Full Solutions
MATH100 December 2011

April 4, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

SOLUTION. When we evaluate the function at $t = -2$ we obtain $0/0$. Since this is a rational function, it means that we can factor $t + 2$ in each of the denominator and numerator:

$$2t^2 + t - 6 = (t + 2)(2t - 3)$$

and

$$t^2 - 4 = (t + 2)(t - 2)$$

Hence

$$\begin{aligned}
\lim_{t\to-2} \frac{2t^2 + t - 6}{t^2 - 4} &= \lim_{t\to-2} \frac{(t + 2)(2t - 3)}{(t + 2)(t - 2)} \\
&= \lim_{t\to-2} \frac{(2t - 3)}{(t - 2)} \\
&= \frac{2(-2) - 3}{(-2) - 2} \\
&= \frac{7}{4}
\end{aligned}$$

Question 1 (b)

SOLUTION. If you try to evaluate the limit directly, you end up with $0/0$, which means we cannot conclude anything for the moment, more work needs to be done. Since we have a square root, we will multiply on each side of the fraction by the conjugate in the hope to simplify things:

$$\begin{aligned}
\lim_{x\to1} \frac{\sqrt{x + 3} - 2}{x - 1} &= \lim_{x\to1} \frac{(\sqrt{x + 3} - 2)}{(x - 1)} \frac{(\sqrt{x + 3} + 2)}{(\sqrt{x + 3} + 2)} \\
&= \lim_{x\to1} \frac{(x + 3) - 4}{(x - 1)(\sqrt{x + 3} + 2)} \\
&= \lim_{x\to1} \frac{x - 1}{(x - 1)(\sqrt{x + 3} + 2)} \\
&= \lim_{x\to1} \frac{1}{(\sqrt{x + 3} + 2)} \\
&= \frac{1}{4}
\end{aligned}$$

Question 1 (c)

SOLUTION. If you try to evaluate the limit directly, you end up with ∞/∞, which means we cannot conclude anything for the moment, more work needs to be done. Here we have

$$
\begin{aligned}
\lim_{x\to\infty} \frac{2x}{\sqrt{9x^2+x-1}} &= \lim_{x\to\infty} \frac{2}{\frac{\sqrt{9x^2+x-1}}{x}} \\
&= \lim_{x\to\infty} \frac{2}{\frac{\sqrt{9x^2+x-1}}{\sqrt{x^2}}} \\
&= \lim_{x\to\infty} \frac{2}{\sqrt{\frac{9x^2+x-1}{x^2}}} \\
&= \lim_{x\to\infty} \frac{2}{\sqrt{9+\frac{1}{x}-\frac{1}{x^2}}} \\
&= \frac{2}{\sqrt{9}} \\
&= \frac{2}{3}.
\end{aligned}
$$

Question 1 (d)

SOLUTION. Our goal is to find *f(3)*. Since *f(x)* is continuous we know that $f(3) = \lim_{x\to 3} f(x)$ and thus finding *f(3)* amounts to finding the limit.
Using the limit laws we have

$$1 = \lim_{x\to 3}(xf(x) + g(x)) = \lim_{x\to 3} x \lim_{x\to 3} f(x) + \lim_{x\to 3} g(x).$$

Using that both functions x and *g(x)* are continuous, we have that

$$\lim_{x\to 3} x = 3$$

and

$$\lim_{x\to 3} g(x) = g(3) = 2.$$

Plugin this into the above equality, we find

$$1 = 3\lim_{x\to 3} f(x) + 2$$

and hence

$$\lim_{x\to 3} f(x) = -\frac{1}{3}.$$

By the first remark above we have

$$f(3) = -\frac{1}{3}.$$

Question 1 (e)

SOLUTION. To find the equation of the tangent line to $y = x^{3.5}$ - $e^{3.5}$ at the point (e, 0), we must first find the derivative of the curve, which is the slope of the tangent line at that point.
First we use the difference rule to separate the two terms.

$$\frac{d}{dx}(x^{3.5} - e^{3.5}) = \frac{d}{dx}(x^{3.5}) - \frac{d}{dx}(e^{3.5})$$

The next step is to recognize that $e^{3.5}$ is a constant, so we know that its derivative is equal to 0 and we can remove that term altogether from the equation. Then we use to power rule to differentiate $x^{3.5}$:

$$\frac{d}{dx}(x^{3.5}) = 3.5x^{3.5-1} = 3.5x^{2.5}$$

By plugging in $x = e$ into the derivative, we find the tangent line at the desired point (e, 0), we get the slope m of the tangent line of equation y-$y_0 = m(x$ - $x_0)$

$$m = 3.5(e^{2.5})$$

From this we can conclude that the equation of the line is:

$$y - y_0 = m(x - x_0) \quad \Rightarrow \quad y - 0 = 3.5e^{2.5}(x - e)$$

Simplifying, we get

$$y = 3.5e^{2.5}(x - e)$$

or

$$y = 3.5e^{2.5}x - 3.5e^{3.5}$$

Question 1 (f)

