Full Solutions
MATH101 April 2006

April 4, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

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Question 1 (a)

SOLUTION. Applying the formula gives
$f_{av} = \frac{1}{\pi - 0}\int_0^{\pi} \cos x \, dx = \frac{1}{\pi}\sin(x)|_0^{\pi} = \frac{\sin(\pi)-\sin(0)}{\pi} = 0$

Question 1 (b)

SOLUTION. First we use the double angle formula
$\cos^2(x) = \frac{1+\cos(2x)}{2}$
to see that

$$\begin{aligned}\int \cos^2(x)\,dx &= \int \frac{1+\cos(2x)}{2}\,dx \\ &= \frac{x}{2} + \frac{1}{2}\int \cos(2x)\,dx \\ &= \frac{x}{2} + \frac{1}{4}\sin(2x) + C\end{aligned}$$

completing the question.

Question 1 (c)

SOLUTION 1. $\int \left(\frac{1}{\sqrt{1-x^2}} + 1 - x^2\right)\,dx = \arcsin(x) + x - \frac{x^3}{3} + C$

SOLUTION 2. We still have
$\int \left(\frac{1}{\sqrt{1-x^2}} + 1 - x^2\right)\,dx = \int \left(\frac{1}{\sqrt{1-x^2}}\right)\,dx + x - \frac{x^3}{3}$
To evaluate the last integral, let $x = \sin\theta$ so that $dx = \cos\theta d\theta$ and thus, we have

$$\begin{aligned}\int \left(\frac{1}{\sqrt{1-x^2}}\right)\,dx &= \int \left(\frac{\cos\theta}{\sqrt{1-\sin^2\theta}}\right)\,d\theta \\ &= \int \left(\frac{\cos\theta}{\cos\theta}\right)\,d\theta \\ &= \int d\theta \\ &= \theta + C \\ &= \arcsin(x) + C\end{aligned}$$

and so
$\int \left(\frac{1}{\sqrt{1-x^2}} + 1 - x^2\right)\,dx = \arcsin(x) + x - \frac{x^3}{3} + C$

Question 1 (d)

SOLUTION. Notice that in our problem, we have
$\Delta x = \frac{8-2}{6} = 1 \qquad x_i = 2 + i$
and so we have

$$
\begin{aligned}
S_6 &= \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6)\right) \\
S_6 &= \frac{1}{3}\left(f(2) + 4f(3) + 2f(4) + 4f(5) + 2f(6) + 4f(7) + f(8)\right) \\
S_6 &= \frac{1}{3}\left(\frac{1}{4} + \frac{4}{9} + \frac{2}{16} + \frac{4}{25} + \frac{2}{36} + \frac{4}{49} + \frac{1}{64}\right) \\
S_6 &= \frac{1}{12} + \frac{4}{27} + \frac{1}{24} + \frac{4}{75} + \frac{1}{54} + \frac{4}{147} + \frac{1}{192}
\end{aligned}
$$

as required.

Question 1 (e)

SOLUTION. Applying the formula gives

$$
\begin{aligned}
V &= \int_a^b \pi f(x)^2\,dx \\
&= \pi \int_1^2 (x + x^2)^2\,dx \\
&= \pi \int_1^2 (x^2 + 2x^3 + x^4)\,dx \\
&= \pi \left(\frac{x^3}{3} + \frac{x^4}{2} + \frac{x^5}{5}\right)\Bigg|_1^2 \\
&= \pi \left(\frac{8}{3} + \frac{16}{2} + \frac{32}{5} - \frac{1}{3} - \frac{1}{2} - \frac{1}{5}\right) \\
&= \pi \left(\frac{7}{3} + \frac{15}{2} + \frac{31}{5}\right) \\
&= \pi \left(\frac{70}{30} + \frac{225}{30} + \frac{186}{30}\right) \\
&= \frac{481\pi}{30}
\end{aligned}
$$

as required.

Question 1 (f)

SOLUTION. The auxiliary polynomial is
$0 = r^2 - r - 2 = (r-2)(r+1)$
and so we have roots
$r = -1, 2$
Thus, the general solution is of the form
$y = Ce^{-x} + De^{2x}$
for constants C, D.

