Full Solutions
MATH103 April 2013

April 4, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

SOLUTION 1. Since the numerator grows faster than the denominator (as $k^k \gg k!$), we get that this sequence diverges.

SOLUTION 2. Since
$\frac{k^k}{k!} = \frac{k \cdot k \cdot k \cdot \ldots \cdot k}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot k} \geq k$
and the sequence defined by $(k)_{k \geq 1}$ diverges, the sequence $\left(\frac{k^k}{k!}\right)_{k \geq 1}$ also diverges.

Question 1 (b)

SOLUTION. Proceeding as in the hint, we have

$$\begin{aligned}
\frac{n^3+2}{\sqrt{n^6+3}+\sqrt[3]{27n^9+1}} &= \frac{n^3(1+\frac{2}{n^3})}{\sqrt{n^6(1+\frac{3}{n^6})}+\sqrt[3]{27n^9(1+\frac{1}{27n^9})}} \\
&= \frac{n^3(1+\frac{2}{n^3})}{n^3\sqrt{(1+\frac{3}{n^6})}+3n^3\sqrt[3]{(1+\frac{1}{27n^9})}} \\
&= \frac{1+\frac{2}{n^3}}{\sqrt{(1+\frac{3}{n^6})}+3\sqrt[3]{(1+\frac{1}{27n^9})}}
\end{aligned}$$

Taking the limit as n tends to infinity in the last equality above yields
$\lim_{n\to\infty} \frac{n^3+2}{\sqrt{n^6+3}+\sqrt[3]{27n^9+1}} = \lim_{n\to\infty} \frac{1+\frac{2}{n^3}}{\sqrt{(1+\frac{3}{n^6})}+3\sqrt[3]{(1+\frac{1}{27n^9})}} = \frac{1}{1+3} = \frac{1}{4}$
and so the sequence converges to one fourth.

Question 1 (c)

SOLUTION. Our problem translates to computing
$\lim\limits_{k\to\infty} a_k = \lim\limits_{k\to\infty} \sum\limits_{n=1}^{k} \frac{\pi}{k} \sin\left(\frac{\pi n}{k}\right)$
To do this, we proceed as in the last hint above and try to compute a $f(x), x_k, \Delta x, a, b$. The leading term suggests we should choose $\Delta x = \frac{\pi}{k} = \frac{b-a}{k}$. Using this, our $x_k = \frac{\pi n}{k} = n\Delta x$ and so a good choice for a and b is $a = 0$ and $b = \pi$
Therefore,

$$\begin{aligned}
\lim_{k\to\infty} \sum_{n=1}^{k} \frac{\pi}{k} \sin\left(\frac{\pi n}{k}\right) &= \int_0^{\pi} \sin(x)\, dx \\
&= -\cos(x)\Big|_0^{\pi} \\
&= -\cos(\pi) + \cos(0) \\
&= -(-1) + 1 = 2
\end{aligned}$$

and so the sum converges to 2.

Question 1 (d)

SOLUTION. This limit is 0 since as n tends to infinity, the denominator tends to infinity.

Question 1 (e)

SOLUTION 1. Let $f(x) = \frac{1}{x \ln x}$. Since this series is continuous, positive and decreasing on $[2, \infty)$, we may apply the integral test to see that the series converges or diverges based on if the following integral converges or diverges:

$\int_2^\infty \frac{1}{x \ln x}$

To compute this, we have that $\int_2^\infty \frac{1}{x \ln x} = \lim_{b\to\infty} \int_2^b \frac{1}{x \ln x}$

Let $u = \ln x$ so $du = \frac{dx}{x}$ and $u(2) = \ln(2)$ and $u(b) = \ln(b)$. This gives

$$\begin{aligned}
\int_2^\infty \frac{1}{x \ln x} &= \lim_{b\to\infty} \int_2^b \frac{1}{x \ln x} \\
&= \lim_{b\to\infty} \int_{\ln(2)}^{\ln(b)} \frac{du}{u} \\
&= \lim_{b\to\infty} \ln|u| \,|_{\ln(2)}^{\ln(b)} \\
&= \lim_{b\to\infty} \ln|\ln(b)| - \ln|\ln(2)|
\end{aligned}$$

As this last limit diverges since $\ln|\ln(b)|$ tends to infinity as b tends to infinity, we have that the series diverges.