SOLUTION. Using the quotient rule we compute

$$\begin{aligned}\left(\frac{x}{x^2-1}\right)' &= \frac{(1)(x^2-1) - (2x)(x)}{(x^2-1)^2} \\ &= \frac{x^2 - 1 - 2x^2}{(x^2-1)^2} \\ &= -\frac{x^2+1}{(x^2-1)^2}\end{aligned}$$

Question 1 (g)

Solution. The two variables that are changing in this problem are the volume of water and the height of the water. These two variables are related in the formula for volume of a cylinder:

$$V = \pi r^2 h$$

For this problem, we know that the radius of the cylinder is 3, so we can put that into our formula to get:

$$V = 9\pi h$$

Since the question is asking about rates of change, we will differentiate the above formula with respect to t.

$$\frac{dV}{dt} = 9\pi \frac{dh}{dt}$$

In this new formula, dh/dt stands for the rate at which the height of the water is changing, which is what we're solving for. We also know that dV/dt stands for the rate at which the volume of water is changing, which is $5\ m^3/min$. So substituting in $dV/dt = 5$ we have:

$$5 = 9\pi \frac{dh}{dt}$$

And solving for dh/dt we get:

$$\frac{dh}{dt} = \frac{5}{9\pi} m/min$$

Question 1 (h)

Solution. In order to find the slope of the tangent line, we need to take the derivative. However, because this expression is not written in the form $y = f(x)$, we will have to use implicit differentiation with respect to x.

Thus we implicitly differentiate the expression with respect to x. Note that the term x^2 can be differentiated directly, the term xy requires the product rule, and the term y^2 requires the chain rule.

$$2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

Since we are finding a specific derivative at the point $(0,2)$, we can go ahead and plug in the values $x=0$ and $y=2$ to get:

$$2(0) + (2) + (0)\frac{dy}{dx} + 2(2)\frac{dy}{dx} = 0$$

$$2 + 4\frac{dy}{dx} = 0$$

If we solve the above expression for the derivative dy/dx we get:

$$\frac{dy}{dx} = \frac{-2}{4} = \frac{-1}{2}$$

So the slope of the tangent line is -1/2.

Question 1 (i)

SOLUTION. Using the fact that

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}},$$

and the chain rule, we get

$$\begin{aligned}\frac{d}{dx}y &= \frac{d}{dx}\sin^{-1}(3x+1)\\ &= \frac{1}{\sqrt{1-(3x+1)^2}}\cdot\frac{d}{dx}(3x+1)\\ &= \frac{3}{\sqrt{1-(3x+1)^2}}.\end{aligned}$$

Question 1 (j)

SOLUTION. In this case we need to use logarithmic differentiation, indeed it is impossible to use the power rule since there is an x^2 as exponent. We cannot differentiate it in the same way we do for exponential functions since we take the power of x and not e or another constant. Hence we need to use logarithmic differentiation. First set

$$y = x^{(x^2)},$$

then taking the logarithm on both sides we find

$$\ln y = \ln(x^{(x^2)}) = x^2\ln(x).$$

Now keeping in mind that y is a function of x, we can take the derivatives on both sides and get

$$\frac{d}{dx}\ln y = \frac{d}{dx}(x^2\ln(x)).$$

The left-hand side is simply

$$\frac{d}{dx}\ln y = \frac{1}{y}\cdot\frac{d}{dx}y = \frac{1}{y}y',$$

while the right-hand side gives, using the product rule,

$$\frac{d}{dx}(x^2\ln(x)) = 2x\ln(x) + x^2\frac{1}{x} = 2x\ln(x) + x.$$

Putting everything back together, we get

$$\frac{1}{y}y' = 2x\ln(x) + x,$$

and thus

$$y' = y(2x\ln(x) + x).$$

Replacing y by $x^{(x^2)}$, we finally get

$$y' = x^{(x^2)}(2x\ln(x) + x).$$

Question 1 (k)

SOLUTION. We compute this derivative using the chain rule twice and the product rule

$$\begin{aligned}\frac{dy}{dt} &= e^{t\cos(2t)} \cdot \frac{d}{dt}(t\cos(2t)) \\ &= e^{t\cos(2t)} \cdot (\frac{d}{dt}(t) \cdot \cos(2t) + t\frac{d}{dt}(\cos(2t))) \\ &= e^{t\cos(2t)} \cdot (1 \cdot \cos(2t) + t(-\sin(2t) \cdot 2)) \\ &= e^{t\cos(2t)} (\cos(2t) - 2t\sin(2t))\end{aligned}$$

Question 1 (l)

SOLUTION 1. We are being told that the instantaneous rate of change of the function is at least of 3 on the interval [-1,2], so the smallest possible value of $f(2)$ is the case where the function has the smallest rate of change, that is a constant rate of change of 3 which means it is a straight line (of slope 3).
And since we are starting at the point (-1,9) with a slope of 3 and a run of 3 (from -1 to 2) we will end up with a rise of 9 and so the minimal value of $f(2)$ is 18.

