Full Solutions
MATH220 April 2011

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Educational Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Exam Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Educational Resources.

Question 1 (a)

Easiness: 90/100

SOLUTION.

$$f^{-1}(D) = \{a \in A \mid f(a) \in D\}$$

and

$$f(C) = \{f(c) \mid c \in C\}.$$

Question 1 (b)

Easiness: 100/100

SOLUTION. The supremum of a set S of real numbers is defined to be the smallest real number M such that every number in S is less than or equal to M.

Question 1 (c)

Easiness: 100/100

SOLUTION. The converse is:
If I buy a new car, then I will win the lottery.

Question 1 (d)

SOLUTION. The negation of the statement is given by

$$\exists x \in \mathbb{R} \text{ s.t. } \forall y \in \mathbb{N}\ x^2 + y \geq x + y^2.$$

Question 1 (e)

Easiness: 100/100

SOLUTION. We say that f is injective if whenever we have $f(x) = f(y)$ for $x, y \in A$ then $x = y$.

Question 1 (f)

Easiness: 100/100

SOLUTION. We say that a sequence $\{a_n\}$ converges to a number L if for all $\epsilon > 0$ there exists an $n \in \mathbb{N}$ such that for all $n \geq N$ we have $|a_n - L| < \epsilon$

Question 1 (g)

Easiness: 80/100

SOLUTION. Let
$S_N = \sum_{n=1}^{N} a_n$
be the Nth partial sum. We say that the series $\sum_{n=1}^{\infty} a_n$ converges provided the sequence $\{S_N\}$ converges to a number L.

Question 1 (h)

Easiness: 100/100

SOLUTION. A set S is countable if there exists a bijection (or a one to one correspondence) between S and

$\mathbb{N}$.

Question 1 (i)

SOLUTION. To prove a statement $S(n)$ is true for all natural numbers n, we use mathematical induction which states the following:
1. Prove that $P(0)$ is true
2. Assume that $P(n)$ is true
2. Prove that $P(n+1)$ is true (likely using information about $P(n)$ being true).
Upon doing this, mathematical induction states that the statement $P(n)$ is true for all natural numbers n.

Question 1 (j)

SOLUTION. $\mathcal{P}(A) = \{\emptyset, \{1\}, \{x\}, \{y\}, \{1, x\}, \{1, y\}, \{x, y\}, \{1, x, y\}\}$

Question 2

Easiness: 95/100

SOLUTION. **Solution to 1.**
The supremum of this set is 5, since it is clearly larger than any number in (-1,5) and no number smaller than 5 could have that property. (If you're not convinced of this, imagine any number smaller than 5, and call it N then there must exist a number between N and 5 so that N cannot be the supremum).
And for a similar reason, the infimum of this set has to be -1.
Solution to 2.
Here the set that we are looking at is all the rational numbers that are between the square root of 3 and the square root of 7. The supremum and infimum of a set do not have to be members of the set, so our intuition isn't bad: the supremum has to be the square root of 7 and the infimum the square root of 3.
Solution to 3.
This set is the intersection of closed intervals that grow. At $n = 1$ it is the set [3,4] and then [2.5, 5], so we have

$$[3, 4] \cap [2.5, 5] \cap [2 + \frac{1}{3}, 6 - \frac{2}{3}] \cap [2 + \frac{1}{4}, 6 - \frac{2}{4}] \cap \ldots = [3, 4]$$

since because we're taking an intersection of larger and larger interval, only the smaller remains. It is now clear that the supremum of that set is 4 and the infimum 3.
Solution to 4.
Here the sets also are growing in size, but this time we are taking a union. Since $1/n$ converges to 0, the left side of the interval converges to 0 and clearly the right side of the interval converges to infinity. This means that

$$\bigcup_{n=1}^{\infty} [\frac{1}{n}, n] = (0, \infty)$$

And so the infimum of this set is 0, while there isn't a supremum since the set does not have an upper bound.

