmatrix} - \frac{3}{50}\begin{pmatrix}4\\5\\-3\end{pmatrix} = \frac{6}{25}e_1 + (-\frac{3}{10})e_2 + \frac{41}{50}e_3,$

so the third column is $\begin{bmatrix}6/25\\-3/10\\41/50\end{bmatrix}$.

We therefore conclude that the standard matrix of T (denoted $[T]$) is

$$[T] = \begin{pmatrix}17/25 & 2/5 & 6/25\\ 2/5 & 1/2 & -3/10\\ 6/25 & -3/10 & 41/50\end{pmatrix}.$$

Is there anyway we can think to check our answer? The original vectors w_1 and w_2 already belong to W and therefore if we project them onto W then nothing should change. For example take w_1 and project

$$T(w_1) = \begin{pmatrix} 17/25 & 2/5 & 6/25 \\ 2/5 & 1/2 & -3/10 \\ 6/25 & -3/10 & 41/50 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$$

and indeed we have it projects onto itself! The same thing would happen if we projected w_2.

Question 12 (j)

SOLUTION. The answer is **true**. Suppose v is an eigenvector of A with corresponding eigenvalue λ. This means that
$Av = \lambda v$.
Then, multiplying by *2*, we find
$(2A)v = (2\lambda)v$.
Hence v is also an eigenvector of *2A* (corresponding to the eigenvalue 2λ).

Question 1

SOLUTION. From the system we are able to get the following matrix. $\begin{bmatrix} 1 & 0 & 4 & -2 & | & 1 \\ -1 & 1 & -7 & 7 & | & 2 \\ 2 & 3 & -1 & c & | & 11 \end{bmatrix}$ By doing Gaussian Elimination, we are able to get the matrix below $\begin{bmatrix} 1 & 0 & 4 & -2 & | & 1 \\ 0 & 1 & -3 & 5 & | & 3 \\ 0 & 0 & 0 & c-11 & | & 0 \end{bmatrix}$
If we look at the 3rd row of this matrix, we can easily see that $x_4 = 0$ for any real value of b, which is to say that for $b \in \mathbb{R}$, we have $x_4 = 0$.
Therefore, the matrix becomes
$\begin{bmatrix} 1 & 0 & 4 & | & 1 \\ 0 & 1 & -3 & | & 3 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$
Which can be solved easily. The general solution is

$$\begin{bmatrix} x1 \\ x2 \\ x3 \\ x4 \end{bmatrix}$$

$= \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -4 \\ 3 \\ 1 \\ 0 \end{bmatrix} * x_3$

Question 3

SOLUTION. Recall that the least squares solution to a linear problem $A\mathbf{x} = \mathbf{b}$ is the solution to the **normal form equation** problem:

$$A^T A\mathbf{x} = A^T\mathbf{b}.$$

Therefore, the first thing we need to do is setup the original linear system problem. From the table we are given t and $P(t)$ data and so we would like each to satisfy the linear equations,

$$
\begin{aligned}
a + 0b &= 5 \\
a + 2b &= 6 \\
a + 4b &= 8 \\
a + 6b &= 9.
\end{aligned}
$$

If we let

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 4 \\ 1 & 6 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 6 \\ 8 \\ 9 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}$$

then solving the 4 linear systems is equivalent to $A\mathbf{x} = \mathbf{b}$. Computing A^TA and $A^T\mathbf{b}$ we have $A^TA = \begin{bmatrix} 4 & 12 \\ 12 & 56 \end{bmatrix}, \quad A^T\mathbf{b} = \begin{bmatrix} 28 \\ 98 \end{bmatrix}.$

We can solve this problem by Gaussian elimination,

$$\left[\begin{array}{cc|c} 4 & 12 & 28 \\ 12 & 56 & 98 \end{array}\right].$$

Subtract 3 multiples of the first row from the second row to get

$$\left[\begin{array}{cc|c} 4 & 12 & 28 \\ 0 & 20 & 14 \end{array}\right].$$

Therefore we have that $b = 7/10$ and

$$4a + 12b = 4a + 42/5 = 28$$

so therefore $a = 49/10$. The line of best fit is given by the equation $P(t) = 49/10 + 7/10t$ and when $t = 7$, $P(t) = 49/10 + 49/10 = 49/5 = 9.8$. Therefore, the population at $t = 7$ will approximately by 9.8. Note that this is a reasonable answer since the population at $t = 6$ was 9.