NOTE: Compare this to the previous question. Even though the sequence of terms converges to zero, the integral here still diverges!

SOLUTION 2. No content found

Question 1 (f)

SOLUTION 1. Using the ratio test, we have that

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|a_{n+1}a_n^{-1}\right| \\
&= \lim_{n\to\infty} \left|\frac{(n+1)(x+2)^{n+1}}{3^{n+1}((n+1)+5)} \cdot \frac{3^n(n+5)}{n(x+2)^n}\right| \\
&= \lim_{n\to\infty} \left|\frac{(n+1)(x+2)}{3(n+6)} \cdot \frac{(n+5)}{n}\right| \\
&= \frac{|x+2|}{3} \lim_{n\to\infty} \frac{(n+1)(n+5)}{n(n+6)} \\
&= \frac{|x+2|}{3} \lim_{n\to\infty} \frac{n^2(1+\frac{1}{n})(1+\frac{5}{n})}{n^2(1+\frac{6}{n})} \\
&= \frac{|x+2|}{3} \lim_{n\to\infty} \frac{(1+\frac{1}{n})(1+\frac{5}{n})}{1+\frac{6}{n}} \\
&= \frac{|x+2|}{3} \cdot (1) \\
&= \frac{|x+2|}{3}
\end{aligned}$$

Now the ratio test tells us that our original series converges when $|x+2|/3 < 1$.

This occurs when $|x+2| < 3$ so whenever the distance from 2 is bounded above by 3.
This occurs when $-5 < x < 1$. Another way to see this is to note that $|x+2| < 3$ is equivalent to $-3 < x+2 < 3$ and subtracting two from each
side gives $-5 < x < 1$.
The ratio test also tells us that the series diverges when $|x+2| > 3$ and this occurs when $x+2 > 3$ or $-(x+2) < 3$.
Isolating for x yields $x < -5$ or $x > 1$.
This leaves only the cases when $|x+2| = 3$, that is, when $x = -5, 1$. To check these, we plug them into the original series. For $x = -5$ we have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^n(n+5)} &= \sum_{n=0}^{\infty} \frac{n(-5+2)^n}{3^n(n+5)} \\ &= \sum_{n=0}^{\infty} \frac{n(-3)^n}{3^n(n+5)} \\ &= \sum_{n=0}^{\infty} \frac{n(-1)^n 3^n}{3^n(n+5)} \\ &= \sum_{n=0}^{\infty} \frac{n(-1)^n}{n+5}\end{aligned}$$

Let's look at the limit of the terms. Since as n tends to infinity, we have that the sequence defined by $a_n = \dfrac{n(-1)^n}{n+5}$ has two subsequences that tend to different values (the even indexed terms limit to 1 since the numerator is positive and the odd terms limit to -1 since the numerator is negative). Therefore, the divergence test tells us that the series diverges when $x = -5$
When $x = 1$, we have that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^n(n+5)} &= \sum_{n=0}^{\infty} \frac{n(1+2)^n}{3^n(n+5)} \\ &= \sum_{n=0}^{\infty} \frac{n3^n}{3^n(n+5)} \\ &= \sum_{n=0}^{\infty} \frac{n}{n+5}\end{aligned}$$

Let's look at the limit of the terms. Since as n tends to infinity, we have that the sequence defined by $a_n = \dfrac{n}{n+5} = \dfrac{1}{1+\frac{5}{n}}$ converges to 1 as n tends to infinity. Therefore, the divergence test tells us that the series diverges when $x = 1$.
Thus the set of x values where this series converges is defined by $-5 < x < 1$.

