Full Solutions
MATH110 December 2011

December 4, 2014

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Educational Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Exam Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Educational Resources.

Question 1 (a)

SOLUTION. When considering limits at infinity of rational functions, only the dominant term matters:

$$\begin{aligned}\lim_{x\to-\infty}\frac{5x^3-2x^2+1}{6x^3+7} &= \lim_{x\to-\infty}\frac{x^3\left(5-\frac{2}{x}+\frac{1}{x^3}\right)}{x^3\left(6+\frac{7}{x^3}\right)}\\ &= \lim_{x\to-\infty}\frac{5-\frac{2}{x}+\frac{1}{x^3}}{6+\frac{7}{x^3}}\\ &= \frac{5}{6}\end{aligned}$$

Here we factored out the leading term (the x^3) on both the top and bottom. That allowed us to cancel it and then evaluate the rest of the limit. The $1/x^n$ terms go to zero as $x \to \infty$.

Question 1 (b)

SOLUTION. If we simply try to plug in the value $x = 0$, we obtain infinity minus infinity, which is undefined (you can't say it is zero, it really depends on the "size" of each infinity in some sense). So we first do a little algebra to rewrite the two fractions as one, using a common denominator:

$$\frac{1}{x(1-2x)}-\frac{1}{x}=\frac{1}{x(1-2x)}-\frac{1-2x}{x(1-2x)}=\frac{1-(1-2x)}{x(1-2x)}=\frac{2x}{x(1-2x)}$$

After canceling the x we obtain something we can compute:

$$\lim_{x\to0}\left(\frac{1}{x(1-2x)}-\frac{1}{x}\right)=\lim_{x\to0}\frac{2}{1-2x}=2$$

Question 1 (c)

SOLUTION. We will consider the function inside the limit as the sum of two functions

$$e^{-x}$$

and $-x$. If we take their limits separately, we get:
$\lim_{x\to\infty} e^{-x} = 0$
$\lim_{x\to\infty} -x$ diverges to $-\infty$
We can see that as x goes to infinity, the e^{-x} term is decreasing towards zero and the $-x$ term grows larger and larger. So, as x goes to infinity the combined value of $e^{-x} - x$ will grow more and more negative. Thus we have:
$\lim_{x\to\infty} e^{-x} - x$ diverges to $-\infty$

Question 2 (a)

SOLUTION. A function f is continuous as a point a if:

1. $f(a)$ is defined

1. $\lim_{x\to a} f(x)$ exists, i.e. $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$ where L is finite.

1. $\lim_{x \to a} = f(a)$

The first and third conditions ensure that the drawing of f has no holes; the second and third conditions ensure that the drawing of f has no jumps.

Question 2 (b)

SOLUTION. The function $g(x)$ is undefined wherever the denominator is zero. Thus we need to know where $\sin(x) = 0$.
Based on the unit circle, we know that $\sin(x)$ is zero at the angles 0, π, 2π, 3π, ... Thus, these points can not be included in the domain of g.
So our domain is:
$\{x : x \neq n\pi\}$ where n is an integer.

Question 2 (c)

SOLUTION. In order for a function to be continuous on the interval $(-\infty, \infty)$ it must be defined everywhere. However, in the previous part of the question, we just showed that g is not defined at an infinite number of points. Thus g is not continuous on $(-\infty, \infty)$.

