Full Solutions
MATH104 December 2012

April 4, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

Solution.

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x} &= \lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}\times\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\\ &= \lim_{x\to 0}\frac{x+1-1}{x(\sqrt{x+1}+1)}\\ &= \lim_{x\to 0}\frac{x}{x(\sqrt{x+1}+1)}\\ &= \lim_{x\to 0}\frac{1}{\sqrt{x+1}+1}\\ &= \frac{1}{2}\end{aligned}$$

$$\text{Therefore, } \lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}=\frac{1}{2}$$

Question 1 (b)

Solution. Following the second hint, we check all the conditions and see if there are any conditions where a needs to take a particular value to satisfy the conditions:
i. $f(1)=a$. So $f(1)$ exists.
ii. To check that the limit at $x=1$ exists, we do not need to bother with left and right-hand limits as the function is defined in the same way on both sides of $x=1$.

$$\begin{aligned}\lim_{x\to 1}\frac{x\exp(x)-\exp(x)}{x^2-3x+2} &= \lim_{x\to 1}\frac{(x-1)\exp(x)}{(x-1)(x-2)}\\ &= \lim_{x\to 1}\frac{\exp(x)}{(x-2)} = -\exp(1) = -e\end{aligned}$$

So we have $\lim_{x\to 1} f(x)$ exists and is equal to $-e$.
iii. The last condition to check is

$$\lim_{x\to 1} f(x) = f(1) \quad\to\quad -e=a$$

So we can see that for the last condition of continuity to be satisfied we must have $a=-e$.
Therefore, $a=-e$.

Question 1 (c)

Solution. Use logarithmic differentiation on this function.

$$\begin{aligned}f(x) &= (x+1)^{\cos(x)}\\ \ln(f(x)) &= \ln((x+1)^{\cos(x)})\\ &= \cos(x)\ln(x+1)\end{aligned}$$

Taking the derivative of this equation with respect to x gives:

$$\begin{aligned}
\frac{d}{dx}\left[\ln(f(x))\right] &= \frac{d}{dx}\left[\cos(x)\ln(x+1)\right] \\
\frac{f'(x)}{f(x)} &= \frac{\cos(x)}{x+1} - \sin(x)\ln(x+1) \\
f'(x) &= f(x)\left(\frac{\cos(x)}{x+1} - \sin(x)\ln(x+1)\right) \\
f'(x) &= (x+1)^{\cos(x)}\left(\frac{\cos(x)}{x+1} - \sin(x)\ln(x+1)\right)
\end{aligned}$$

Question 1 (d)

SOLUTION. Taking the derivative of $h(x)$ and using the chain rule, we get:

$$h'(x) = \exp(f(x))f'(x) + 2f(x)f'(x)$$

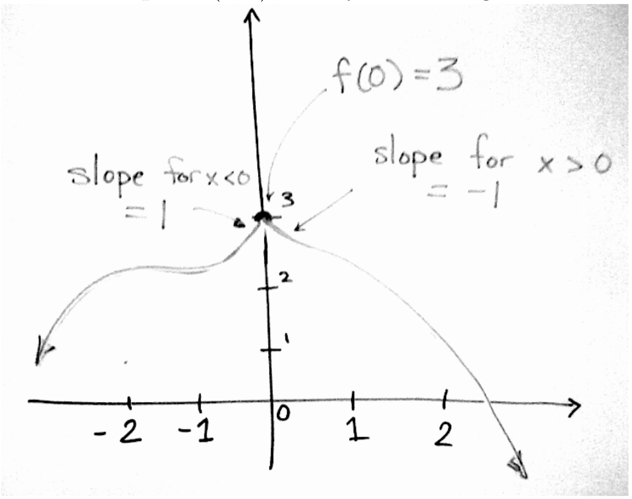
Substituting $x = 1$ and using the given values of $f(1)$ and $f'(1)$ gives:

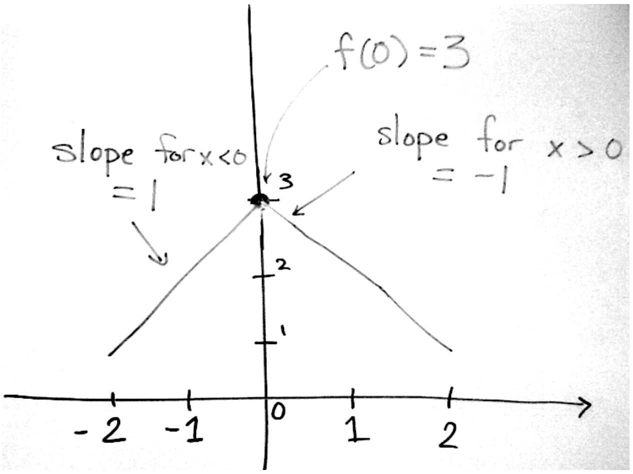
$$\begin{aligned}
h'(1) &= \exp(f(1))f'(1) + 2f(1)f'(1) \\
&= 5\exp(2) + 2(2)(5) \\
&= 5e^2 + 20
\end{aligned}$$

Therefore, $h'(1) = 5e^2 + 20$.