Solution 2. This is a second solution for checking the endpoints portion of the question. After finding the radius of convergence as in Solution 1, this leaves only the cases when $|x+2| = 3$. To check these (simultaneously), we check this value in the original series with absolute values. We have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{n|x+2|^n}{3^n(n+5)} &= \sum_{n=0}^{\infty} \frac{n3^n}{3^n(n+5)} \\ &= \sum_{n=0}^{\infty} \frac{n}{n+5}\end{aligned}$$

Let's look at the limit of the terms. Since as n tends to infinity, we have that the sequence defined by $a_n = \dfrac{n}{n+5} = \dfrac{1}{1+\frac{5}{n}}$ converges to 1 as n tends to infinity. This means the summands $\dfrac{n(x+2)^n}{3^n(n+5)}$ does not converge to 0 any time $|x+2| = 3$. Therefore, the divergence test tells us that the series diverges at the endpoints.
Thus the set of x values where this series converges is defined by $-5 < x < 1$.

Question 2 (a)

Solution 1. Let $u = \ln x$ so that $du = \dfrac{dx}{x}$, $u(e^2) = \ln(e^2) = 2$ and $u(1) = \ln(1) = 0$. Then, we have
$\displaystyle\int_1^{e^2} \frac{\ln x}{x}\,dx = \int_0^2 u\,du = \frac{u^2}{2}\bigg|_0^2 = \frac{2^2}{2} - \frac{0^2}{2} = 2$
completing the problem.

Solution 2. Proceed by integration by parts. Let $u = \ln x$ and $dv = \dfrac{dx}{x}$. Then $du = \dfrac{dx}{x}$ and $v = \ln x$ so applying integration by parts gives
$\displaystyle I_a = \int_1^{e^2} \frac{\ln x}{x}\,dx = (\ln x)^2\bigg|_1^{e^2} - \int_1^{e^2} \frac{\ln x}{x}\,dx = \ln(e^2)^2 - \ln(1)^2 - I_a = 4 - I_a$
As I_a now appears on both sides, we bring the value over to get $2I_a = 4$ or that $I_a = 2$ completing the problem.

Question 2 (b)

Solution. Proceeding as in the hint, let $u = \arctan\left(\frac{1}{x}\right)$ and $dv = dx$. Then via the chain rule, we have $du = \dfrac{\frac{-1}{x^2}}{\frac{1}{x^2}+1} = \dfrac{-1}{x^2+1}$ and $v = x$ so integrating by parts gives us

$$\begin{aligned} I_b &= \int \arctan\left(\tfrac{1}{x}\right) dx \\ &= x\arctan\left(\tfrac{1}{x}\right) - \int \frac{-x}{x^2+1}\,dx \end{aligned}$$

Now let $w = x^2+1$ so that $dw = 2x\ dx$ and thus the above becomes after substitution

$$\begin{aligned} I_b &= x\arctan\left(\tfrac{1}{x}\right) + \int \frac{x}{x^2+1}\,dx \\ &= x\arctan\left(\tfrac{1}{x}\right) + \int \frac{dw}{2w} \\ &= x\arctan\left(\tfrac{1}{x}\right) + \frac{1}{2}\cdot\ln|w| + C \\ &= x\arctan\left(\tfrac{1}{x}\right) + \frac{1}{2}\cdot\ln|x^2+1| + C \end{aligned}$$

Question 2 (c)

Solution 1. First, note that the integral above is improper with a singularity at $x = 1$. Splitting up the integral there, we see that

$$\int_0^2 \frac{1}{x^2-1}\,dx = \int_0^1 \frac{1}{x^2-1}\,dx + \int_1^2 \frac{1}{x^2-1}\,dx$$
$$= \lim_{b\to 1^-}\int_0^b \frac{1}{x^2-1}\,dx + \lim_{a\to 1^+}\int_a^2 \frac{1}{x^2-1}\,dx$$

Now, let $\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$ by the partial fraction decomposition. This gives via cross multiplication $1 = A(x+1) + B(x-1)$

METHOD 1: Plugging in $x = -1$ shows that $B = -\frac{1}{2}$ and plugging in $x = 1$ shows that $A = \frac{1}{2}$.

METHOD 2: Expanding the right yields $1 = A - B + (A+B)x$

Since the coefficients of the constant term are equal and the coefficients of the x term must be equal, we have

$A - B = 1$

and

$A + B = 0$

Adding the two equations gives $2A = 1$ so $A = \frac{1}{2}$ as above and subtracting the two equations gives $-2B = 1$ and so

$B = \frac{-1}{2}$.