Question 5

Solution. To find a basis for W, we take these 4 vectors and form a matrix A: A $= \begin{bmatrix} 1 & 3 & 0 & 1 \\ -3 & 1 & 0 & 2 \\ 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix}$

and take rref(A) to get the reduced row echelon form of A: rref(A) $= \begin{bmatrix} 1 & 0 & 0 & -0.5 \\ 0 & 1 & 0 & 0.5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Since only the first 2 columns are pivot, a basis for W is

$$\left\{ \begin{pmatrix} 1 \\ -3 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

To find a basis for the orthogonal complement of W, we need find the null space of the matrix B: B = $\begin{bmatrix} 1 & -3 & 1 & -2 \\ 3 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$

rref(B) = $\begin{bmatrix} 1 & 0 & 0.4 & -0.2 \\ 0 & 1 & -0.2 & 0.6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Therefore, a basis for the orthogonal complement of W is

$$\left\{ \begin{pmatrix} -0.4 \\ 0.2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.2 \\ -0.6 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Question 6

Solution. Set up Markov System

$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 7/10 & 6/10 \\ 3/10 & 4/10 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$

Find the eigenvalues, and get $\lambda_1 = 1/10$ $\lambda_2 = 1$

which yields the eigenvectors $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

The solution can now be written as $\begin{pmatrix} x_k \\ y_k \end{pmatrix} = c_1(0.1)^k \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2(1)^k \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

As $k \to \infty$ we have $\begin{pmatrix} x_\infty \\ y_\infty \end{pmatrix} = c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Using initial conditions to find c_1, c_2 $x_0 = 0, y_0 = 3$

At k = 0, $\begin{pmatrix} 0 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Solving for c_1 and c_2, we get

$c_1 = -2 \quad c_2 = 1$

and thus our steady state solution is $\begin{pmatrix} x_\infty \\ y_\infty \end{pmatrix} = (1) \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and our limiting values would be $= \binom{2}{1}$

Question 7 (a)

Solution. We first note that as T is a reflection mapping, it leaves the length of a vector unchanged. This is important as it allows us to determine the eigenvalues and eigenvectors without ever having to explicitly determine what T is.

Since T leaves the length of a vector unchanged, the only eigenvalues are *1* and *-1*.

The eigenvalue *1* corresponds to a vector which is invariant under the reflection; this vector must therefore lie on the line $x_1 + 3x_2 = 0$. Now the vector with coordinates $x_1 = 3, x_2 = -1$ lies on the line $x_1 + 3x_2 = 0$, so we conclude $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue *1*.

The eigenvalue *-1* corresponds to vectors which, under the reflection, has a *180* degree reversal in orientation. Such vectors necessarily must lie on the line *perpendicular* to the line $x_1 + 3x_2 = 0$. The equation for this perpendicular line is given by $3x_1 - x_2 = 0$, and the vector $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ lies on this perpendicular line. Therefore $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue *-1*.

Hence, we conclude $\left\{ \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^2$ consisting of eigenvectors of T.

Question 7 (b)

SOLUTION. From part (a), we know that *1* is an eigenvalue of A with eigenvector $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$, and *-1* is an eigenvalue of A with eigenvector $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

If we form the matrix $P = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}$ whose columns are the eigenvectors, with corresponding diagonal matrix $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, we know that the matrix A decomposes via the standard matrix diagonalization $A = P\Lambda P^{-1}$.

Upon computing $P^{-1} = \frac{1}{10}\begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$, we multiply the three matrices together to find A:

$A = \frac{1}{10}\begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 4 & -3 \\ -3 & -4 \end{pmatrix}$.