SOLUTION 2. Alternatively, you can use the MVT (Mean Value Theorem), which states that if *f(x)* is defined and continuous on the interval *[a,b]* and differentiable on *(a,b)*, then there is at least one number c in the interval *(a,b)* (that is $a < c < b$) such that
$f'(c) = \frac{f(b)-f(a)}{b-a}$
If we assume $f(x)$ is differentiable on *(-1,2)* and differentiable and continuous on the interval *[-1,2]*, we can use the MVT which gives:
$\frac{f(2)-f(-1)}{2-(-1)} = f'(c)$
For some value c in the interval *(-1,2)*. However, we know that $f'(x) \geq 3$ for any value between -1 and 2, thus, $f'(c) \geq 3$. Adding this, plugging in known values, and simplifying gives:
$\frac{f(2)-9}{3} \geq 3$
Solving for *f(2)* we get $f(2) \geq 18$, thus the minimal value of *f(2)* is 18.

Question 1 (m)

SOLUTION. The strength of Newton's method is on root finding. We're asked to approximate

$$x = 2^{1/6}$$

which we can write as a root finding problem,

$$x^6 - 2 = 0.$$

With this in mind, let

$$f(x) = x^6 - 2.$$

Taking the derivative gives

$$f'(x) = 6x^5.$$

We start with $x_1 = 1$ as our first iteration and so if we apply Newton's Method, we get,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{-1}{6} = 1 + \frac{1}{6} = \frac{7}{6}.$$

If we actually compute $2^{1/6}$ we get

$$2^{1/6} \approx 1.122462048$$

and so $\frac{7}{6} = 1.166666667$ is already in good agreement.

Question 1 (n)

SOLUTION. The limit definition of the derivative at a point $x{=}a$ is:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

We notice that the term $cos(\pi/3 + h)$ in the statement of the question resembles the $f(x{+}h)$ term in the limit definition of the derivative. If this is the case, $f(x) = cos(x)$ and $a = \pi/3$.
We now test these guesses using the limit definition of the derivative. If $f(x) = cos(x)$ and $a = \pi/3$, then

$$\begin{aligned} f'\left(\frac{\pi}{3}\right) &= \lim_{h \to 0} \frac{\cos(\frac{\pi}{3} + h) - \cos(\frac{\pi}{3})}{h} \\ &= \lim_{h \to 0} \frac{\cos(\frac{\pi}{3} + h) - \frac{1}{2}}{h} \end{aligned}$$

If we multiply both sides by 2 we get

$$2f'\left(\frac{\pi}{3}\right) = \lim_{h \to 0} \frac{2\cos(\frac{\pi}{3} + h) - 1}{h}$$

This is the expression we were trying to find, so our guess of the function $f(x)$ and the value a were correct

$$f(x) = cos(x)$$

and $a = \pi/3$.

Question 2 (a)

SOLUTION. Newton's Law of Cooling which states that

$$T(t) = T_a + (T_0 - T_a)e^{-kt}.$$

Here, the initial temperature of the body T_0 is 33°C; the temperature of the environment T_{a} is given to be 21°C; and the temperature of the body after 1 hour ($T(1)$) is 31°C. Plugging in these numbers give us

$$31 = 21 + (33 - 21)e^{-k}$$

Using this equation, we can solve for k:

$$\begin{aligned} e^{-k} &= \frac{31-21}{33-21} = \frac{5}{6} \\ -k &= \ln(5/6) \\ k &= -\ln(5/6) \end{aligned}$$

And so now we have found the equation that gives us the temperature of the body at any time t:

$$T(t) = 21 + 12e^{t\ln(5/6)}$$

and we would like to know the time of death, which is the time at which the body temperature was 37°C. For this, we simply solve

$$T(t) = 37$$

that is

$$\begin{aligned} 21 + 12e^{t\ln(5/6)} &= 37 \\ 12e^{t\ln(5/6)} &= 16 \\ e^{t\ln(5/6)} &= 16/12 \\ t\ln(5/6) &= \ln(4/3) \\ t &= \frac{\ln(4/3)}{\ln(5/6)} \\ t &\approx -1.578 \end{aligned}$$

Therefore, the police arrived $(\ln(3/4))/(\ln(5/6))$ hours after the murder, that is, roughly an hour and a half.

