Full Solutions
MATH220 December 2011

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How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

Easiness: 5.0/5

SOLUTION. De Morgan's laws are, for sets $X, Y \subseteq Z$ the following.

$$(X \cup Y)^c = X^c \cap Y^c$$

and

$$(X \cap Y)^c = X^c \cup Y^c$$

Here the c indicates the complement of the set.

Question 1 (b)

Easiness: 4.0/5

SOLUTION. The set X is the *intersection* of all of the sets S_α; that is, X is the set of all x such that, for all $\alpha \in I$ we have $x \in S_\alpha$.
The set Y is the *union* of all of the sets S_α; that is, Y is the set of all y such that there exists an $\alpha \in I$ such that $y \in S_\alpha$.

Question 1 (c)

Easiness: 5.0/5

SOLUTION. Two sets A and B are said to have the same cardinality if there exists a bijection from one set to the other.
More precisely, that means there must exist at least one function $f\colon A \to B$ such that the function is injective and surjective; that is, for each element a of the set A there exists a unique element b in the set B such that $f(a) = b$ (that condition is the injectivity) AND for each element b of the B there exists a unique element a in the set A such that $f(a) = b$ (and this is the condition for surjectivity).

Question 1 (d)

Easiness: 5.0/5

SOLUTION. Since the composition $\alpha \circ \beta$ is given by doing β first and then α, we must have that

$$\begin{aligned} 1 \mapsto 2 \mapsto 3 \\ 2 \mapsto 1 \mapsto 2 \\ 3 \mapsto 3 \mapsto 4 \\ 4 \mapsto 4 \mapsto 1 \end{aligned}$$

and so we have that $\alpha \circ \beta$ is given by

$$\alpha \circ \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$$

Question 1 (e)

Easiness: 4.5/5

SOLUTION. A function f has an inverse if and only if it is *bijective*. That is, it has an inverse if and only if it is

1. Injective (or one-to-one)

2. Surjective (or onto).

In such a case, for every $y \in B$, there is a unique $x \in A$ such that $f(x) = y$. Using this fact, we define f^{-1} by the rule

$$f^{-1}(y) = x$$

where x is the unique element of the set A such that $f(x) = y$. From the injectivity and surjectivity, this is well defined. Moreover,

$$f\big(f^{-1}(y)\big) = f(x) = y$$

and

$$f^{-1}\big(f(x)\big) = f^{-1}(y) = x$$

and so this really is the inverse of f.

Question 1 (f)

Easiness: 3.6/5

SOLUTION. Strong mathematical induction is a method to prove a statement $S(n)$ is true for all natural numbers n. It is given by the following.
Suppose that for every natural number n, that we have the implication

$$\big(\forall\, k < n, S(k)\big) \Rightarrow S(n)$$

Then for every natural number n, the statement $S(n)$ is true.

Question 2 (a)

Easiness: 5.0/5

SOLUTION. We note that $P(4)$ is the statement "4 is divisible by 4", which is true. We also note that $Q(4)$ is the statement "17 is divisible by 3" which is false. As such, the implication

$$P(4) \Rightarrow Q(4)$$

does not hold, and so the statement

$$\forall\, n \in \mathbb{N}, P(n) \Rightarrow Q(n)$$

is false.

Question 2 (b)

Easiness: 5.0/5

SOLUTION.

- The statement

$$P(3) \Rightarrow Q(4)$$

is *"3 is divisible by 4 implies that 17 is divisible by 3"*.

- The **converse** is

$$Q(4) \Rightarrow P(3)$$

hence in words it is *"17 is divisible by 3 implies that 3 is divisible by 4"*.

- The **contrapositive** is

$$\neg Q(4) \Rightarrow \neg P(3)$$

In words this is *"17 is not divisible by 3 implies that 3 is not divisible by 4"*.