Question 1 (e)

SOLUTION. There are infinitely many possible solutions to this problem. As long you sketch a continuous function that has slope equal to 1 the near $x = 0$ on the left, a slope equal to -1 near $x = 0$ on the right and touches the point $(0, 3)$, then you should get full marks. Click the images below to see a couple of examples.





Question 1 (f)

SOLUTION. Evaluate the first and second derivatives of $y(x)$:

$$y'(x) = re^{rx}, \quad y''(x) = r^2 e^{rx}.$$

Now substitute them into the given equation:

$$\begin{aligned} y''(x) - 4y'(x) + 4y(x) &= 0 \\ r^2 e^{rx} - 4re^{rx} + 4e^{rx} &= 0 \\ (r^2 - 4r + 4)e^{rx} &= 0 \end{aligned}$$

Thus, either $e^{rx} = 0$ for any x which does not happen since the exponential function never evaluates to 0, or $r^2 - 4r + 4 = 0$. Factoring the quadratic gives

$$r^2 - 4r + 4 = (r-2)^2 = 0.$$

Thus, the value of r that satisfies the given equation is $r = 2$.

Question 1 (g)

SOLUTION. For our problem, we have that $a = 1$ and $f(x) = \arctan(x)$. Evaluating the derivatives, we get

$$f'(x) = \frac{1}{1+x^2}, \quad f''(x) = \frac{-2x}{(1+x^2)^2}$$

Thus,

$$f(1) = \arctan(1) = \frac{\pi}{4}, \quad f'(1) = \frac{1}{2}, \quad f''(1) = -\frac{1}{2}$$

and so the 2nd order Taylor polynomial is

$$T_2(x) = \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2.$$

Question 1 (h)

SOLUTION. Following the hint, we first find a suitable value for M. To do this, we need the second derivative. Computing gives

$$\begin{aligned} f(x) &= \cos(x) \\ f'(x) &= -\sin(x) \\ f''(x) &= -\cos(x) \end{aligned}$$

and thus

$$|f''(x)| = |-\cos(x)| \leq 1 = M$$

where the above bound holds for any value of x. Plugging into the upper bound on the error in a linear approximation as given by the hint, we have
$|R_1(0.25)| \leq \frac{M}{2}|0.25-0|^2 = \frac{1}{2}|0.25-0|^2 = \frac{1}{32}$
completing the question.

Question 1 (i)

SOLUTION. Proceeding as in the hints, we compute the 20th derivative to be

$$\begin{aligned} p'(q) &= 3e^{3q} \\ p''(q) &= 3^2e^{3q} \\ p'''(q) &= 3^3e^{3q} \\ &\vdots \\ p^{(20)}(q) &= 3^{20}e^{3q} \end{aligned}$$

Now using the second hint, we see that
$a_{20} = \dfrac{p^{(20)}(1)}{20!} = \dfrac{3^{20}e^3}{20!}$
which completes the problem.

Question 1 (j)

SOLUTION. Recall the formula for continuously compounded interest,

$$A = Pe^{rt}$$

where A, P, r are the future value, principal and growth rate respectively and t is the time since the investment was made. The rate of growth of the investment is simply the time derivative of the future value (i.e. dA/dt). Evaluating the time derivative of A gives:

$$\frac{d}{dt}A = \frac{d}{dt}\left(Pe^{rt}\right) = rPe^{rt}.$$

By the information given in the question, we want to solve for the time t when the $dA/dt = 10,000$ dollars/year, given $P = 100,000$ dollars and $r = 0.07$ yr^{-1}.

$$\begin{aligned}
10000 &= (0.07)(100000)e^{0.07t} \\
10000 &= 7000e^{\frac{7}{100}t} \\
\frac{10}{7} &= e^{\frac{7}{100}t} \\
\ln\left(\frac{10}{7}\right) &= \frac{7}{100}t \\
\to \quad t &= \frac{100}{7}\ln\left(\frac{10}{7}\right) \quad (\approx 5.10)
\end{aligned}$$

Therefore, we would need to wait (100/7)ln(10/7) years for our initial investment to be growing at a rate of \$10,000 per year so we can retire.

Question 1 (k)

Solution. Using the hints, we have

$$\begin{aligned}
\epsilon(p) &= \frac{p \cdot q'(p)}{q(p)} \\
\epsilon(10) &= \frac{10 \cdot q'(10)}{q(10)} = \frac{10(-4))}{2} = -20
\end{aligned}$$

where the values of $q(10) = 2$ are read from the given degree 2 Taylor polynomial expansion and similarly for $q'(10)$.

Question 1 (l)

Solution. The fact that $f'(a)$ exists is equivalent to saying that the function $f(x)$ is differentiable at $x = a$. Recall that a function must be continuous at a point to be differentiable at that point.
For the function to be continuous at a point $x = a$, the condition

$$\lim_{x \to a} f(x) = f(a)$$

must be satisfied. Therefore, (i) is the answer.

