Full Solutions
MATH152 April 2012

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How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

SOLUTION. For the first plane S_1 we know that its normal is $\mathbf{n} = [2, 1, -1]$ as this is provided. We're also told that the point $P = [1, -1, 0]$ is on the plane. Using this information we can easily get the equation for S_1 via

$$\begin{aligned} 0 &= \mathbf{n} \cdot (\mathbf{x} - \mathbf{P}) \\ &= [2, 1, -1] \cdot [x_1 - 1, x_2 + 1, x_3] \\ &= 2x_2 + x_2 - x_3 - 1. \end{aligned}$$

We have to work a little harder to get the equation for the plane of S_2. However, notice we are given three points that are on the plane. Let $\mathbf{u} = \vec{AB}$ be the vector connecting A to B and $\mathbf{v} = \vec{AC}$ be the vector connecting A to C. We therefore have that,

$$\begin{aligned} \mathbf{u} &= B - A = [-2, 2, 1] \\ \mathbf{v} &= C - A = [-1, 1, 1]. \end{aligned}$$

We know that the normal to the plane is orthogonal to all vectors on the plane. Since $\mathbf{u}$ and $\mathbf{v}$ are on the plane, if we take their cross product, we will get a vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ and thus be orthogonal to the plane. Call this vector $\mathbf{m}$,

$$\begin{aligned} \mathbf{m} &= \mathbf{u} \times \mathbf{v} \\ &= [-2, 2, 1] \times [-1, 1, 1] \\ &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & 1 \\ -1 & 1 & 1 \end{pmatrix} \\ &= [1, 1, 0]. \end{aligned}$$

We can now use this normal vector along with any point on the plane to get the equation. We will use $\mathbf{a} = [1, 0, 0]$,

$$0 = \mathbf{m} \cdot (\mathbf{x} - \mathbf{a}) = [1, 1, 0] \cdot [x_1 - 1, x_2, x_3] = x_1 + x_2 - 1.$$

Therefore, we get that the equation for S_1 is $2x_1 + x_2 - x_3 - 1 = 0$ while the equation for S_2 is $x_1 + x_2 - 1 = 0$.

Question 1 (b)

SOLUTION. From 1(a) we saw that the equations for the planes S_1 and S_2 are

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 1 \\ x_1 + x_2 &= 1. \end{aligned}$$

This precisely the equation form of the line since any x_1, x_2 and x_3 that satisfy this will be on both planes and therefore be on the line of intersection.

Question 1 (c)

SOLUTION 1. From 1(b) we saw that the system of equations the line must satisfy is given by

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 1 \\ x_1 + x_2 &= 1. \end{aligned}$$

In order to find the parametric form we solve this system. First we put it in augmented form

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 1 & 1 & 0 & 1 \end{array}\right]$$

and then perform row operations to reduce the matrix. First we will swap row 1 and row 2 so that the first pivot (row 1, column 1) has value 1.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 2 & 1 & -1 & 1 \end{array}\right]$$

and then we will subtract 2 multiples of row 1 from row 2 to place a zero below the pivot,

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -1 & -1 & -1 \end{array}\right].$$

We will then multiply the second row by -1 to put a 1 in the pivot position there (row 2, column 2),

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

and finally we will subtract 1 multiple of row 2 from row 1 to put a 0 above the pivot,

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right].$$

Notice that because we have more columns than rows, then we have a free variables (the matrix is rank deficient). Let the free variable be x_3, i.e. let $x_3 = t$. Then from our reduced matrix problem we have that $x_1 = t$ and that $x_2 + t = 1$ or $x_2 = 1 - t$. Therefore we can write that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

This parametrically describes the line.

SOLUTION 2. From 1(a) we have that the normal directions to the two planes are

$$\begin{aligned} \mathbf{n} &= [2, 1, -1] \\ \mathbf{m} &= [1, 1, 0] \end{aligned}$$

We know that the line must lie on both planes and therefore the direction of the line must be orthogonal to both normals. Therefore we can find it by taking the cross product,

$$\mathbf{n} \times \mathbf{m} = [2, 1, -1] \times [1, 1, 0] = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{pmatrix} = [1, -1, 1].$$

We then must identify a point on the line. To do this we attempt to find one of the intercepts with the coordinate planes (one of x_1, x_2, or x_3 being zero). Set x_1 to be zero, from the plane equation for S_2, this tells us that $x_2 = 1$. Then using the plean equation for S_1, we get that $x_3 = 0$. Therefore, the point $(0, 1, 0)$ is on the line. This tells us that we can write the equation of the line as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

which is the parametric equation of the line.