Now that we have the coefficients, we note that

$\frac{1}{x^2-1} = \frac{\frac{1}{2}}{x-1} + \frac{\frac{-1}{2}}{x+1}$

and go back to the integral to see that

$$\begin{aligned}
\int_0^2 \frac{1}{x^2-1}\,dx &= \int_0^1 \frac{1}{x^2-1}\,dx + \int_1^2 \frac{1}{x^2-1}\,dx \\
&= \lim_{b\to 1^-}\int_0^b \frac{1}{x^2-1}\,dx + \lim_{a\to 1^+}\int_a^2 \frac{1}{x^2-1}\,dx \\
&= \lim_{b\to 1^-}\int_0^b \left(\frac{\frac{1}{2}}{x-1} + \frac{\frac{-1}{2}}{x+1}\right)\,dx + \lim_{a\to 1^+}\int_a^2 \left(\frac{\frac{1}{2}}{x-1} + \frac{\frac{-1}{2}}{x+1}\right)\,dx \\
&= \lim_{b\to 1^-} \tfrac{1}{2}\ln|x-1| - \tfrac{1}{2}\ln|x+1|\Big|_0^b + \lim_{a\to 1^+} \tfrac{1}{2}\ln|x-1| - \tfrac{1}{2}\ln|x+1|\Big|_a^2 \\
&= \lim_{b\to 1^-} \tfrac{1}{2}\ln|b-1| - \tfrac{1}{2}\ln|b+1| - (\tfrac{1}{2}\ln|1| - \tfrac{1}{2}\ln|1|) \\
&\quad + \lim_{a\to 1^+} \tfrac{1}{2}\ln|1| - \tfrac{1}{2}\ln|3| - (\tfrac{1}{2}\ln|a-1| - \tfrac{1}{2}\ln|a+1|)
\end{aligned}$$

Since $\lim_{b\to 1^-} \frac{1}{2}\ln|b-1|$ diverges, we have that the entire integral diverges and thus the integral does not exist.

SOLUTION 2. First, note that the integral above is improper with a singularity at $x = 1$. Splitting up the integral there, we see that

$$\int_0^2 \frac{1}{x^2-1}\,dx = \int_0^1 \frac{1}{x^2-1}\,dx + \int_1^2 \frac{1}{x^2-1}\,dx$$
$$= \lim_{b\to 1^-}\int_0^b \frac{1}{x^2-1}\,dx + \lim_{a\to 1^+}\int_a^2 \frac{1}{x^2-1}\,dx$$

Now, let $\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$ by the partial fraction decomposition. We note that

$$\int_0^1 \frac{1}{x^2-1}\,dx = \int_0^1 \frac{A}{x-1}\,dx + \int_0^1 \frac{B}{x+1}\,dx$$

and

$$\int_0^1 \frac{A}{x-1}\ dx$$

doesn't seem to converge since it behaves like $\int_{-1}^{0} \frac{1}{x}\ dx$, so the whole integral does not converge. (In particular, we don't even need to worry about what A and B are, since they are non-zero constants!) Let's prove this rigorously.

$$\begin{aligned}
\int_0^1 \frac{A}{x-1}\ dx &= A \cdot \lim_{b\to 1^-} \int_0^b \frac{1}{x-1}\ dx \\
&= A \cdot \lim_{b\to 1^-} \ln|x-1|\Big|_0^b \\
&= A \cdot \left(\lim_{b\to 1^-} \ln|b-1| - \ln|0-1| \right)
\end{aligned}$$

Since $\lim\limits_{b\to 1^-} \ln|b-1|$ diverges, we have that the integral diverges and thus the entire integral does not exist.

Question 2 (d)

SOLUTION. The integral will become less complicated if we make a substitution $u = \ln x$.