Question 3

SOLUTION. First, we are of course interested in the values of the function at x = -4 and x = 4. We have:

$$f(-4) = (-4)^3 - 15(-4) + 1 = -64 + 60 + 1 = -3$$
$$f(4) = 4^3 - 15 \cdot 4 + 1 = 64 - 60 + 1 = 5$$

So, using the Intermediate Value Theorem (since this polynomial is clearly continuous everywhere) we can guarantee the existence of one root. But not three.
To show there are three roots, we need to show that this cubic polynomial crosses the x-axis again. If we just find the local extrema (if there are three roots, there should be one local maximum followed by one local minimum) and show that they are in our interval, we could refine our use of the Intermdeiate Value Theorem. Let's find out where these extrema are. First, we compute the derivative of the function f:

$$f'(x) = 3x^2 - 15$$

And then find out the critical points:

$$f'(x) = 0 \iff 3x^2 - 15 = 0 \iff x^2 = 5 \iff x = \pm\sqrt{5}$$

We actually only need to know what is the value of the function f at these two points:

$$f(-\sqrt{5}) = -5\sqrt{5} + 15\sqrt{5} + 1 = 5\sqrt{5} + 1 > 0$$
$$f(\sqrt{5}) = 5\sqrt{5} - 15\sqrt{5} + 1 = -10\sqrt{5} + 1 < 0$$

(If you are not sure how we know that the second value is negative, consider that $>$ 2 so -10 $<$ -20).
So we can now apply the Intermediate Value Theorem three times, on the intervals [-4, -], [-,] and [, 4]] to show that there must be a root in each of these intervals.
Note that we need not use these intervals precisely. As long as you can identify three intervals similar to the ones above, i.e.

1. The three intervals are contained in [-4,4].
2. The three intervals don't overlap.
3. If for one endpoint of an interval, $f > 0$ then for the other endpoint, $f < 0$.

Then you can use the intermediate value theorem in the same way as described above to show there is a root in each interval.

Question 4 (a)

SOLUTION. We simply input the function in the definition and compute the limit:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{(x+h)+1} - \frac{1}{x+1}}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{(x+h+1)} - \frac{1}{(x+1)}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{x+1}{(x+h+1)(x+1)} - \frac{x+h+1}{(x+h+1)(x+1)}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\cdot\frac{(x+1)-(x+h+1)}{(x+h+1)(x+1)} \\
&= \lim_{h\to 0} \frac{1}{h}\cdot\frac{-h}{(x+h+1)(x+1)} \\
&= \lim_{h\to 0} -\frac{1}{(x+h+1)(x+1)} \\
&= -\frac{1}{(x+1)(x+1)} \\
&= -\frac{1}{(x+1)^2}
\end{aligned}$$

Question 4 (b)

SOLUTION 1. Using the chain rule is much faster. Rewrite the function as

$$f(x) = (x+1)^{-1}$$

Then its derivative simply is

$$f'(x) = (-1)(x+1)^{-2}\cdot 1 = -\frac{1}{(x+1)^2}$$

(The multiplication by 1 represents the derivative of $x + 1$.)

SOLUTION 2. Using the quotient rule works but is usually more messy. If you can avoid using this rule and use the chain rule as demonstrated in the first solution, do it. Using the quotient rule, the solution looks like this:

$$f'(x) = \frac{0\cdot(x+1) - 1\cdot 1}{(x+1)^2} = -\frac{1}{(x+1)^2}$$

Question 5 (a)

Solution. Since this is a quotient, we will be using the quotient rule. For this, we need to have the derivative of both the denominator and the numerator.
The numerator is simply the cosine function and we know that:

$$(\cos(x))' = -\sin(x)$$

For the denominator, we will need to use the chain rule and obtain:

$$\left(\sin^2(x)\right)' = 2\sin(x)\cos(x)$$

So now we can use the quotient rule and obtain:

$$\begin{aligned}
f'(x) &= \frac{(\cos(x))'(\sin^2(x)) - (\cos(x))(\sin^2(x))'}{(\sin^2(x))^2} \\
&= \frac{(-\sin(x))(\sin^2(x)) - (\cos(x))(2\sin(x)\cos(x))}{\sin^4(x)} \\
&= \frac{-\sin^3(x) - 2\sin(x)\cos^2(x)}{\sin^4(x)} \\
&= \frac{-\sin^2(x) - 2\cos^2(x)}{\sin^3(x)} \\
&= -\frac{\sin^2(x) + 2\cos^2(x)}{\sin^3(x)} \\
&= -\frac{1 + \cos^2(x)}{\sin^3(x)}
\end{aligned}$$

Where the last equality was obtained using that $\sin^2(x) + \cos^2(x) = 1$.