Question 1 (m)

Solution. To determine the values of t where $f(t)$ is increasing, we must solve for the critical points of $f(t)$ and examine the sign of the derivative in between the critical points, solving $f'(t) = 0$ gives

$$\begin{aligned}
f'(t) &= 0 \\
2t\exp(t) + t^2\exp(t) &= 0 \\
(t^2 + 2t)\exp(t) &= 0.
\end{aligned}$$

Since the exponential function is never zero for any value of t, we can solve the above equation by solving

$$\begin{aligned} t^2 + 2t &= 0 \\ t(t+2) &= 0, \quad \to \quad t = 0, -2 \end{aligned}$$

Now we need to evaluate the sign of $f'(t)$ in the intervals $(-\infty, -2), (-2, 0), (0, \infty)$. Taking test points $t = -3, -1, 1$ and plugging them into $f'(t)$ gives:

$$f'(-3) = 3e^{-3} > 0, \quad f'(-1) = -e^{-1} < 0, \quad f'(1) = 3e > 0.$$

So $f'(t)$ is positive on $(-\infty, -2), (0, \infty)$ and negative on $(-2, 0)$.
Therefore, $f(t)$ is increasing for $t < -2$, $t > 0$ and decreasing for $0 < t < 2$.

Question 1 (n)

Solution. This question requires multiple uses of the chain rule.

$$f(x) = \ln(1 + \ln(1 + \ln(x)))$$

Recognizing that the inner function is $u = h(x) = 1 + \ln(1 + \ln(x))$ and the outer function is $g(u) = \ln(u)$, we apply the chain rule and get

$$\begin{aligned} f'(x) &= g'(h(x)) \cdot h'(x) \\ &= \frac{1}{1 + \ln(1 + \ln(x))} \cdot h'(x). \end{aligned}$$

Now we need to use the chain rule again to evaluate $h'(x)$. The function $h(x)$ has inner function $v = k(x) = 1 + \ln(x)$ and outer function $j(v) = 1 + \ln(v)$. Thus, $h'(x)$ is given by

$$\begin{aligned} h'(x) &= j'(k(x)) \cdot k'(x) \\ &= \frac{1}{1 + \ln(x)} \cdot k'(x) \\ &= \frac{1}{1 + \ln(x)} \cdot \frac{1}{x} \end{aligned}$$

Thus putting the pieces together, we write the expression for the derivative of f:

$$f'(x) = \frac{1}{1 + \ln(1 + \ln(x))} \cdot \frac{1}{1 + \ln(x)} \cdot \frac{1}{x}.$$

Question 1 (o)

Solution. Define the function $f(x)$ as

$$f(x) = 5^x - 10x - 7.$$

Notice that this function is continuous for all values of x. Hence, we need only find a closed interval where the sign of $f(x)$ changes and invoke the intermediate value theorem to get the desired conclusion.

Consider $x = 2$ and $x = 3$. If we take these two points and plug them into $f(x)$, we get

$$f(2) = 5^2 - 10(2) - 7 = -2 < 0$$
$$f(3) = 5^3 - 10(3) - 7 = 88 > 0$$

and so the sign of $f(x)$ changes over the closed interval $[2, 3]$.
Therefore, there exists a value c in $[2, 3]$ such that

$$f(c) = 5^c - 7c - 10 = 0$$

by the Intermediate Value Theorem.

Question 2 (a)

SOLUTION. Since we are given the derivative, it is easy to see that the values of the derivative where it is equal to zero or undefined.
The derivative is equal to zero where $x = 0$ and where $x^2 - 9 = 0$, so at $x = \pm 3$. The derivative is undefined where $x^2 - 3 = 0$ so at the points $x = \pm\sqrt{3}$.
Notice that the derivative's denominator is always positive since it is a positive number times a square. So when checking for sign changes, we need only to look at the derivative's numerator, that is, the function $2x^2(x^2 - 9)$ (note we could check the denominator as well, but this trick saves a lot of valuable time on an exam!). Let's look at each interval individually.
Case 1

$$(-\infty, -3)$$

On this interval, we have that at the point $x = -4$ that the numerator is $2(-4)^2((-4)^2 - 9) = 224 > 0$ and so the function is increasing on this interval.
Case 2

$$(-3, -\sqrt{3})$$

On this interval, we have that at the point $x = -2$ that the numerator is $2(-2)^2((-2)^2 - 9) = -40 < 0$ and so the function is decreasing on this interval.
Case 3

$$(-\sqrt{3}, 0)$$

On this interval, we have that at the point $x = -1$ that the numerator is $2(-1)^2((-1)^2 - 9) = -16 < 0$ and so the function is decreasing on this interval.
Case 4

$$(0, \sqrt{3})$$

On this interval, we have that at the point $x = 1$ that the numerator is $2(1)^2((1)^2 - 9) = -16 < 0$ and so the function is decreasing on this interval.
Case 5

$$(\sqrt{3}, 3)$$

On this interval, we have that at the point $x = 2$ that the numerator is $2(2)^2((2)^2 - 9) = -40 < 0$ and so the function is decreasing on this interval.
Case 6

$$(3, \infty)$$

On this interval, we have that at the point $x = 4$ that the numerator is $2(4)^2((4)^2 - 9) = 224 > 0$ and so the function is increasing on this interval.
This completes the problem.