First substitution If $u = ln\ x$ then $du = (1/x)\ dx$ so $dx = x\ du = e^u\ du$. For the last equality we solved $u = \ln x$ for x. Therefore,

$$I = \int \sin(\ln x)\ dx = \int e^u \sin u\ du$$

First integration by parts To find this integral, we will need to integrate by parts twice. Let $f = e^u$ with $df = e^u\ du$ and $dg = \sin u\ du$ with $g = -\cos u$. Then

$$\begin{aligned}
I = fg - \int gdf &= -e^u \cos u - \int e^u(-\cos u)\ du \\
&= -e^u \cos u + \int e^u \cos u\ du
\end{aligned}$$

Second integration by parts We repeat the process on the second integral, this time with $f = e^u$ and $df = e^u\ du$, and $dg = \cos u\ du$ with $g = \sin u$. Now we find:

$$I = -e^u \cos u + (e^u \sin u - \int e^u \sin u\ du)$$

Inspect the result We recognize the integral above as I! We can now solve for I:

$$I = -e^u \cos u + e^u \sin u - I$$

so

$$2I = e^u \sin u - e^u \cos u$$

Then

$$I = \frac{1}{2}e^u(\sin u - \cos u) + C$$

We added the arbitrary constant C because we are computing an indefinite integral.

Bring x back Finally, the integral we desire can be found by replacing u by $\ln x$ so that $e^u = x$:

$$I = \frac{1}{2}x(\sin(\ln x) - \cos(\ln x)) + C$$

Question 3 (a)

SOLUTION. Notice that $F(x)$ is an antiderivative of $p(x)$. If we use integration by parts, we let $u = x$ and $dv = p(x)dx$ so that $du = dx$ and $v = F(x)$ so that

$$\begin{aligned}\overline{x} &= \int_0^a xp(x)\,dx \\ &= xF(x)\Big|_0^a - \int_0^a F(x)\,dx \\ &= aF(a) - (0)F(0) - \int_0^a F(x)\,dx\end{aligned}$$

Now, since the probability distribution is only defined on [0,a] then the cumulative probability satisfies *F(0)=0* since nothing can happen before 0 and *F(a)=1* since by x=a, there is a 100% chance that our event happened. Therefore,

$$\begin{aligned}\overline{x} &= a - \int_0^a F(x)\,dx \\ &= \int_0^a 1 - \int_0^a F(x)\,dx \\ &= \int_0^a (1 - F(x))\,dx\end{aligned}$$

If we missed the second last step, we could have instead simplified the other equation to see that $\overline{x} = \int_0^a (1 - F(x))\,dx = a - \int_0^a F(x)$ and thus the two sides are equal. Either way this completes the proof.

Question 3 (b) i

SOLUTION. Firstly, notice that in order for the denominator to be well defined, we require that $1 - x > 0$ (strictly greater than 0 since the denominator of a fraction cannot be 0). This gives us that b has value 1 if it is to be maximal. Now, for a to be minimal, we notice that the denominator is always positive if a is less than 1 so that gives no restriction. However the numerator has to be positive in order to have a pdf and so we require that a is greater than 0. Lastly, the area condition on a pdf shows us that
$1 = \int_0^1 \frac{cx}{\sqrt{1-x}} = \lim_{q\to 1^-} \int_0^q \frac{cx}{\sqrt{1-x}}$
Let $u = 1 - x$ so that $du = -dx$, $u(0) = 1$, $u(q) = 1 - q$ and substituting gives

$$
\begin{aligned}
1 &= \lim_{q\to 1^-} \int_0^q \frac{cx}{\sqrt{1-x}} \\
&= \lim_{q\to 1^-} \int_1^{1-q} \frac{-c(1-u)}{\sqrt{u}} \\
&= \lim_{q\to 1^-} c\int_{1-q}^{1} (\sqrt{u}^{-1} - \sqrt{u})\,du \\
&= \lim_{q\to 1^-} c(2\sqrt{u} - \frac{2u^{3/2}}{3})\Big|_{1-q}^{1} \\
&= \lim_{q\to 1^-} \big(c(2\sqrt{1} - \frac{2(1)^{3/2}}{3}) - c(2\sqrt{1-q} - \frac{2(1-q)^{3/2}}{3})\big) \\
&= c\left(2 - \frac{2}{3}\right) \\
&= \frac{4c}{3}
\end{aligned}
$$

and isolating for c gives us that $c = 3/4$.

