Full Solutions
MATH101 April 2011

April 5, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

Easiness: 5.0/5

Solution 1. We notice that the term x^2 is almost the derivative of what is inside of the square root (up to a constant term) and so we will use a change of variable.
Let $u = x^3 + 1$. Then we have:

$$\frac{du}{dx} = 3x^2$$

and hence

$$du = 3x^2\,dx$$

or to match what we have in our integral:

$$\frac{1}{3}\,du = x^2\,dx$$

Since this is a definite integral, we also need to change the bounds of integration. If $x = -1$ then $u = 0$ and if $x = 0$ then $u = 1$. This allows us to write:

$$\int_{-1}^{0} x^2\sqrt{1+x^3}\,dx = \int_0^1 \frac{1}{3}\sqrt{u}\,du$$

Which is an easy integral to compute:

$$\begin{aligned}\int_0^1 \frac{1}{3}\sqrt{u}\,du &= \frac{1}{3}\int_0^1 u^{1/2}\,du = \frac{1}{3}\left[\frac{2}{3}u^{3/2}\right]_0^1 \\ &= \frac{1}{3}\left(\frac{2}{3}\cdot 1^{3/2} - \frac{2}{3}\cdot 0^{3/2}\right) = \frac{2}{9}.\end{aligned}$$

Solution 2. For such an integral, you do not have to do the change of variable if you can directly figure out the correct antiderivative. Since you have the x^2 term outside of the square root, you expect the antiderivative to be something like:

$$(1+x^3)^{3/2}$$

Taking the derivative of this, you will see how to modify the constants to make it work:

$$\frac{d}{dx}\left[(1+x^3)^{3/2}\right] = \frac{3}{2}(1+x^3)^{1/2}\,3x^2 = \frac{9}{2}\,x^2\sqrt{1+x^3}$$

And so we know that an antiderivative of $x^2\sqrt{1+x^3}$ will be $\frac{2}{9}(1+x^3)^{3/2}$. Which allows us to conclude:

$$\int_{-1}^{0} x^2\sqrt{1+x^3}dx = \left[\frac{2}{9}(1+x^3)^{3/2}\right]_{-1}^{0} = \frac{2}{9}$$

Question 1 (b)

Easiness: 4.2/5

SOLUTION. Using hint 2 provided, we may immediatley rewrite $f(t)$ as

$$f(t) = -\int_2^{t^4} \sqrt{1+x^3}dx$$

We would like to use hint 1, but this only works if the upper limit of integration is t, which in this case is not true. So define for the time being

$$g(u) = -\int_2^{u} \sqrt{1+x^3}dx.$$

From hint 1, we see immediately that $g'(u) = -\sqrt{1+u^3}$. We also see that $f(t) = g(t^4)$. Using the chain rule we can then compute that

$$f'(t) = g'(t^4)\frac{du}{dt} = -\sqrt{1+(t^4)^3}(4t^3) = -4t^3\sqrt{1+t^{12}}.$$

Question 1 (c)

Easiness: 4.7/5

SOLUTION. The average value of a function $f(x)$ over an interval $[a, b]$ is given by

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

In this case, this is given by

$$\frac{1}{2}\int_0^2 xe^x\,dx$$

Let us then compute the indefinite integral

$$\int xe^x\,dx$$

which we can compute via integration by parts. If we let $u = x$ and $dv = e^x dx$, then we find that

$$\begin{aligned} u &= x & v &= e^x \\ du &= dx & dv &= e^x dx \end{aligned}$$

and so it follows that

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C = (x-1)e^x + C$$

If we then put that into the definition of the average value, we find that the average value is given by

$$\frac{1}{2}\int_0^2 xe^x\,dx = \frac{1}{2}(x-1)e^x|_0^2 = \frac{1}{2}\big((2-1)e^2 - (0-1)e^0\big) = \frac{1}{2}(e^2+1)$$

Question 1 (d)

Easiness: 1.5/5

SOLUTION. The depth of the water is 2 ft, so we divide the tank wall into n horizontal strips of height $\Delta x = (2-0)/n$. The width of every strip is 2 ft, so the area of each strip is $A_i = 2\Delta x$. The hydrostatic force on a strip at depth x_i^* is

$$F_i = \rho g x_i^* A_i = \rho g x_i^* (2\Delta x).$$

We are given that the weight density of the fluid is $\rho g = 50\ lb/ft^3$. To get the total force on the wall, we add up all the forces F_i and take the limit as $n \to \infty$:

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} F_i = 2\rho g \lim_{n\to\infty} \sum_{i=1}^{n} x_i^* \Delta x = 2\rho g \int_0^2 x dx = 2 \cdot 50 \cdot 2 = 200\ lb.$$

We integrate from 0 to 2 because the water extends from depth 0 ft to 2 ft. This is also why we set $\Delta x = (2-0)/n$.