Question 2 (b)

SOLUTION. We know from Question 2 (a) that the police arrived -ln(4/3)/ln(5/6) hours after the murder (we add a negative sign because we look at this as a positive amount of time now).
So the question is whether -ln(4/3)/ln(5/6) is larger than 1 or not. Let's see if this is meaningful or not. We know that $\ln(x)$ is positive if x is larger than 1. So ln(4/3) is positive and ln(5/6) is negative. Since -ln(5/6) = ln(6/5), we can say that the (positive) amount of time it took for the police to get to the crime scene is ln(4/3)/ln(6/5).

Which of 4/3 or 6/5 is larger? Well, they are the same as 20/15 and 18/15 respectively, so the denominator is smaller than the numerator and likewise if we take a logarithm, hence $\ln(4/3) > \ln(6/5)$ and so

$$\frac{\ln(4/3)}{\ln(6/5)} > 1$$

which means that the police arrived MORE than an hour after the murder, that is, the murder took place BEFORE 9:00PM.

Question 3 (a)

SOLUTION. If $s = f($"t *) is the function describing the position of the particle, then its derivative f '(*t") is its velocity and its second derivative the acceleration. This means we know that

$$f''(t) = a(t)$$

and so

$$f''(t) = 12t - 30$$

Since this is a polynomial in t of degree one, we know that the velocity must be a polynomial of degree 2:

$$f'(t) = at^2 + bt + c$$

whose derivative is 12t-30

$$\frac{d}{dt}\left(at^2 + bt + c\right) = 2at + b$$

Hence we have

$$2a = 12 \qquad \text{and} \qquad b = -30$$

And so the velocity is

$$f'(t) = 6t^2 - 30t + c$$

for some constant c. Since the velocity after 2 seconds is 0, we actually are being told that f'(2)=0 and hence

$$6 \cdot 4 - 30 \cdot 2 + c = 0 \qquad \Rightarrow c = 36$$

And so the velocity is

$$f'(t) = 6t^2 - 30t + 36$$

Now, the velocity being the derivative of the position function, we can deduce that it has to be a polynomial of degree 3 and write

$$f(t) = At^3 + Bt^2 + Ct + D$$

and computing its derivative gives us

$$f'(t) = 3At^2 + 2Bt + C$$

Hence

$$3A = 6 \qquad \text{and} \qquad 2B = -30 \qquad \text{and} \qquad C = 36$$

So the position function is

$$f(t) = 2t^3 - 15t^2 + 36t + D$$

and we can find out the value of D since we are told that $f(0) = 0$ which gives

$$D = 0$$

And so finally, we obtain that the position function is:

$$f(t) = 2t^3 - 15t^2 + 36t$$

Question 3 (b)

SOLUTION. We are given acceleration and asked to argue about distance. Since the derivative of a position function is velocity and the derivative of a velocity function is acceleration, we with to antidifferentiate our original function twice. This was done in part (a) giving the answer of

$$f(t) = 2t^3 - 15t^2 + 36t$$

As this is a position function, we need to check when the particle reverses direction. This occurs potentially at critical points. We know the derivative of $f(t)$ is $v(t)$ and so, setting this equal to zero and solving gives

$$0 = f'(t) = 6t^2 - 30t + 36 = 6(t^2 - 5t + 6) = 6(t-3)(t-2)$$

and hence the critical points are at $t = 2, 3$. Since we only care about time up to $t = 3$, we only need to check whether the point at $t = 2$ is a maximum or minimum. Using the second derivative test we see that

$$f''(2) = v'(2) = a(2) = 12(2) - 30 = -6 < 0$$

and thus t=*2* is a maximum of f. In particular, this means that our particle changes directions at t=2. Thus, we calculate the total distance traveled up to t=2 first then from t=2 until t=3:

$$\begin{aligned} f(0) &= 2(0)^3 - 15(0)^2 + 36(0) \\ &= 0 \end{aligned}$$

$$\begin{aligned} f(2) &= 2(2)^3 - 15(2)^2 + 36(2) \\ &= 2(8) - 15(4) + 72 \\ &= 16 - 60 + 72 \\ &= 28 \end{aligned}$$

$$\begin{aligned} f(3) &= 2(3)^3 - 15(3)^2 + 36(3) \\ &= 2(27) - 15(9) + 108 \\ &= 54 - 135 + 108 \\ &= 27 \end{aligned}$$

The total distance traveled from t=0 to t=2 is:

$$|f(2) - f(0)| = |28 - 0| \qquad = 28$$

Then, we can calculate the total distance traveled from t=2 to t=3:

$$|f(3) - f(2)| = |27 - 28| \qquad = 1$$

We add these two values together to obtain the total distance traveled in the first three seconds, which we find to be 29 metres.