Question 1 (d)

SOLUTION. The area of the parallelogram spanned by two vectors is the magnitude of their cross product. The area of a triangle is half the area of a parallelogram. The parallelogram form by A, B, and C is spanned by $\vec{AB}$ and $\vec{AC}$ which in 1(a), we already determined were

$$\vec{AB} = [-2, 2, 1]$$
$$\vec{AC} = [-1, 1, 1].$$

We need the cross product of this, but as we saw in 1(a), this is just the normal vector to the plane S_2 and therefore $\vec{AB} \times \vec{AC} = \mathbf{m} = [1, 1, 0]$. Therefore the area of the triangle is

$$\text{area} = \frac{1}{2}||\vec{AB} \times \vec{AC}|| = \frac{1}{2}||\mathbf{m}|| = \frac{1}{2}\sqrt{1^2 + 1^2 + 0^2} = \frac{1}{2}\sqrt{2}.$$

Question 1 (e)

SOLUTION. The parallelepiped formed by the vectors AB, AC and AP is given by (the absolute value of) the determinant of the three vectors. From 1(a) and 1(c), we have that

$$\vec{AB} = [-2, 2, 1]$$
$$\vec{AC} = [-1, 1, 1]$$

and so all we need is $\vec{AP}$,

$$\vec{AP} = P - A = [0, -1, 0].$$

Therefore, the volume of the parallelepiped is given by

$$\begin{aligned} \text{volume} &= \left| \det \begin{pmatrix} -2 & 2 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right| \\ &= \left| -1 \det \begin{pmatrix} -2 & 1 \\ -1 & 1 \end{pmatrix} \right| \\ &= |-2 + 1| = |-1| = 1. \end{aligned}$$

Question 2 (a)

SOLUTION. The matrix A should be composed of the coefficients of the variables on the left hand sides of the equations. Thus

$$A = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 3 & -6 \\ 3 & 2 & -4 \end{bmatrix}$$

Next, we note that to compose the $\mathbf{x}$ on the right to the matrix A. The right hand side has to be $\mathbf{b}$. Thus

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

As a sanity check, perform the matrix multiplication so that you can verify that this works.

Question 2 (b)

SOLUTION. We obtain the homogeneous problem by setting a=b=c=0. We can then solve the system by Gaussian elimination. First we switch the second and first row so that we have a 1 as the first entry. Doing this we get,

$$\left[\begin{array}{ccc|c} 1 & 3 & -6 & 0 \\ 2 & -1 & 2 & 0 \\ 3 & 2 & -4 & 0 \end{array}\right]$$

Next we want zeros in the rest of the entries of the first column. To do this we subtract 2 multiples of row 1 from row 2 and 3 multiples of row 1 from row 3

$$\left[\begin{array}{ccc|c} 1 & 3 & -6 & 0 \\ 0 & -7 & 14 & 0 \\ 0 & -7 & 14 & 0 \end{array}\right]$$

Now divide the second row by -7 to get a 1 in the second column of the second row

$$\left[\begin{array}{ccc|c} 1 & 3 & -6 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -7 & 14 & 0 \end{array}\right]$$

Next subtract 3 multiples of row 2 from row 1 and add 7 multiples of row 2 to row 3

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system is now in reduced row echelon form and we can now solve the system. From the first two rows we get

$$\begin{aligned} x \qquad\qquad &= 0 \\ y \quad -2z &= 0 \end{aligned}$$

The third row is a little more interesting. No matter what we multiply the third row with, we will get the zero vector which implies that we have a free variable. This free variable can't be x since we know that x=0 and so it will be either y or z. Let's choose it to be the z variable and set

$$z = t$$

to be a parameter. Using the relation from row 2 then we get

$$y = 2z = 2t$$

We can therefore write that the solution to the homogeneous problem is

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

for any number t.