Question 1 (e)

Easiness: 2.9/5

SOLUTION. For $n \geq 2$, we have
$\left|\frac{\sin(n)}{\ln(n)}\right| \leq \frac{1}{\ln(n)}$ and $\lim_{n\to\infty} \frac{1}{\ln(n)} = 0..$
So, by the Squeeze Theorem, $\left\{\frac{\sin(n)}{\ln(n)}\right\}_{n=2}^{\infty}$ converges to 0.

Question 1 (f)

Easiness: 4.1/5

SOLUTION. The series can be rewritten as
$\sum_{n=1}^{\infty} \frac{7\cdot 3^{2n-1}}{10^n} = \sum_{n=1}^{\infty} \frac{7}{3} \cdot \left(\frac{9}{10}\right)^n$.
This is a geometric series with common ratio $r = 9/10 < 1.$, so it converges. Recall that a convergent geometric series has sum
$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$.
Note that to use this formula we need to adjust the series to start at $n = 0$. We get
$\sum_{n=1}^{\infty} \frac{7\cdot 3^{2n-1}}{10^n} = \sum_{n=1}^{\infty} \frac{7}{3} \cdot \left(\frac{9}{10}\right)^n = \sum_{n=0}^{\infty} \frac{7}{3} \cdot \frac{9}{10} \left(\frac{9}{10}\right)^n = \frac{7}{3} \cdot \frac{9}{10} \cdot \frac{1}{1-\frac{9}{10}} = \frac{7}{3} \cdot \frac{9}{10} \cdot \frac{1}{\frac{1}{10}} = \frac{7}{3} \cdot 9 = 21.$

Question 1 (g)

Easiness: 1.6/5

SOLUTION. We want to find the smallest k such that

$$\left|\sum_{n=1}^{\infty} \frac{1}{n^3} - \sum_{n=1}^{k} \frac{1}{n^3}\right| \leq \frac{1}{20000}.$$

Or, equivalently,

$$\left|\sum_{n=k+1}^{\infty} \frac{1}{n^3}\right| \leq \frac{1}{20000}.$$

Since all of the terms in the sum are positive, the sum itself is positive. So, the absolute value signs are unnecessary. Note that

$$\sum_{n=k+1}^{\infty} \frac{1}{n^3}$$

can be interpreted as the right Riemann sum for the integral

$$\int_k^{\infty} \frac{1}{x^3} dx.$$

Since $1/x^3$ is monotonically decreasing, the right Riemann sum is an underestimation for the area under the curve. As step size for the Riemann sum $\Delta x = 1$ is used.
Therefore, we have

$$\sum_{n=k+1}^{\infty} \frac{1}{n^3} < \int_k^{\infty} \frac{1}{x^3} dx,$$

and hence it suffices to find the minimal value of k such that the integral above is $\leq$ 20,000.
But the integral

$$\int_k^{\infty} \frac{1}{x^3} dx = \left[-\frac{1}{2x^2}\right]_k^{\infty} = \frac{1}{2k^2}.$$

Hence

$$\sum_{n=k+1}^{\infty} \frac{1}{n^3} < \int_k^{\infty} \frac{1}{x^3} dx = \frac{1}{2k^2} \leq \frac{1}{20000}.$$

The minimal such k is $k = 100$ (in fact, for this choice of k, the right $\leq$ becomes an $=$).
Therefore we must take the first 100 terms of the sum for the approximation to be accurate to within 1/20000.

Question 1 (h)

Easiness: 4.2/5

SOLUTION. The alternating series test says that, given a series

$$\sum_{n=1}^{\infty} (-1)^n a_n$$

where a_n is a positive sequence such that $\lim_{n\to\infty} a_n = 0$, then we have that the series converges. In our case, so long as $p > 0$, this satisfies the conditions of the alternating series test and so it converges. For $p \leq 0$, the sequence $1/n^p$ does not converge to zero, and so the sequence diverges.
Thus the series converges for all $p > 0$.

Question 1 (i)

Easiness: 4.7/5

SOLUTION 1. We will use the ratio test to prove that this series is absolutely convergent. Each term in the series is

$$a_n = \frac{(-2011)^n}{n!}$$

and so we have that

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(-2011)^{n+1}}{(n+1)!}}{\frac{(-2011)^n}{n!}} = \frac{-2011n!}{(n+1)!} = -\frac{2011}{n+1}$$

We see then that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0 < 1$, and so the series converges absolutely.