As a quick check of our work, we should verify that indeed

$A\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and $A\begin{pmatrix} 1 \\ 3 \end{pmatrix} = -\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

Question 8

SOLUTION. To find a formula for A^k, we first need to find matrices P and D such that $A = P{*}D{*}P^{-1}$, where D is a diagonal matrix, whose diagonal entries are eigenvalues of A, and their corresponding eigenvectors are the columns of the matrix P.

Finding Eigenvalues

$$(A - I\lambda) = \begin{bmatrix} -1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix} \Rightarrow (-1-\lambda)(4-\lambda) - (3*2) = \lambda^2 - 3\lambda - 10$$

$= (\lambda - 5)(\lambda + 2)) \Rightarrow \lambda_1 = 5, \lambda_2 = -2$

Finding Eigenvectors

$$A\vec{v} = \lambda\vec{v} \Rightarrow (A - I\lambda)\vec{v} = \vec{0}$$

$(A - I\lambda_1)\vec{v_1} = \begin{bmatrix} -6 & 2 \\ 3 & -1 \end{bmatrix}\vec{v_1} = \vec{0} \Rightarrow 3v_{11} = v_{21} \Rightarrow \vec{v_1} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$(A - I\lambda_2)\vec{v_2} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\vec{v_2} = \vec{0} \Rightarrow -v_{12} = 2v_{22} \Rightarrow \vec{v_2} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

So we have $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$, $P = \begin{bmatrix} \vec{v_1} & \vec{v_2} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$, and $P^{-1} = \frac{\begin{bmatrix} -1 & -2 \\ -3 & 1 \end{bmatrix}}{(-1)*1-2*3} = \begin{bmatrix} 1/7 & 2/7 \\ 3/7 & -1/7 \end{bmatrix}$

Since $P * P^{-1} = I, A^k = (P*D*P^{-1})^k = (P*D*P^{-1})*(P*D*P^{-1})*...*(P*D*P^{-1})*(P*D*P^{-1})$
$= P*D*(P^{-1}*P)*D*(P^{-1}*P)*D*...*(P^{-1}P)*D*P^{-1} = P*D^k*P^{-1}$.

Hence, $A^k = P*D^k*P^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 5^k & 0 \\ 0 & (-2)^k \end{bmatrix}\begin{bmatrix} 1/7 & 2/7 \\ 3/7 & -1/7 \end{bmatrix}$.

Question 9 (a)

SOLUTION. We first find rref(A),

R1-2R2 R3+R1 R1 divided by 3 R2-2R1

we get $V = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
where x2 = -x4 x3 = -x4
so, $\begin{pmatrix} x1 \\ x2 \\ x3 \\ x4 \end{pmatrix} = x4 \begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}$

thus the basis for $Nul(A) = Nul(V) = \begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}$

Question 9 (b)

SOLUTION. We first look back at our rref(A), $V = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ from this we can see that the pivot columns are the 2nd and 3rd columns of the matrix, corresponding to the columns in the original matrix A, we get $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and these are the basis for $Col(A)$

Question 9 (c)

SOLUTION. To find the coordinate vector of $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ relative to the basis of $Col(A)$, we place the basis for $Col(A)$ and the coordinate vector in an augmented matrix

$$\left.\begin{array}{cc|c} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & 2 \end{array}\right.$$

R1-R2 R2-R1 R3-R2 R2 divided by 3 R1+R2

$$\left.\begin{array}{cc|c} 1 & 0 & -1/3 \\ 0 & 1 & 2/3 \\ 0 & 0 & 0 \end{array}\right.$$

so we have x1 = -1/3 x2 = 2/3

Thus, the coordinate vecctor of $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ relative to basis of $Col(A)$ is $\binom{-1/3}{2/3}$

Question 9 (d)

SOLUTION. To find the the dimension of $Null(A^T)$, we can use the rank-nullity theorem $dim(N(A^T)) = n - r(A)$ where n is the number of rows of matrix A, and r(A) is the rank of matrix A.
so, we have $dim(N(A^T)) = 3 - 2 = 1$
the dimension of $Null(A^T)$ is therefore 1.

