Full Solutions
MATH103 April 2014

April 22, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

SOLUTION 1. Let $F(x) = \int_0^x \sin t\, dt$ be an antiderivative of $\sin x$. Setting $y = x^2$ this means that $F(y) = F(x^2) = \int_0^{x^2} \sin t\, dt = \int_0^y \sin t\, dt$. By the fundamental theorem of calculus it holds that $F'(y) = \sin y$. Hence, applying the chain rule first and then the fundamental theorem of calculus we find that

$$\begin{aligned}
\frac{d}{dx}F(x^2) &= \frac{d}{dx}F(y) \\
&= \frac{dy}{dx} \cdot \frac{d}{dy}F(y) \\
&= 2x \sin y \\
&= 2x \sin(x^2)
\end{aligned}$$

The correct final answer is A.

SOLUTION 2. Alternatively, notice that

$$\begin{aligned}
\frac{d}{dx}\left[\int_0^{x^2} \sin t\, dt\right] &= \frac{d}{dx}\left[-\cos(t)|_0^{x^2}\right] \\
&= \frac{d}{dx}\left[(-\cos(x^2) + \cos(0))\right] \\
&= \sin(x^2)(2x) + 0 \\
&= 2x \sin(x^2)
\end{aligned}$$

Question 1 (b) ii

SOLUTION 1. The quantity $\int_0^{10} tp(t)\mathrm{d}t$ is the *mean* of a random quantity with probability density p. Since p is a density on the interval $[0, 10]$, the random quantity can take values in $[0, 10]$. Thus, the mean is also a value in $[0, 10]$.

SOLUTION 2. Multiplying the inequality $0 \le t \le 10$ by $p(t)$ yields $0 \le tp(t) \le 10p(t)$. Integrating this gives $0 \le \int_0^{10} tp(t)\mathrm{d}t \le 10 \int_0^{10} p(t)\mathrm{d}t = 10$.

Question 1 (b) i

SOLUTION. By the fundamental theorem of calculus, $f'(x) = \frac{d}{dx}\int_2^x p(t)\mathrm{d}t = p(x)$. Thus, $f'(5) = p(5)$. But p is a probability density function, so $p(x) \ge 0$ for all x. In particular, $f'(5) = p(5) \ge 0$.

Question 1 (c) iii

SOLUTION. As seen in the previous part, at time $t = 1$, the car is back at its initial position. During the interval $[1, 1.5]$, the velocity curve is above the t-axis, hence the velocity is positive. Thus, the car is moving away from its initial position during this entire time interval. Since the time $5/4 = 1.25$ occurs within this interval, the car is moving away from its initial position at that time.
Note: A common mistake that students tend to make in this type of question is interpreting the negative slope of the graph at (1.25,1.5) as the car returning to its original position. Keep in mind, the graph is illustrating the velocity of the car, not its displacement.

Question 1 (c) ii

SOLUTION 1. The car's distance from its starting point at time t is given by the total area between the graph and the t-axis accumulated by time t (where regions of the graph underneath the t-axis count towards negative area). We can see from the graph that the area accumulated over the time interval $[0, 0.5]$ is the same as that accumulated in the interval $[0.5, 1]$, but the latter is counted as the negative of the former. Thus, the total area accumulated over $[0, 1]$ is 0.

SOLUTION 2. The car's distance from its starting point at time t is given by the integral $\int_0^t v(s)\mathrm{d}s$. When $t = 1$, we get
$\int_0^1 v(s)\mathrm{d}s = \int_0^1 \sin(2\pi s)\mathrm{d}s = -\frac{1}{2\pi}\cos(2\pi s)\big|_0^1 = -\frac{1}{2\pi}(\cos(2\pi) - \cos(0)) = 0,$
since $\cos(0) = \cos(2\pi) = 1$

Question 1 (c) i

SOLUTION 1. When $t = 0$, we see from the graph that $v(t) = v(0) = 0$.

SOLUTION 2. $v(0) = \sin(2\pi \cdot 0) = \sin(0) = 0$.

Question 1 (d) iii

SOLUTION. If $x(0)$ is in $[0, 1]$, then in order for $x(t)$ to lie outside of $[0, 1]$, by continuity and the Intermediate Value Theorem, there must a time $s < t$ such that $x(s) = 0$ or $x(s) = 1$. But since 0 and 1 are steady states, if this occurs then $x(t)$ will be 0 (respectively, 1) for all $t \geq s$.