SOLUTION 2. As per the second hint, we note that the series is given by

$$\sum_{n=0}^{\infty} \frac{(-2011)^n}{n!} = e^{-2011}$$

Since the function e^x converges absolutely over the whole real line, it follows that this series does as well.

Question 1 (j)

Easiness: 4.6/5

SOLUTION. We could compute all of the derivatives by hand, but this would be a long and arduous process. This is not an efficient use of exam time! As such, we use the known series of $\sin(x)$ to note that

$$\sin(x^3) = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots$$

It follows then that

$$x^3 \sin(x^3) = x^3 \left(x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots \right) = x^6 - \frac{x^{12}}{3!} + \frac{x^{18}}{5!} - \cdots$$

and so the first three nonzero terms of the Maclurian series are

$$x^6 - \frac{x^{12}}{3!} + \frac{x^{18}}{5!}$$

Question 2 (a)

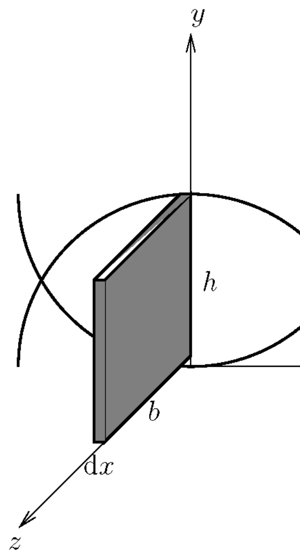
Easiness: 5.0/5

SOLUTION. The curves $y = x$ and $y = x^3$ intersect at the points $(-1, -1)$, $(0, 0)$, $(1, 1)$. By symmetry, the area of the region bounded by the curves between $x = -1$ and $x = 0$ is equal to the area of the region bounded by the curves between $x = 0$ and $x = 1$. So the total area will be equal to two times the area of the region between $x = 0$ and $x = 1$.
Between $x = 0$ and $x = 1$, we have $x \geq x^3$; that is, the curve $y = x$ is above the curve $y = x^3$. So the area bounded by the curves and between $x = 0$ and $x = 1$ is
$\int_0^1 (\text{upper} - \text{lower})dx = \int_0^1 (x - x^3)dx$
$= \left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_0^1 = \left(\frac{1^2}{2} - \frac{1^4}{4}\right) - \left(\frac{0}{2} - \frac{0}{4}\right) = \frac{1}{4}$.
The total area of the region is two times this:

$A = 2 \cdot \frac{1}{4} = \frac{1}{2}$.

Question 2 (b)

Easiness: 2.5/5



SOLUTION. If we follow the hint, we can draw a diagram similar to the one pictured on the right We have drawn the axis so that we see the xy plane in its familiar 2D position.

The first thing we have to consider is the x limits of where our picture begins and ends. To do this, we find the intersection points of our two parabolas. Therefore we want to find the x such that

$x^2 = 2 - x^2$

which occurs when $x = \pm 1$. Notice from our diagram that $x = -1$ is the left endpoint and $x = 1$ is the right endpoint. As we are told in the question, the cross section in the z-direction is a square which we can think of as a rectangular prism with a very small depth, $\mathrm{d}x$. Therefore, the volume of any one square, $\mathrm{d}V$, is given by

$\mathrm{d}V = bh\mathrm{d}x$.

Since the cross section is indeed a square we have that $b = h$ and this equals the height between the two parabolas. Therefore,

$b = h = (2 - x^2) - x^2 = 2 - 2x^2$

and thus for any x we have that the volume of the rectangle is,

$\mathrm{d}V = h^2\mathrm{d}x = (2 - 2x^2)^2\mathrm{d}x = 4(1 - x^2)^2\mathrm{d}x$.

To get the total volume we have to add up all the little square volumes that we would sweep out from the start of our figure at $x = -1$ to the end at $x = 1$. However, there are an infinite number of x values in this range and so what would normally be a sum, we write as an integral. Therefore,