Question 2 (c)

SOLUTION. If we multiply the matrix A to the vector [1,1,1] we get

$$\begin{bmatrix} 2 & -1 & 2 \\ 1 & 3 & -6 \\ 3 & 2 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

and therefore a=3, b=-2, and c=1. If we set those values permanently then we know that [1,1,1] is a particular solution to the problem. However, from part(b) we have that the solution to the homogeneous problem was

$$\mathbf{x}_h = t \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

and therefore we can write that the general solution $\mathbf{x}$ is the sum of the particular and homogeneous solution,

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$

Question 3 (a)

SOLUTION. Let's start with $\mathbf{e}_1$. We could solve these problems by inspection but let's, at least for this case, show how one could go about solving this schematically. We want to find a, b, c such that

$$\mathbf{e}_1 = a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3$$
$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Writing the above in matrix form we get

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

In general we will always get a matrix with columns that are the vectors we are trying to combine. If we row-reduce this matrix (try it for yourself!) we will get

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Hence $a = 1$, $b =$ -1, $c = 0$, and so finally we can write

$$\mathbf{e}_1 = \mathbf{v}_1 - \mathbf{v}_2.$$

We want to emphasize that it is perfectly fine to inspect the solution visually, however this technique will always work if you are stuck. For the other vectors we will write down the answers by inspection. For $\mathbf{e}_2$, we have

$$\mathbf{e}_2 = \mathbf{v}_1 - \mathbf{v}_3$$

and for $\mathbf{e}_3$ we have

$$\mathbf{e}_3 = -\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3.$$

Question 3 (b)

SOLUTION. The linear transformation acting on the vector $\mathbf{e}_1$ will get given by

$$T(\mathbf{e}_1) = A\mathbf{e}_1.$$

From part (a) we have that

$$\mathbf{e}_1 = \mathbf{v}_1 - \mathbf{v}_2.$$

In the question we are given the eigenvalues for these eigenvectors so we know

$$A\mathbf{v}_1 = 1\mathbf{v}_1$$
$$A\mathbf{v}_2 = 2\mathbf{v}_2$$

Therefore

$$T(\mathbf{e}_1) = A\mathbf{e}_1 = A\mathbf{v}_1 - A\mathbf{v}_2 = \mathbf{v}_1 - 2\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$

Question 3 (c)

SOLUTION. For any matrix A and basis vector $\mathbf{e}_i$ computing

$$A\mathbf{e}_i$$

obtains column i of A. If A is our linear transformation matrix then computing $A\mathbf{e}_1$, $A\mathbf{e}_2$, and $A\mathbf{e}_3$ will give us the three columns of A and hence we'll know the matrix. From part (b), we already determined the transformation on the vector $\mathbf{e}_1$,

$$A\mathbf{e}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

and so this is the first column of A. To get the other relations we can use our work from part (a) where we obtained the basis vectors in terms of the eigenvectors. For $\mathbf{e}_2$ we have

$$\mathbf{e}_2 = \mathbf{v}_1 - \mathbf{v}_3$$

and so

$$A\mathbf{e}_2 = A\mathbf{v}_1 - A\mathbf{v}_3 = \mathbf{v}_1 - 3\mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$$

where we have used the eigenvalue/eigenvector relationships. This is the second column of A. Finally for $\mathbf{e}_3$ we have from part (a),

$$\mathbf{e}_3 = -\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$$

and so

$$A\mathbf{e}_2 = -A\mathbf{v}_1 + A\mathbf{v}_2 + A\mathbf{v}_3 = -\mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

gives us the third column of A. Therefore we have that the linear transformation matrix A is

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 1 \\ -1 & -2 & 4 \end{bmatrix}.$$

Question 4 (a)

SOLUTION. To determine the eigenvalues, λ_1, λ_2, of A, we solve for the characteristic polynomial of A for λ:

$$\begin{aligned}
\det(A - \lambda I) &= 0 \\
\det\left(\begin{bmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{bmatrix}\right) &= 0 \\
(3-\lambda)^2 - 4 &= 0 \\
(3-\lambda)^2 &= 4 \\
3-\lambda &= \pm 2 \\
\lambda &= -(-3 \pm 2) \\
\lambda &= 1 \text{ or } 5
\end{aligned}$$

Thus, the eigenvalues of $A$are $\lambda_1 = 1,\ \lambda_2 = 5$.