Question 1 (g)

SOLUTION. The auxiliary polynomial is

$$r^2 - 2r + 2$$

Solving for r when the above is set to 0 yields

$$r = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2} = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

where $i = \sqrt{-1}$. Thus, the general solution is

$$y = e^x (C\cos(x) + D\sin(x))$$

Question 1 (h)

SOLUTION. $W = \int_0^{10} x^{3/2}\,dx = \frac{2x^{5/2}}{5}\Big|_0^{10} = \frac{2\sqrt{100000}}{5} = 40\sqrt{10}$

Question 1 (i)

SOLUTION 1. Evaluating directly, we see that

$$\begin{aligned}\int_1^\infty \frac{dx}{1+2x} &= \lim_{b\to\infty} \int_1^b \frac{dx}{1+2x} \\ &= \lim_{b\to\infty} \frac{\ln|1+2x|}{2}\Big|_1^b \\ &= \lim_{b\to\infty} \frac{\ln|1+2b|}{2} - \frac{\ln(3)}{2}\end{aligned}$$

and this last limit diverges.

SOLUTION 2. Following hints 3 and 4, we want to show that

$$\frac{1}{1+2x} \geq C\frac{1}{x}.$$

A fraction is larger when the denominator is smaller, hence we can use

$$1 + 2x \leq x + 2x = 3x$$

to write

$$\frac{1}{1+2x} \geq \frac{1}{3x}.$$

Thus, by the integral comparison test,

$$\int_1^\infty \frac{1}{1+2x}\,dx \geq \frac{1}{3}\int_1^\infty \frac{1}{x}\,dx$$

However, the latter integral diverges by the integral p-test ($p = 1$), and thus the former integral also diverges.

Question 1 (j)

SOLUTION. Starting with the Maclaurin series for e^y, we have

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + .. = \sum_{n=0}^{\infty} \frac{y^n}{n!}$$

Plugging in -t^2, we have

$$e^{-t^2} = 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + .. = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}$$

Multiplying by t^2, we have

$$t^2 e^{-t^2} = t^2 - t^4 + \frac{t^6}{2!} - \frac{t^8}{3!} + .. = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+2}}{n!}$$

Now integrating gives

$$\begin{aligned}\int t^2 e^{-t^2} &= \int t^2 - t^4 + \frac{t^6}{2!} - \frac{t^8}{3!} + .. \, dt = \sum_{n=0}^{\infty} \int \frac{(-1)^n t^{2n+2}}{n!} \, dt \\ &= C + \frac{t^3}{3} - \frac{t^5}{5} + \frac{t^7}{7 \cdot 2!} - \frac{t^9}{9 \cdot 3!} + ... = C + \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+3}}{(2n+3)n!}\end{aligned}$$

Plugging in endpoints, we have

$$\int_0^x t^2 e^{-t^2} = \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7 \cdot 2!} - \frac{x^9}{9 \cdot 3!} + ... = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n+3)n!}$$

and so the first three nonzero terms are
$\frac{x^3}{3}, -\frac{x^5}{5}, \frac{x^7}{14}$

Question 1 (k)

SOLUTION. Using the fundamental theorem of calculus and the chain rule, we have

$$\begin{aligned}f'(x) &= \frac{d}{dx} \int_x^{x^2} \sqrt{t^4+1} \, dt \\ &= \frac{d}{dx} \int_x^0 \sqrt{t^4+1} \, dt + \frac{d}{dx} \int_0^{x^2} \sqrt{t^4+1} \, dt \\ &= \frac{d}{dx} - \int_0^x \sqrt{t^4+1} \, dt + \frac{d}{dx} \int_0^{x^2} \sqrt{t^4+1} \, dt \\ &= -\sqrt{x^4+1} + 2x\sqrt{x^8+1}\end{aligned}$$

.
Plugging in $x = 1$ yields

$$\begin{aligned} f'(1) &= -\sqrt{1^4+1} + 2(1)\sqrt{(1)^8+1} \\ &= -\sqrt{2} + 2\sqrt{2} \\ &= \sqrt{2} \end{aligned}$$

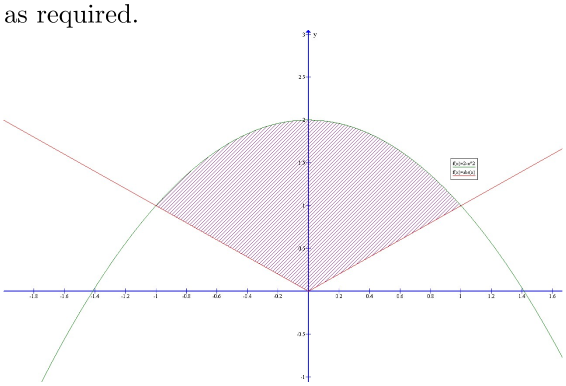
completing the question.