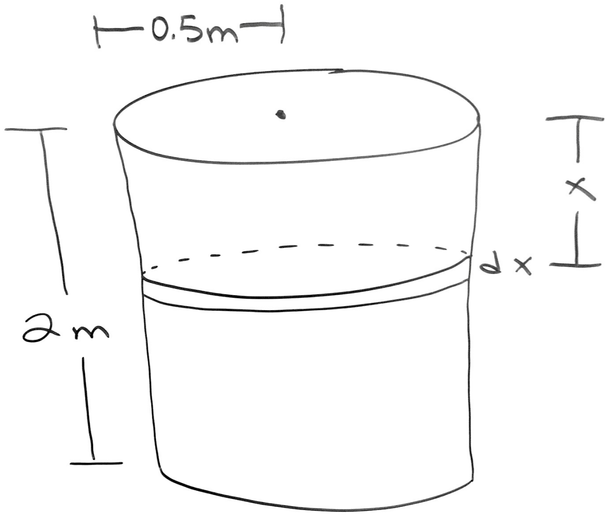
$$\begin{aligned} V = \int \mathrm{d}V &= \int_{-1}^{1} 4(1 - x^2)^2 \mathrm{d}x \\ &= 4\int_{-1}^{1} (x^4 - 2x^2 + 1)\mathrm{d}x \\ &= \frac{64}{15}. \end{aligned}$$

Therefore, the volume of the solid is $V = \frac{64}{15}$.

Question 3 (a)

Easiness: 4.8/5

SOLUTION. To solve this problem, we consider moving up the liquid in the cylinder slice by slice.



Consider the slice of the cylinder of height dx as in the picture. Its volume is given by $A\ dx$ where a is the area of the disc. This is

$$\pi(0.5)^2\,dx$$

Its mass is thus

$$1000\pi(0.5)^2\,dx = 250\pi\,dx$$

Since the density of the liquid is 1000 kg/m^3. The work to lift the slice to the top is *force* multiplied by *distance* which is

$$9.8 \cdot 250\pi\,dx \cdot x = 9.8 \cdot 250\pi x\,dx$$

where x is the distance from the top of the cylinder to the slice.
Hence the total work is given by integrating this from the top to the bottom of the cylinder, that is from 0 to 2 and we obtain the integral

$$W = \int_0^2 9.8 \cdot 250\pi x\,dx$$

Question 3 (b)

Easiness: 4.4/5

SOLUTION. First we compute the area A of the shape. This is given by

$$\begin{aligned} A &= \int_{-\pi/2}^{\pi/2} 1 + \sin(x)dx \\ &= x\big|_{-\pi/2}^{\pi/2} - \cos\big|_{-\pi/2}^{\pi/2} \\ &= \pi - \cos(\pi/2) + \cos(-\pi/2) \\ &= \pi \end{aligned}$$

The coordinates of the centroid are computed using the formulas

$$\bar{x} = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x(1+\sin(x))dx$$

and

$$\bar{y} = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \frac{1}{2}(1+\sin(x))^2 dx$$

We will compute $\bar{x}$ first.

$$\begin{aligned}\bar{x} &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x + x\sin(x)dx \\ &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} xdx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\sin(x)dx \\ &= 0 + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\sin(x)dx\end{aligned}$$

The first integral is zero because the function is odd. For the second we can use integration by parts.

$$\begin{aligned}\int x\sin(x)dx &= -x\cos(x) + \int \cos(x)dx \\ &= -x\cos(x) + \sin(x) + C\end{aligned}$$

That means if we plug in the limits for the definite integral we get

$$\begin{aligned}\bar{x} &= 1/\pi(-x\cos(x) + \sin(x))\big|_{-\pi/2}^{\pi/2} \\ &= 1/\pi(\sin(\pi/2) - \sin(-\pi/2)) \\ &= 2/\pi\end{aligned}$$

Now we compute $\bar{y}$

$$\begin{aligned}\bar{y} &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \frac{1}{2}(1+\sin(x))^2 dx \\ &= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} 1 + 2\sin(x) + \sin(x)^2 dx \\ &= \frac{1}{2\pi}\Big(\int_{-\pi/2}^{\pi/2} 1dx + 2\int_{-\pi/2}^{\pi/2} \sin(x)dx + \int_{-\pi/2}^{\pi/2} \sin(x)^2 dx\Big) \\ &= \frac{1}{2} + \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \sin(x)^2 dx\end{aligned}$$

To compute this we note $\sin(x)^2 = \frac{1-\cos(2x)}{2}$

$$\begin{aligned}\int \sin(x)^2 dx &= \int \frac{1-\cos(2x)}{2} dx \\ &= \frac{x}{2} - \frac{1}{4}\sin(2x) + C\end{aligned}$$

Hence

$$\bar{y} = \frac{1}{2} + \frac{1}{2\pi}(\frac{x}{2} - \frac{1}{4}\sin(2x))_{-\pi/2}^{\pi/2}$$
$$= \frac{3}{4}$$

Thus the centroid is $(\frac{2}{\pi}, \frac{3}{4})$

Question 4 (a)