Question 10

SOLUTION. (i) find inverse of A
To see if inverse matrix exists, we need to verify if determinant is not zero.
Matrix is upper triangular, therefore we can simply take the product of the diagonal entries as the determinant of the matrix, so $det(A) = 1 * 1 * 1 = 1 \neq 0$

Therefore, inverse matrix exists.
We need $A^{-1}A = I$, so with

$$\left.\begin{array}{ccc|ccc} 1 & 3 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right.$$

$R1 + R3$ $R2 - 2R3$ $R1 - 3R2$
then we get,

$$\left.\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3 & 7 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right.$$

Therefore, inverse of A is $\begin{bmatrix} 1 & -3 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

(ii) Find inverse of AA^T we get $AA^T = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 1 & -1 \\ 1 & 5 & 2 \\ -1 & 2 & 1 \end{bmatrix}$

Again, We need $A^{-1}A = I$, so with

$$\left.\begin{array}{ccc|ccc} 11 & 1 & -1 & 1 & 0 & 0 \\ 1 & 5 & 2 & 0 & 1 & 0 \\ -1 & 2 & 1 & 0 & 0 & 1 \end{array}\right.$$

Swap 2nd and the 1st rows, eliminate 1st column R2-11R1 R3+R1 R2 divided by -54, eliminate 2nd column R3-7R2 R1-5R2 R3 divided by (1/54), eliminate 3rd column R1-(-7/54)R3 R2-(23/54)R3
yields

$$\left.\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3 & 7 \\ 0 & 1 & 0 & -3 & 10 & -23 \\ 0 & 0 & 1 & 7 & -23 & 54 \end{array}\right.$$

Thus the inverse of AA^T is $\begin{bmatrix} 1 & -3 & 7 \\ -3 & 10 & -23 \\ 7 & -23 & 54 \end{bmatrix}$

Question 11 (a)

SOLUTION. To find a nonzero vector v such that Av = 2v, let v = $[a, b, c]^T$, we have $\begin{matrix} Av = 2v \\ \begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2a \\ 2b \\ 2c \end{bmatrix} \end{matrix}$

Therefore, we can get $\begin{matrix} a + 2c = 2a \\ a - b + c = 2b \\ a = 2c \end{matrix}$ solving for the system, we get a = 2c, c = b, and so a = 2b.

Thus, the nonzero vector v = $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

Question 11 (b)

SOLUTION. To find the eigenvalues of matrix A = $\begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ we have

$(A - \lambda I) = \begin{bmatrix} 1 - \lambda & 0 & 2 \\ 1 & -1 - \lambda & 1 \\ 1 & 0 & -\lambda \end{bmatrix}$

Set the Determinant = 0, we have

$$\begin{aligned}(1-\lambda)((-\lambda)(-1-\lambda))+2(0-(-1-\lambda))&=0\\(1-\lambda)(\lambda+\lambda^2)+2+2\lambda&=0\\\lambda^3-3\lambda-2&=0\\(\lambda+1)^2(\lambda-2)&\\\text{Hence,}\lambda_1=\lambda_2=-1,\lambda_3=2&\end{aligned}$$

Question 11 (c)

SOLUTION. To Find a matrix P such that $P^{-1}AP$ is diagonal, we need to find the matrix P formed by the eigenvectors of A. Using Matlab commands to find the eigenvectors, we write [V,L]=eig(A). Then we have

$$V = \begin{bmatrix} 0 & 0.8165 & -0.7071 \\ 1 & 0.4082 & 0 \\ 0 & 0.4082 & 0.7071 \end{bmatrix}$$

Since V is formed by the eigenvectors of A, the matrix P = V.

Question 12 (a)

SOLUTION. The answer is **true**.
The Cayley Hamilton theorem tells us that the eigenvalues are roots of $q(\lambda) = \lambda^2 + 1 = 0$. Thus, exactly one of $\lambda + i$ or $\lambda - i$ contains an odd number of factors in the matrix $p(\lambda)\det(A - \lambda I)$. This means the constant term of $p(\lambda)$ (which is the product of the roots) must be strictly non-real, a contradiction since the matrix was assumed to be real.