Question 2 (b)

SOLUTION. Since we are given the second derivative, it is easy to see that the derivative is 0 at $x = 0$. The derivative is undefined when $x^2 - 3 = 0$ so at the points $x = \pm\sqrt{3}$. Let's look at each interval individually.
Case 1

$$(-\infty, -\sqrt{3})$$

On this interval, we have that at the point $x = -2$ that the second derivative is $\frac{4(-2)((-2)^2+9)}{((-2)^2-3)^3} = (-8)(13) < 0$ and so the function is concave down on this interval.
Case 2

$$(-\sqrt{3}, 0)$$

On this interval, we have that at the point $x = -1$ that the second derivative is $\frac{4(-1)((-1)^2+9)}{((-1)^2-3)^3} = \frac{-40}{-8} = 5 > 0$ and so the function is concave up on this interval.
Case 3

$$(0, \sqrt{3})$$

On this interval, we have that at the point $x = 1$ that the second derivative is $\frac{4(1)((1)^2+9)}{((1)^2-3)^3} = \frac{40}{-8} = -5 < 0$ and so the function is concave down on this interval.
Case 4

$$(\sqrt{3}, \infty)$$

On this interval, we have that at the point $x = 2$ that the second derivative is $\frac{4(2)((2)^2+9)}{((2)^2-3)^3} = (8)(13) > 0$ and so the function is concave up on this interval.
This completes the problem.

Question 2 (c)

SOLUTION. By part(a) and part(b) of this problem we see that the function changes from decreasing to increasing at $x = -3$, so there is a local maximum there, and the function changes from decreasing to increasing at $x = 3$, so there is a local minimum at *x=3*. The function changes concavity at $x = 0$, so there is an inflection point at *x = 0*.

(Notice that $x = \pm\sqrt{3}$ are not points in the domain and hence cannot be extrema or inflection points.)

Question 2 (d)

SOLUTION 1. We know that $f(x)$ does not exist where its denominator equals 0:

$$3x^2 - 9 = 0 \quad \to \quad x = \pm\sqrt{3}.$$

So there are possible vertical asymptotes at $\pm\sqrt{3}$. To confirm that there is a vertical asymptote at $x = \sqrt{3}$, we must show that

$$\lim_{x\to\sqrt{3}^-} f(x) = +\infty \text{ or } -\infty, \quad \text{or} \quad \lim_{x\to\sqrt{3}^+} f(x) = +\infty \text{ or } -\infty.$$

This is easy to show because as $x \to \sqrt{3}$ from the left (from values of x less than $\sqrt{3}$), the numerator of $f(x)$ approaches $6\sqrt{3}$ and the denominator approaches 0 from the left (negative values). So we know that

$$\lim_{x\to\sqrt{3}^-} f(x) = -\infty.$$

Similarly we can show that the following hold:

$$\lim_{x\to\sqrt{3}^-} f(x) = +\infty, \qquad \lim_{x\to-\sqrt{3}^-} f(x) = -\infty, \qquad \lim_{x\to-\sqrt{3}^+} f(x) = +\infty,$$

Therefore, there are two vertical asymptotes

$$x = \sqrt{3}, x = -\sqrt{3}.$$

To find the slant asymptote, we can use long division.

$$\begin{array}{rr} & 2x/3 \\ \cline{2-2} 3x^2 - 9 & \big) 2x^3 + 0x \\ & \underline{2x^3 - 6x} \\ & 6x \end{array}$$

Thus $f(x) = \frac{2x}{3} + \frac{6x}{3x^2-9}$.
Therefore, y = (2/3)x is a slant asymptote of f(x).

SOLUTION 2. Proceed as in solution 1 to find the vertical asymptotes.
If $y = \frac{2}{3}x$ is a slant (oblique) asymptote to $f(x)$ then the difference between $f(x)$ and the slant asymptote has to vanish when x gets very small or very large:

$$\lim_{x\to\infty}\left(f(x) - \frac{2}{3}x\right) = 0, \quad \text{or} \quad \lim_{x\to-\infty}\left(f(x) - \frac{2}{3}x\right) = 0$$

We will show that the first equation is true.

$$
\begin{aligned}
\lim_{x\to\infty}\left(f(x)-\frac{2}{3}x\right) &= \lim_{x\to\infty}\left(\frac{2x^3}{3x^2-9}-\frac{2}{3}x\right)\\
&= \lim_{x\to\infty}\left(\frac{2x^3}{3x^2-9}-\frac{2x(x^2-3)}{3(x^2-3)}\right)\\
&= \lim_{x\to\infty}\left(\frac{2x^3}{3x^2-9}-\frac{2x^3-6x}{3x^2-9}\right)\\
&= \lim_{x\to\infty}\left(\frac{6x}{3x^2-9}\right)\\
&= \lim_{x\to\infty}\left(\frac{6/x}{3-9/x^2}\right)\\
&= 0
\end{aligned}
$$

Similarly, it can be shown that

$$
\lim_{x\to-\infty}\left(f(x)-\frac{2}{3}x\right)=0.
$$

Therefore, y = (2/3)x is a slant asymptote of f(x).

