Full Solutions
MATH102 December 2014

April 22, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question A 01

SOLUTION. THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!

As the tangent line to $f(x)$ at $x = c$ is $y = a + b(x - c)$, then we know $f(c) = a$ and $f'(c) = b$ from the definition of tangent line. If $g(x)$ is the inverse function of $f(x)$, then $g(a) = c$ and $g'(a) = \frac{1}{b}$. Knowing a point (a, c) on the tangent line and the slope $\frac{1}{b}$, we can write down the equation of the tangent line, which is answer (A) $y = c + \dfrac{1}{b}(x - a)$.

Question A 02

SOLUTION. THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!

From the definition of the derivative of $\sin(x)$ at $x = 0$ we know $\sin'(0) = \lim_{x\to 0} \frac{\sin(x)-\sin(0)}{x-0} = \lim_{x\to 0} \frac{\sin(x)}{x}$, and we also know $(\sin(x))' = cos(x)$, $(\sin(0))' = \cos(0) = 1$. Using the L'Hopital's rule we find $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$. Then to make the original function $f(x)$ continuous, we just need $a = \lim_{x\to 0} \frac{\sin(x)}{x} = 1$. It gives $a = 1$ which is answer (B). Note: This question can also be solved using the Squeeze Theorem.

Question A 03

SOLUTION. THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!

We will pair each graph of y' with the graph of corresponding function y by the sign and the magnitude of y'.
First look at diagram (A). It shows the y' is always positive in the domain of interest, which means function y must be a increasing function. Hence we know the solution sketch corresponding to (A) might be (2) or (4). Next we notice y' has a local minimum in the mediate y value which means the increasing rate of y achieves its lowest value when the value of y is about in the middle, which indicates (2) is the correct choice. Similarly, the plot of y' in (B) indicates y is an increasing function and its increasing rate first increases before the middle point and then decreases again. The only correct pairing is (B)(4).
Next, from y' in (C) we know y is a decreasing function. Its decreasing rate first increases as the absolute value of y' reaches its maximum in the middle and then decreases. Only (1) matches it.
Finally, from (D) we get y is a decreasing function but its decreasing rate slows down in the middle, which matches (3).
So the correct answer is (b).

Question A 04

SOLUTION. THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!

First we observe there are two steady states (where y'=0) $y = 0$ and $y = 1$. Then from the direction of those vectors, we find $y = 0$ is unstable and $y = 1$ is stable for $y < 1$, unstable for $y > 1$. To obtain the correct stability of those steady states, we must have

$$\begin{aligned} y' &< 0, \quad \text{when } y < 0, \\ y' &> 0, \quad \text{when } 0 < y < 1, \\ y' &> 0, \quad \text{when } 1 < y, \end{aligned}$$

The first observation eliminates options a) and d). The third observation rules out option b). Option e) is not valid because a steady state must exist when $y = 1$. Thus the only possible answer is (c).

Question A 05

SOLUTION 1. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

By observing the three given differential equations, we have the following interpretations

1. Healthy cells have a constant increasing rate P.
2. Healthy cells can become infected when they encounter infected virus(from the $\pm aCV$ term in equations of C and I).
3. Virus is produced at a rate proportional to the current infected cell density(from the βI term in the equation for V).
4. Healthy cells, infected cells and virus die at a rate proportional to its own density(from the $-\gamma_1 C$, $-\gamma_2 I$ and $-\gamma_3 V$ terms).

Thus the correct statement is (b).

SOLUTION 2. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

Another way through which this question can be solved involves analyzing the relationship between the variables of the equations.

1. In the first equation, change in the number of healthy cells ($\frac{dC}{dt}$) can be described as dependent on two variables (C and V) and three constants(P, α, γ_1). This observation rules out option a) as it indicates the change in the density of healthy immune cells ($C(t)$) is only dependent on the density of healthy immune cells and the density of virus in the blood of the patient, and not the density of the infected cells ($I(t)$). (The correct answer of this question may also be deduced from this equation)
2. Similarly, for the second question, change in the number of infected immune cells can be described as dependent on three variables (C,V, and I). Based on the equation, it can be deduced that an increase in V will lead to an increase in $I(t)$, and thus, option c) can be ruled out.
3. The third equation illustrates that change in viral density is dependent on two variables (I and V). Based on the equation, option c) can be eliminated as an increase in V will lead to smaller $\frac{dV}{dt}$. Option e) can also be ruled out as $\frac{dV}{dt}$ does not dependent on the variable C.