Question 1 (d) ii

SOLUTION. If the initial population is 0, then it will be 0 forever, as 0 is a steady state.

Question 1 (d) i

SOLUTION. Putting together the differential equation $\frac{dx}{dt} = x(1-x)(x-1/2)$ and the steady state equation $\frac{dx}{dt} = 0$, we get $x(x-1)(x-1/2) = 0$. For the left-hand side to be 0, one of its factors must be 0. Thus, $x = 0$, $x - 1 = 0$, or $x - 1/2 = 0$. The answer is all of D, E and F.

Question 1 (e) ii

SOLUTION. The series is
$1 + 0 + \frac{1}{2} + 0 + 0 + \frac{1}{3} + \cdots = 1 + \frac{1}{2} + \frac{1}{3} + \cdots,$
which is just the harmonic series. But the harmonic series diverges (say by using the p-series test if need be)! Note the difference between a sequence and a series.

Question 1 (e) i

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

If we expect that the sequence converges to zero then we can ignore the terms that already are zero. The remaining terms (let's call them b_n) are

$$b_n = \{1, 1/2, 1/3, 1/4, 1/5, \dots\} = \{1/n\}.$$

To determine if the sequence converges we compute

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

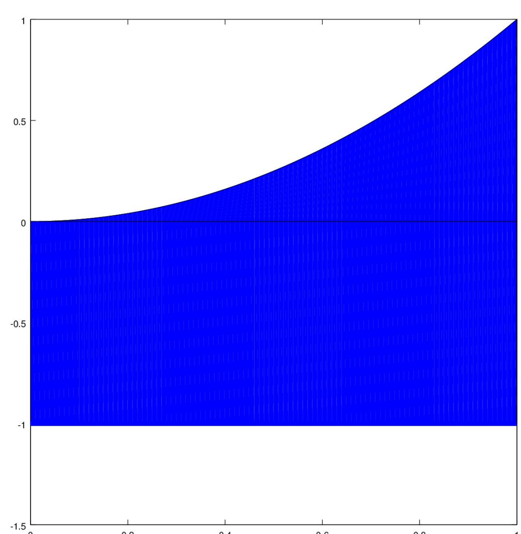
and indeed we see the sequence converges to the value zero. Since all the remaining terms in a_n are already exactly zero then a_n must converge to zero as well.

Question 2 (a)

SOLUTION. The arc length is given by
$\int_1^4 \sqrt{1 + (f'(x))^2}\mathrm{d}x$.
By the fundamental theorem of calculus, $f'(x) = \sqrt{x^3 - 1}$. Thus,
$\sqrt{1 + (f'(x))^2} = \sqrt{1 + (x^3 - 1)} = \sqrt{x^3} = x^{3/2}$.
It follows that the arc length is
$\int_1^4 x^{3/2}\mathrm{d}x = \frac{2}{5}x^{5/2}\big|_1^4 = \frac{2}{5}(4^{5/2} - 1^{5/2}) = \frac{2}{5}(32 - 1) = \frac{62}{5}$.

Question 2 (b)

SOLUTION. For each $x, 0 \le x \le 1$, the point $(x, f(x))$ is rotate to form a circle of radius
$r(x) = f(x) - (-1) = f(x) + 1 = x^2 + 1$.
The area of this circle is thus $\pi r(x)^2 = \pi(x^2 + 1)^2 = \pi(x^4 + 2x^2 + 1)$. The volume V of the surface formed by putting together all these circles is just the integral of their respective areas:
$V = \int_0^1 \pi(x^4 + 2x^2 + 1)\mathrm{d}x = \pi\left[\frac{1}{5}x^5 + \frac{2}{3}x^3 + x\right]_0^1 = \pi\left(\frac{1}{5} + \frac{2}{3} + 1\right) = \frac{28}{15}\pi$.



Question 3 (a)

SOLUTION. This probability is given by $\int_5^\infty e^{3-t}\mathrm{d}t = \lim_{b\to\infty}\int_5^b e^{3-t}\mathrm{d}t = -\lim_{b\to\infty} e^{3-t}\big|_5^b = -\lim_{b\to\infty}(e^{3-b}-e^{-2}) = e^{-2}$.