Question 5 (b)

Solution. Here we will be using the chain rule:

$$(p(q(x)))' = p'(q(x)) \cdot q'(x)$$

Here we have that

$$p(x) = e^x \quad q(x) = \sqrt{x}$$

And so

$$\begin{aligned}
g'(x) &= p'(q(x)) \cdot q'(x) \\
&= e^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\right) \\
&= \frac{e^{\sqrt{x}}}{2\sqrt{x}}
\end{aligned}$$

Question 5 (c)

SOLUTION. Here we have several compositions going on. So a first step, would be to compute the derivative of

$$\ln(x^2 + 2x)$$

We do this using the chain rule:

$$(p(q(x)))' = p'(q(x)) \cdot q'(x)$$

Here we have that

$$p(x) = \ln(x) \qquad q(x) = x^2 + 2x$$

and

$$p'(x) = \frac{1}{x} \qquad q'(x) = 2x + 2$$

And so:

$$(\ln(x^2 + 2x))' = \frac{1}{x^2 + 2x}(2x + 2) = \frac{2x + 2}{x^2 + 2x}$$

And so now, we can use the chain rule again to compute the derivative of

$$\ln(\ln(x^2 + 2x))$$

Using:

$$(p(q(x)))' = p'(q(x)) \cdot q'(x)$$

Where

$$p(x) = \ln(x) \qquad q(x) = \ln(x^2 + 2x)$$

and we have

$$p'(x) = \frac{1}{x} \qquad q'(x) = \frac{2x + 2}{x^2 + 2x}$$

And so we obtain

$$h'(x) = \frac{1}{\ln(x^2 + 2x)} \cdot \frac{2x + 2}{x^2 + 2x}$$

Question 6

SOLUTION. In order for $f(x)$ to be differentiable at a, we have two requirements: that the function be continous and derivatives from the right and left of a point are the same.
Before we start, let us note that f is surely, as a composition of differentiable functions, differentiable at all points expect $x = a$. Hence we can focus our study on the point $x = a$.
We begin with the second condition: that the derivative from the right $(x < a)$ must be the same as the derivative from the left $(x > a)$. This means that we are trying to find the value a such that

$$-\frac{1}{2}\sin a = 2a$$

Simplifying a bit, this is the same as

$$-\sin a = 4a$$

If we think about the function $-\sin x$ we know that it is bounded between -1 and 1. If a was greater than 1/4, say, 1/2, the left side would be less than one, while the right side would already be greater than 2. So a must be between -1/4 and 1/4 for this equation to hold. In fact, the only place that the graphs of $-\sin x$ and $4x$ intersect is at $x = 0$. So $a = 0$.
This satisfies the condition that the derivatives match; now we must ensure that $f(x)$ is continuous. The function is defined everywhere, so we must check that the limit at $x = a$ exists.
The limit from the left of $x = 0$ is

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{2}\cos x = \frac{1}{2}$$

The limit from the right of $x = 0$ is

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 + b = b$$

These two limits must be equal, so we have $b = 1/2$.

Question 7 (a)

SOLUTION. As mentioned in the hint, a point is on a curve if its coordinates satisfy the curve equation. So we simply compute:

$$f(4) = 4^2 - 6\cdot 4 = 16 - 24 = -8$$

Since -8 is not -12, we can conclude that the point P is not on this curve.