Question 2 (e)

SOLUTION. The function is decreasing between $(-\sqrt{3},\sqrt{3})$ by part (a) and so this eliminates graphs a,b,c,d,f,h,i,j,k,l.
This leaves only e and g. The slant asymptote on g is incorrect and hence this leave the correct answer e.

Question 3

SOLUTION. In this optimization problem, we want to maximize our revenue, so our objective function is

$$
\begin{aligned}
R &= (\text{number of nuts sold})(\text{price per nut})\\
&\quad+(\text{number of bolts sold})(\text{price per bolt})\\
&= (x)(-3x+500)+(y)(-y+300)\\
&= 500x+300y-3x^2-y^2.
\end{aligned}
$$

Since nuts and bolts weigh 0.5kg each and we only have 100kg of steel, our constraint becomes

$$
0.5x+0.5y=100.
$$

Using the constraint, we can express the objective function in terms of just one variable. Solving for y gives $y=200-x$. Substituting into R we get,

$$
\begin{aligned}
R &= 500x+300(200-x)-3x^2-(200-x)^2\\
&= 20000+600x-4x^2
\end{aligned}
$$

We will only concern ourselves with values of $x>0$ since we can't sell negative numbers of nuts. To determine the critical points, solve $R'(x)=0$ for x.

$$R'(x) = 600 - 8x = 0$$
$$x = \frac{600}{8} = 75.$$

Since this is the only critical point of R, if we can show that $R''(75) < 0$, then R has a global maximum at $x = 75$.
We have that

$$R''(x) = -8$$

and so $R''(x) < 0$ for all x. Therefore, $R''(75) = -8 < 0$, and R has a global maximum at $x = 75$. The corresponding value of bolts is $y = 125$. Therefore, we should sell 75 nuts and 125 bolts to maximize revenue.

Question 4 (a)

SOLUTION 1. To determine q when $p = 4$ is simply a matter of substituting $p = 4$ into the demand equation and solving for q.

$$\begin{aligned} q^2 + p^{3/2} + 2p &= 20 \\ q^2 + 8 + 8 &= 20 \\ q^2 &= 4 \\ q &= \pm 2. \end{aligned}$$

We ignore the negative solution since it is not possible to have negative demand. Therefore $q = 2$.
To determine dq/dp, we use implicit differentiation.

$$\begin{aligned} \frac{d}{dp}\left(q^2 + p^{3/2} + 2p\right) &= \frac{d}{dp}(20) \\ 2q\frac{dq}{dp} + \frac{3}{2}p^{1/2} + 2 &= 0 \\ \frac{dq}{dp} &= \frac{1}{2q}\left(-2 - \frac{3}{2}p^{1/2}\right) \end{aligned}$$

Substituting the values $p = 4$, $q = 2$ into the expression gives the value of dq/dp:

$$\frac{dq}{dp} = \frac{1}{4}(-2 - 3) \quad \rightarrow \quad \frac{dq}{dp} = -\frac{5}{4}$$

SOLUTION 2. Instead of implicitly differentiating, we first solve the equation for q and then differentiate normally.

$$\begin{aligned} q^2 + p^{3/2} + 2p &= 20 \\ q^2 &= 20 - p^{3/2} - 2p \\ q &= \sqrt{20 - p^{3/2} - 2p} \end{aligned}$$

Then

$$\frac{dq}{dp} = \frac{-(3/2)p^{1/2} - 2}{2\sqrt{20 - p^{3/2} - 2p}}$$

Substituting in $p = 4$ gives

$$\frac{dq}{dp} = \frac{-(3/2)(4)^{1/2} - 2}{2\sqrt{20 - (4)^{3/2} - 2(4)}} = \frac{-(3/2)(2) - 2}{2\sqrt{20 - 8 - 8}} = \frac{-5}{4}$$

Question 4 (b)

Solution. Using the hint and the first part, we have that
$\epsilon(p) = \dfrac{4 \cdot q'(4)}{q(4)} = \dfrac{4(-5/4)}{2} = \dfrac{-5}{2} < -1$
As the value is less than -1, we see that the revenue will decrease.