Question 3 (b)

SOLUTION. The mean is given by
$\int_3^\infty tp(t)\mathrm{d}t = \int_3^\infty te^{3-t}\mathrm{d}t = \lim_{b\to\infty}\int_3^b te^{3-t}\mathrm{d}t$.
Let $u = t$ and $\mathrm{d}v = e^{3-t}\mathrm{d}t$. Then $\mathrm{d}u = \mathrm{d}t$ and $v = \int e^{3-t}\mathrm{d}t = -e^{3-t}$. Thus, by integration by parts,
$\int te^{3-t}\mathrm{d}t = uv - \int v\mathrm{d}u = -te^{3-t} + \int e^{3-t}\mathrm{d}t = -te^{3-t} - e^{3-t} = -e^{3-t}(t+1)$.
It follows that the mean is
$-\lim_{b\to\infty} e^{3-t}(t+1)\big|_3^b = -\lim_{b\to\infty}(e^{3-b}(b+1) - e^{3-3}(3+1)) = -(0-4) = 4$.

Question 3 (c)

SOLUTION 1. Since $s = \ln(1+t)$, we have $\mathrm{d}s = \frac{1}{1+t}\mathrm{d}t$, and so

$$\begin{aligned} q(s)\mathrm{d}s = p(t)\mathrm{d}t &\Rightarrow \frac{q(s)}{1+t}\mathrm{d}t = e^{3-t}\mathrm{d}t \\ &\Rightarrow q(s) = (1+t)e^{3-t}. \end{aligned}$$

But $s = \ln(1+t)$ implies $t = e^s - 1$, so we get
$q(s) = (1+t)e^{3-t} = (1+e^s-1)e^{3-(e^s-1)} = e^s e^{4-e^s} = e^{s-e^s+4}$.

SOLUTION 2. For an alternative solution, let's use the cumulative density function

$$\begin{aligned} F(t) &= \int_3^t p(x)\,dx = \int_3^t e^{3-x}\,dx \\ &= -e^{3-x}\big|_3^t = 1 - e^{3-t} \end{aligned}$$

Connecting the cdf of the juvenile to the lifetime of the salmon, $s = \ln(t+1)$, that is, $t = e^s - 1$ we find the cdf of the lifetime of a salmon as

$$\begin{aligned} G(s) &= F(e^s - 1) \\ &= 1 - e^{3-e^s+1} = 1 - e^{4-e^s} \end{aligned}$$

Finally, to find the pdf we take the derivative

$$q(s) = G'(s) = -e^{4-e^s}(-e^s) = e^{4+s-e^s}$$

Question 4 (a)

SOLUTION 1. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

By definition of the integral (see Hint 3),

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x)\mathrm{d}x.$$

Comparing to the original sum $\sum_{k=1}^{n}\left(\frac{k}{n}\right)^2 \frac{1}{n}$ we have that $f(x) = x^2$. Therefore,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 = \int_0^1 x^2\mathrm{d}x = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}.$$

SOLUTION 2. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

There is an alternative solution if you remember the sum of squares:
$\sum_{k=1}^{n} k^2 = \frac{2n^3+3n^2+n}{6}$
With this we quickly find

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^2\frac{1}{n} &= \lim_{n\to\infty}\frac{1}{n^3}\sum_{k=1}^{n}k^2\\ &= \lim_{n\to\infty}\frac{1}{n^3}\frac{2n^3+3n^2+n}{6}\\ &= \lim_{n\to\infty}\frac{2+3\frac{1}{n}+\frac{1}{n^2}}{6}\\ &= \frac{1}{3}.\end{aligned}$$

Question 4 (b)

SOLUTION. We have
$\lim_{n\to\infty} a_n = 1 + \lim_{n\to\infty}\frac{n}{(n+1)n(n-1)\ldots2\cdot1}$.
Cancelling the common factor n in the numerator and denominator, we get
$\frac{n}{(n+1)n(n-1)\ldots2\cdot1} = \frac{1}{(n+1)(n-1)\ldots2\cdot1}$.
As $n\to\infty$, the numerator (1) of the last expression is constant, while the denominator $((n+1)(n-1)\ldots2\cdot1)$ tends to ∞. Thus,
$\lim_{n\to\infty}\frac{n}{(n+1)n(n-1)\ldots2\cdot1} = \lim_{n\to\infty}\frac{1}{(n+1)(n-1)\ldots2\cdot1} = 0$.
We conclude that $\lim_{n\to\infty} a_n = 1+0 = 1$.