Question 3 (a)

Easiness: 60/100

SOLUTION 1. **This is the proof as I worked it out, *Solution 2* presents it as you can polish it and write it nicely to impress your teacher or nerdy friends.**
I fix myself an $\epsilon > 0$, that means, it's as if I have a fixed value in my mind like $\epsilon = 0.01$. I need to show that if n is large enough (that is, larger than some number I call N_ϵ) then I will have that the term a_n is

ϵ-close to my number L (which here I'm already told should be 1). In other words, if n is large enough, then I should have

$$|a_n - 1| < \epsilon$$

Here I know the sequence really well, I'm told that

$$a_n = \frac{n^2 + 3n + 1}{n^2}$$

So I can rewrite the not-so-good-looking term $|a_n - 1|$ into something much more presentable by using a little algebra:

$$\begin{aligned} |a_n - 1| &= \left| \frac{n^2 + 3n + 1}{n^2} - 1 \right| \\ &= \left| \frac{n^2 + 3n + 1}{n^2} - \frac{n^2}{n^2} \right| \\ &= \left| \frac{n^2 + 3n + 1 - n^2}{n^2} \right| \\ &= \left| \frac{3n + 1}{n^2} \right| \end{aligned}$$

And since n is a positive integer, the above fraction is clearly always positive so the absolute value is here totally useless. So I'll rewrite

$$|a_n - 1| = \frac{3n + 1}{n^2}$$

Now it seems indeed that I can make this as small as I want as long as n is large enough. I still need to be precise about this and explicitly guarantee how big n has to be for the term $|a_n - 1|$ to be less than ϵ. Here are two ways to do this:
First way: I can brutally solve

$$\frac{3n + 1}{n^2} < \epsilon$$

It gives me

$$3n + 1 < \epsilon n^2$$

which I can rewrite as

$$0 < \epsilon n^2 - 3n - 1$$

Now since ϵ is some fixed number, the above is just a quadratic equation (something like $\epsilon x^2 - 3x - 1$ with an ugly coefficient in front of the x^2 term). And since it corresponds to a positive parabola (I like to think them as *smiling* parabolas) I know for sure that it will be positive (above the x-axis) for any value larger than its second x-intercept (if there are any x-intercepts). The x-intercepts are the solutions of the equation $\epsilon x^2 - 3x - 1 = 0$ so using the quadratic formula I find

$$x = \frac{3 \pm \sqrt{9 + 4\epsilon}}{2\epsilon}$$

The larger solution has to be the one using the "+" sign, so I get that

$$0 < \epsilon n^2 - 3n - 1 \quad \text{if} \quad n > \frac{3 + \sqrt{9 + 4\epsilon}}{2\epsilon}$$

Which is my N_ϵ. You can try for yourself, if you chose $\epsilon = 0.01$ then the above gives me $N = 300.33$ and for any value of n larger than 301 you can see that $(3n + 1)/n^2 < 0.01$.
Second way: as so often in math, there is another way that is easier, but involves being a little bit more smart about all this. Since I don't care that the value of N_ϵ is the best possible, just one that works, I can do the following manipulation

$$\frac{3n+1}{n^2} = \frac{3}{n} + \frac{1}{n^2}$$

And now since n is some positive integer, it is always true that $n < n^2$ and hence

$$\frac{1}{n^2} < \frac{1}{n}$$

Hence

$$\frac{3}{n} + \frac{1}{n^2} < \frac{3}{n} + \frac{1}{n} = \frac{4}{n}$$

And so we get something much simpler to deal with

$$|a_n - 1| = \frac{3n+1}{n^2} < \frac{4}{n}$$

And so now we just wonder when is it true that

$$\frac{4}{n} < \epsilon$$

well... clearly when

$$\frac{4}{\epsilon} < n$$

And so for the value $N_\epsilon = 4/\epsilon$ we can guarantee the convergence of the sequence. You can again verify this for yourself that if you chose a value $\epsilon = 0.01$ then you will get here $N = 400$ which we know from the above work is way large enough to guarantee that for any value of n larger than that, we have $|a_n - 1| < 0.01$. In other words this is much simpler, we just lost some precision on how big the number N_ϵ has to be, but to prove convergence, we really don't care so we might as well make our life easier.
Note: this was the work done *as you go*, which corresponds to how you would find out how to prove that the given sequence converges. Now that we have done that work, we can do like fancy mathematicians and present our work more nicely and concisely. To see that, check out ***Solution 2***.