Question 2 (a)

SOLUTION. A picture is included below.
First, find the points of intersection.
$2 - x^2 = |x|$
Take the positive values of x first and solving gives
$0 = 2 - x^2 - |x| = -x^2 - x + 2 = -(x-1)(x+2)$
and so $x = 1$ is a solution here (notice that we discard $x = -2$ since we started with only the positive x values first). Now, taking the negative x values gives
$0 = 2 - x^2 - |x| = -x^2 + x + 2 = -(x+1)(x-2)$
and so $x = -1$ is the other solution. Now, we can draw the picture (see below). The picture clearly shows that the upper function is 2-x^2, and that the function $2 - x^2 - |x|$ is an even function. So we see that it suffices to compute
$A = 2\int_0^1 (2 - x^2 - |x|)\,dx = 2\int_0^1 (2 - x^2 - x)\,dx$
Evaluating gives

$$\begin{aligned} A &= 2\int_0^1 (2 - x^2 - x)\,dx \\ &= 2\left(2x - \frac{x^3}{3} - \frac{x^2}{2}\right)\Bigg|_0^1 \\ &= 2\left(2 - \frac{1}{3} - \frac{1}{2}\right) \\ &= 3 - \frac{2}{3} \\ &= \frac{7}{3} \end{aligned}$$

as required.



Question 2 (b)

SOLUTION. Plugging in all the values gives

$$\begin{aligned}\bar{y} &= \frac{3}{7}\int_{-1}^{1}\frac{1}{2}\left((2-x^2)^2-(|x|)^2\right)dx\\ &= \frac{3}{14}\int_{-1}^{1}\left(4-4x^2+x^4-x^2\right)dx\\ &= \frac{3}{14}\int_{-1}^{1}(4-5x^2+x^4)dx\end{aligned}$$

completing the question.

Question 2 (c)

SOLUTION. We first compute the derivative of $y = (2/3)x^{3/2}$ and see that

$$\frac{dy}{dx} = \sqrt{x}$$

Plugging this into the arc length formula gives

$$\int_0^3 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^3 \sqrt{1+\sqrt{x}^2}\,dx = \int_0^3 \sqrt{1+x}\,dx$$

Now, let $u = 1+x$ so that $du = dx$ and $u(0) = 1$ and $u(3) = 4$. Then

$$\begin{aligned}\int_0^3 \sqrt{1+x}\,dx &= \int_1^4 \sqrt{u}\,du\\ &= \left.\frac{2u^{3/2}}{3}\right|_1^4\\ &= \frac{2}{3}\left(4^{3/2}-1\right)\\ &= \frac{2}{3}(8-1)\\ &= \frac{14}{3}\end{aligned}$$

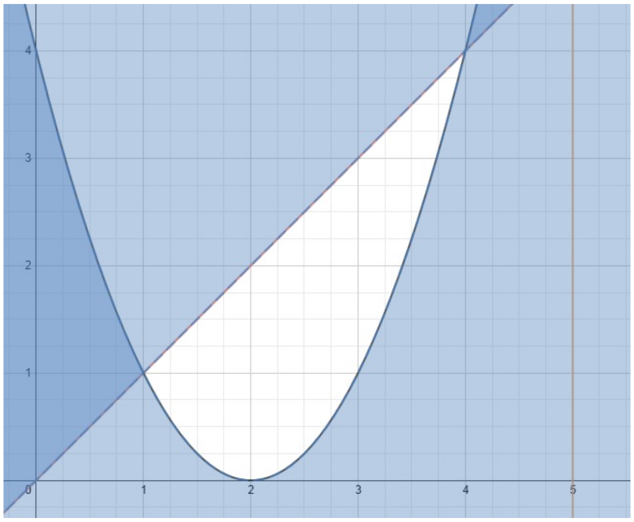
completing the question.