Question 4 (c)

SOLUTION. First, write the product as a ratio:
$n\ln\left(1+\frac{2}{n}\right) = \frac{\ln\left(1+\frac{2}{n}\right)}{\frac{1}{n}}$.
Note that both the numerator and denominator in this ratio tend to 0 as $n\to\infty$ (since $\ln(1+2/n)\to\ln(1) = 0$). That is,
$\frac{\lim_{n\to\infty}\ln\left(1+\frac{2}{n}\right)}{\lim_{n\to\infty}\frac{1}{n}}$
is an indeterminate form. Thus, we can apply l'Hôpital's rule to get
$\lim_{n\to\infty} n\ln\left(1+\frac{2}{n}\right) = \lim_{n\to\infty}\frac{\ln\left(1+\frac{2}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty}\frac{\frac{d}{dn}\ln\left(1+\frac{2}{n}\right)}{\frac{d}{dn}\frac{1}{n}}$.

Now
$\frac{d}{dn}\ln\left(1+\frac{2}{n}\right) = \frac{-2n^{-2}}{1+\frac{2}{n}}$,
while
$\frac{d}{dn}\frac{1}{n} = -n^{-2}$.
Thus,
$\frac{\frac{d}{dn}\ln\left(1+\frac{2}{n}\right)}{\frac{d}{dn}\frac{1}{n}} = \frac{\left(\frac{-2n^{-2}}{1+\frac{2}{n}}\right)}{-n^{-2}} = \frac{2}{1+\frac{2}{n}}$.
Therefore,
$\lim_{n\to\infty}\frac{\frac{d}{dn}\ln\left(1+\frac{2}{n}\right)}{\frac{d}{dn}\frac{1}{n}} = \lim_{n\to\infty}\frac{2}{1+\frac{2}{n}} = \frac{2}{1+\lim_{n\to\infty}\frac{2}{n}} = \frac{2}{1} = 2$.
Note: l′Hôpital′s rule is only applicable when its conditions are met by the numerator, the denominator, and even the ratio itself.

Question 5 (a)

SOLUTION. A steady state solution is a solution y of the differential equation, in this case $\frac{dy}{dt} = (1-y)^3$, such that $\frac{dy}{dt} = 0$. Putting together the right-hand sides of these two equations yields $(1-y)^3 = 0$. Thus, $y = 1$.

Question 5 (b)

SOLUTION. Use the method of separation of variables. First, re-arrange the differential equation:
$\frac{dy}{dt} = (1-y)^3 \Rightarrow \frac{\mathrm{d}y}{(1-y)^3} = \mathrm{d}t$.
Next, integrate both sides. On the right-hand side, we have
$\int \mathrm{d}t = t + C$,
where C is a constant. On the left-hand side, we have
$\int \frac{\mathrm{d}y}{(1-y)^3}$.
We can perform the integral, for instance, by substitution. Letting $u = 1-y$, so that $\mathrm{d}u = -\mathrm{d}y$, or equivalently $\mathrm{d}y = -\mathrm{d}u$, we get
$\int \frac{\mathrm{d}y}{(1-y)^3} = \int \frac{-\mathrm{d}u}{u^3} = \frac{1}{2u^2} = \frac{1}{2(1-y)^2}$.
We have neglected to include the constant because it has already been taken into account above. Thus, we have found that
$\frac{1}{2(1-y)^2} = t + C$.
To determine the constant, use the fact that $y(0) = \frac{1}{2}$. Plugging this into the equation above yields
$\frac{1}{2\left(1-\frac{1}{2}\right)^2} = 0 + C$,
i. e. $2 = C$. Thus,
$\frac{1}{2(1-y)^2} = t + 2$.
Solving for y gives
$y = 1 - \frac{1}{\sqrt{2t+4}}$.
Note that we have taken the positive square root as we require the solution y to lie in the interval $[0, 1]$.