SOLUTION 2. This is the *polished* proof that we obtained in ***Solution 1*** and resembles more what mathematicians present in books.
Fix a value of $\epsilon > 0$ and let

$$N_\epsilon = \frac{4}{\epsilon}$$

Then, if $n > N_\epsilon$ it holds that

$$\begin{aligned}
|a_n - 1| &= \left|\frac{n^2+3n+1}{n^2} - 1\right| \\
&= \frac{3n+1}{n^2} \\
&= \frac{3}{n} + \frac{1}{n^2} \\
&< \frac{3}{n} + \frac{1}{n} \\
&= \frac{4}{n} \\
&< \frac{4}{N_\epsilon} \\
&< \epsilon
\end{aligned}$$

which concludes our proof.

Question 3 (b)

Easiness: 20/100

SOLUTION. Here, the limit being 0 means that we simply need to show that given any $\epsilon > 0$ there exists a number N_ϵ such that

$$\left|\frac{1-2\cos(n)}{n}\right| < \epsilon$$

for any value of n larger than N_ϵ.
Since we have that

$$-1 \le \cos(n) \le 1$$

we can multiply everywhere by -2 and obtain that

$$2 \ge -2\cos(n) \ge -2$$

and add 1 everywhere to obtain

$$3 \ge 1 - 2\cos(n) \ge -1$$

which now allows us to conclude about the size of $1 - 2\cos(n)$ in absolute value:

$$|1 - 2\cos(n)| \le 3$$

So we can conclude that

$$\begin{aligned}
\left|\frac{1-2\cos(n)}{n}\right| &= \frac{|1-2\cos(n)|}{n} \\
&\le \frac{3}{n} \\
&< \epsilon
\end{aligned}$$

for any value of n that is larger than $N_\epsilon = 3/\epsilon$; this concludes our proof.

Question 4

Easiness: 70/100

SOLUTION 1. Using truth tables.
We write the truth table for each of the two statements that we would like to compare.

P	**Q**	**R**	$\mathbf{Q \vee R}$	$\mathbf{P \to (Q \vee R)}$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	F	T

Table 1: table for $\mathbf{P \to (Q \vee R)}$

And

P	**Q**	**R**	$\mathbf{P \to Q}$	$\mathbf{P \to R}$	$\mathbf{(P \to Q) \vee (P \to R)}$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	F	F
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	T	T	T

Table 2: table for $\mathbf{(P \to Q) \vee (P \to R)}$

Since the last column is the same for both tables (and since we order the truth of P, Q and R in the same order) we can conclude that the two statements are logically equivalent.

SOLUTION 2. Another solution consists of using an equivalent statement for the implication:

$$A \Rightarrow B \equiv \neg A \vee B$$

So the statement

$$P \Rightarrow (Q \vee R) \equiv (P \Rightarrow Q) \vee (P \Rightarrow R)$$

becomes

$$\neg P \vee (Q \vee R)) \equiv (\neg P \vee Q) \vee (\neg P \vee R)$$

To prove this statement, we'll just show that the left hand side (LHS) is equal to the right hand side (RHS). For the LHS, we get

$$\neg P \vee (Q \vee R)) \equiv \neg P \vee Q \vee R$$

since disjunction (the logical OR) is associative (that means doesn't care about brackets).
For the RHS we get

$$(\neg P \vee Q) \vee (\neg P \vee R) \equiv \neg P \vee Q \vee R$$

since again the disjunction is associative and having twice the ¬P term is useless.
We obtained that LHS = RHS which proves the statement.