Easiness: 2.8/5

SOLUTION. Set the following change of variable

$$u = \sqrt{x}$$

then we have

$$x = u^2$$

and so

$$dx = 2u\,du.$$

Making this substitution in the integral gives

$$\int \frac{\sqrt{x}}{x-1}dx = \int \frac{u}{u^2-1}\,2u\,du = 2\int \frac{u^2}{u^2-1}du.$$

Now we use the method of partial fractions. We have

$$\begin{aligned}
\int \frac{\sqrt{x}}{x-1}dx &= 2\int \frac{u^2}{u^2-1}du \\
&= 2\int \frac{u^2-1+1}{u^2-1}du \\
&= 2\int \left(1 + \frac{1}{u^2-1}\right)du \\
&= 2u + \int \frac{2}{u^2-1}du \\
&= u + \int \frac{2}{(u-1)(u+1)}du.
\end{aligned}$$

We want to find numbers A and B such that

$$\frac{2}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1}.$$

Multiplying both sides by $(u-1)(u+1)$ yields

$$2 = Au + A + Bu - B.$$

Equating coefficients of powers of u on both sides gives

$$\begin{cases} 0 = A + B \\ 2 = A - B \end{cases}$$

From the first of these equations we see that $A = -B$. So the second equation becomes

$$2 = 2A.$$

Hence $A = 1$ and $B = -A = -1$. Therefore, we have

$$\frac{2}{(u-1)(u+1)} = \frac{1}{u-1} - \frac{1}{u+1}.$$

Using this to rewrite the last integral above, we find

$$\begin{aligned} \int \frac{\sqrt{x}}{x-1} dx &= 2u + \int \frac{2}{(u-1)(u+1)} du \\ &= 2u + \int \frac{1}{u-1} du - \int \frac{1}{u+1} du \\ &= 2u + \ln|u-1| - \ln|u+1| + C \\ &= 2\sqrt{x} + \ln|\sqrt{x} - 1| - \ln|\sqrt{x} + 1| + C. \end{aligned}$$

Question 4 (b)

Easiness: 2.8/5

SOLUTION 1. For the change of variable $u = 1 + x^2$, we have

$$du = 2x\, dx.$$

So

$$x^3 \sqrt{1+x^2} dx = x^2 \sqrt{1+x^2}\, x\, dx = x^2 \sqrt{u}\, \frac{du}{2}$$

To write the remaining x^2 in terms of u, we note that since $u = 1 + x^2$ we have $x^2 = u - 1$. Thus

$$\begin{aligned} \int x^3 \sqrt{1+x^2} dx &= \int (u-1)\sqrt{u}\, \frac{du}{2} \\ &= \int (u-1) u^{1/2}\, \frac{du}{2} \\ &= \frac{1}{2} \int (u^{3/2} - u^{1/2}) du \\ &= \frac{1}{2} \left(\frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right) + C \\ &= \frac{1}{2} \left(\frac{2}{5} (1+x^2)^{5/2} - \frac{2}{3} (1+x^2)^{3/2} \right) + C \\ &= \frac{1}{5} (1+x^2)^{5/2} - \frac{1}{3} (1+x^2)^{3/2} + C. \end{aligned}$$

Solution 2. This integral could be done with a trig substitution as well. We notice inside the square root that we have $1+x^2$. This motivates a substitution

$$x = \tan(\theta)$$

so that we can make use of the identity

$$1 + \tan^2\theta = \sec^2\theta.$$

With this substitution we also have

$$\mathrm{d}x = \sec^2\theta \mathrm{d}\theta.$$

Putting this into our integral

$$\begin{aligned}\int x^3\sqrt{1+x^2}\mathrm{d}x &= \int \tan^3\theta\sqrt{\sec^2\theta}\sec^2\theta\mathrm{d}\theta \\ &= \int \tan^3\theta\sec^3\theta\mathrm{d}\theta\end{aligned}$$

where we have used the identity from above. This trig integral is of the form

$$\int \tan^m\theta\sec^n\theta\mathrm{d}\theta$$

with m and n odd. We want to write this in a form that we can easily integrate. If we let $u = \sec\theta$ then we have hope because we notice the derivative $\sec\theta\tan\theta$ is also there. Making this substitution with

$$\mathrm{d}u = \sec\theta\tan\theta\mathrm{d}\theta$$

we get

$$\begin{aligned}\int \tan^3\theta\sec^3\theta\mathrm{d}\theta &= \int \tan^2\theta(\tan\theta\sec\theta)\sec^2\theta\mathrm{d}\theta \\ &= \int(\sec^2\theta-1)\sec^2\theta(\tan\theta\sec\theta)\mathrm{d}\theta \\ &= \int(u^2-1)u^2\mathrm{d}u = \int u^4-u^2\mathrm{d}u \\ &= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C.\end{aligned}$$