Question 4 (a)

SOLUTION. The general forumla for a second degree Taylor polynomial is:

$$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

To find this for $f(x)$, we need to first find $f'(x)$ and $f''(x)$:

$$f'(x) = (\sqrt[3]{x})' = (x^{1/3})' = \frac{1}{3}x^{-2/3}$$

$$f''(x) = (\frac{1}{3}x^{-2/3})' = -\frac{2}{9}x^{-5/3}$$

And now we can evaluate at $a = 8$ to get

$$f(8) = 8^{1/3} = 2$$

$$f'(8) = \frac{1}{3}8^{-2/3} = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$$

$$f''(8) = -\frac{2}{9}8^{-5/3} = -\frac{2}{9} \cdot \frac{1}{2^5} = -\frac{2}{9 \cdot 32} = -\frac{1}{144}$$

We can now plug these in to the equation for the second Taylor polynomial and obtain:

$$T_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$$

(This is simplified enough for the purpose of Taylor polynomials).

Question 4 (b)

SOLUTION. Lagrange's Remainder Formula states that:

$$\begin{aligned} R_n(x) &= f(x) - T_n(x) \\ &= \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{(n+1)} \end{aligned}$$

Where c is a number in $[x, a]$.
When $R_n(x)$ is positive, then $T_n(x)$ is an underestimate; if it is negative, then $T_n(x)$ is an overestimate. For this problem,

$$\begin{aligned} f(x) &= \sqrt[3]{x} = x^{1/3} \\ f'(x) &= \frac{1}{3x^{2/3}} \\ f''(x) &= -\frac{2}{9x^{5/3}} \\ f'''(x) &= \frac{10}{(27x^{8/3})} \\ (x-a)^{(n+1)} &= (8.1-8)^3 = 0.001 \end{aligned}$$

Since $f(8.1)$ and $(x-a)^{n+1}$ are both positive, we can conclude that $R_n(8.1)$ is also positive since the factorial is positive as well. This means that:

$$T_2(8.1) < \sqrt[3]{8.1}$$

Question 5 (a)

SOLUTION 1. Critical points (numbers) are roots of the derivative. Let's compute this derivative:

$$f'(x) = \frac{6}{5}x^{-4/5} + \frac{6}{5}x^{1/5}$$

And now find the roots:

$$\frac{6}{5}x^{-4/5} + \frac{6}{5}x^{1/5} = 0$$

We can divide by 6/5

$$x^{-4/5} = -x^{1/5}$$

Now, clearly $x = 0$ is a solution. To find possibly others, we now assume that x isn't zero and now multiply by $x^{4/5}$

$$1 = -x$$

And obtain a second solution. So we can conclude that there are two critical points: $x = 0$ and $x =$ -1.

Solution 2. Critical points (numbers) are roots of the derivative. Let's compute this derivative:

$$f'(x) = \frac{6}{5}x^{-4/5} + \frac{6}{5}x^{1/5}$$

And now we can do a little algebra on the function and rewrite as

$$\begin{aligned} f'(x) &= \frac{6}{5}x^{-4/5} + \frac{6}{5}x^{1/5} \\ &= \frac{6}{5}x^{1/5}(x^{-1} + 1) \end{aligned}$$

And so the critical points are solutions of either

$$x^{1/5} = 0 \quad \Longleftrightarrow \quad x = 0$$

or

$$x^{-1} + 1 = 0 \quad \Longleftrightarrow \quad x = -1$$

And so we have two critical points: $x = 0$ and $x =$ -1.

Question 5 (b)

Solution. The intervals of decrease are the intervals on which the derivative of the function f is negative. Given that

$$\begin{aligned} f'(x) &= \frac{6}{5}x^{-4/5} + \frac{6}{5}x^{1/5} \\ &= \frac{6x^{-4/5}(1 + x)}{5} \\ &= \frac{6(x + 1)}{5x^{4/5}} \end{aligned}$$

Our critical points occur when $x = -1, 0$, that is, points on the function where the derivative is undefined or zero. Writing out a sign chart, we see that when $x < -1$ we have that $f'(x) < 0$. When $-1 < x < 0$ we have that $f'(x) > 0$. When $0 < x$ we have that $f'(x) > 0$. Hence the function is decreasing on $(-\infty, -1)$.