Question 5 (c)

SOLUTION. The problem amounts to solving the equation $y(t) = 0.8$ for t. Since we found in (b) that $y(t) = 1 - \frac{1}{\sqrt{2t+4}}$,

$$
\begin{aligned}
y(t)=0.8 \Rightarrow 1-\frac{1}{\sqrt{2t+4}} &= 0.8\\
\Rightarrow \frac{1}{\sqrt{2t+4}} &= 0.2\\
\Rightarrow \sqrt{2t+4} &= 5\\
\Rightarrow 2t+4 &= 25\\
\Rightarrow t &= \frac{21}{2}.
\end{aligned}
$$

Question 6 (a)

Solution 1. Since $-1 \le \cos(e^n) \le 1$ for any n,
$0 \le \frac{9}{n^2} \le \frac{\cos(e^n)+10}{n^2} \le \frac{11}{n^2}$.
Thus, by the comparison test, $\sum_{n=1}^{\infty}\left(\frac{\cos(e^n)+10}{n^2}\right)$ converges if $\sum_{n=1}^{\infty}\frac{11}{n^2}$. But
$\sum_{n=1}^{\infty}\frac{11}{n^2} = 11\sum_{n=1}^{\infty}\frac{1}{n^2}$
converges by the p-test with $p=2$.

Solution 2. Carry out the same steps as in the previous solution, but use the integral test to confirm convergence of $\sum_{n=1}^{\infty}\frac{11}{n^2}$. That is, since
$\int_1^{\infty}\frac{11}{x^2}\mathrm{d}x = -11x^{-1}|_1^{\infty} = 11 < \infty$,
$\sum_{n=1}^{\infty}\frac{11}{n^2}$ converges.

Question 6 (c)

Solution. Use the integral test. That is, we must integrate the function $f(x)=\frac{1}{x(\ln x)^{103}}$ over an interval $[a,\infty)$ where it is well-defined and monotone decreasing. Since $f(x)$ is undefined at $x=1$, we must choose $a>1$. For any such choice of a, $f(x)$ will be well-defined and decreasing on the resulting interval $[a,\infty)$. For instance, choose $a=2$ and consider the integral
$\int_2^{\infty}\frac{1}{x(\ln x)^{103}}\mathrm{d}x$.
To evaluate this integral, use substitution. Let $u=\ln x$, so $\mathrm{d}u=\frac{1}{x}\mathrm{d}x$. Note that since x ranges from 1 to ∞, the variable $u=\ln x$ ranges from $\ln 2$ to ∞. Thus,
$\int_2^{\infty}\frac{1}{x(\ln x)^{103}}\mathrm{d}x = \int_{\ln 2}^{\infty}\frac{1}{u^{-103}}\mathrm{d}u = \frac{1}{-102}u^{-102}\Big|_{\ln 2}^{\infty} = \frac{1}{102}(\ln 2)^{-102} < \infty$
and it follows that the corresponding sum
$\sum_{k=1}^{\infty}\frac{1}{k(\ln k)^{103}}$
converges.

Question 7 (a)

Solution. Recall that
$e^x = \sum_{n=0}^{\infty}\frac{1}{n!}x^n$.
Note: If you have forgotten the series expansion of e^x (at 0), recall that, in general, such an expansion has the form
$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$.
For $f(x)=e^x$, we have $f^{(1)}(x)=f'(x)=e^x$ and it follows that $f^{(n)}(x)=e^x$ for all n. Thus, $f^{(n)}(0)=1$ and we get
$e^x=\sum_{n=0}^{\infty}\frac{1}{n!}x^n$

as above.
Now replacing x by $-x^2$ in the above expansion yields
$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$.
Since only even powers of x appear in this series, $a_n = 0$ whenever n is odd. In particular, $a_{11} = 0$.
On the other hand, $a_{10}x^{10} = a_{10}x^{2n}$ for $n = 5$. But the coefficient of x^{2n} is $\frac{(-1)^n}{n!}$. For $n = 5$, this yields $a_{10} = \frac{(-1)^5}{5!} = -\frac{1}{120}$.

Question 7 (b)

Solution. By part (a), (with t replacing x)
$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{2n}$.
Integrating term-by-term yields
$\int_0^x e^{-t^2} \mathrm{d}t = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^x t^{2n} \mathrm{d}t = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{1}{2n+1} x^{2n+1}$.
Since only odd powers appear in this series, $b_{12} = 0$.
On the other hand, $b_{11}x^{11} = b_{11}x^{2n+1}$ for $n = 5$. But the coefficient of x^{2n+1} is $\frac{(-1)^n}{n!} \cdot \frac{1}{2n+1}$. For $n = 5$, this yields $b_{11} = \frac{(-1)^5}{5!} \cdot \frac{1}{2(5)+1} = -\frac{1}{11 \cdot 5!}$.