Question 7 (b)

SOLUTION. We have two points:

$$(4, -12) \qquad (a, a^2 - 6a)$$

So the slope, m, of the line going through these two points is the ratio of rise (change in the y-coordinates) over run (change in the x-coordinates):

$$m = \frac{(-12) - (a^2 - 6a)}{4 - a} = -\frac{a^2 - 6a + 12}{4 - a}$$

Question 7 (c)

SOLUTION. First we take the derivative of the function.

$$f'(x) = 2x - 6$$

To find the slope of the tangent line at the point (a, a^2 - 6a), we simply need to plug the value $x = a$ into the derivative. This gives:

$$f'(a) = 2a - 6$$

So the slope of the tangent line is *2a - 6*.

Question 7 (d)

SOLUTION. This question is asking which lines through P are also tangent lines to the curve. In order for a line through P to be a tangent line, it must pass through a point on the curve. We found the slope of such a line in part (b). Furthermore, the slope between that point and P must be equal to the value of the derivative at that point. which is what we found in part (c).
We know that the slope of a line between P *(4, -12)* and a point (a, a^2 - 6a) on the curve is:

$$-\frac{a^2 - 6a + 12}{4 - a}$$

The slope of the tangent line at the same point on the curve, (a, a^2 - 6a) is:

$$2a - 6$$

We set these two slopes equal to each other and solve for a.

$$\begin{aligned}
-\frac{a^2 - 6a + 12}{4 - a} &= 2a - 6 \\
-(a^2 - 6a + 12) &= (2a - 6)(4 - a) \\
-a^2 + 6a - 12) &= -2a^2 + 14a - 24 \\
a^2 - 8a + 12) &= 0 \\
(a - 6)(a - 2) &= 0
\end{aligned}$$

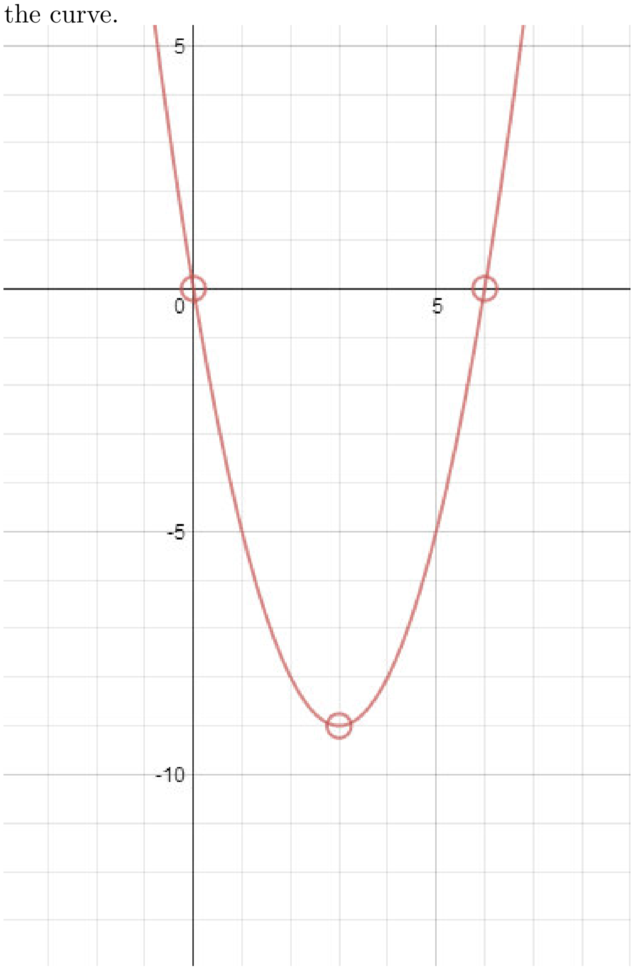
So a = *2* and a = *6*. Plugging these back into our original slopes (either one will work, since we have found the values where the two are equal) we get the two slopes of 6 and -2.