Question 5 (c)

Solution 1. From the initial form of the derivative, as computed in part (a),

$$f'(x) = \frac{6}{5}x^{-4/5} + \frac{6}{5}x^{1/5}$$

we find that the second derivative is

$$\begin{aligned} f''(x) &= \left(\frac{6}{5}\right)\left(-\frac{4}{5}\right)x^{-9/5} + \left(\frac{6}{5}\right)\left(\frac{1}{5}\right)x^{-4/5} \\ &= \frac{6}{25}\left(-4x^{-9/5} + x^{-4/5}\right) \\ &= \frac{6}{25x^{9/5}}(-4+x) \end{aligned}$$

SOLUTION 2. Using the compact form of the derivative, as computed in part (b),

$$f'(x) = \frac{6(x+1)}{5x^{4/5}}$$

we find the second derivative by taking the quotient rule:

$$\begin{aligned} f''(x) &= \frac{6}{5}\left(\frac{x^{4/5} - \frac{4}{5}x^{-1/5}(x+1)}{x^{8/5}}\right) \\ &= \frac{6x^{-1/5}(x - \frac{4}{5}x - \frac{4}{5})}{5x^{8/5}} \\ &= \frac{6(\frac{1}{5}x - \frac{4}{5})}{5x^{9/5}} \\ &= \frac{6(x-4)}{25x^{9/5}} \end{aligned}$$

Question 5 (d)

SOLUTION. From part (c), we know that

$$f''(x) = \frac{6(x-4)}{25x^{9/5}}$$

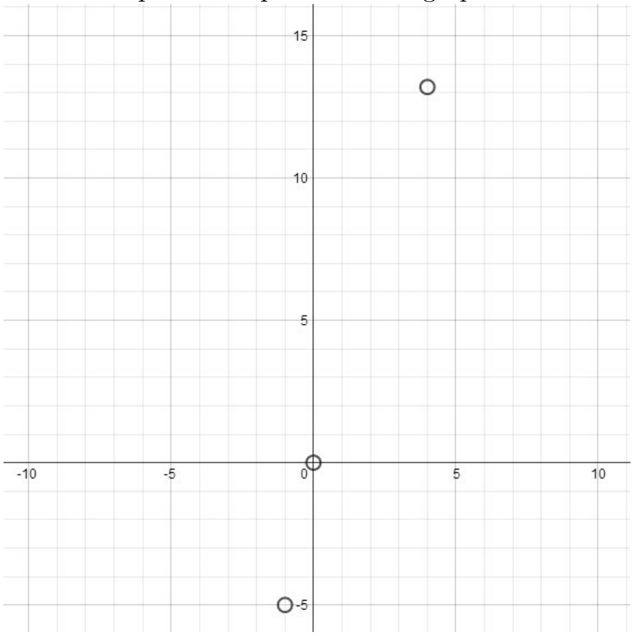
Notice that the potential inflection points are at x=0,4, where $f''(x)$ is either zero or undefined. Doing a sign chart for this function yields the following. When $x < 0$ we have that $f''(x) > 0$. When $0 < x < 4$ we have that $f''(x) < 0$. When $x > 4$ we have that $f''(x) > 0$. Hence, the function is concave up on $(-\infty, 0) \cup (4, \infty)$. *Note: Since the sign of $f''(x)$ changes at both points, both $x = 0$ and $x = 4$ are inflection points. Even though $f''(x)$ is not defined at $x = 0$.*

Question 5 (e)

SOLUTION. From parts (a) and (d), we know that the critical values for the first and second derivative of $f(x)$ are $x = -1, x = 0$, and $x = 4$. We first calculate the corresponding y-values for the x-values named above ...

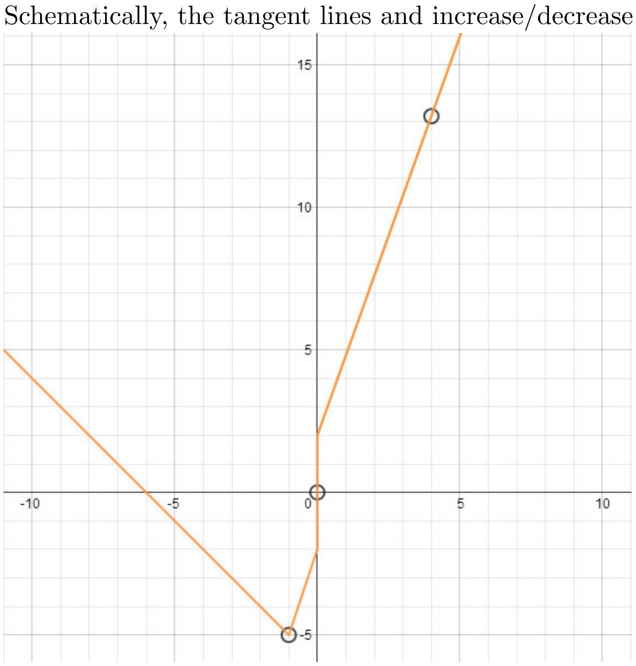
- $x = -1$, $y = f(-1) = 6(-1)^{1/5} + (-1)^{6/5} = -6 + 1 = -5$
- $x = 0$, $y = f(0) = 6(0)^{1/5} + (0)^{6/5} = 0$
- $x = 4$, $y = f(4) = 6(4)^{1/5} + (4)^{6/5} \approx 13.2$

...and then plot these points on the graph.