Question 4 (b)

SOLUTION. Refer to the previous part for the eigenvalue computations. For the eigenvalue $\lambda = 1$

$$A - I = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

and from this we see that the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector. For the $\lambda = 5$ eigenvalue,

$$A - 5I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}.$$

and from this we see that the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector.

Question 4 (c)

SOLUTION. The matrix M is made up of the eigenvectors of A and so

$$M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

The matrix D is a diagonal matrix of the eigenvalues listed in the order that we list the eigenvectors in M. Therefore,

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}.$$

We can also compute M^{-1} by recalling that for a 2x2 matrix Q such that

$$Q = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

that

$$Q^{-1} = \frac{1}{\det Q} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Since det(M)=2, we have that

$$M^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}.$$

Multiplying everything together we get

$$MDM^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

as we were expecting.

Question 4 (d)

SOLUTION. From part (c), we have that $A = MDM^{-1}$ where

$$M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

Taking powers yields

$$\begin{aligned} A^k &= (MDM^{-1})^k \\ &= MDM^{-1} \cdot MDM^{-1} \cdot \ldots \cdot MDM^{-1} \\ &= MD^kM^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 5^k \end{bmatrix}\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \\ &= \left(\frac{1}{2}\right)\begin{bmatrix} 1+5^k & -1+5^k \\ -1+5^k & 1+5^k \end{bmatrix} \end{aligned}$$

Question 5 (a)

SOLUTION.

$$B^T B\mathbf{x} = B^T\mathbf{c} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

or, in other words,

$$\begin{bmatrix} 3 & 10 \\ 10 & 38 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 12 \\ 46 \end{bmatrix}$$

So the augmented matrix is

$$\begin{bmatrix} 3 & 10 & | & 12 \\ 10 & 38 & | & 46 \end{bmatrix}$$

The vector $\mathbf{c} = [2, 4, 6]^T$ is projected to the colum space of the matrix B (ColB) which is the plane spanned by the column vectors of B, namely

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

Question 5 (b)

SOLUTION. We begin by taking our matrix and applying row operations. First, we take 100 times the first row and subtract it from the second row to turn the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 100 & 200 & 300 & 401 \\ 0 & p & p+1 & p+2 \\ 0 & -p & -2p+1 & -3p+2 \end{bmatrix}$$

into

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & p & p+1 & p+2 \\ 0 & -p & -2p+1 & -3p+2 \end{bmatrix}$$

Next we subtract $p+2$ times the second row to the third row and also add $-3p+2$ times the second row to the fourth row giving

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & p & p+1 & 0 \\ 0 & -p & -2p+1 & 0 \end{bmatrix}$$

Next, we add the third row to the fourth row to get

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & p & p+1 & 0 \\ 0 & 0 & -p+2 & 0 \end{bmatrix}$$

Swap the second row with the third row and the resulting third row with the fourth row (so this introduces no change in the sign since we swap twice and so multiply the determinant $(-1)^2 = 1$) giving

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & p & p+1 & 0 \\ 0 & 0 & -p+2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now we take determinants and notice that the determinant of A is the same as the determinant of the above matrix. Since the above matrix is upper triangular, the determinant can be taken by multiplying the entries on the diagonal to give

$$\det(A) = \det\left(\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & p & p+1 & 0 \\ 0 & 0 & -p+2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = p(-p+2)$$

This is zero when $p = 0$ or when $p = 2$.