Question 7 (e)

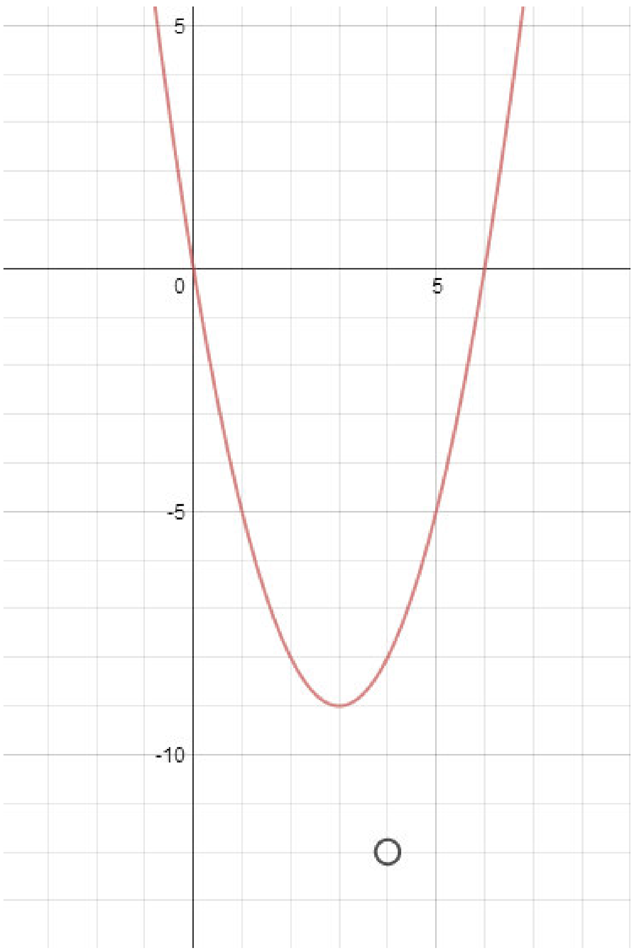
SOLUTION.

- $f(x) = x^2 - 6x$

As suggested in the hint, first find the roots of the function.
$x^2 - 6x = 0$
$x(x-6) = 0$
$f(x)$ has roots at $x = 0$ and $x = 6$. Using this fact, or by calculating where the derivative of f is zero, we can deduce that the minimum of $f(x)$ is at $x = 3$. To find the y-value at this point we simply plug in $x = 3$ to get $y = (3)^2 - 6(3) = -9$. Plotting the roots and the minimum on the graph should allow you to sketch the curve.