Question 2 (d)

SOLUTION. First, we draw a picture. Setting the line and parabola equal to each other, we see that the points of intersection are at

$$\begin{aligned}(x-2)^2 &= x\\ x^2-4x+4 &= x\\ x^2-5x+4 &= 0\\ (x-4)(x-1) &= 0\end{aligned}$$

and so x=1 or x=4. This gives the following picture.



The white area is what we want to rotate. Let x be a point between 1 and 4. Then the radius of a shell is r = 5-x. The height of a shell is $h = x - (x-2)^2$. Thus, when we rotate about x=5, the height will move a distance of $2\pi r$ and hence, the volume of revolution is

$$\begin{aligned}
V &= \int_1^4 2\pi(5-x)(x-(x-2)^2)\,dx \\
&= \int_1^4 2\pi(5-x)(x-x^2+4x-4)\,dx \\
&= \int_1^4 2\pi(5-x)(-x^2+5x-4)\,dx \\
&= \int_1^4 2\pi(5(-x^2+5x-4)-x(-x^2+5x-4))\,dx \\
&= \int_1^4 2\pi(-5x^2+25x-20+x^3-5x^2+4x)\,dx \\
&= \int_1^4 2\pi(x^3-10x^2+29x-20)\,dx \\
&= 2\pi(x^4/4-10x^3/3+29x^2/2-20x)\Big|_1^4 \\
&= 2\pi(4^4/4-10(4)^3/3+29(4)^2/2-20(4) \\
&\quad -((1)^4/4-10(1)^3/3+29(1)^2/2-20(1))) \\
&= 2\pi(64-640/3+29(8)-80-(1/4-10/3+29/2-20)) \\
&= 45\pi/2
\end{aligned}$$

Question 3 (a)

SOLUTION. We proceed via the hint. Let $u = x^3+1$ so that $du = 3x^2dx$ and so

$$
\begin{aligned}
\int \frac{x^2}{(x^3+1)^{101}}\,dx &= \frac{1}{3}\int \frac{du}{u^{101}} \\
&= \frac{1}{3}\cdot\frac{1}{-100u^{100}} + C \\
&= \frac{1}{-300(x^3+1)^{100}} + C
\end{aligned}
$$

completing the question.

Question 3 (b)

SOLUTION. We proceed by partial fractions. Let
$\frac{x^2+2}{x^3+x} = \frac{x^2+2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+(Bx+C)x}{x^2+1}$
where above we used the fact that x^2+1 is irreducible (its discriminant is $-4<0$). Thus, we know that
$x^2+2 = A(x^2+1)+(Bx+C)x$
Plugging in $x=0$ gives
$2 = 0^2+2 = A(0^2+1)+(B(0)+C)(0) = A$
and so $A=2$. Plugging in $x=1$ gives
$3 = 1^2+2 = A(1^2+1)+(B(1)+C)(1) = 2A+B+C = 4+B+C$
and so $B+C=-1$. Plugging in $x=-1$ gives
$3 = (-1)^2+2 = A((-1)^2+1)+(B(-1)+C)(-1) = 2A+B-C = 4+B-C$
and so $B-C=-1$. Adding the last two equations and dividing by 2 gives $B=-1$ and so $C=0$ Hence, we have

$$
\begin{aligned}
\int \frac{x^2+2}{x^3+x}\,dx &= \int \frac{2dx}{x} - \int \frac{x}{x^2+1}\,dx \\
&= 2\ln|x| - \int \frac{x}{x^2+1}\,dx
\end{aligned}
$$

To solve the last integral, use substitution. Let $u=x^2+1$ so that $du=2xdx$ and so

$$
-\int \frac{x}{x^2+1}\,dx = -\frac{1}{2}\int \frac{du}{u} = -\frac{1}{2}\ln|u| + C = -\frac{\ln|x^2+1|}{2} + C
$$

and so

$$
\begin{aligned}
\int \frac{x^2+2}{x^3+x}\,dx &= 2\ln|x| - \int \frac{x}{x^2+1}\,dx \\
&= 2\ln|x| - \frac{\ln|x^2+1|}{2} + C
\end{aligned}
$$

Question 3 (c)