Question 12 (b)

SOLUTION. This is **false**. To construct a counterexample, we need to think of an invertible matrix which cannot be diagonalized, i.e. an invertible matrix which has only one linearly independent eigenvector. The first invertible matrix which comes to mind is the identity matrix,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

however, this is not a counterexample as it is diagonal. However, consider the simple modification

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

A is invertible as its determinant is *1*, however, A is **not** diagonalizable as the only linearly independent solution to the eigenvector equation

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and so A has only one linearly independent eigenvector corresponding to its eigenvalue *1*.

Therefore, we conclude $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is a counterexample to the statement.

Question 12 (c)

SOLUTION. The answer is **true**.
We have $A = -A^T$ and thus, since determinants are transpose invariant, we have $\det(A) = -\det(A)$ or equivalently that $2\det(A) = 0$ and hence that $\det(A) = 0$.

Question 12 (d)

SOLUTION. The answer is **true**. Let $\mathbf{0}$ denote the zero vector in $\mathbb{R}^3$. Then we can write $\mathbf{0} = 0 \cdot v + 0 \cdot w$;

that is, $\mathbf{0}$ is a linear combination of v and w.

Question 12 (e)

SOLUTION. The answer is **false**.
The motivation for why it should be false is as follows: Let v_1, v_2 be two eigenvectors of A with respective eigenvalues λ_1, λ_2. Then $A(v_1 + v_2) = Av_1 + Av_2 = \lambda_1 v_1 + \lambda_2 v_2$. So unless $\lambda_1 = \lambda_2$, $v_1 + v_2$ will **not** be an eigenvector.
The above observation will guide us in our construction of a counterexample. Consider
$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

The eigenvalues of A are *1* and *2* with corresponding eigenvectors $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
However, as discussed above, $A(v_1 + v_2) = v_1 + 2v_2$. So $v_1 + v_2$ is not an eigenvector of A.

Question 12 (f)

SOLUTION. The answer is **true**. Suppose that
$a_1 v_1 + a_2 v_2 + a_3 v_3 = 0$,
where $a_1, a_2, a_3 \in \mathbb{R}$. We want to show that $a_1 = a_2 = a_3 = 0$.
Applying T to the above equation, we find
$T(a_1 v_1 + a_2 v_2 + a_3 v_3) = a_1 T(v_1) + a_2 T(v_2) + a_3 T(v_3) = 0$.
Since we are given that $T(v_1), T(v_2)$, and $T(v_3)$ are linearly independent, the above equality forces $a_1 = a_2 = a_3 = 0$, which is what we wanted to show.
Therefore, v_1, v_2, v_3 are linearly independent.

Question 12 (g)

SOLUTION. The answer is **true**.
If A is not invertible, then A has a non-trivial nullspace, i.e. there is a non-zero vector v such that $Av = 0$. This precisely means that the columns of A are linearly dependent.

Question 12 (h)

SOLUTION. The answer is **false**. Consider the matrix
$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.
A has two distinct eigenvalues, *0* and *1*. However, clearly A has only one linearly independent column, so $\text{Rank}(A) = 1 < 2 = \#$ of distinct eigenvalues.

Question 12 (i)

SOLUTION. The answer is **false**.
Let $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, and $v_3 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ denote the given vectors.
A matrix A has eigenvectors v_1, v_2, v_3 with corresponding respective eigenvalues *1, -1, 4* if and only if $A = P\Lambda P^{-1}$, where
$P = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 3 \end{pmatrix}$ and $\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

Thus the matrix A exists if and only if P is invertible (that is, if and only if v_1, v_2, v_3 are linearly independent). We can calculate $\det P = 1 \neq 0$. Therefore, P is invertible and the desired matrix A will be given by $A = P\Lambda P^{-1}$.

Remark: By computing the inverse (not necessary to answer the question), we can determine the explicit form of A as

$$A = \begin{pmatrix} -9 & 2 & 8 \\ -3 & 1 & 3 \\ -13 & 2 & 12 \end{pmatrix}.$$

Good Luck for your exams!