The correct statement is therefore b).

Question A 06

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

From the graph we observe the y-intercept of the line is $\frac{1}{2}$ and suppose the slope of the line is k. Then we can write down the equation of the line $y = \frac{1}{2} + kt$, But notice the y-axis is $\ln(C(t))$, then we have

$$\begin{aligned} \ln(C(t)) &= \frac{1}{2} + kt, \\ C(t) &= e^{\frac{1}{2}+kt}, \\ \text{or } C(t) &= e^{\frac{1}{2}}e^{kt}, \end{aligned}$$

where the only possible answer is (d).

Question A 07

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

From the equation $\frac{dT}{dt} = k(E - T)$, we observe the steady state temperature is E, the ambient temperature, which means $T(t)$ will approach E as time goes to infinity. From the graph we see object (1) and (3) were kept in the same ambient temperatures and (2) was kept in a different ambient temperature(so (a) is wrong). And as plots (1), (2) and (3) are all of the same value when $t = 0$, then they have the same initial temperatures(so (b) is wrong).
From the equation we know the constant k controls how fast $T(t)$ approaches E, the larger the k, the faster the rate. So comparing plots of (1) and (3) we find the k for (1) was larger than the value of k for (3) as (1) approaches to its steady state faster than (3)(so (c),(d) are wrong).
This leaves the only choice (e).

Question A 08

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

Let $y = \sin(x)$ then the problem becomes finding x such that $\sin(x) = 0.4$ or $\sin(x) - 0.4 = 0$. According to Newton$'$s method, to find roots of $f(x) = 0$, the formula is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.
Here we have $f(x) = \sin(x) - 0.4$, then $f'(x) = \cos(x)$ and the above formula becomes

$$x_{n+1} = x_n - \frac{\sin(x_n) - 0.4}{\cos(x_n)},$$

which is answer (C).

Question B 01

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

First we calculate $f'(x)$, which is $f'(x) = (-2x + 1)e^{-x^2+x}$, So when $f'(x) = 0$, $x = 1/2$ is the only critical point. To determine whether it is a maximum or minimum, we calculate the second derivative $f''(x)$, which has the expression $f''(x) = ((2x - 1)^2 - 2)e^{-x^2+x}$. As $f''(\frac{1}{2}) < 0$, then we know $x = \frac{1}{2}$ is a max. To find the inflection point, we solve $f''(x) = 0$ and obtain $x_1 = \frac{1-\sqrt{2}}{2}, \quad x_2 = \frac{1+\sqrt{2}}{2}$. Notice the two points divide the whole domain into three intervals and in each of those intervals we have

$$\begin{aligned} f''(x) > 0, &\quad x < x_1, \\ f''(x) < 0, &\quad x_1 < x < x_2, \\ f''(x) > 0, &\quad x_2 < x, \end{aligned}$$

So $f''(x)$ change signs on both sides of x_1 and x_2. So $x_{1,2}$ are inflection points.

Question B 02

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

According to given information, we write down the equation describing the rate of change of the mussel population as $\frac{dP}{dt} = kP$, where k is a constant. Solving it we obtain $P(t) = P(0)e^{kt}$. With $P(0) = 100$ and $k = 0.5\ln(2)$, it becomes $P(t) = 100e^{0.5\ln(2)t} = 100e^{\ln(\sqrt{2})t} = 100(\sqrt{2})^t$.
When $P = 800$, solving for t we get $100(\sqrt{2})^t = 800$ and hence $t = 6$.