Question 3

Easiness: 5.0/5

SOLUTION. Using both hints, we can rewrite the statement into equivalent statements as follow

$$\begin{aligned}
[(P \Rightarrow Q) \Rightarrow R] \vee (\neg P \vee Q) &\equiv [(\neg P \vee Q) \Rightarrow R] \vee (\neg P \vee Q) \\
&\equiv \neg(\neg P \vee Q) \vee R \vee (\neg P) \vee Q \\
&\equiv (P \wedge \neg Q) \vee R \vee (\neg P) \vee Q \\
&\equiv [P \vee R \vee (\neg P) \vee Q] \wedge [(\neg Q) \vee R \vee (\neg P) \vee Q]
\end{aligned}$$

The last equivalent statement is of the type A AND B with

$$\begin{aligned}
A &= P \vee R \vee (\neg P) \vee Q \\
B &= (\neg Q) \vee R \vee (\neg P) \vee Q
\end{aligned}$$

which both are tautologies since they contain P OR *not* P and respectively *not* Q OR Q. So if both A and B are tautologies, then so is the conjunction of them into the statement A AND B.

Question 4

Easiness: 2.6/5

SOLUTION. We want to show that

$$f(C \cap D) \subseteq f(C) \cap f(D)$$

and that

$$f(C) \cap f(D) \subseteq f(C \cap D)$$

Let us start with the first.
Suppose that $y \in f(C \cap D)$; that is, there is some $x \in C \cap D$ such that $f(x) = y$. Since $x \in C$ and $x \in D$, it follows that $y \in f(C)$ and that $y \in f(D)$, i.e. that $y \in f(C) \cap f(D)$ as desired.
Note: This proof in no way uses injectivity; this is a general statement about all functions.
Now we prove the second statement. Suppose that $y \in f(C) \cap f(D)$. That is, $y \in f(C)$ and $y \in f(D)$. Equivalently, there is some $x_1 \in C$ such that $f(x_1) = y$, and some $x_2 \in D$ such that $f(x_2) = y$. We claim that $x_1 = x_2$, and hence $x_1 = x_2 \in C \cap D$. But this follows immediately from the injectivity of f. It follows now, since $f(x_1) = y$ and $x_1 \in C \cap D$ that $y \in f(C \cap D)$ as claimed.

Question 5 (a)

Easiness: 3.0/5

SOLUTION. A set A is denumerable if it is in bijective correspondence with the natural numbers. That is, it is denumerable if and only if there exists a bijective function

$$f : A \to \mathbb{N}$$

Note: Mathematicians think of this function as a *counting* of all the elements of the set A. Given a counting, that is such a bijection f the *first* element of the set A is the one that is mapped to 1 by the function, the *second* element is the one mapped to 2 and so on. Observe that we need to require the function to be bijective to guarantee that all the elements are counted and are counted in a unique way (we don't want to have two elements mapped to the same natural number).

Question 5 (b)

Easiness: 3.2/5

SOLUTION 1. The core idea of this solution is to say that if the set A is denumerable, then we can remove any element a from this set and obtain a proper subset which is also denumerable. We will show below how to precisely prove that this subset is denumerable.
Since the set A is denumerable, there exists a bijection from that set to the set of natural numbers. Let f be this bijection, that is

$$f : A \to \mathbb{N}$$

Let a be the element of the set A which is mapped to the integer 1, that is

$$a = f^{-1}(1)$$

Then define the set B to be the subset containing all the elements of the set A except for the element a, that is

$$B = A \backslash \{a\}$$

Note that B is a proper subset of A. To show this subset is denumerable, define a function g to the natural numbers as follow

$$\begin{aligned} g : B &\to \mathbb{N} \\ b &\mapsto f(b) - 1 \end{aligned}$$

Notice that each element of the set B is mapped by f to an integer greater or equal to 2 (since we removed a who precisely was the integer mapped to 1), so the function g simply assigns the natural number just below. We claim that this g is bijective.
To show surjectivity, notice that each natural number m is the image by the function g of the element which was mapped to m+1 by the function f. We can write this as

$$m = g(f^{-1}(m + 1))$$

For injectivity, suppose that you have elements b and"b"'of the set Bsuch that

$$g(b) = g(b')$$

By definition of the function g this means that

$$f(b) - 1 = f(b') - 1$$

and so that

$$f(b) = f(b')$$

And since we know that the function f is a bijection (and in particular is injective) we can conclude that b = b*'*, and thus the function g is injective.
Since g is both surjective and injective it is a bijection and thus the set B is denumerable.

Solution 2. Another way to find a denumerable proper subset goes as follow:

- use the fact that the set A has a bijection with the natural numbers to define a proper subset of all the elements mapped to an even natural number,
- realize that this will be a proper and denumerable subset of the set A.