Question 7 (c)

Solution. Substituting $y(x) = \sum_{n=0}^{\infty} c_n x^n$ into $\frac{dy}{dx} = 2 + x^2 y$ yields
$\sum_{n=1}^{\infty} n c_n x^{n-1} = 2 + \sum_{n=0}^{\infty} c_n x^{n+2}$.
Writing out the first few terms explicitly on both sides, we have
$c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \cdots = 2 + c_0x^2 + c_1x^3 + \cdots$.
Comparing the coefficients of x^3 on both sides, we see that $4c_4 = c_1$. But comparing the coefficients of $x^0 (= 1)$ on both sides, we see that $c_1 = 2$. Thus, $4c_4 = 2$, i. e. $c_4 = \frac{1}{2}$.
Now comparing the coefficients of x^2 on both sides, we get $3c_3 = c_0$. Since c_0 does not appear on the left-hand side, we determine it instead by using the given initial condition $y(0) = 1$. That is, setting $x = 0$ in $y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + \cdots$, we get $y(0) = c_0$. Thus, $y(0) = 1$ implies $c_0 = 1$. Putting this all together, we get $3c_3 = c_0 = 1$, i. e. $c_3 = \frac{1}{3}$.

Question 8 (a)

Solution. Integrating term-by-term, we have

$$\begin{aligned}
\int_0^{1/2} f(x) \mathrm{d}x &= \sum_{n=0}^{\infty} (3n+2) \int_0^{1/2} x^{3n+1} \mathrm{d}x \\
&= \sum_{n=0}^{\infty} (3n+2) \left. \frac{1}{3n+2} x^{3n+2} \right|_0^{1/2} \\
&= \sum_{n=0}^{\infty} 2^{-(3n+2)}.
\end{aligned}$$

We want to use the fact that
$a \sum_{n=0}^{\infty} r^n = a \frac{1}{1-r}$
for $|r| < 1$. To do so, write
$\sum_{n=0}^{\infty} 2^{-(3n+2)} = 2^{-2} \sum_{n=0}^{\infty} (2^{-3})^n$.
We are thus in the situation above with $a = 2^{-2}$ and $r = 2^{-3}$. Hence,
$2^{-2} \sum_{n=0}^{\infty} (2^{-3})^n = 2^{-2} \frac{1}{1-2^{-3}} = \frac{1}{4} \cdot \frac{8}{7} = \frac{2}{7}$.

Question 8 (b)

SOLUTION. Let $a_n = \frac{n+1}{5^n(n+2)}$ be the coefficient of $(x+1)^n$ in the series. By the ratio test, the series converges for all x such that $\lim_{n\to\infty}\left|\frac{a_{n+1}(x+1)^{n+1}}{a_n(x+1)^n}\right| < 1$. Now

$$\begin{aligned}\frac{a_{n+1}(x+1)^{n+1}}{a_n(x+1)^n} &= \frac{n+2}{5^{n+1}(n+3)}\cdot\frac{5^n(n+2)}{(n+1)}(x+1)\\ &= \frac{(n+2)^2}{5(n+3)(n+1)}(x+1)\\ &= \frac{n^2+4n+4}{5n^2+20n+15}(x+1).\\ &= \frac{(1/n^2)(1+4/n+4/n^2)}{(1/n^2)(5+20/n+15/n^2)}(x+1)\end{aligned}$$

Thus, as n approaches infinity
$\lim_{n\to\infty}\left|\frac{a_{n+1}(x+1)^{n+1}}{a_n(x+1)^n}\right| = \frac{1}{5}|x+1|$
and so the series converges whenever $\frac{1}{5}|x+1| < 1$. But

$$\begin{aligned}\frac{1}{5}|x+1| < 1 &\Rightarrow -1 < \frac{1}{5}(x+1) < 1\\ &\Rightarrow -5 < x+1 < 5\\ &\Rightarrow -6 < x < 4.\end{aligned}$$

Lastly, we check the boundary cases where $x=-6$ or $x=4$ (the ratio test is indeterminate for these cases).
When $x=-6$,
$a_n = \frac{(n+1)(-5)^n}{5^n(n+2)} = \frac{n+1}{n+2}(-1)^n$.
Thus, $\lim_{n\to\infty} a_n \neq 0$ (in fact, a_n oscillates and does not converge), so the series diverges by the divergence test.
When $x=4$,
$a_n = \frac{(n+1)5^n}{5^n(n+2)} = \frac{n+1}{n+2}$,
so $\lim_{n\to\infty} a_n = 1 \neq 0$ and the series diverges once again. We conclude that the only values for which the series converges are $-6 < x < 4$.