Question B 03

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

In this problem we know the point is at $a = \frac{\pi}{4}$ and the function value $f(a) = f(\frac{\pi}{4}) = \tan(\frac{\pi}{4}) = 1$. To calculate $f'(\frac{\pi}{4})$ we need the derivative of $f(x)$, which is $f'(x) = (\tan(x))' = \sec^2(x)$, and it follows $f'(\frac{\pi}{4}) = \sec^2(\frac{\pi}{4}) = 2$. So the equation of the tangent line is $L(x) = 1 + 2(x - \frac{\pi}{4})$.

Question B 04

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

The general expression of a cosine function is $T(t) = A + B\cos(\omega t + \varphi)$. Suppose A, B are both positive. Here the lowest temperature is 2^o C and highest is 10^o C, then
$A - B = 2, A + B = 10$,
so $A = 6$ and $B = 4$. $2\pi/\omega$ is the period of this function. As it describes the temperature changes throughout a day which lasts for 24 hours, we have $\frac{2\pi}{\omega} = 24, \quad \omega = \frac{2\pi}{24}$. From the problem statement we know the lowest temperature is obtained at 7:00, by symmetry the highest is around 19:00. Then to shift the maxima from 0 to 19 we subtract t by 19. So $\varphi = -19\frac{2\pi}{24}$ and the final function is $T(t) = 6 + 4\cos(\frac{2\pi}{24}t - \frac{2\pi}{24}19)$. Note the expression of the solution is not unique, it can be written as $T(t) = 6 - 4\cos(\frac{2\pi}{24}t - \frac{2\pi}{24}7)$, or $T(t) = 6 + 4\cos(\frac{2\pi}{24}t + \frac{2\pi}{24}5)$,

Question B 05

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

To approximate $y(1/4)$, we use the Euler's method $y_{n+1} = y_n + \Delta t f(y_n)$. with a single step $\Delta t = 1/4$ and $n = 0$, where $\frac{dy}{dt} = f(y)$. As $y_0 = y(0) = 2$, we have

$$\begin{aligned} y(1/4) = y_1 &= y_0 + \Delta t \cdot 2(1 - y_0) \\ &= 2 + \frac{1}{4} \cdot 2(1-2) \\ &= \frac{3}{2}. \end{aligned}$$

Question B 06

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

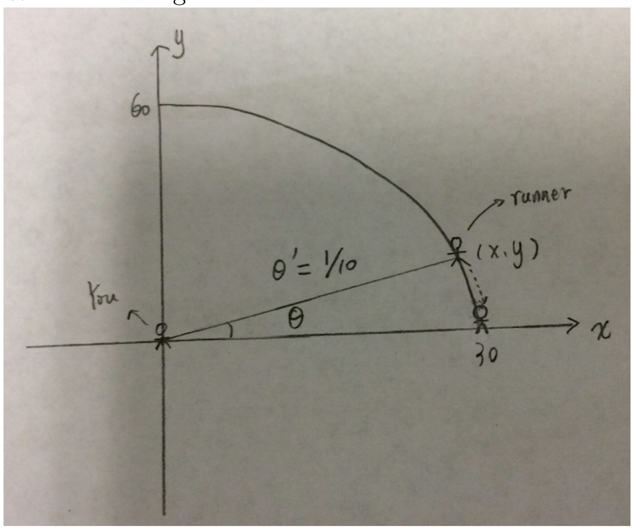
From the definition of linear approximation we know that, for x close to a, we can approximation function value $f(x)$ by the following form $f(x) \approx L(x) = f(a) + f'(a)(x - a)$. Here in this problem we have $f(x) = \ln(x), \quad f'(x) = \frac{1}{x}, \quad a = 1, \quad x = 0.95$, Substitute them into above form we get

$L(0.95) = 0 + 1(0.95 - 1) = -0.05.$

Question B 07

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

We draw a diagram as shown:



Suppose the finish line is located on the positive side of the x-axis. Assume the angle between the line pass through the runner and the origin and the positive x-axis is θ, then we have $\tan(\theta) = \frac{y}{x}$, where $(x,\ y)$ is the runner's position. When the runner crosses the finish line, $\theta = 0, \quad \theta' = \frac{1}{10}$, We also get $x' = 0$ as the elliptical track is perpendicular to the x-axis at the finish point so there is no change in x. Thus, at the finish point, the velocity of the runner can be described solely as the change in the y-axis. Also it follows $y' = v$. Further, we obtain $x = 30$ from the equation of the ellipse $x^2 + \dfrac{y^2}{4} = 900$(when $y = 0$, $x = \pm 30$). Then we differenciate $\tan(\theta)$ with respect to time to get

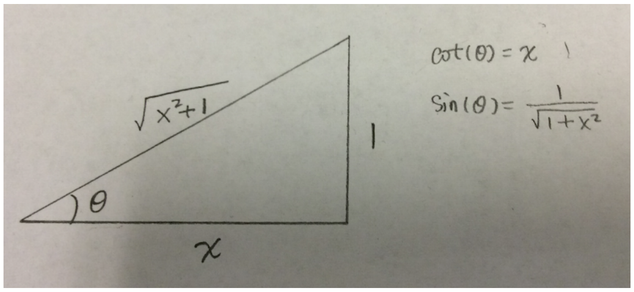
$$\begin{aligned}
\sec^2(\theta)\theta' &= \frac{y'x - yx'}{x^2} \\
1 \cdot \theta' &= \frac{y'x - yx'}{x^2} \\
\theta' &= \frac{v \cdot 30 - 0}{30^2} \\
&= \frac{v}{30}
\end{aligned}$$

Then $v = 30\theta' = 3m/s$
Note: The third line of the equation above is where the observation that $x' = 0$ is applied into the question as indicated by the substitution of y' with v, the variable of x with 30, and the rate of x' with 0.

Question C 01

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

Let $x = \cot(\theta)$, we can rewrite it as $x = \frac{\cos(\theta)}{\sin(\theta)}$, With implicit differentiation we take the derivative of above expression with respect to x on both sides to get



$$1 = \frac{-\sin^2(\theta) - \cos^2(\theta)}{\sin^2(\theta)} \cdot \theta',$$
$$1 = -\frac{1}{\sin^2(\theta)} \cdot \theta'$$

So we obtain $\theta' = -\sin^2(\theta) = -\frac{1}{1+x^2}$, as shown in the following diagram
Note: the triangle drawn on the right diagram can be solely constructed from the equation of $x = \cot(\theta)$

Question C 02 (a)

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

By the definition of steady state, we look for the value of c such that $\frac{dc}{dt} = 0$, or $P(c) - D(c) = 0$. Thus we need to solve

$$P_m \frac{c^2}{k^2 + c^2} = rc,$$
$$\text{or } c(rc^2 - P_m c + k^2 r) = 0,$$

Substitute values of P_m, k and r and simplify the expression, we obtain $\frac{1}{5}c(c-5)(c-20) = 0$, which gives $c = 0,\ 5,\ 20$ are the steady states.

Question C 02 (b)

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

To determine the stability of each steady state, we just need to find the increasing and decreasing regions. As c represents the concentration of a substance, we only need to consider the positive half of the real axis. The three steady states divide it into three intervals

$$\begin{aligned} c \in (0,5), &\quad P(c) - D(c) < 0 \quad \text{decreasing},\\ c \in (5,20), &\quad P(c) - D(c) > 0 \quad \text{increasing},\\ c \in (20,\infty), &\quad P(c) - D(c) < 0 \quad \text{decreasing}, \end{aligned}$$

If we choose a initial condition in the interval $c_0 \in (0,5)$, as $P(c) - D(c)$ is negative, c would decrease and approach the steady state $c = 0$ as time goes on. So $c = 0$ is stable. Similarly, we find $c = 5$ is unstable and $c = 20$ is stable.

Question C 02 (c)

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

If $c(0) = 8$ then it falls in the region $(5, 20)$ and $P(c) - D(c) > 0$. So c would increase with time and finally approach the steady state $c = 20$.