Question 6 (a)

SOLUTION 1. A set of vectors is linearly dependent if $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ has more solutions than just the trivial solution $c_1 = c_2 = c_3 = 0$.
To determine which values of a will cause the vectors to be linearly dependent, we write the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ in its matrix-vector form

$$M\mathbf{c} = \mathbf{0}, \quad \text{where} \quad M = \begin{bmatrix} 1 & 0 & a \\ 2 & 1 & 4 \\ -1 & 3 & 5 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

If the row-reduced echelon form of M has a row of zeros, there are non-zero values of $\mathbf{c}$ that will satisfy $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$. Row-reducing M as follows:

$$M = \begin{bmatrix} 1 & 0 & a \\ 2 & 1 & 4 \\ -1 & 3 & 5 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & 4-2a \\ 0 & 3 & 5+a \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & 4-2a \\ 0 & 0 & -7+7a \end{bmatrix},$$

we can see that the last row will be a row of zeros if $a = 1$. Therefore, the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ arelinearly dependent if $a = 1$.

SOLUTION 2. As a quick alternative to the first solution, recall that the columns of a matrix, M, are *linearly dependent* if the determinant of M is equal to zero. Hence, we define

$$M = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \begin{bmatrix} 1 & 0 & a \\ 2 & 1 & 4 \\ -1 & 3 & 5 \end{bmatrix},$$

which has the determinant

$$\det(M) = -7 + 7a.$$

The determinant equal to zero when $a = 1$. Therefore, the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ arelinearly dependent if $a = 1$.

Question 6 (b)

SOLUTION. By the equation $A = C^{-1}BC$, we can see that A, B are similar matrices.
One of the properties of similar matrices is that their determinants are equal. Hence,

$$\text{Det}(A) = \text{Det}(B) = 12$$

Question 6 (c)

Solution.

$$\begin{array}{lll} (1) & A^6 = I & \text{(True)} \\ (2) & A^{-1} = A & \text{(False)} \\ (3) & B^3 = I & \text{(False)} \\ (4) & B^{-1} = B & \text{(True)} \\ (5) & C^{-1} = C & \text{(False)} \\ (6) & C^2 = C & \text{(True)} \end{array}$$

The quick solution to this problem involves using some understanding about how applying the matrices above to any arbitrary vector changes the orientation of that vector. The reasoning is explained in the following points:
(1): Applying the matrix A to a vector once will rotate the vector counterclockwise by 60 degrees. Applying the matrix A to a vector twice (i.e. A^2) will rotate the vector counterclockwise by 120 degrees... Applying it to a vector 6 times (i.e. A^6) will rotate the vector counterclockwise by 360 degrees... giving us the original vector back, just as applying the identity matrix I would do. Hence (1) is true.
(2): If $v = Au$, then v the 60 degree counterclockwise rotated version of u. Also if $v = Au$, then $u = A^{-1}v$, which tells us that A^{-1} is the matrix corresponding to 60 degree clockwise rotation. So A, A^{-1} are not the same matrix operator and (2) is false.
(3): Since the matrix B will reflect a vector across the line $y = \sqrt{3}x$, applying B twice to a vector will reflect the vector across the same line twice, resulting in the original vector. Hence applying the matrix B three times to a vector will reflect it across the line. This is not equivalent to multiplying the vector by I and so (3) is false.
(4): If $B^{-1} = B$, then $I = B^2$ which, as we mentioned in (3), is true since reflecting a vector across the line twice gives the original vector. Therefore, (4) is true.
(5): Projections are non-invertible operations (since you cannot uniquely determine the original vector from its projection) and since C is a projection, C^{-1} does not even exist. So (5) is false.
(6): Since C is a projection, it satisfies (6) by definition and so (6) is true.

Question 6 (d)

Solution. (1) It always has at least one solution. (FALSE)
(2) There is either no solution or infinitely many solutions. (TRUE)
(3) If a solution exists, then there is precisely a 2-parameter family of solutions. (FALSE)
(4) If the coefficient matrix has rank k, then the associated homogeneous system as a k-parameter family of solutions. (FALSE)
The justification for each answer is given below.
(1) is FALSE.
Consider the following counter-example of three equations with five unknowns:

$$x = 1, \quad x = 2, \quad x + y + z + u + v = 3$$

Clearly, this system is inconsistent. Hence (1) is FALSE by counter-example.
(2) is TRUE.
As we saw in (1), it is possible that there is no solution. For the second part of the statement, assume that a solution exists. This solution cannot be unique since any row-reduced echelon form of the coefficient matrix admitting a solution has at least two free parameters (i.e. Rank of the matrix is at most 3, but there are 5 parameters). Hence, if any solution exists, then infinitely many solutions must exist. Therefore (2) is true.
(3) is FALSE.
Consider the following example:

$$\begin{aligned} x+y+z+u+v&=0\\ x+y+z+u+v&=0\\ x+y+z+u+v&=0 \end{aligned}$$

In this example, there is a four-parameter family of solutions since the associated coefficient matrix has rank 1.
(4) is FALSE.
See (1) and (3) for counterexamples.