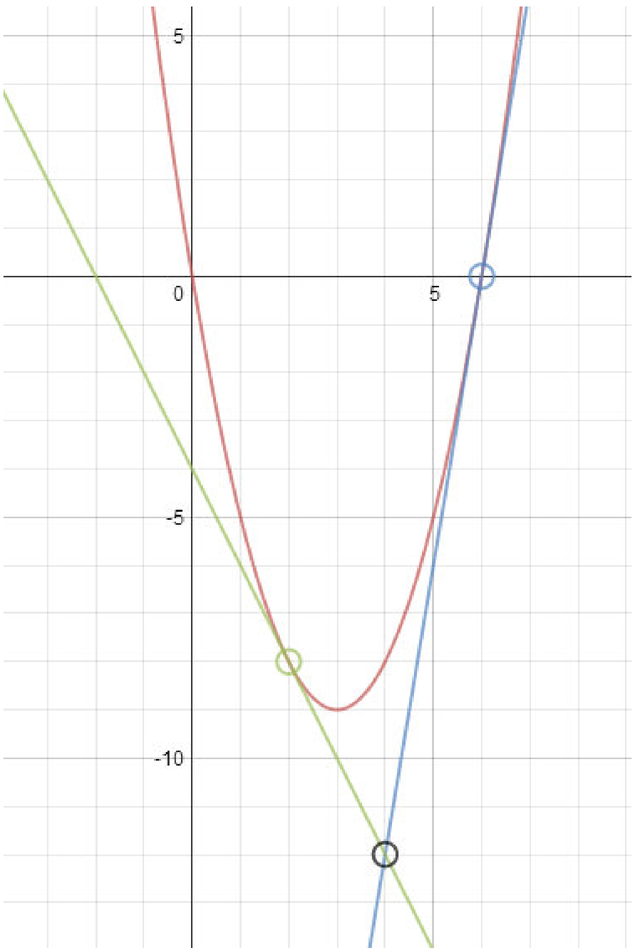


- $P = (4, -12)$



- Tangent lines that pass through P

We know there are two lines. Both pass through the point $(4, -12)$ and their two slopes are -2 and 6 (from part c). The equation of the two lines are thus
$y + 12 = -2(x - 4)$
$y + 12 = 6(x - 4)$
Or in $y = mx + b$ form
$y + 12 = -2x - 4$
$y = 6x - 36$
Alternatively, you know that the line with slope -2 passes through P and the point *(2, f(2)) = (2, -8)*, while the line with slope 6 passes through P and the point *(6, f(6)) = (6, 0)*. Drawing these points and then connecting them with lines will also produce the correct picture.



Question 8 (a)

SOLUTION. If a is different than zero, then the given curve is a polynomial of degree 7. Let

$$f(x) = (ax + b)^7$$

then its derivative (using the chain rule) is

$$f'(x) = 7(ax + b)^6 \cdot a = 7a(ax + b)^6$$

and so the slope of the tangent line to this curve at the point $x = b/a$ is

$$f'(b/a) = 7a(a\left(\frac{b}{a}\right) + b)^6 = 7a(2b)^6 = 2^6 \cdot 7ab^6$$

And the y coordinate of that point is $f(b/a)$ which is

$$f(b/a) = (a\left(\frac{b}{a}\right) + b)^7 = (2b)^7 = 2^7b^7$$

So using the standard equation

$$y = m(x - x_0) + y_0$$

for a line of slope m and going through the point $(x_0,\, y_0)$ we obtain

$$y = 2^6 \cdot 7ab^6(x - \frac{b}{a}) + 2^7 b^7$$

which you can simplify a bit and obtain

$$y = 2^6 \cdot 7ab^6 x - 2^6 \cdot 5b^7$$

or if you compute things out

$$y = 448ab^6 x - 320b^7$$

You don't need to do the last two steps during a test usually.

Question 8 (b)

SOLUTION. If $a = 0$, then the curve we are looking at is the curve

$$y = (0x + b)^7 = b^7$$

which is simply a horizontal line at the height b^7 and so its tangent line is itself, that is, the line $y = b^7$.

Question 9

SOLUTION. The best way to approach this function is as a piecewise function. The statement of the problem suggests that there will need to be at least two pieces to the final function: one for $x > 0$ and one for $x < 0$. We will start by considering the $x > 0$ piece.

We know for $x > 0$ that $f'(x) = -2$. The only function that has -2 as its derivative is the line with slope -2. So we know that this piece of our function will look something like $y = -2x + b$. Looking at the second requirement of the function, we know that the limit of this function as it approaches 0 must be -1. Thus our line should intersect the y-axis at -1 and one piece of our function will be y = *-2x - 1*.

Now we consider the piece for $x < 0$. We know that the limit approaching x = *0* must be 2 and that the limit to negative infinity must be 0. There are many functions that satisfy the first condition; an easy example is y = *2*. There are also lots of functions satisfying the second condition; two examples are e^x and $\frac{1}{x}$. There are multiple ways to continue. We could break this piece of the function into two more pieces, using the examples I've cited above to get the following piecewise:

$$f(x) = \begin{cases} e^x & x \leq -1 \\ 2 & -1 < x \leq 0 \\ -2x - 1 & x > 0 \end{cases}$$

A more complex solution would be to take a function that with the appropriate limit as x goes to negative infinity, like $y = -\frac{1}{x}$ and shift it so that its y-intercept is 2. This would produce a piecewise like this:

$$f(x) = \begin{cases} -\frac{1}{x - 1/2} & x \leq 0 \\ -2x - 1 & x > 0 \end{cases}$$