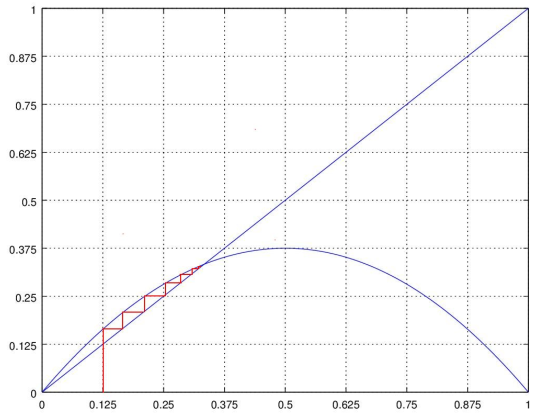
Question 9 (a)

SOLUTION. The fixed points are solutions to $F(x) = x$, i. e. $\frac{3}{2}x(1-x) = x$. Thus, either $x=0$ or $\frac{3}{2}(1-x) = 1$, i. e. $x = \frac{1}{3}$.
Now a fixed point x is stable if $|F'(x)| < 1$ and unstable if $|F'(x)| > 1$. Since $F'(x) = \frac{3}{2}(1-2x)$, we have

$$\begin{aligned}|F'(0)| &= \frac{3}{2} > 1\\ \left|F'\left(\frac{1}{3}\right)\right| &= \frac{1}{2} < 1.\end{aligned}$$

Thus 0 is an unstable fixed point and 1/3 is a stable fixed point.

Question 9 (b)



Solution.
The purpose of a cobweb plot is to determine graphically the sequence of points $x_0, F(x_0), F(F(x_0)), \ldots$ determined by iterating F (for some initial condition x_0). Starting at x_0, our first goal is to determine $F(x_0)$. To do so, begin by tracing a vertical line from $(x_0, 0)$ to $(x_0, F(x_0))$.
Once we have found $(x_0, F(x_0))$ (see the first hint), we want to determine where $F(x_0)$ lies on the x-axis (so that we can repeat this procedure). To do so, we begin by finding $(F(x_0), F(x_0))$. Since the x- and y-coordinates of this point agree with each other, it lies on the line $y = x$. Thus, we sketch the line $y = x$ and determine the point on this line with height $F(x_0)$.
To find the point on the line $y = F(x)$ of height $F(x_0)$, take the point $(x_0, F(x_0))$ found in Hint 1 and draw a horizontal line towards the line $y = x$. The intersection of these two lines is $(F(x_0), F(x_0))$.
Once we have found $(F(x_0), F(x_0))$ (see Hint 3), we can get determine the point $F(x_0)$ on the x-axis simply by drawing a vertical line from this point to the x-axis. We can now repeat this procedure (replacing x_0 by $F(x_0)$, $F(x_0)$ by $F(F(x_0))$, etc.).

Question 9 (c)

Solution. From the cobweb plot in the previous part, we see that $\lim_{n\to\infty} a_n \approx \frac{1}{3}$ when $a_0 = \frac{1}{8}$. At the very least, we see that $\lim_{n\to\infty} a_n \neq 0$. Thus, the series $\sum_{n=0}^{\infty} a_n$ diverges by the divergence test.

Question 10 (a)

Solution. Let $y = \sqrt{x}$ so that $\mathrm{d}y = \frac{1}{2\sqrt{x}}\mathrm{d}x$. Since x ranges from 1 to e^2, y ranges from $\sqrt{1} = 1$ to $\sqrt{e^2} = e$. Thus,
$\int_1^{e^2} \frac{\ln\sqrt{x}}{\sqrt{x}}\mathrm{d}x = \int_1^e 2\ln y\mathrm{d}y$.
Next use integration by parts. Let $u = 2\ln y$ and $\mathrm{d}v = \mathrm{d}y$ so that $\mathrm{d}u = \frac{2}{y}\mathrm{d}y$ and $v = y$. Then
$\int_1^e \ln y\mathrm{d}y = 2y\ln y|_1^e - \int_1^e 2\mathrm{d}y = 2e - 2y|_1^e = 2e - (2e - 2) = 2$.