Question 6 (e)

SOLUTION. (1) YES. (2) YES. (3) NO. (4) NO.
Justification for each answer is given below. Remember: A transformation T is linear if the following hold:
i) T(u + v) = T(u) + T(v)
ii) T(cu) = cT(u)
for any vectors u,v and any constant c.
(1) YES. The reflection at the x = 0 plane results in the component of the vector parallel to x being mapped to its negative while all other components are unchanged. For a formal proof, let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. If we let M be the transformation in question, then

$$M(\mathbf{u}) = \begin{bmatrix} -u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$

If we check that the above conditions i), ii) hold, we see that they are satisfied:

$$\begin{aligned} \text{i)} \quad M(\mathbf{u}+\mathbf{v}) &= \begin{bmatrix} -(u_1+v_1) \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix} \\ &= \begin{bmatrix} -u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} -v_1 \\ v_2 \\ v_3 \end{bmatrix} \\ &= M(\mathbf{u}) + M(\mathbf{v}) \end{aligned}$$

$$\text{ii)} \quad M(c\mathbf{u}) = \begin{bmatrix} -cu_1 \\ cu_2 \\ uc_3 \end{bmatrix} = c \begin{bmatrix} -u_1 \\ u_2 \\ u_3 \end{bmatrix} = cM(\mathbf{u})$$

Therefore, the reflection in the x = 0 line/plane is a linear transformation.
(2) YES. We need only check the conditions i), ii) above to confirm T is linear

$$\begin{aligned} \text{i)} \quad T(\mathbf{u}+\mathbf{v}) &= \begin{bmatrix} (u_1+v_1)+2(u_2+v_2) \\ -3(u_1+v_1) \end{bmatrix} \\ &= \begin{bmatrix} u_1+2u_1 \\ -3u_1 \end{bmatrix} + \begin{bmatrix} v_1+2v_1 \\ -3v_1 \end{bmatrix} \\ &= T(\mathbf{u}) + T(\mathbf{v}) \end{aligned}$$

$$\text{ii)} \quad T(c\mathbf{u}) = \begin{bmatrix} cu_1 + 2cu_1 \\ -3cu_1 \end{bmatrix} = c \begin{bmatrix} u_1 + 2u_1 \\ -3u_1 \end{bmatrix} = cT(\mathbf{u})$$

Therefore, T is linear.
(3) NO. In this case, we can see right way that R fails condition i)

$$\text{i)} \quad R(\mathbf{u}+\mathbf{v}) = \begin{bmatrix} 2(u_1+v_1) - 5(u_2+v_2) \\ \cos^2(u_1+v_1) \end{bmatrix}$$

If we let $u_1 = \pi$, $v_1 = -\pi$, then we see

$$\cos^2(u_1+v_1) = \cos^2(\pi - \pi) = (1)^2 = 1$$
$$\cos^2(u_1) + \cos^2(v_1) = \cos^2(\pi) + \cos^2(-\pi) = (-1)^2 + (-1)^2 = 2$$

Hence

$$\begin{aligned} \text{i)} \quad R(\mathbf{u}+\mathbf{v}) &= \begin{bmatrix} 2(u_1+v_1) - 5(u_2+v_2) \\ \cos^2(u_1+v_1) \end{bmatrix} \\ &\neq \begin{bmatrix} 2u_1 - 5u_2 \\ \cos^2(u_1) \end{bmatrix} + \begin{bmatrix} 2v_1 - 5v_2 \\ \cos^2(v_1) \end{bmatrix} \\ &= R(\mathbf{u}) + R(\mathbf{v}) \end{aligned}$$

Therefore, R is NOT linear.
(4) NO. In this case, we can see right way that S fails condition ii)

$$\text{ii)} \quad S(c\mathbf{u}) = \begin{bmatrix} -cu_1 \\ (cu_2)(cu_3) \\ cu_3 + cu_1 \end{bmatrix} = \begin{bmatrix} -cu_1 \\ c^2 u_2 u_3 \\ c(u_3 + u_1) \end{bmatrix} \neq cS(\mathbf{u})$$

Therefore, S is NOT linear.