Question 10 (a)

Solution. We need to make sure to use both the product rule since we have $2x^2$ multiplied by $sin(1/x)$ and then use the chain rule to deal with $sin(1/x)$.
We obtain:

$$\begin{aligned} f'(x) &= 1 + 4x \cdot \sin(1/x) + 2x^2 \cdot \cos(1/x) \cdot (-x^{-2}) \\ &= 1 + 4x \sin(1/x) - 2\cos(1/x) \end{aligned}$$

Question 10 (b)

Solution. We are trying to find points c in an interval *(-d, d)* such that $f'(c) = -1$. To begin, we set the derivative (from part (a)) equal to -1 and isolate the terms with the variable x.
$-1 = 1 + 4x\sin(1/x) - 2\cos(1/x)$
$-2 = 4x\sin(1/x) - 2\cos(1/x)$
$-1 = 2x\sin(1/x) - \cos(1/x)$
The question is asking us to find an infinite number of points that satisfy this equation. We now use the information from the hint. If we can find a value of x such that $\sin(1/x) = 0$, and for the same value of x, $\cos(1/x) = 1$, we will have found a solution.
We already know that $\sin(x) = 0$ when
$x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$.
So in order for $\sin(1/x) = 0$,
$x = \pm\frac{1}{\pi}, \pm\frac{1}{2\pi}, \pm\frac{1}{3\pi}, \ldots$.
(Note that x = 0 cannot be a solution because it makes the function undefined.)
Thus we have a possible solution of the form
$x = \frac{1}{n\pi}$
where n is a non-zero integer. For this list of x-values, $\sin(1/x) = 0$. However, we also need $\cos(1/x) = 1$ in order to satisfy our equation. If we plug our potential solution into the $\cos(1/x)$ function, we get:
$\cos\left(\frac{1}{1/(n\pi)}\right) = \cos(n\pi)$
This expression is only equal to 1 if n is even. Thus we must modify our list of solutions to the following:
$x = \frac{1}{m\pi}$
Where m is an even integer (other than zero).
Because there are an infinite number of even integers, we have found an infinite number of solutions c such that $f'(c) = -1$
The last part of the question asks us to show that, given an arbitrary interval *(-d, d)* for $d > 0$, an infinite number of our list of solutions will fall inside that interval. The key to this section of this problem is noticing that our list of solutions is a fraction, and as the denominator of the fraction can be arbitrarily large (some integer times pi), it can be arbitrarily close to zero. Thus, no matter which d is chosen to be the endpoint of our interval, there will be some even integer, say M, such that
$-d < -\frac{1}{M\pi} < 0 < \frac{1}{M\pi} < d$
And for all even m larger than M, the value of $\frac{1}{m\pi}$ will also fall between -d and d, meaning we have an infinite number of solutions inside the interval (-d, d).

Question 11

Solution. We compute the first few derivatives of $f(x) = x\ln(x)$ and look for a pattern.

$$\begin{aligned} f'(x) &= \ln(x) + 1 \\ f''(x) &= 1/x \\ f'''(x) &= -1/x^2 \\ f^{(4)}(x) &= 2/x^3 \\ f^{(5)}(x) &= -6/x^3 \end{aligned}$$

After the second derivative, we notice that the derivative is a fraction, with a constant in the numerator and a power of x in the denominator. In fact, we can write this pattern as

$$f^{(n)}(x) = \frac{(n-2)!(-1)^n}{x^{n-1}}$$

for all $n \geq 2$.
The only derivative that is different is the first one. Thus, our piecewise function describing the derivative is:
: f^{(n)}(x) = \begin{cases} \ln(x) & n = 1 \\ \frac{ (n-2)! (-1)^n }{x^{n-1}} & n > 1 \end{cases}

Good Luck for your exams!