Question 5

Easiness: 100/100

SOLUTION 1. We will first show that this function is *injective*.
Suppose that $f(x) = f(y)$ for some $x, y \in \mathbb{R} - \{1\}$. First, we note that if $f(x) = 0$ then $x = 0$. So we can suppose that $x \neq 0 \neq y$. Then we would have

$$\begin{aligned} \frac{2x}{x-1} = \frac{2y}{y-1} &\iff \frac{y-1}{y} = \frac{x-1}{x} \\ &\iff 1 - \frac{1}{y} = 1 - \frac{1}{x} \\ &\iff \frac{1}{y} = \frac{1}{x} \end{aligned}$$

which is true if and only if $x = y$, and so the function is injective.
To show that the function is *surjective*, we need to "invert" the function. That is, we want to show that for each $r \in \mathbb{R} - \{2\}$ that we can find some x_r so that $f(x_r) = r$. Expanding this yields

$$\begin{aligned} \frac{2x_r}{x_r - 1} &= r \\ 2x_r &= r(x_r - 1) \\ 2x_r &= rx_r - r \\ 2x_r - rx_r &= -r \\ x_r(2-r) &= -r \\ x_r &= \frac{-r}{2-r} \\ x_r &= \frac{r}{r-2} \end{aligned}$$

Hence we can choose $x_r = \frac{r}{r-2}$. In such a case, we have

$$\begin{aligned} f(x_r) &= \frac{2\frac{r}{r-2}}{\frac{r}{r-2} - 1} \\ &= \frac{2\frac{r}{r-2}}{\frac{r-(r-2)}{r-2}} \\ &= \frac{2r}{r-(r-2)} = r \end{aligned}$$

as desired. (Note that this final justification is included only as a sanity check and is not needed in a solution to obtain full marks).

Solution 2. Recall that a function is bijective if and only if it has an inverse function, that is, a function $g(x)$ such that both
$f(g(x)) = x$
and that
$g(f(x)) = x$
I claim that for
$f(x) = \frac{2x}{x-1}$
the function

$$g : \mathbb{R} - \{2\} \to \mathbb{R} - \{1\}$$
$$x \mapsto \frac{x}{x-2}$$

is an inverse function for $f(x)$. To prove this, we proceed directly. Notice that

$$\begin{aligned} f(g(x)) &= \frac{2(\frac{x}{x-2})}{(\frac{x}{x-2}) - 1} \\ &= \frac{2(\frac{x}{x-2})}{(\frac{x-x+2}{x-2})} \\ &= \frac{2x}{x-2} \cdot \frac{x-2}{2} \\ &= x \end{aligned}$$

and that

$$\begin{aligned} g(f(x)) &= \frac{\frac{2x}{x-1}}{(\frac{2x}{x-1}) - 2} \\ &= \frac{\frac{2x}{x-1}}{(\frac{2x-2(x-1)}{x-1})} \\ &= \frac{2x}{x-1} \cdot \frac{x-1}{2} \\ &= x \end{aligned}$$

and this completes the proof. To see where $g(x)$ came from, check out solution 1.

Question 6

Easiness: 100/100

Solution 1. Let us first assume that $n \equiv 3 \pmod 5$. This is equivalent to the statement that $5 \mid (n-3)$ or that there exists some integer k so that $n - 3 = 5k$ or that $n = 5k + 3$. In such a case, we have that

$$3n + 1 = 3(5k + 3) + 1 = 5(3k) + 10 = 5(3k + 2)$$

and so since $3k + 2$ is an integer, it follows that $5 \mid (3n + 1)$.
So suppose now that $5 \mid (3n + 1)$; that is, there is some integer ℓ so that $5\ell = 3n + 1$. From this we can conclude that

$$5\ell = 3n + 1 \iff 5(2\ell) = 6n + 2 = 5n + n + 5 - 3 = 5(n + 1) + n - 3$$

Shifting the first of those terms to the other side we obtain that

$$5(2\ell - n - 1) = n - 3$$

Since $2\ell - n - 1$ is an integer, the result is proven.