Question 10 (b)

Solution. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

Following the suggestion in the question, we will integrate by parts. It is much easier to differentiate $f(x)$ than integrate it (we'd have to integrate the integral) and so we let

$$u = f(x) = \int_1^x \sin(t^2)\mathrm{d}t.$$

The remaining part must be $\mathrm{d}v$ and so therefore $\mathrm{d}v = 1\mathrm{d}x$. This is fairly straightforward to integrate and so we get

$$v = x.$$

To differentiate u we will use the fundamental theorem of calculus which states that

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(y)\mathrm{d}y = f(x).$$

Applying this to our function yields

$$\mathrm{d}u = \sin(x^2)\mathrm{d}x.$$

Using the information we have obtained with integration by parts formula,

$$\int_a^b u\mathrm{d}v = uv|_a^b - \int_a^b v\mathrm{d}u$$

we get

$$\begin{aligned}\int_0^1 f(x)\mathrm{d}x &= \left[x\int_1^x \sin(t^2)\mathrm{d}t\right]_0^1 - \int_0^1 x\sin(x^2)\mathrm{d}x \\ &= (1)\int_1^1 \sin(t^2)\mathrm{d}t - (0)\int_1^0 \sin(t^2)\mathrm{d}t - \int_0^1 x\sin(x^2)\mathrm{d}x \\ &= -\int_0^1 x\sin(x^2)\mathrm{d}x,.\end{aligned}$$

The simplification comes from the fact that

$$\int_a^a g(t)\mathrm{d}t = 0$$

for any function $g(t)$. In our particular case, $g(t) = \sin(t^2)$ and $a = 1$. The last integral can be evaluated by substitution. Let $y = x^2$ so that $\mathrm{d}y = 2x\mathrm{d}x$. If we change variables, we have to be careful with the bounds. When $x = 0$ then $y = 0^2 = 0$ and when $x = 1$ then $y = 1^2 = 1$. Therefore the bounds don't change and the integral becomes

$$-\int_0^1 x\sin(x^2)\mathrm{d}x = -\frac{1}{2}\int_0^1 \sin y\mathrm{d}y = \frac{1}{2}\cos y\Big|_0^1 = \frac{1}{2}(\cos 1 - 1).$$

Question 10 (c)

SOLUTION. Let $x = \tan t$, so $\mathrm{d}x = \sec^2 t\mathrm{d}t$. Since x ranges from 0 to ∞, t ranges from 0 to $\pi/2$. Thus,

$$\begin{aligned}
\int_0^\infty \left(\frac{1}{\sqrt{x^2+1}}\right)^3 \mathrm{d}x &= \int_0^{\pi/2} \left(\frac{1}{\sqrt{\tan^2 t+1}}\right)^3 \sec^2 t\mathrm{d}t \\
&= \int_0^{\pi/2} \left(\frac{1}{\sqrt{\sec^2 t}}\right)^3 \sec^2 t\mathrm{d}t \\
&= \int_0^{\pi/2} \frac{1}{\sec t}\mathrm{d}t \\
&= \int_0^{\pi/2} \cos t\mathrm{d}t \\
&= \sin t|_0^{\pi/2} \\
&= 1.
\end{aligned}$$

Question 11

SOLUTION. By the divergence test, it is necessary (though perhaps not sufficient) that $b_n \to 0$. Thus, the map $F(x) = \beta\frac{x}{1+x}$ must at least have a stable fixed point at 0. We thus begin by find the fixed points of F. Solving $F(x) = x$ yields $x = 0$ or $x = \beta - 1$. To check stability, note that $F'(x) = \frac{\beta}{(1+x)^2}$. Since $F'(0) = \beta > 0$, we require $\beta \le 1$ for 0 to be stable; we will deal with the indeterminate case $\beta = 1$ later.
If $\beta < 1$, we use the ratio test to see that
$\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\left|\frac{\beta}{1+b_n}\right| = \beta$.
We conclude by the ratio test that the series converges for $\beta < 1$ and diverges for $\beta > 1$.
In the case $\beta = 1$, we have $b_{n+1} = \frac{b_n}{1+b_n}$. With $b_0 = 1$, looking at the first few terms of the resulting sequence, we see that $b_1 = 1/2, b_2 = 1/3, b_3 = 1/4$ and so on. We thus see that $b_n = 1/(n+1)$. But then the series is the harmonic series, which diverges.
Thus, the series converges for $0 < \beta < 1$.

Good Luck for your exams!