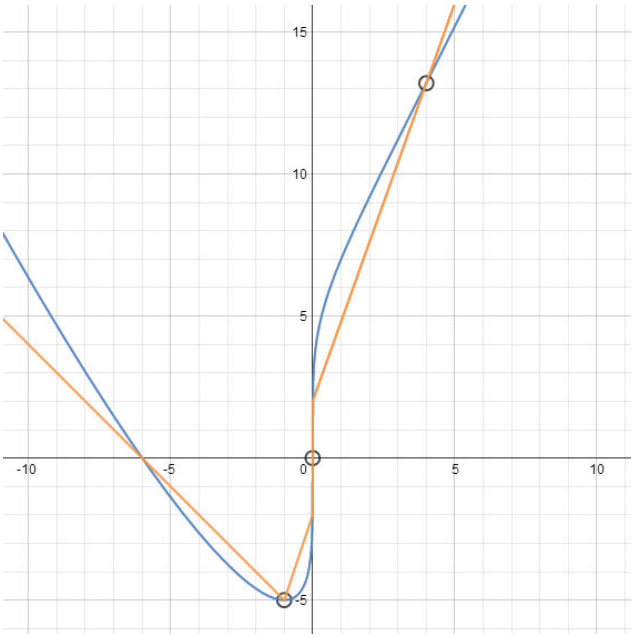
From part (b), we know the function is decreasing until the point $(-1, -5)$ and then increasing. Thus the function has a minimum at that point. Because the derivative is undefined at $(0, 0)$, it has a vertical tangent line there.

Schematically, the tangent lines and increase/decrease of the function looks like this.

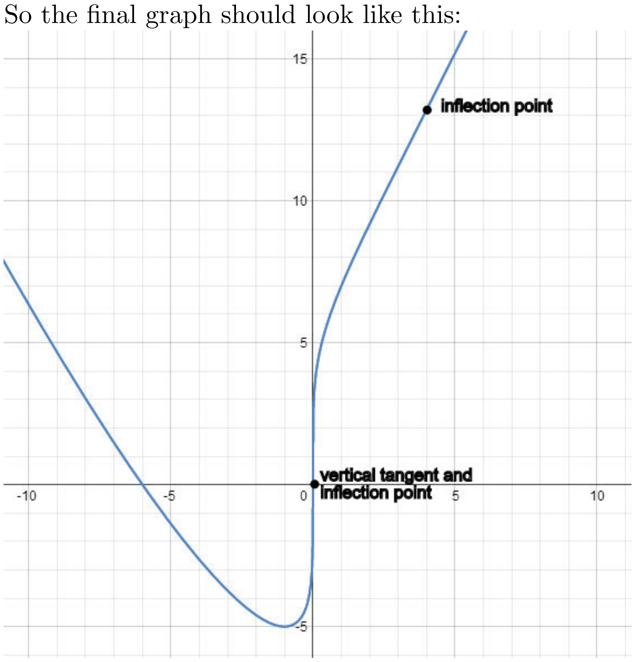


However, we know the function isn't made up of straight lines - it's more curvy than that. From part (d) we know that the function has an inflection point at $x = 0$ and $x = 4$. Furthermore we know that the function is concave up for $x < 0$ and $x > 4$ and is concave down for $0 < x < 4$. This tells us what our curves should look like.

If we superimpose the curve on top of the straight lines it looks something like this:



So the final graph should look like this:



Question 6

SOLUTION. We will need the following formulas:
The volume of the box is given by

$$V = lwh$$

Where l is the length of the base of the box, w is the width, and h is the height.
The material will form the outside of the box, which is the same as its surface area. The surface area of an open box is given by

$$SA = 2lh + 2wh + lw$$

With l, w, and h as before.
We also know that the length of the base of the box is twice its width. In an equation, this means

$$l = 2w$$

We can plug this into our volume and surface area formulas to get

$$V = 2w^2h$$

$$SA = 6wh + 2w^2$$

We know one more fact. Because we have $24m^2$ of material, the formula for surface area must be equal to 24, or

$$24 = 6wh + 2w^2$$

Isolating for h yields

$$h = \frac{24 - 2w^2}{6w}$$

(NOTE: In this step we are assuming that the width is not equal to zero, which makes sense given the physical limitations of our problem, that is, boxes have positive dimensions). Plugging this into our formula for volume yields

$$V = 2w^2h = 2w^2 \cdot \frac{24 - 2w^2}{6w} = \frac{w(24 - 2w^2)}{3} = 8w - \frac{2}{3}w^3$$

We wish to optimize V. Now that is has only one variable, differentiating yields

$$V' = 8 - 2w^2$$

Setting this equal to 0 and solving gives $w^2 = 4$ and thus that $w = 2$ (where we note again that dimensions must be positive). Looking at the derivative of volume, a sign chart tells us that$V'(w) > 0$ when $w < 2$ and $V'(w) > 0$ when $w < 2$. Hence, we have a maximum at w=2. The volume of the box with this width is

$$V(2) = 8(2) - \frac{2}{3}(2)^3 = 16 - \frac{16}{3} = \frac{32}{3}$$

and this is the maximal volume as required.