SOLUTION 2. Let us first show that if n is congruent to 3 mod 5, then $3n+1$ is divisible by 5, that is, $3n+1$ is congruent to 0 mod 5.
Since we know that

$$n \equiv 3 \pmod 5$$

We can substitute this in the expression of $3n+1$ and obtain that

$$\begin{aligned} 3n + 1 &\equiv 3(3) + 1 \pmod 5 \\ &\equiv 10 \pmod 5 \\ &\equiv 0 \pmod 5 \end{aligned}$$

Which proves that $3n+1$ is divisible by 5.
Now, let's prove the converse, that is that if $3n+1$ is divisible by 5, then n is congruent to 3 mod 5.
We start with the fact that is $3n+1$ is divisible by 5, then it is congruent to 0 mod 5 and write

$$3n + 1 \equiv 0 \pmod 5$$

Which we can rewrite as

$$\begin{aligned} 3n &\equiv -1 \pmod 5 \\ &\equiv 4 \pmod 5 \end{aligned}$$

We would like to get some information about n, not $3n$ but division in modular arithmetic is tricky. In this case, since 3 and 5 are coprime, we know it can be done: the inverse of 3 mod 5 is 2 since $2{\cdot}3{=}6$ which is 1 mod 5. So we multiply both sides of the equation by 2 and obtain

$$6n \equiv 8 \pmod 5$$

And so since 6 is 1 mod 5 and 8 is 3 mod 5 we can conclude that

$$n \equiv 3 \pmod 5$$

Which shows that n is congruent to 3 mod 5 and finishes our proof.

Question 7 (a)

Easiness: 80/100

SOLUTION. We will prove this by induction.
The first case, $n = 1$ is easy to go through, indeed

$$2^1 - 1 = 2 - 1 = 1 = a_1$$

so the first term of the sequence matches the general formula. You can do as many initial cases as you want, but remember, you have to at least one to get your argument started.
Then comes the general case. We say: assume that the statement is true for all values of n up to m. We will show that the statement is also true for $n = m + 1$.
By definition of the sequence a_n, we know that

$$a_{m+1} = 2a_m + 1$$

And we also assume that the statement is true up to $n = m$ hence

$$a_m = 2^m - 1$$

Combining these two facts together, we obtain that

$$\begin{aligned} a_{m+1} &= 2a_m + 1 \\ &= 2(2^m - 1) + 1 \\ &= 2^{m+1} - 2 + 1 \\ &= 2^{m+1} - 1 \end{aligned}$$

which is the claimed statement to be proved and so concludes our proof.

Question 7 (b)

SOLUTION. We will prove this by induction on n.
First, for $n = 1$, since $b_1 = 2$ we clearly have that

$$1 \leq b_1 \leq 2$$

Now, let us assume that the statement is true up to $n = m$, that is

$$1 \leq b_n \leq 2 \quad \text{ for } n = 1, \dots, m$$

and let us show that

$$1 \leq b_{m+1} \leq 2$$

By definition of the sequence b_n we have that

$$b_{m+1} = \frac{b_m + \sqrt{b_m}}{2}$$

Since $b_m \leq 2$ we have that

$$\sqrt{b_m} \leq \sqrt{2} \leq 2$$

and so

$$b_{m+1} = \frac{b_m + \sqrt{b_m}}{2} \leq \frac{2+2}{2} = 2$$

And since $b_m \geq 1$ we have that

$$\sqrt{b_m} \geq 1$$

and so

$$b_{m+1} = \frac{b_m + \sqrt{b_m}}{2} \geq \frac{1+1}{2} = 1$$

These two arguments show that

$$1 \leq b_{m+1} \leq 2$$

and hence conclude our proof.