Question 4 (c)

Solution. Proceeding as in the hints, recall that $R = pq$ and thus

$$\frac{dR}{dt} = q\frac{dp}{dt} + p\frac{dq}{dt}$$

Plugging in $\frac{dR}{dt} = 0.15$ and $p = 4$ and $q = 2$ gives

$$0.15 = 2\frac{dp}{dt} + 4\frac{dq}{dt}$$

Notice that we have two unknowns and one equation so we'll need to find another equation to get our target value of $\frac{dp}{dt}$. This means we need to eliminate $\frac{dq}{dt}$. To do this we can differentiate the original demand equation given in the problem with respect to t,

$$\frac{d}{dt}\left(q^2 + p^{3/2} + 2p\right) = \frac{d}{dt}20$$
$$2q\frac{dq}{dt} + \frac{3\sqrt{p}}{2}\frac{dp}{dt} + 2\frac{dp}{dt} = 0.$$

Plugging in the information above gives

$$2(2)\frac{dq}{dt} + \frac{3\sqrt{4}}{2}\frac{dp}{dt} + 2\frac{dp}{dt} = 0$$

and simplifying

$$4\frac{dq}{dt} + 5\frac{dp}{dt} = 0$$

This gives us our second equation. Subtracting the second equation from the first gives

$$3\frac{dp}{dt} = -0.15$$

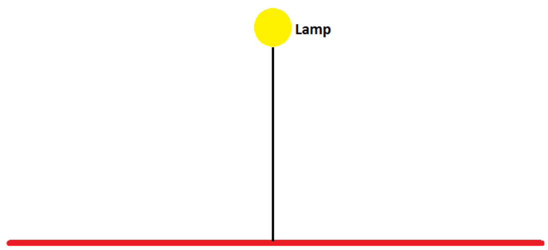
and thus

$$\frac{dp}{dt} = -0.05$$

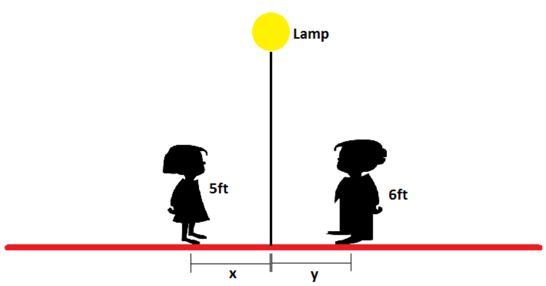
So the rate in price must be changed by a decrease of 5 cents (0.05 dollars) per hour completing the question.

Question 5 (a)

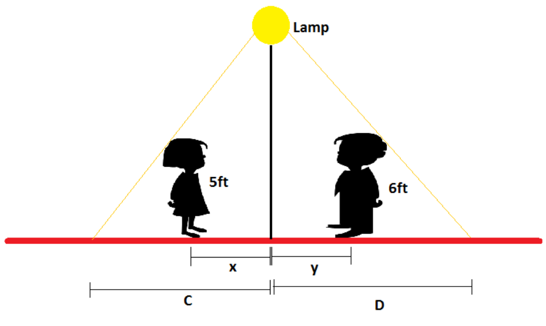
SOLUTION. Consider a 15ft tall lamp at the origin of a grid (so that everyone to the right is a positive distance away and everyone to the left is a negative distance away).



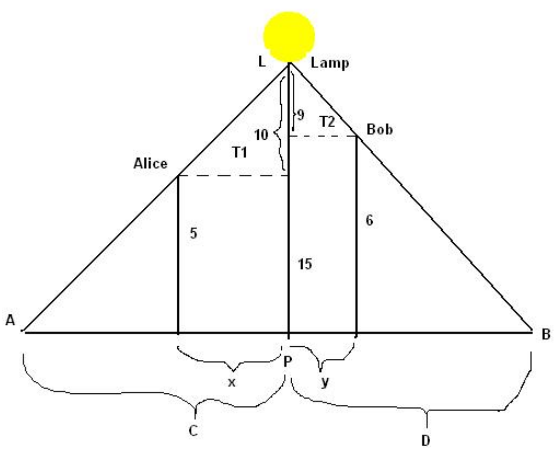
Let x be the distance that Alice is to the **left** of the lamp. We know that she is 5ft tall (which is shorter than the lamp post). Next, let y be the distance that Bob is to the **right** of the lamp. He is 6ft tall and so he is also shorter than the lamp but slightly taller than Alice.



Rays of light will shine from the lamp, touch Alice and Bob and eventually reach the ground (forming a big triangle with the lamp and ground). Label the distance between where the light hits the ground for Alice and the lamp as C and for Bob, D.