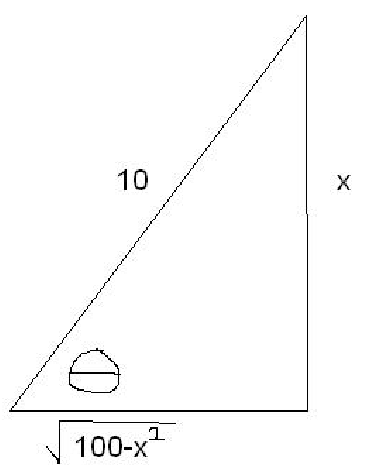
SOLUTION. Let $x=10\sin\theta$ so that $dx=10\cos\theta d\theta$. Then we have

$$\begin{aligned}\int \frac{x^2}{(100-x^2)^{3/2}}\,dx &= \int \frac{100\sin^2\theta(10\cos\theta d\theta)}{(100-100\sin^2\theta)^{3/2}}\\ &= \int \frac{1000(\sin^2\theta)(\cos\theta)d\theta}{(100\cos^2\theta)^{3/2}}\\ &= \int \frac{1000(\sin^2\theta)(\cos\theta)d\theta}{1000\cos^3\theta}\\ &= \int \frac{\sin^2\theta d\theta}{\cos^2\theta}\\ &= \int \tan^2\theta d\theta\\ &= \int (\sec^2\theta-1)d\theta\\ &= \tan\theta-\theta+C\end{aligned}$$

Now we isolate for the original variables. Using the diagram below (created from noting the original condition on x and by using Pythagorean theorem), we have

$$\begin{aligned}\int \frac{x^2}{(100-x^2)^{3/2}}\,dx &= \tan\theta-\theta+C\\ &= \frac{x}{\sqrt{100-x^2}}-\arcsin\left(\frac{x}{10}\right)+C\end{aligned}$$

completing the proof.



Question 3 (d)

SOLUTION. We proceed by integration by parts. Let

$$\begin{aligned} u &= e^{2x} & v &= -\frac{\cos(2x)}{2}\\ du &= 2e^{2x}dx & dv &= \sin(2x)dx\end{aligned}$$

and so the integral becomes

$$\int e^{2x}\sin(2x)\,dx = -e^{2x}\frac{\cos(2x)}{2} + \int e^{2x}\cos(2x)\,dx$$

We use parts again. Let

$$\begin{aligned} u &= e^{2x} & v &= \frac{\sin(2x)}{2} \\ du &= 2e^{2x}dx & dv &= \cos(2x)dx \end{aligned}$$

Then, we have

$$\begin{aligned} \int e^{2x}\sin(2x)\,dx &= -e^{2x}\frac{\cos(2x)}{2} + \int e^{2x}\cos(2x)\,dx \\ &= -e^{2x}\frac{\cos(2x)}{2} + \left(\frac{e^{2x}\sin(2x)}{2} - \int e^{2x}\sin(2x)\,dx\right) \\ &= -e^{2x}\frac{\cos(2x)}{2} + \frac{e^{2x}\sin(2x)}{2} - \int e^{2x}\sin(2x)\,dx \end{aligned}$$

This last integral on the right is the same as the integral on the left (up to a constant). Hence, we have

$$\begin{aligned} \int e^{2x}\sin(2x)\,dx + \int e^{2x}\sin(2x)\,dx &= -e^{2x}\frac{\cos(2x)}{2} + \frac{e^{2x}\sin(2x)}{2} + C \\ 2\int e^{2x}\sin(2x)\,dx &= -e^{2x}\frac{\cos(2x)}{2} + \frac{e^{2x}\sin(2x)}{2} + C \\ \int e^{2x}\sin(2x)\,dx &= -\frac{e^{2x}\cos(2x)}{4} + \frac{e^{2x}\sin(2x)}{4} + D \end{aligned}$$

where $D = 2C$. This completes the question.

Question 4 (a)

SOLUTION. First, by separating variables, we have
$\frac{dy}{y^4} = (x+1)^2 dx = (x^2+2x+1)dx$
Then, integrating both sides yields
$\frac{1}{-3y^3} = \frac{x^3}{3} + x^2 + x + C$
Under the initial condition $y(0) = -1$, we have
$\frac{1}{3} = C$
and so we have
$\frac{1}{-3y^3} = \frac{x^3}{3} + x^2 + x + \frac{1}{3}$
and simplifying, we have
$y = \frac{1}{\sqrt[3]{-x^3-3x^2-3x-1}} = \frac{1}{\sqrt[3]{-(x+1)^3}} = \frac{-1}{x+1}$
completing the question.