Question 7 (a)

SOLUTION. The probability of being at the i^{th} state at the n^{th} time step is given by the i^{th} entry in the vector $\mathbf{p}^{(n)}$, where

$$\mathbf{p}^{(n)} = P^n \mathbf{p}^{(0)}$$

and $\mathbf{p}^{(0)}$ is the initial distribution of probability.
In this case, $\mathbf{p}^{(0)} = [0, 0, 1]^T$ since we are starting in the third state with probability 1. We want to compute the 3^{rd} entry of $\mathbf{p}^{(2)}$,

$$\mathbf{p}^{(2)} = P^2 \mathbf{p}^{(0)} = \begin{bmatrix} 1/4 & 1/4 & 1/4 \\ 3/4 & 3/4 & 1/4 \\ 0 & 0 & 1/2 \end{bmatrix}^2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/4 & 1/4 & 1/4 \\ 3/4 & 3/4 & 1/2 \\ 0 & 0 & 1/4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}$$

Therefore, the probability of being in the third state at the second step is 1/4.

Question 7 (b)

Solution 1. The eigenvalues of P satisfy

$$\mathrm{Det}(P - \lambda I) = 0.$$

Performing the calculation of the determinant using the co-factor method gives

$$\begin{aligned}\mathrm{Det}(P - \lambda I) &= \mathrm{Det}\left(\begin{bmatrix} 1/4-\lambda & 1/4 & 1/4 \\ 3/4 & 3/4-\lambda & 1/4 \\ 0 & 0 & 1/2-\lambda \end{bmatrix}\right) \\ &= (-1)^{3+3}(1/2-\lambda)\left((1/4-\lambda)(3/4-\lambda) - 3/16\right) \\ &= (1/2-\lambda)(\lambda^2-\lambda) \\ &= \lambda(1/2-\lambda)(\lambda-1).\end{aligned}$$

Solving $\mathrm{Det}(P - \lambda I) = 0$ gives the eigenvalues:

$$\lambda = 0, 1/2, 1.$$

Solution 2. If you want to be clever, you don't have to do any calculation to find the eigenvalues of P.

- P is a 3x3 matrix and hence has at most 3 eigenvalues.
- Since P is a transition matrix we know that $\lambda_1 = 1$ must be an eigenvalue.
- Since the first and second row (or first and second colum) of P are identical, P is singular. Singular matrices always have an eigenvalue $\lambda_2 = 0$.
- The only non-zero entry in the third row of P is the diagonal entry 1/2. Hence $Pe_3 = (1/2)e_3$ and therefore $\lambda_3 = 1/2$ is the last missing eigenvalue.

We can double check our answer by verifying that the sum of eigenvalues equals the sum of diagonal entries:

$$\lambda_1 + \lambda_2 + \lambda_3 = 3/2 = P_{11} + P_{22} + P_{33}$$

Question 7 (c)

Solution. The stationary probability density, $\bar{\mathbf{p}}$, satisfies the equation:

$$\bar{\mathbf{p}} = P\bar{\mathbf{p}}.$$

We notice that $\bar{\mathbf{p}}$ must be the eigenvector with corresponding eigenvalue $\lambda = 1$. Thus, we need to solve for this eigenvector (making sure all the entries sum to 1 in order for it to be a probability distribution) using the equation $(P - I)\bar{\mathbf{p}} = \mathbf{0}$

$$\begin{bmatrix} -3/4 & 1/4 & 1/4 \\ 3/4 & -1/4 & 1/4 \\ 0 & 0 & -1/2 \end{bmatrix} \bar{\mathbf{p}} = \mathbf{0} \quad \rightarrow \quad \bar{\mathbf{p}} = \begin{bmatrix} 1/4 \\ 3/4 \\ 0 \end{bmatrix}.$$

Therefore, the stationary probabilities are probability $1/4$ of being in state 1, probability $3/4$ of being in state 2, and zero probability of being in state 3.