Question 3 (b) ii

SOLUTION. We seek to compute
$\overline{x} = \int_0^1 xp(x)\,dx = \frac{3}{4}\int_0^1 \frac{x^2}{\sqrt{1-x}}\,dx = \frac{3}{4}\lim_{q\to 1^-}\int_0^q \frac{x^2}{\sqrt{1-x}}\,dx$
Let $u = 1-x$ so that $du = -dx$, $u(0) = 1$, $u(q) = 1-q$ and substituting gives

$$
\begin{aligned}
\overline{x} &= \int_0^1 xp(x)\,dx \\
&= \frac{3}{4}\int_0^1 \frac{x^2}{\sqrt{1-x}}\,dx \\
&= \frac{3}{4}\lim_{q\to 1^-}\int_0^q \frac{x^2}{\sqrt{1-x}}\,dx \\
&= \frac{3}{4}\lim_{q\to 1^-}\int_1^{1-q} \frac{-(1-u)^2}{\sqrt{u}}\,du \\
&= \frac{3}{4}\lim_{q\to 1^-}\int_{1-q}^{1} \frac{(1-u)^2}{\sqrt{u}}\,du \\
&= \frac{3}{4}\lim_{q\to 1^-}\int_{1-q}^{1} \frac{1-2u+u^2}{\sqrt{u}}\,du \\
&= \frac{3}{4}\lim_{q\to 1^-}\int_{1-q}^{1} \sqrt{u}^{-1} - 2\sqrt{u} + u^{3/2}\,du \\
&= \frac{3}{4}\lim_{q\to 1^-}\left(2\sqrt{u} - \frac{4u^{3/2}}{3} + \frac{2u^{5/2}}{5}\right)\Big|_{1-q}^{1} \\
&= \frac{3}{4}\lim_{q\to 1^-}\left(\left(2\sqrt{1} - \frac{4(1)^{3/2}}{3} + \frac{2(1)^{5/2}}{5}\right) - \left(2\sqrt{1-q} - \frac{4(1-q)^{3/2}}{3} + \frac{2(1-q)^{5/2}}{5}\right)\right) \\
&= \frac{3}{4}\left(2 - \frac{4}{3} + \frac{2}{5}\right) \\
&= \frac{4}{5}
\end{aligned}
$$

Question 4 (a)

SOLUTION. Using the formula in the hint, we have

$$\begin{aligned} V &= \pi \int_1^\infty \frac{1}{x^2}\,dx \\ &= \lim_{b\to\infty} \pi \int_1^b \frac{1}{x^2}\,dx \\ &= \lim_{b\to\infty} \pi(-x^{-1})\Big|_1^b \\ &= \lim_{b\to\infty} \pi(-b^{-1} - (-(1)^{-1})) \\ &= \pi \end{aligned}$$

Question 4 (b)

SOLUTION. In our scenario, the surface area is $A = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + (\frac{dy}{dx})^2}\,dx$

As mentioned in the hint, the square root must always be at least 1 on the domain $[1, \infty)$ since $\frac{1}{x}\sqrt{1 + (\frac{dy}{dx})^2} = \frac{1}{x}\sqrt{1 + (-\frac{1}{x^2})^2} = \frac{1}{x}\sqrt{1 + \frac{1}{x^4}} > \frac{1}{x}$

Now, since

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln|x|\Big|_1^b = \lim_{b\to\infty} \ln|b| - \ln(1)$$

diverges, we have that our integral diverges by the integral comparison test. Thus the surface area diverges. NOTE: Even though you can fill the horn with a finite amount of paint, you cannot paint the inside with a finite amount of paint.

Question 5 (a)