With this notation then the length of Alice's shadow, for example, should be C-x. To get a better understanding of how we could see the geometry of the problem better and how to go about solving it, it is often useful to simplify the diagram. To do this we will replace our characters with sticks since their dimensions don't affect the problem as far as we are concerned. An example of what a more mathematically-ready diagram should look like is below:

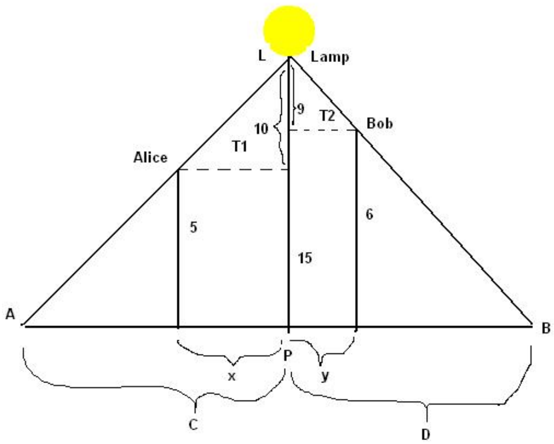


The distance between the tips of the shadows is C+D and since we know this is decreasing at 11ft/s, we could write

$$\frac{d}{dt}(C + D) = -11.$$

Question 5 (b)

SOLUTION 1. The diagram from part a has been recopied for ease of reading



Proceeding via the hints, we are interested in solving for $\frac{dy}{dt}$. We first note that triangles T_2 and triangle BPL are similar (they share an angle and both contain a 90 degree angle). Thus, we have that

$$\frac{y}{9} = \frac{D}{15}.$$

Multiplying by 15, reducing and differentiating with respect to time gives

$$\frac{5}{3}\frac{dy}{dt} = \frac{dD}{dt}.$$

To solve for $\frac{dy}{dt}$ we will need $\frac{dD}{dt}$, the speed at which Bob's shadow tip is changing. To do this we consider that the distance between the two tips is decreasing by 11ft/s and so

$$\frac{dC}{dt} + \frac{dD}{dt} = -11.$$

To determine $\frac{dC}{dt}$ we turn to the similar triangle problem for Alice. Triangles T_1 and triangle APL are similar (they share an angle and both contain a 90 degree angle). Thus, we have that

$$\frac{x}{10} = \frac{C}{15}$$

Multiplying by 15, reducing and differentiating with respect to time gives

$$\frac{3}{2}\frac{dx}{dt} = \frac{dC}{dt}$$

We are given that $\frac{dx}{dt} = -4$ and so we have that

$$\frac{dC}{dt} = -6.$$

Therefore we have that

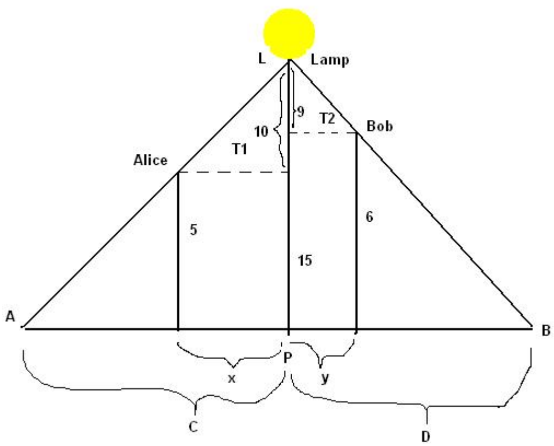
$$\frac{dD}{dt} = -11 - \frac{dC}{dt} = -5$$

and finally then that

$$\frac{dy}{dt} = \frac{3}{5}\frac{dD}{dt} = -\frac{3}{5}(5) = -3$$

and so Bob is walking **towards** the lamppost at 3ft/s. Notice the sign we get makes sense because we are told that each person is walking towards the lamp in the question!

SOLUTION 2. The diagram from part a has been recopied for ease of reading



Proceeding via the hints, we are interested in solving for $\frac{dy}{dt}$. We first note that triangles $BBob$ and triangle BPL are similar (they share an angle and both contain a 90 degree angle). Thus, we have that

$$\frac{D-y}{6} = \frac{D}{15}.$$

Multiplying by 30, reducing and differentiating with respect to time gives

$$\frac{dD}{dt} = \frac{5}{3}\frac{dy}{dt}.$$

To solve for $\frac{dy}{dt}$ we will need $\frac{dD}{dt}$, the speed at which Bob's shadow tip is changing. To do this we consider that the distance between the two tips is decreasing by 11ft/s and so

$$\frac{dC}{dt} + \frac{dD}{dt} = -11.$$

To determine $\frac{dC}{dt}$ we turn to the similar triangle problem for Alice. Triangles $AAlice$ and triangle APL are similar (they share an angle and both contain a 90 degree angle). Thus, we have that

$$\frac{C-x}{5} = \frac{C}{15}$$

Multiplying by 15, reducing and differentiating with respect to time gives

$$\frac{3}{2}\frac{dx}{dt} = \frac{dC}{dt}$$

We are given that $\frac{dx}{dt} = -4$ and so we have that

$$\frac{dC}{dt} = -6.$$

Therefore we have that

$$\frac{dD}{dt} = -11 - \frac{dC}{dt} = -5$$

and finally then that

$$\frac{dy}{dt} = \frac{3}{5}\frac{dD}{dt} = -\frac{3}{5}(5) = -3$$

and so Bob is walking **towards** the lamppost at 3ft/s. Notice the sign we get makes sense because we are told that each person is walking towards the lamp in the question!