Question C 02 (d)

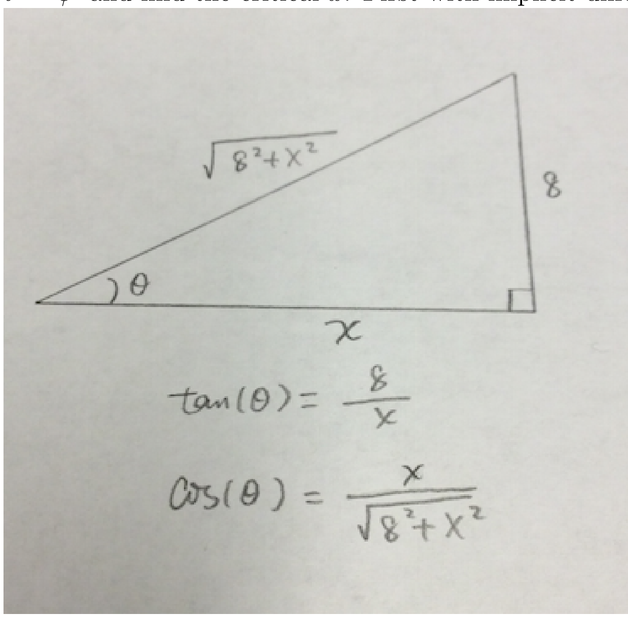
SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

Same as in C2(a), to solve the steady states, we just need to solve the following equation $c(rc^2 - P_m c + k^2 rc) = 0$, Obviously $c = 0$ is one of the steady states and other steady states are roots of the quadratic equation $rc^2 - P_m c + k^2 rc = 0$. To get only two steady state, from the quadratic formula we need to have $b^2 - 4ac = 0$ and in this case $(P_m)^2 - 4 \cdot r \cdot k^2 r = 0$. Thus with the given values of P_m and k we have $r^2 = \frac{P_m^2}{4k^2} = \frac{1}{16}$, To obtain positive steady state c, we get $r = \frac{1}{4}$.

Question C 03

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

As indicated in the hint, we define θ as the angle to the top of the screen* and ϕ as the angle to the bottom of the screen. Then $\tan(\theta) = \frac{8}{x}$, $\quad \tan(\phi) = \frac{2}{x}$. Maximizing the visual angle occupied by the screen is equivalent to maximizing the angle $\theta - \phi$. As θ and ϕ are both functions of x, we just need to calculate the derivative $\theta' - \phi'$ and find the critical x. First with implicit differentiation we calculate



$$
\begin{aligned}
(\tan(\theta))' &= (\frac{8}{x})' \\
\theta' \sec^2(\theta) &= -\frac{8}{x^2} \\
\theta' &= -\frac{8}{x^2}\cos^2(\theta).
\end{aligned}
$$

Note: It is a common mistake to interpret the angle ϕ as the angle that is below the angle θ and not the angle that is within the angle θ.

With the diagram on the right
we obtain $\cos(\theta) = \frac{x}{\sqrt{8^2+x^2}}$. So $\theta' = -\frac{8}{x^2}\frac{x^2}{8^2+x^2} = -\frac{8}{8^2+x^2}$. Similarly, we calculate $\phi' = -\frac{2}{2^2+x^2}$. Hence, $\theta' - \phi' = -\frac{8}{8^2+x^2} + \frac{2}{2^2+x^2} = \frac{-8(4+x^2)+2(64+x^2)}{(64+x^2)(4+x^2)} = \frac{-6x^2+96}{(64+x^2)(4+x^2)}$.
By setting $\theta' - \phi' = 0$ we solve $x^2 = 96/6 = 16$, $x = 4$ as x represents the distance and must be positive. To verify it is a maximum, we use the first derivative test

$$\theta' - \phi'\Big|_{x=3} = \frac{-6\cdot 9+96}{(64+3^2)(4+3^2)} = \frac{96-54}{(64+3^2)(4+3^2)} > 0,$$
$$\theta' - \phi'\Big|_{x=5} = \frac{-6\cdot 25+96}{(64+5^2)(4+5^2)} = \frac{96-150}{(64+5^2)(4+5^2)} < 0.$$

So the angle is maximized at $x = 4$.

Good Luck for your exams!