Question 7 (c)

SOLUTION. We will prove this statement by induction on n.
First, for $n = 1$. By definition of the sequence we have that

$$\begin{aligned} b_2 &= \frac{b_1 + \sqrt{b_1}}{2} \\ &= \frac{2 + \sqrt{2}}{2} \\ &\leq \frac{2+2}{2} \\ &= 2 \\ &\leq b_1 \end{aligned}$$

which proves the first step of our induction.
Let us now assume that the statement is true for all values of n up to m and let us show that the statement holds for $n = m + 1$.
In part (b) we showed that

$$b_n \geq 1 \quad \text{for all } n \in \mathbb{N}$$

hence we can guarantee that

$$\sqrt{b_n} \leq b_n \quad \text{for all } n \in \mathbb{N}$$

and so, we have that

$$\begin{aligned} b_{m+1} &= \frac{b_m + \sqrt{b_m}}{2} \\ &\leq \frac{b_m + b_m}{2} \\ &= b_m \end{aligned}$$

which concludes our proof.

Question 8

SOLUTION. **For (a):** First we can rewrite this statement without the symbols in slightly more meaningful English.

For any prime number $\boldsymbol{m}$ *and for any prime number* $\boldsymbol{n}$, $\boldsymbol{m}+\boldsymbol{n}$ *is prime.*

This is clear false since adding two primes doesn't always give a prime, for example 3+5=8.
For (b): we get

for any prime $\boldsymbol{m}$ *there exists a prime* $\boldsymbol{n}$ *such that* $\boldsymbol{m}+\boldsymbol{n}$ *is prime.*

For this to be true, it has to be true for any choice of prime **m**. Consider for example **m**=7. Can we add a prime to that number so that the sum is prime as well? There are two types of primes: the even ones (actually, 2 is the only even prime) and the odd ones. If we add 2 to 7, we get 9 which isn't prime, so that doesn't work. If we add any odd number to 7, we get an even number (clearly not 2) so clearly not a prime either. So no prime can be added to 7 to make it prime and hence the statement *(b)* is false.
For (c): we get

There exists a prime $\boldsymbol{m}$ *such that for any prime* $\boldsymbol{n}$ *we have* $\boldsymbol{m}+\boldsymbol{n}$ *is prime as well.*

This is fairly similar to the statement *(b)* except that here we want to be able to always add the same prime to any prime to get a new one. In other words, it is even harder to make this statement true and hence it is false. Indeed, we showed above that no prime can be added to 7 to make it prime again, so here 7 is a case of a value of **n** for which we show that no prime **m** would do the trick.
For (d): we get

There exists a prime $\boldsymbol{m}$ *and there exists a prime* $\boldsymbol{n}$ *such that* $\boldsymbol{m}+\boldsymbol{n}$ *is prime.*

This sounds much more reasonable and it is. This statement is true since we can actually even show values of **m** and **n** that work. Consider for example **m**=2 and **n** = 3. Then **m**+**n**=5 which is prime. So such values exist and so statement *(d)* is true.
Note: actually in *(d)*, you should easily convince yourself that one of the prime has to be 2 and so the other has to be what is called a twin prime, that is, a prime with the property that if you add 2 to it it is prime as well (those are, except for 2 and 3, the closest primes you might get and that's why we call them twin primes). For example, 5 and 7 are twin primes and so are 11 and 13; 17 and 19; 29 and 31. We do not know if there are infinitely many twin primes but we suppose it is the case (this is what we call the The twin prime conjecture).

Question 9 (a)

SOLUTION. We are told that the function $g \circ f$ is surjective and we want to show that the function g is surjective as well. For this, let us consider an element in the image of g, that is an element a in the set A (since g sends elements from B to elements in A).
To show the surjectivity we need to show that there is at least one element b in the set B which is mapped to a by the function g. Now the function $g \circ f$ sends elements of the set A to the set A and is surjective. So there must exist an element"a"' in the set A such that

$$(g \circ f)(a') = a$$

But since

$$(g \circ f)(a') = g(f(a'))$$

we can say that the element *f(a')*, which is an element of the set B is the element b that we are looking for since it is mapped by g to the element a as requested. This explains why g is surjective.