Question 4 (b)

SOLUTION. We proceed as in the hint. Multiplying both sides by e^{2x} gives
$e^{2x}y'' + 2e^{2x}y' = e^{2x}\cos x$

Now, the left hand side is just $(e^{2x}y')'$ and so
$(e^{2x}y')' = e^{2x}\cos x$
Integrating both side, we see that the integral of the right hand side above is difficult - so we compute it separately. We proceed by integration by parts. Let

$$\begin{aligned} u &= e^{2x} & v &= \sin x \\ du &= 2e^{2x}dx & dv &= \cos x dx \end{aligned}$$

and so the integral becomes

$$\int e^{2x}\cos x\,dx = e^{2x}\sin x - 2\int e^{2x}\sin x\,dx$$

We use parts again. Let

$$\begin{aligned} u &= e^{2x} & v &= -\cos x \\ du &= 2e^{2x}dx & dv &= \sin x dx \end{aligned}$$

Then, we have

$$\begin{aligned} \int e^{2x}\cos x\,dx &= e^{2x}\sin x - 2\int e^{2x}\sin x\,dx \\ &= e^{2x}\sin x - 2\left(-e^{2x}\cos x + 2\int e^{2x}\cos x\,dx\right) \\ &= e^{2x}\sin x + 2e^{2x}\cos x - 4\int e^{2x}\cos x\,dx \end{aligned}$$

This last integral on the right is the same as the integral on the left (up to a constant). Hence, we have

$$\begin{aligned} \int e^{2x}\cos x\,dx + 4\int e^{2x}\cos x\,dx &= e^{2x}\sin x + 2e^{2x}\cos x + C \\ 5\int e^{2x}\cos x\,dx &= e^{2x}\sin x + 2e^{2x}\cos x + C \\ \int e^{2x}\cos x\,dx &= \frac{e^{2x}\sin x}{5} + \frac{2e^{2x}\cos x}{5} + D \end{aligned}$$

where $D = \frac{C}{5}$.
After this gigantic diversion, we return to the point. We had
$(e^{2x}y')' = e^{2x}\cos x$
and integrating both sides, we get
$e^{2x}y' = \displaystyle\int e^{2x}\cos x\,dx = \frac{e^{2x}\sin x}{5} + \frac{2e^{2x}\cos x}{5} + D$
Dividing both sides by e^{2x}, we have
$y' = \dfrac{\sin x}{5} + \dfrac{2\cos x}{5} + De^{-2x}$
Now, we solve for D. We plug in the initial condition $y'(0) = -1$ and see that
$-1 = y'(0) = \dfrac{\sin(0)}{5} + \dfrac{2\cos(0)}{5} + De^{-2(0)} = \dfrac{2}{5} + D$
and so isolating, we see that $D = -\frac{7}{5}$. Again integrating both sides of y' gives

$$
\begin{aligned}
y &= \int \frac{\sin x}{5} + \frac{2\cos x}{5} - \frac{7}{5}e^{-2x}\,dx \\
&= \frac{-\cos x}{5} + \frac{2\sin x}{5} + \frac{7e^{-2x}}{10} + E
\end{aligned}
$$

where E is another constant. To compute E, we use the other initial condition that $y(0) = 1$ to see that

$$1 = y(0) = \frac{-\cos(0)}{5} + \frac{2\sin(0)}{5} + \frac{7e^{-2(0)}}{10} + E = -\frac{1}{5} + \frac{7}{10} + E$$

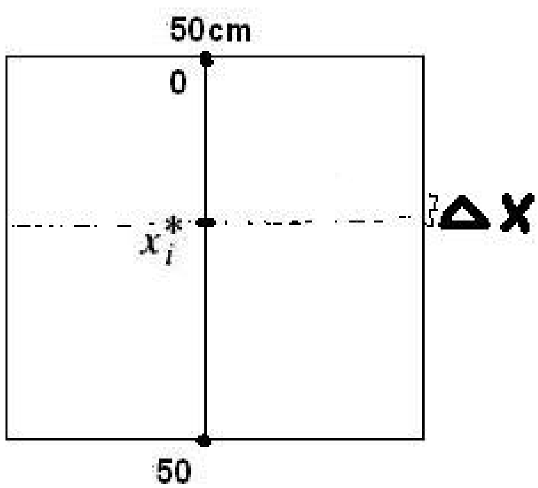
and so $E = \frac{1}{2}$. Combining gives

$$y = \frac{-\cos x}{5} + \frac{2\sin x}{5} + \frac{7e^{-2x}}{10} + \frac{1}{2}$$

and we are finished!