SOLUTION. The steady states are given by solving

$$x = 4x^3 - 12x^2 + 8x$$

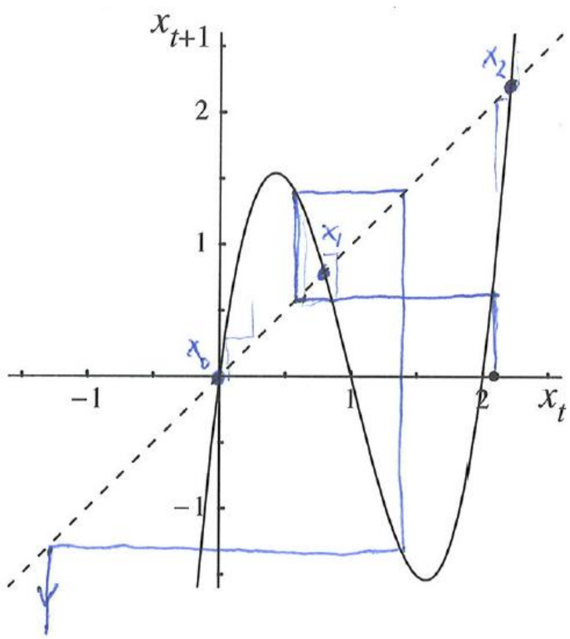
Subtracting x from both sides yields

$$0 = 4x^3 - 12x^2 + 7x = x(4x^2 - 12x + 7)$$

Using the quadratic formula gives

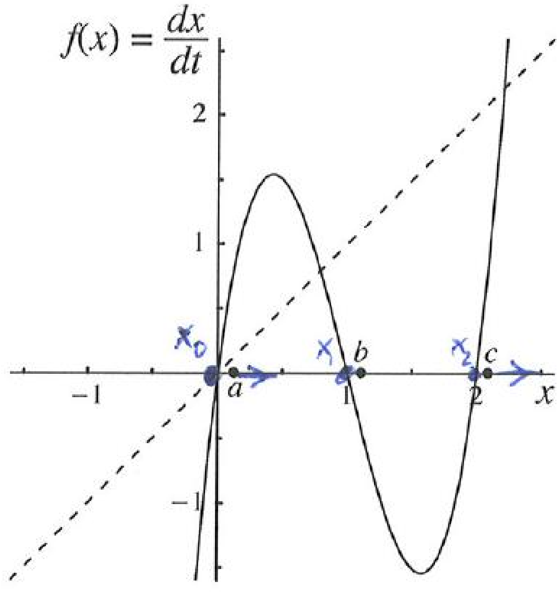
$x = \frac{12\pm\sqrt{144-4(4)(7)}}{8} = \frac{12\pm\sqrt{32}}{8} = \frac{3\pm\sqrt{2}}{2}$

Analytically to check stability, we would need to take the derivative of our function and then plug in the fixed points and check that the absolute value of the derivatives in the fixed points is less than 1 (this would show that the fixed point is stable). It's easier to notice that graphically, none of the fixed points are the limit of an iterative sequence and so none of the fixed points are stable.



Question 5 (b)

Solution. Setting the derivative to 0 and solving gives
$0 = 4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2) = 4x(x-1)(x-2)$
and thus, the steady states are $x = 0, 1, 2$. At the point a, the derivative is positive and so the function $x(t)$ is increasing and thus approaches the steady state 1 (the first steady states bigger than a). At b, the dervative is negative hence the function is negative and thus approaches the steady state 1. At c, the derivative is positive and so the function is increasing and as t approaches infinity, the function $x(t)$ approaches infinity.



Question 6 (a)

SOLUTION. Solving first for the cdf in terms of θ, using that $p(t) = 0$ for $t <$ $-\pi/4$, we have

$$F(\theta) = \int_{-\pi/4}^{\theta} \frac{2}{\pi}\,dt = \frac{2}{\pi}t\Big|_{-\pi/4}^{\theta}$$
$$= \frac{2}{\pi}\left(\theta - \frac{-\pi}{4}\right)$$

To find the cdf in terms of x we note that, from the diagram, $\tan(\theta) = \frac{x}{2}$. Solving for θ gives $\theta = \arctan(\frac{x}{2})$ and thus we have

$$F(x) = \frac{2}{\pi}\left(\arctan\left(\frac{x}{2}\right) + \frac{\pi}{4}\right)$$

completing the problem.

Question 6 (b)

SOLUTION. Taking the derivative of the cdf found in part a we have

$$p(x) = F'(x) = \left(\frac{2}{\pi}\left(\arctan\left(\frac{x}{2}\right) + \frac{\pi}{4}\right)\right)'$$
$$= \frac{2}{\pi}\cdot\left(\frac{1}{1+\left(\frac{x}{2}\right)^2}\cdot\frac{1}{2} + 0\right)$$
$$= \frac{1}{\pi}\cdot\frac{4}{4+x^2} = \frac{4}{\pi(4+x^2)}$$

completing the question.