We now have to substitute back,

$$u = \sec\theta = \sec\arctan x = \sqrt{x^2+1}.$$

Therefore we get

$$\int x^3\sqrt{x^2+1}\mathrm{d}x = \frac{1}{5}\left(\sqrt{x^2+1}\right)^5 - \frac{1}{3}\left(\sqrt{x^2+1}\right)^3 + C$$

Question 4 (c)

Easiness: 3.7/5

SOLUTION. First notice that attempting to evaluate this integral is long and tedious. Instead we attempt to find an integral which we know converges or diverges, and compare our integral to that integral. Observe that on the given interval, $[1,\infty)$ we have that $x^2+x > x^2$ and so $\frac{\sqrt{x}}{x^2+x} < \frac{\sqrt{x}}{x^2}$. Hence,

$$\begin{aligned}\int_1^\infty \frac{\sqrt{x}}{x^2+x}dx &< \int_1^\infty \frac{\sqrt{x}}{x^2}dx \\ &= \int_1^\infty \frac{1}{x^{3/2}}dx\end{aligned}$$

and since we know that the integral $\int_1^\infty \frac{1}{x^p}dx$ converges whenever $p>1$, we know that our integral does indeed converge since $3/2 > 1$.

Question 4 (d)

Easiness: 3.8/5

SOLUTION. By the integral test, this sum is convergent if and only if the integral

$$\int_2^\infty \left(\frac{1}{\ln x}\right)^2 \frac{1}{x}\,dx$$

is also convergent. Using the substitution $u = \ln x$,

$$\begin{aligned}&\int_2^\infty \left(\frac{1}{\ln x}\right)^2 \frac{1}{x}\,dx \\ &= \lim_{b\to\infty}\int_2^b \left(\frac{1}{\ln x}\right)^2 \frac{1}{x}\,dx \\ &= \lim_{b\to\infty}\int_{\ln 2}^{\ln b} \frac{1}{u^2}\,du \\ &= \lim_{b\to\infty} \left.-\frac{1}{u}\right|_{\ln 2}^{\ln b} \\ &= \lim_{b\to\infty}\left(-\frac{1}{\ln b}+\frac{1}{\ln 2}\right) \\ &= \frac{1}{\ln 2} < \infty.\end{aligned}$$

Hence, the sum is convergent.

Question 5 (a)

Easiness: 3.3/5

SOLUTION. We will denote by $S(t)$ the amount of sugar in the tank at time t (where t is in minutes). The differential equation that we are building should look like

S'(t) = (rate at which sugar is added to the tank) - (rate at which sugar is spilled out of the tank)

Every minute, there is 1 L of honey poured in the tank, that litre contains 1kg of sugar; and 9 L of sugar solution are also poured in, containing 900 g of sugar (since each litre of sugar solution contains 100 g of sugar). Hence all in all, 1.9 kg of sugar is poured in the tank every minute.

Now, we still have to deal with the water that is spilling on the floor. The question asks us to assume that the sugar solution and honey are instantly mixed to the rest of the tank, so when dealing with the fluid going out of the tank, we can assume it has the same concentration of sugar as in the rest of the tank. What is that concentration? Well there are *S(t)* kilograms of sugar in the tank at that time and since the tank is full, there is 200 L of liquid in it, so the concentration of sugar in the tank at time t is *S(t)*/200 kg/L. Since we are told that the fluid is spilling at an instantaneous rate of 10 L per minute, this means that every minute, there are

$$10 \cdot \frac{S(t)}{200} = \frac{1}{20} S(t)$$

kilograms of sugar spilled out of the tank.
Putting these two bit of information together, we obtain that the amount of sugar in the tank must satisfy the differential equation

$$\frac{dS}{dt} = 1.9 - \frac{1}{20} S(t).$$

Question 5 (b)

Easiness: 3.0/5

SOLUTION. Recall that the differential equation is given by

$$\frac{dS}{dt} = 1.9 - \frac{1}{20} S$$

We use separation of variables and bring all terms with S to the left hand side, and all terms involving t to the right hand side. So we multiply both sides with dt and divide by $1.9 - \frac{1}{20} S$:

$$\frac{dS}{1.9 - \frac{1}{20} S} = dt$$

Let's now integrate both sides:

$$\int \frac{dS}{1.9 - \frac{1}{20} S} = \int dt$$

The integral on the right hand side is simply $t + C_1$. To solve the left hand side, use the substitution $u = 1.9 - \frac{1}{20} S,\ dS = -20\, du$ and obtain

$$\int \frac{dS}{1.9 - \frac{1}{20} S} = \int \frac{-20\, du}{u} = -20 \ln |u| = -20 \ln \left| 1.9 - \frac{S}{20} \right| = t + C_1.$$