Question 5

SOLUTION. Start by placing a coordinate system on your picture with 0 on top. We break the problem up into two parts. First, we take care of the square.
Choose a sample point x_i^* somewhere between 0 and 50 on your picture. Draw the horizontal line across at this point. A picture is included below.

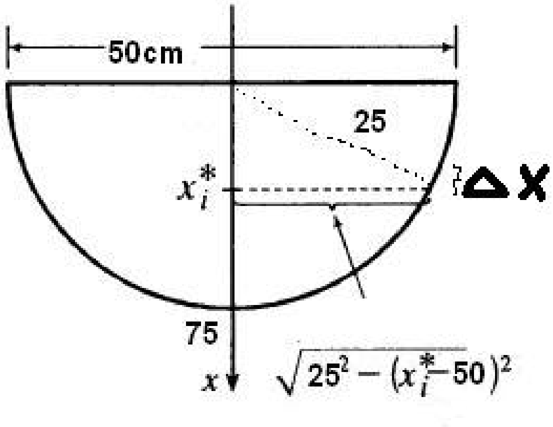


The area of this horizontal strip is given by
$A_i = 50\Delta x$.
The pressure on this strip is given by
$P_i = \rho g x_i^*$.
where $\rho = 1000$ is the density of water and $g = 9.8$ is the acceleration due to gravity. Thus, the force on this strip is
$F_i = P_i A_i = 50\rho g x_i^* \Delta x$.
The total force is
$F_{sq} = \lim_{n\to\infty} \sum_{i=1}^{n} F_i = \lim_{n\to\infty} \sum_{i=1}^{n} 50\rho g x_i^* \Delta x$.
This can be expressed as the following integral

$F_{sq} = \int_0^{50} 50\rho g x\, dx$

Now we compute the force on the semicircle and add it to this force to get the total force. To make the notation easier, I will reuse some of the letters from above.

Choose a sample point x_i^* somewhere between 50 and 75 on your picture. Draw the horizontal line across at this point. A picture is included below.



To compute the length of the horizontal line, draw the radius as shown in the picture. Then the Pythagorean theorem tells us that the length across is double $\sqrt{25^2 - (x_i^* - 50)^2}$. Thus, the area of this strip is given by

$A_i = 2\sqrt{25^2 - (x_i^* - 50)^2}\Delta x.$

The pressure on this strip is given by

$P_i = \rho g x_i^*.$

Thus, the force on this strip is

$F_i = P_i A_i = 2\sqrt{25^2 - (x_i^* - 50)^2}\rho g x_i^* \Delta x.$

The total force is

$F_{circ} = \lim_{n\to\infty} \sum_{i=1}^n F_i = \lim_{n\to\infty} \sum_{i=1}^n 2\sqrt{25^2 - (x_i^* - 50)^2}\rho g x_i^* \Delta x.$

This can be expressed as the following integral

$F_{circ} = \int_{50}^{75} 2\sqrt{25^2 - (x - 50)^2}\rho g x\, dx$

Adding the two integrals together, we get

$$\begin{aligned}
F_{tot} &= F_{sq} + F_{circ} \\
&= \int_0^{50} 50\rho g x + \int_{50}^{75} 2\sqrt{25^2 - (x-50)^2}\rho g x\, dx \\
&= \left.\frac{50\rho g x^2}{2}\right|_0^{50} + \int_0^{25} 2\sqrt{25^2 - x^2}\rho g (x + 50)\, dx \\
&= 25\rho g (50)^2 + \int_0^{25} 2\sqrt{25^2 - x^2}\rho g x\, dx + \int_0^{25} 100\sqrt{25^2 - x^2}\rho g\, dx
\end{aligned}$$