Question 6 (a)

SOLUTION. $f(1) = 1$

(Explanation: We are given that the curve $x^2 + y^3 - 2xy = 0$ passes through the point $(1, 1)$. Near this point, the curve is the graph of a function, $y = f(x)$. When $x = 1$, $y = f(x) = 1$.)

$f'(1)$ is the same thing as $\frac{dy}{dx}$ evaluated at the point $(x, y) = (1, 1)$. In order to find this, we use implicit differentiate the equation $x^2 + y^3 - 2xy = 0$ with respect to x.

$$2x + 3y^2\frac{dy}{dx} - 2y - 2x\frac{dy}{dx} = 0$$

In order to solve for $\frac{dy}{dx}$, we first factorize,

$$2x - 2y + (3y^2 - 2x)\frac{dy}{dx} = 0.$$

Then subtract $2x - 2y$ from both sides and divide by $(3y^2 - 2x)$ to obtain

$$\frac{dy}{dx} = \frac{2y - 2x}{3y^2 - 2x}.$$

When $x = y = 1$,

$$\frac{dy}{dx} = \frac{2(1) - 2(1)}{3(1)^2 - 2(1)} = 0.$$

Hence, $f'(1) = 0$.

Question 6 (b)

SOLUTION. From part (a), we have

$$2x + 3y^2\frac{dy}{dx} - 2y - 2x\frac{dy}{dx} = 0$$

Differentiating again gives

$$2+3\left(2y\left(\frac{dy}{dx}\right)^2+y^2\frac{d^2y}{dx^2}\right)-2\frac{dy}{dx}-2\left(\frac{dy}{dx}+x\frac{d^2y}{dx^2}\right)=0$$

Isolating for the second derivative gives

$$\begin{aligned}2+6y\left(\frac{dy}{dx}\right)^2+3y^2\frac{d^2y}{dx^2}-4\frac{dy}{dx}-2x\frac{d^2y}{dx^2}&=0\\2-4\frac{dy}{dx}+6y\left(\frac{dy}{dx}\right)^2+(3y^2-2x)\frac{d^2y}{dx^2}&=0\\\frac{d^2y}{dx^2}&=\frac{-2+4\frac{dy}{dx}-6y\left(\frac{dy}{dx}\right)^2}{3y^2-2x}\end{aligned}$$

Now we plug in $x=y=1$ into this equation. From part (a), we know that the derivative at 1 is 0 and thus, we get

$$\frac{d^2y}{dx^2}=\frac{-2+4(0)-6(1)(0)^2}{3(1)^2-2(1)}=\frac{-2}{1}=-2$$

and so $f''(1)=-2$

Question 6 (c)

SOLUTION. The linear approximation formula is

$$L(x)=f(a)+f'(a)(x-a).$$

The number a is the value we are approximating around and thus in our case a=1. Since we are trying to approximate f(1.02) we take x=1.02. Using the information from part (a) we have that

$$f(1.02)\approx L(1.02)=f(1)+f'(1)(1.02-1)=1+0(0.02)=1.$$

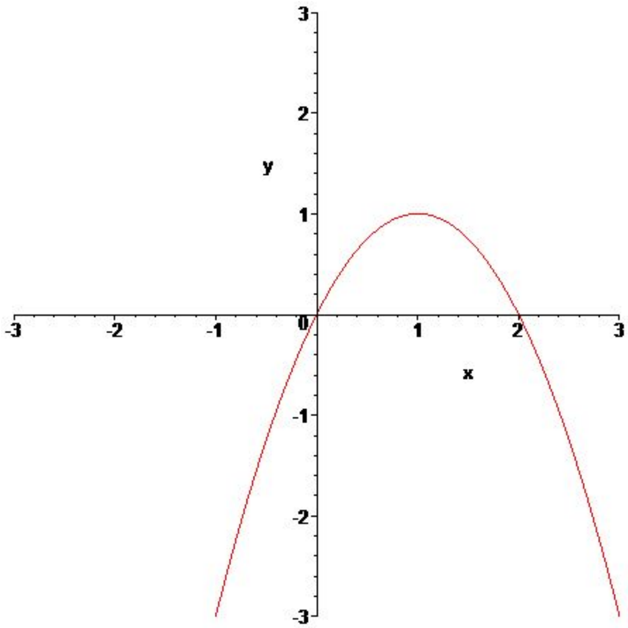
This gives the estimate that $f(1.02)=1$. From part (b), the second derivative at x=1 is negative and therefore we see that the function is concave down at this point. This tells us that points near x=1 are smaller than what the tangent line predicts and therefore we expect that our estimate is an overestimate.

Question 6 (d)

SOLUTION. Plugging in the values from part (a) and part (b) into the second degree Taylor polynomial (recalling the a value is 1) gives

$$\begin{aligned}T_2(x)&=f(1)+f'(1)(x-1)+f''(1)(x-1)^2/2\\&=1+(0)(x-1)+(-2)(x-1)^2/2\\&=1-(x-1)^2\\&=-x^2+2x\\&=x(2-x)\end{aligned}$$

The graph is given below.



Good Luck for your exams!