Solving the above for S yields

$$
\begin{aligned}
-20 \ln \left|1.9-\frac{S}{20}\right| &= t+C_1 \\
\ln \left|1.9-\frac{S}{20}\right| &= -\frac{t}{20}-\frac{C_1}{20} \\
\left|1.9-\frac{S}{20}\right| &= e^{-\frac{t}{20}-\frac{C_1}{20}} \\
1.9-\frac{S}{20} &= \pm e^{-\frac{C_1}{20}} e^{-\frac{t}{20}} \\
S &= 20\left(1.9-C_2 e^{-t/20}\right),
\end{aligned}
$$

with a different constant $C_2 = \pm e^{-C_1/20}$. Now we use the initial condition $S(0) = 0$ to solve for C_2:

$$0 = 20\left(1.9 - C_2 e^0\right) = 20(1.9 - C_2),$$

hence $C_2 = 1.9$ and therefore the final answer is

$$S(t) = 20\left(1.9 - 1.9 e^{-t/20}\right).$$

Question 5 (c)

Easiness: 3.0/5

SOLUTION. Pluggin in $t=20$ we obtain

$$
\begin{aligned}
S(20) &= 20\left(1.9 - 1.9 e^{-20/20}\right) \\
&= 20\left(1.9 - \frac{1.9}{e}\right) \\
&= 38 - \frac{38}{e} \\
&\leq 38 - 10 < 30.
\end{aligned}
$$

Remember that $e \approx 2.7$ and hence $\frac{38}{e} > 10$.
That means that after 20 minutes there is less than 30kg of sugar in the tank. Thus the solution cannot be sold.

Question 6 (a)

Easiness: 3.4/5

SOLUTION. We have the series expansion

$$\frac{1}{1+x^3} = 1 - x^3 + x^6 - x^9 + \cdots = \sum_{n=0}^{\infty} (-x)^{3n}$$

and so, integrating term by term we find that

$$\int \frac{1}{1+x^3} dx = \sum_{n=0}^{\infty} \int (-x)^{3n} dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+1}}{3n+1}$$

It only remains to find the radius of convergence. The computing the limit in the ratio test yields

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{x^{3(n+1)+1}}{3(n+1)+1}}{(-1)^n\frac{x^{3n+1}}{3n+1}}\right| \\
&= \lim_{n\to\infty}\left|\frac{\frac{x^{3n+4}}{3n+4}}{\frac{x^{3n+1}}{3n+1}}\right| \\
&= \lim_{n\to\infty}\frac{|x|^{3n+4}}{|x|^{3n+1}}\frac{3n+1}{3n+4} \\
&= \lim_{n\to\infty}|x|^3\frac{n(3+1/n)}{n(3+4/n)} \\
&= \lim_{n\to\infty}|x|^3\frac{3+1/n}{3+4/n} \\
&= |x|^3
\end{aligned}$$

Now the ratio test tells us that the series converges whenever $|x|^3 < 1$ and thus our radius of convergence is 1.

Question 6 (b)

Easiness: 4.5/5

SOLUTION. We saw from part (a) that we have the series expansion

$$\frac{1}{1+x^3} = 1 - x^3 + x^6 - x^9 + \dots$$

and so we can write that

$$\begin{aligned}
\int_0^{0.1}\frac{1}{1+x^3}\,dx &= \int_0^{0.1}dx - \int_0^{0.1}x^3\,dx + \int_0^{0.1}x^6\,dx - \int_0^{0.1}x^9\,dx + \dots \\
&= 0.1 - \frac{(0.1)^4}{4} + \frac{(0.1)^7}{7} - \frac{(0.1)^{10}}{10} + \dots
\end{aligned}$$

where the $(n+1)^{\text{st}}$ term is

$$\frac{(-1)^n(0.1)^{3n+1}}{3n+1}$$

Since this is an alternating sequence which converges to zero, we have that the error when neglecting everything past the $(n+1)^{\text{st}}$ term. Thus

$$\begin{aligned}
|\text{error}| &\le \left|\frac{(-1)^n(0.1)^{3n+1}}{3n+1}\right| \\
&= \frac{0.1^{3n+1}}{3n+1} \\
&= \frac{10^{-(3n+1)}}{3n+1}
\end{aligned}$$

if we want this to be less than 10^{-9}, then we can choose $n = 3$, where the error will be less than

$$\frac{10^{-(9+1)}}{9+1} = 10^{-11}$$

which works fine since it is clearly even less than 10^{-9}.