Details of this proof are described here.
Since the set A is denumerable there must exist a bijection g: $A \to \mathbf{N}$ from that set to the natural numbers. We define the set B to be all the elements of the set A that are mapped to an even number, that is

$$B = \{a \in A \mid g(a) \text{ is even}\}$$

We need to show that the set B is a proper subset of A and is denumerable. It is a proper subset since it doesn't contain any element that are mapped to an odd number. To show it is denumerable, we define the function

$$\begin{aligned} h : B &\to \mathbb{N} \\ b &\mapsto \frac{g(b)}{2} \end{aligned}$$

and show that it is a bijection.
It is surjective since for any natural number m there is an element of the set A which maps to the number $2m$ and since that's an even integer, that element of the set A is a member of the subset B, let's call it b for now. Then since $g(b) = 2m$, then $h(b) = m$ and thus the function h is surjective.
To show injectivity, consider two elements b and"b"'of the set B such that

$$h(b) = h(b')$$

by definition of the function h we have that

$$\frac{g(b)}{2} = \frac{g(b')}{2}$$

and thus

$$g(b) = g(b')$$

but since g is a bijection, it is also an injection and we can conclude that $b = b'$, and so the function h is injective.

Question 6 (a)

Easiness: 3.3/5

SOLUTION 1. We start our induction at $n = 1$. In that case we have that $2n$ - 1 = 1 and so the sum is just 1 alone which is a perfect square.
We can check $n = 2$ if we want as well, in that case $2n$ - 1 = 4 - 1 = 3 and so the sum is 1 + 3 = 4 which is the square of 2.
For $n = 3$ we have that 1 + 3 + 5 = 9 which is the square of 3; we now see something which seems to be a pattern, that is for $n = k$ the sum will end up being the square of the number k. And so we conjecture that

$$1 + 3 + 5 + \ldots + (2n - 1) = n^2$$

We have proved above that this is true for n = 1, 2 and 3. We will use induction to prove that is holds for all values of n. Since we have the first step(s) done, we now assume that the conjecture holds for $n = k$ and we will show that it still holds for $n = k+1$. Indeed

$$\begin{aligned}\underbrace{1 + 3 + 5 + \ldots + (2k - 1)}_{=k^2} + (2(k + 1) - 1) &= k^2 + (2k + 2 - 1)\\ &= k^2 + 2k + 1\\ &= (k + 1)^2\end{aligned}$$

This shows that our conjecture is true for $n = k+1$ and thus concludes our proof.

SOLUTION 2. *The following proof is not valid for the purpose of this exam question since it doesn't use induction, but it is of the level of this course, so we add it here for completeness.*
Consider playing with square pieces of paper, or colouring squares in a lined piece of paper. If you look at a n by n square, it is made of n^2 squares. The next square you can built, which will have now $n+1$ squares on each side will contain $(n+1)^2$ squares, but how many squares were added to it? Well you need to add a layer on say the right side, so n additional squares and also a layer on the top side, so another n additional squares and now you are also missing one square at the top right corner, so yet another square. In total, to go from the square with n little squares on the side to the next one with $n+1$ squares on each side you need to add $2n+1$ squares.
So if we start with 1 square, we need to add 3 squares to get a 2 by 2 squares and we need to add 5 squares to that to get a 3 by 3 square and we need to add 7 squares to get a 4 by 4 square and hence if we sum $1 + 3 + 5 + \ldots + (2n - 1)$ squares we will get a n by n square. Which is exactly what we wanted to show.

SOLUTION 3. *The following proof is not valid for the purpose of this exam question since it doesn't use induction, but it is of the level of this course, so we add it here for completeness.*
Another proof can be obtained by simply playing with the formula that sums consecutive integers, that is

$$1 + 2 + 3 + 4 + 5 + 6 + \cdots + k = \sum_{i=1}^{k} i = \frac{k(k + 1)}{2}$$

And we just use the fact that

$$\begin{aligned}1 + 3 + 5 + \cdots + (2n - 1) = &(1 + 2 + 3 + \ldots + (2n - 1) + (2n))\\ &- (2 + 4 + 6 + \ldots + (2n))\end{aligned}$$