The last integral can be recognized as the [scaled] area of a quarter circle of a circle with radius 25. The first integral we can let $u = 25^2 - x^2$. When $x = 0$, then u= 625 and when $x = 25$ then u=0 and $du = -2xdx$. Hence,

$$
\begin{aligned}
F_{tot} &= 25\rho g(50)^2 - \int_{625}^{0} \sqrt{u}\rho g\, du + 100\rho g \frac{\pi(25)^2}{4} \\
&= 25\rho g(50)^2 + \int_{0}^{625} \sqrt{u}\rho g\, du + 25\rho g\pi(25)^2 \\
&= 25\rho g(50)^2 + \left.\frac{2u^{3/2}\rho g}{3}\right|_0^{625} + \rho g\pi(25)^3 \\
&= 25\rho g(50)^2 + \frac{2(25)^3\rho g}{3} + \rho g\pi(25)^3
\end{aligned}
$$

which is our final answer.

Question 6

SOLUTION. *Probability was not covered this year, hence this is not material for the April 2012 exam.*
Since $f(x) = 0$ for $x \leq 0$ it follows that $m > 0$. Now we can directly compute m:

$$
\begin{aligned}
\frac{1}{2} &= \int_{-\infty}^{m} f(x)\, dx \\
&= \int_{-\infty}^{0} 0\, dx + \int_{0}^{m} 2xe^{-x^2}\, dx
\end{aligned}
$$

We use the substitution $u = x^2$, $du = 2x\ dx$. Then $u(0) = 0$ and $u(m) = m^2$. Hence

$$
\begin{aligned}
\frac{1}{2} &= \int_{-\infty}^{m} f(x)\, dx \\
&= \int_{-\infty}^{0} 0\, dx + \int_{0}^{m} 2xe^{-x^2}\, dx \\
&= \int_{0}^{m^2} e^{-u}\, du \\
&= -e^{-u}\Big|_0^{m^2} \\
&= -e^{-m^2} + e^0 \\
&= 1 - e^{-m^2}.
\end{aligned}
$$

In other words, m satisfies

$$
\begin{aligned}
\frac{1}{2} &= 1 - e^{-m^2} \\
e^{-m^2} &= \frac{1}{2} \\
-m^2 &= \ln(2^{-1}) \\
m^2 &= \ln 2 \\
m &= \pm\sqrt{\ln 2}
\end{aligned}
$$

Note that we can rule out the solution $m = -\sqrt{\ln 2}$ since we already noted that m is positive.
Hence, the solution is $m = \sqrt{\ln 2}$.

Question 7

SOLUTION. We proceed as in the hint (specifically hint 3). Notice that
$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{(i+n)(i+2n)} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\cdot\frac{1}{(\frac{i}{n}+1)(\frac{i}{n}+2)}$
From this, it is clear that $\Delta x = \frac{1}{n}$ and that we should probably set
$x_i = 1 + \frac{i}{n}$
and so $a = 1$ and thus, combining this with the Δx information, we have
$\frac{1}{n} = \Delta x = \frac{b-a}{n} = \frac{b-1}{n}$
and solving for b gives $b = 2$. Hence, we have
$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{(i+n)(i+2n)} = \int_1^2 \frac{dx}{x(x+1)}$
We've reduced the problem to solving an integral. This integral is ripe for the partial fraction algorithm so let's apply that. Let
$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{A(x+1)+Bx}{x(x+1)}$
and this gives rise to
$1 = A(x+1) + Bx$
Plugging in $x = 0$ in the above yields
$1 = A((0)+1) + B(0) = A$
and so $A = 1$. Plugging in $x = -1$ into the equation gives
$1 = A((-1)+1) + B(-1) = -B$
and so $B = -1$. Thus, we have

$$\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{(i+n)(i+2n)} &= \int_1^2 \frac{dx}{x(x+1)} \\
&= \int_1^2 \frac{dx}{x} - \int_1^2 \frac{dx}{x+1} \\
&= \ln|x|\big|_1^2 - \ln|x+1|\big|_1^2 \\
&= \ln(2) - \ln(1) - (\ln(3) - \ln(2)) \\
&= 2\ln(2) - \ln(3)
\end{aligned}$$

completing the question.

Good Luck for your exams!