Question 7

Easiness: 4.5/5

SOLUTION. Since $f(a+i(b-a)/N) = f(-1+3i/N) = -1+3i/N-1$ we follow the hints, to obtain

$$\begin{aligned}\int_{-1}^{2}(x-1)\,dx &= \lim_{N\to\infty}\sum_{i=1}^{N} f(-1+3i/N)\frac{3}{N}\\ &= \lim_{N\to\infty}\sum_{i=1}^{N}(-1+3i/N-1)\frac{3}{N}\\ &= \lim_{N\to\infty}\frac{3}{N}\sum_{i=1}^{N}\left(-2+\frac{3i}{N}\right)\\ &= \lim_{N\to\infty}\left(-\frac{6}{N}\sum_{i=1}^{N}1+\frac{9}{N^2}\sum_{i=1}^{N}i\right)\end{aligned}$$

Since

$$\sum_{i=1}^{N} i = N \quad \text{and} \quad \sum_{i=1}^{N} i = \frac{N(N+1)}{2},$$

we can continue our computation

$$\begin{aligned}\int_{-1}^{2}(x-1)\,dx &= \lim_{N\to\infty}\left(-\frac{6}{N}N+\frac{9}{N^2}\frac{N(N+1)}{2}\right)\\ &= -6+\frac{9}{2}\lim_{N\to\infty}\frac{N+1}{N}\\ &= -6+\frac{9}{2}\\ &= -\frac{3}{2}.\end{aligned}$$

You can always compare your answer to the computation of the integral directly and see that the answers match:

$$\begin{aligned}\int_{-1}^{2}(x-1)\,dx &= \frac{x^2}{2}-x\bigg|_{-1}^{2}\\ &= \frac{4}{2}-2-\left(\frac{1}{2}+1\right)\\ &= -\frac{3}{2}.\end{aligned}$$

Question 8 **Easiness: 4.5/5**

SOLUTION. To find the radius of convergence, we use the ratio test. Let $a_n = \frac{\ln(n+2)}{|x-2|^n}$.

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{|x-2|^{n+1}}{\ln(n+1+2)} \cdot \frac{\ln(n+2)}{|x-2|^n} \\ &= |x-2|\frac{\ln(n+2)}{\ln(n+3)} \to |x-2| \text{ as } n \to \infty.\end{aligned}$$

To see that $\frac{\ln(n+2)}{\ln(n+3)} \to 1$ as $n \to \infty$, note that

$$\left|\frac{\ln(n+2)}{\ln(n+3)} - 1\right| = \left|\frac{\ln(n+2) - \ln(n+3)}{\ln(n+3)}\right| = \left|\frac{\ln\left(\frac{n+2}{n+3}\right)}{\ln(n+3)}\right|$$

Since logarithm is continuous on its domain, the numerator satisfies

$$\lim_{n\to\infty} \ln\left(\frac{n+2}{n+3}\right) = \ln\left(\lim_{n\to\infty}\frac{n+2}{n+3}\right) = \ln\left(\lim_{n\to\infty}\frac{1+2/n}{1+3/n}\right) = \ln(1) = 0.$$

In the denominator, $\ln(n+3) \to \infty$ as $n \to \infty$.
We conclude that
$\left|\frac{\ln(n+2)}{\ln(n+3)} - 1\right| \to 0$; that is, $\frac{\ln(n+2)}{\ln(n+3)} \to 1$, as $n \to \infty$.
By the ratio test, it follows that the series is absolutely convergent if and only if $|x-2| < 1$. Therefore, the radius of convergence is equal to 1.
In order to determine the interval of convergence, it remains to determine whether the series is convergent at the endpoints, $x = 2 \pm 1$.
At $x = 3$, the series is equal to

$$\sum_{n=1}^{\infty} \frac{(3-2)^n}{\ln(n+2)} = \sum_{n=1}^{\infty} \frac{1}{\ln(n+2)}.$$

By comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$, this series is divergent. (To compare, note that $\frac{1}{\ln(n+2)} > \frac{1}{n+2} \geq \frac{1}{3n}$.)
At $x = 1$ the series is equal to

$$\sum_{n=1}^{\infty} \frac{(1-2)^n}{\ln(n+2)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+2)}$$

By the alternating series test, this converges.
Hence, the interval of convergence is $\boxed{[1,3)}$.

Good Luck for your exams!