Or if we use sum notations

$$\sum_{i=1}^{n}(2i-1) = \sum_{i=1}^{2n} i - \sum_{i=1}^{n} 2i$$

Now using the formula above we obtain that

$$\sum_{i=1}^{2n} i = \frac{2n(2n+1)}{2} = n(2n+1) = 2n^2 + n$$

and

$$\sum_{i=1}^{n} 2i = 2\sum_{i=1}^{n} i = 2\frac{n(n+1)}{2} = n(n+1) = n^2 + n$$

And so

$$1 + 3 + 5 + \cdots + (2n-1) = (2n^2+n) - (n^2+n) = n^2$$

which proves that it is a perfect square.

Question 6 (b)

Easiness: 1.0/5

SOLUTION. First, we prove the claim true when $n = 1$. In this case,

$$12^{2n-1} + 11^{n+1} = 12^{2(1)-1} + 11^{1+1} = 12 + 121 = 133$$

which is clearly divisible by 133. We assume the claim is true for $n = k$. For $n = k+1$, we have

$$\begin{aligned}
12^{2(k+1)-1} + 11^{(k+1)+1} &= 12^{2k+1} + 11^{k+2} \\
&= 12^2 \cdot 12^{2k-1} + 11 \cdot 11^{k+1} \\
&= 144 \cdot 12^{2k-1} + 11 \cdot 11^{k+1} \\
&= (133+11)12^{2k-1} + 11 \cdot 11^{k+1} \\
&= 133 \cdot 12^{2k-1} + 11 \cdot 12^{2k-1} + 11 \cdot 11^{k+1} \\
&= 133 \cdot 12^{2k-1} + 11(12^{2k-1} + 11^{k+1})
\end{aligned}$$

Now, in the last line, the first summand is divisible by 133 and the second summand is also divisible by 133 by the induction hypothesis. Hence we must have that $12^{2(k+1)-1} + 11^{(k+1)+1}$ is divisible by 133 as required.

Question 7 (a)

Easiness: 3.0/5

SOLUTION. Multiplying by any non-zero integer (other than 1 or -1) will do the trick. For example, multiplying by 2 gives the map

$$\begin{aligned}
f : \mathbb{Z} &\to \mathbb{Z} \\
n &\mapsto 2n
\end{aligned}$$

Proof: *recall that this question didn't ask for a proof, just the answer. Since the proofs are of a difficulty that you are expected to be able to handle in this course, we present them here for completeness.*
The function is injective since

$$2n = 2m \iff n = m$$

and is not surjective since clearly no odd integer will be in the image of the function f.

Question 7 (b)

Easiness: 5.0/5

SOLUTION. If we divide all the even integers by two, we obtain all the integers. To turn this into a map, we just need to say what we'll do with the odd integers (anything would work, for example mapping all the odd integers to the number 4 or as we present below, map them to the rounded half).

$$\begin{aligned} g : \mathbb{Z} &\to \mathbb{Z} \\ n &\mapsto \begin{cases} \frac{n}{2} & \text{if } n \text{ is even;} \\ \frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Proof: *recall that this question didn't ask for a proof, just the answer. Since the proofs are of a difficulty that you are expected to be able to handle in this course, we present them here for completeness.*
This function is clearly surjective since any integer n is the image of the integer $2n$. Now the function is not injective since $g(3) = g(2) = 1$ (or more generally since $g(2k+1) = g(2k) = k$ for any integer k).

Question 8

Easiness: 1.0/5

SOLUTION. By saying that we look at primes larger than 4, what we really mean is we look at all the primes except 2 and 3 (which makes sense since $2^2 = 4$ and $3^2 = 9$ which is 3 mod 6; so these two don't work anyway).
Now any prime p other than 2 and 3 will then clearly NOT be a multiple of two and NOT be a multiple of 3. This means that

$$p \equiv 1 \text{ or } 5 \pmod 6$$

since integers that are 0, 2 or 4 mod 6 are all even and since an integer that is 3 mod 6 is a multiple of 3.
But since

$$1^2 \equiv 1 \pmod 6$$

and

$$5^2 = 25 \equiv 1 \pmod 6$$

we can conclude that if p is a prime larger than 4, then its square is always 1 mod 6.

Good Luck for your exams!