Question 7

SOLUTION. Using the definition of a derivative, we see that $f'(1) = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1}$
Since the functions differ on either side of 1, we will want to use one sided limit to attempt to evaluate the limit. If the one sided limits are equal, then the derivative exists. Otherwise, the derivative does not exist. Firstly, notice that $f(1) = 2(1) + 1 = 3$. From the left, we have

$$\begin{aligned}
\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1} &= \lim_{x\to 1^-}\frac{4x-\frac{\sin^2(2x-2)}{3x-3}-1-3}{x-1} \\
&= \lim_{x\to 1^-}\frac{4x-4}{x-1} - \lim_{x\to 1^-}\frac{\frac{\sin^2(2x-2)}{3(x-1)}}{x-1} \\
&= \lim_{x\to 1^-}\frac{4(x-1)}{x-1} - \lim_{x\to 1^-}\frac{\sin^2(2x-2)}{3(x-1)^2} \\
&= 4 - \lim_{x\to 1^-}\frac{\sin^2(2x-2)}{3(x-1)^2}
\end{aligned}$$

Now, to evaluate the last limit, recall that $\lim_{x\to 0}\frac{\sin(x)}{x}=1$ hence $\lim_{x\to 1}\frac{\sin(2x-2)}{2x-2}=1$ and so, multiplying top and bottom by 4, we have

$$\begin{aligned}
\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1} &= 4 - \lim_{x\to 1^-}\frac{\sin^2(2x-2)}{3(x-1)^2} \\
&= 4 - \lim_{x\to 1^-}\frac{4\sin^2(2x-2)}{3\cdot 4(x-1)^2} \\
&= 4 - \lim_{x\to 1^-}\frac{4\sin^2(2x-2)}{3\cdot (2x-2)^2} \\
&= 4 - \frac{4}{3}\left(\lim_{x\to 1^-}\frac{\sin(2x-2)}{2x-2}\right)^2 \\
&= 4 - \frac{4}{3} \\
&= \frac{8}{3}
\end{aligned}$$

As for the other side, we have

$$\begin{aligned}
\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1} &= \lim_{x\to 1^+}\frac{2x+1-3}{x-1} \\
&= \lim_{x\to 1^+}\frac{2x-2}{x-1} \\
&= 2
\end{aligned}$$

As the left and right hand limits do not agree, we have that the derivative does not exist at 1.

Question 8

Solution. We want to use the intermediate value theorem, as suggested by the hints. To do this we have to combine the two given functions f and $y = 2x + 3$ to one. So define

$$h(x) = f(x) - (2x+3)$$

for x in the interval [0,1]. The function f crosses the line $y = 2x + 3$ when both functions are equal, i.e. at zeros of the function h. Since we are asked to show they cross at least one, we have to show that the function h has at least one zero. In other words, we need to show that there exists a value c in [0,1] such that $h(c) = 0$.
To show this we need to make use of the fact that the range of the function f is in [3,5], i.e.

$$3 \leq f(x) \leq 5$$

for any value of x in the domain [0,1] of the function f. This implies that

$$h(0) = f(0) - (2(0) + 3) = f(0) - 3 \geq 3 - 3 = 0.$$

and

$$h(1) = f(1) - (2(1) + 3) = f(1) - 5 \leq 5 - 5 = 0.$$

Now, in the case that either $h(0)$ or $h(1)$ actually takes the value 0, we are done because we have found a zero of h.
In the case that neither takes the value zero, then we know that

$$h(1) < 0 < h(0)$$

This is where the Intermediate Value Theorem shines: h is a continuous function on [0,1], hence for any value N between $h(0)$ and $h(1)$ there exists a number $c = c(N)$ such that

$$h(c) = N.$$

Since we have shown that $h(0)$ is negative and $h(1)$ is positive, we can choose that value N to be exactly zero and thus the Intermediate Value Theorem guarantees the existence of a number c in the interval [0,1] such that $h(c) = 0$.
Conclusion: At this value c we have that

$$0 = h(c) = f(c) - (2c + 3)$$

or, in other words, $f(c) = 2c+3$, which is what we had to show.

Good Luck for your exams!