Question 7 (d)

SOLUTION. The stationary distribution is the eigenvector corresponding to eigenvalue 1. The other two eigenvectors correspond to eigenvalues with absolute value strictly less than 1. This means that only the component of the initial distribution that is parallel to the stationary distribution will maintain its length over repeated applications of P. Since the 3x3 matrix P has three distinct eigenvalues, every vector of initial conditions can be composed to a sum of eigenvectors. Hence, all components of the initial distribution that are not parallel to the stationary distribution will decay with repeated applications of P.
Hence, all initial distributions eventually converge to the stationary distribution computed in part (c).

Question 8 (a)

SOLUTION. To calculate the eigenvalues of the matrix $A = \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix}$, we find λ such that

$$\begin{aligned} 0 &= \det \begin{pmatrix} -1-\lambda & -2 \\ 2 & -1-\lambda \end{pmatrix} \\ &= (-1-\lambda)^2 + 4 \\ &= \lambda^2 + 2\lambda + 5 \end{aligned}$$

To calculate λ, we use the quadratic formula,
$\lambda_{1,2} = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$
where $\sqrt{-1} = i$.
Hence, the eigenvalues $\lambda_{1,2}$ are $-1+2i$ and $-1-2i$.

Question 8 (b)

SOLUTION. We first calculate the eigenvector x of the eigenvalue $-1-2i$,
$0 = \begin{pmatrix} -1-(-1-2i) & -2 \\ 2 & -1-(-1-2i) \end{pmatrix} x = \begin{pmatrix} 2i & -2 \\ 2 & 2i \end{pmatrix} x = \begin{pmatrix} 2ix_1 - 2x_2 \\ 2x_1 + 2ix_2 \end{pmatrix} = 0$
Considering the first line, we obtain

$$\begin{aligned} 2ix_1 - 2x_2 &= 0 \\ x_2 &= ix_1 \end{aligned}$$

Notice that there is a free variable which is common when finding eigenvectors. Therefore, we can choose $x_1 = 1$, then $x_2 = i$ and have the eigenvector $x = \begin{pmatrix} 1 \\ i \end{pmatrix}$.
For the second eigenvector y of the eigenvalue $-1+2i$, we calculate
$0 = \begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix} y = \begin{pmatrix} -2iy_1 - 2y_2 \\ 2y_1 - 2iy_2 \end{pmatrix} = 0$
Considering the first line, we obtain

$$\begin{aligned} -2iy_1 - 2y_2 &= 0 \\ y_2 &= -iy_1 \end{aligned}$$

There is a free choice like before and we choose $y_1 = 1$, then $y_2 = -i$ and have the eigenvector $y = \begin{pmatrix} 1 \\ -i \end{pmatrix}$.

Question 8 (c)

SOLUTION. The complex solution $\vec{\mathbf{z}}(t)$ that we found in (a) and (b) is

$$\begin{aligned} \vec{\mathbf{z}}(t) &= e^{-t}e^{2ti}\begin{pmatrix} 1 \\ -i \end{pmatrix} \\ &= e^{-t}(\cos 2t + i\sin 2t)\begin{pmatrix} 1 \\ -i \end{pmatrix} \\ &= e^{-t}\begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix} + ie^{-t}\begin{pmatrix} \sin 2t \\ -\cos 2t \end{pmatrix} \end{aligned}$$

From this we can read off the real-valued form as

$$\vec{\mathbf{x}}(t) = C_1 e^{-t}\begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix} + C_2 e^{-t}\begin{pmatrix} \sin 2t \\ -\cos 2t \end{pmatrix}$$

Question 8 (d)

SOLUTION. Since e^{-t} goes to 0 as $t \to \infty$, the solution goes to 0 as well.

Good Luck for your exams!