Question 9 (b)

SOLUTION 1. We showed in part (a) that the function g will always be surjective, this question asks us to give an example to illustrate why the function f doesn't have to be necessarily.
For example, consider the case where the set A is all the natural numbers including zero: A={ 0, 1, 2, ... } and the set B is all the integers: B = { ..., -2, -1, 0, 1, 2, ... }. Now the function f just maps a natural number to itself, so

$$f(n) = n \quad \text{for all } n = 0, 1, 2, \ldots$$

and the function g maps an integer to its absolute value, so

$$g(k) = |k| \quad \text{for all } k \in \mathbb{Z}$$

Then the function $g \circ f$ is clearly surjective since it takes a natural number back to itself. That is, $(g \circ f)(a) = a$ for all $a \in A$.
But the function f isn't surjective since it doesn't map to any negative integer (no negative number has a pre-image under the function f). Notice that as shown in part (a) the function g is, and has to be, surjective.

SOLUTION 2. Another, minimal example is the following:
Let A = {a}, and B = {1,2}. Then define

$$f : A \to B, \quad f(a) = 1, \qquad g : B \to A, \quad g(1) = g(2) = a.$$

Then, clearly

$$(g \circ f) : A \to A, \quad (g \circ f)(a) = g(f(a)) = g(1) = a,$$

and thus $g \circ f$ is surjective.
However, f is not surjective, since $2 \in B$ does not have a pre-image in A under *f*.
"Notice again, that g is, and needs to be, surjective for $g \circ f$ to be surjective.

Question 9 (c)

SOLUTION. The example that we constructed in part (b) showed a case where the function f is not surjective and so it cannot be possible for $f \circ g$ to be surjective either. For example, in that case, no negative number is in the image of the function f and so no negative number can be in the image of $f \circ g$ either.

Question 10 (a)

SOLUTION. By definition of the supremum of a set, $\sup(B)$ is the smallest real number that is larger or equal to all the numbers in the set B. Since all the numbers of the set B are also numbers of the set A, we can conclude that $\sup(A)$ is also larger or equal to all the numbers of the set B (since it is larger or equal to all numbers in the set A) but since $\sup(B)$ is the smallest real number with that property we can conclude that

$$\sup(B) \leq \sup(A)$$

Now $\inf(A)$ is a number that is less or equal than any number in the set A and so since B is a subset, it is in particular less or equal to any number in the set B. Since $\sup(B)$ is larger than any number in the set B we can conclude that

$$\inf(A) \leq \sup(B)$$

Note: *we could actually prove the more interesting and stronger statement:*

$$\inf(A) \leq \inf(B) \leq \sup(B) \leq \sup(A).$$

Question 10 (b)

SOLUTION. The sequence $\{a_n\}$ is bounded, which means that there exists a real number M such that

$$|a_n| \leq M \quad \text{for all } n \geq 1$$

Hence the supremum of all the numbers a_n is at most M and by definition of the numbers b_n we have that

$$|b_n| \leq M \quad \text{for all } n \geq 1$$

or in other words, the sequence $\{b_n\}$ is bounded as well. That sequence will be converging if it is decreasing, that is if

$$b_{n+1} \leq b_n \quad \text{for all } n \geq 1$$

Which we can easily show to be true. Indeed, if we denote by

$$B_n = \{a_m \mid m \in \mathbb{N} \text{ s.t. } m \geq n\} \quad \text{for all } n \geq 1$$

then

$$b_n = \sup(B_n) \quad \text{for all } n \geq 1$$

and clearly

$$B_{n+1} \subseteq B_n \quad \text{for all } n \geq 1$$

and by part (a) we have that

$$b_{n+1} = \sup(B_{n+1}) \leq \sup(B_n) = b_n$$

This concludes our proof.
Advanced note: *We call the limit of this convergent sequence the limit superior of the original sequence* $\{a_n\}$.

Good Luck for your exams!