Question 6 (c)

SOLUTION 1. Notice that
$$xp(x) = \frac{2x}{4+x^2}$$
is an odd function. The interval in question comes from noting that $x = 2\tan(\theta)$ and so $a = 2\tan(-\pi/4) = -2$ and $b = 2\tan(\pi/4) = 2$ and so $[a, b] = [-2, 2]$ is symmetric about the origin. Therefore,
$$\overline{x} = \int_a^b xp(x)\,dx = 0$$
by symmetry of an odd function.

SOLUTION 2. As in the previous solution, the interval in question comes from noting that $x = 2\tan(\theta)$ and so $a = 2\tan(-\pi/4) = -2$ and $b = 2\tan(\pi/4) = 2$. To solve the integral
$$\overline{x} = \int_a^b xp(x)\,dx = \int_{-2}^{2} xp(x)\,dx = \int_{-2}^{2} \frac{2x}{4+x^2}\,dx$$
we use a substitution given by $u = 4+x^2$ so that $du = 2xdx$, $u(-2) = 4+(-2)^2 = 8$, $u = 4+(2)^2 = 8$ and so

$$
\begin{aligned}
\overline{x} &= \int_a^b xp(x)\,dx = \int_{-2}^2 xp(x)\,dx \\
&= \int_{-2}^2 \frac{2x}{4+x^2}\,dx \\
&= \int_8^8 \frac{du}{u} \\
&= \ln|u|\Big|_8^8 \\
&= 0
\end{aligned}
$$

Question 7 (a)

SOLUTION. Setting the derivative to 0 gives
$0 = 2M\big(1 - \frac{M}{K}\big)$
and so either $M = 0$ or $1 - \frac{M}{K} = 0$, that is $M = K$.

Question 7 (b)

SOLUTION. As given by the hint, the new differential equation is given by
$\frac{dM}{dt} = 2M\big(1 - \frac{M}{K}\big) - hM$
Factoring out a $2M$ gives
$\frac{dM}{dt} = 2M\big(1 - \frac{M}{K} - \frac{h}{2}\big)$
Setting the derivative in this differential equation to 0 gives either $M = 0$ or $1 - \frac{M}{K} - \frac{h}{2} = 0$ and isolating for M gives $M = K\big(1 - \frac{h}{2}\big)$

Question 7 (c)

SOLUTION. As $\alpha = 2$, we have that $h^* = 2$ and in fact, $h < h^*$ must be true in order for the population to survive.

Question 7 (d)

SOLUTION. As mentioned in the hint above, we are looking at the modified differential equation given by
$\frac{dM}{dt} = 2M\big(1 - \frac{M}{K}\big) - H$
Setting the derivative to be 0, we see that
$0 = 2M\big(1 - \frac{M}{K}\big) - H = 2M - \frac{2}{K}M^2 - H$
Using the quadratic formula, we see that

$$
\begin{aligned}
M &= \frac{-2 \pm \sqrt{4 - 4(\frac{-2}{K})(-H)}}{2(\frac{-2}{K})} \\
&= \frac{-2K \pm K\sqrt{4(1 - \frac{2H}{K})}}{-4} \\
&= \frac{-2K \pm 2K\sqrt{1 - \frac{2H}{K}}}{-4} \\
&= \frac{K \mp K\sqrt{1 - \frac{2H}{K}}}{2} \\
&= \frac{K \mp \sqrt{K^2 - 2HK}}{2}
\end{aligned}
$$

(Note any of the last three lines would be an acceptable answer)

Question 7 (e)

SOLUTION 1. Setting the derivative to 0 in

$$\frac{dM}{dt} = 2M\big(1 - \frac{M}{K}\big) - H$$

with $M = K/4$ we have that

$$0 = 2 \cdot \frac{K}{4}\left(1 - \frac{\frac{K}{4}}{K}\right) - H = \frac{K}{2} \cdot \frac{3}{4} - H$$

Isolating for H gives $H = \dfrac{3}{8}K$

SOLUTION 2. Using the solution from part d. and plugging in $M = K/4$, we have

$$\frac{K}{4} = M = \frac{K \mp K\sqrt{1 - \frac{2H}{K}}}{2}$$

Multiplying by 2 and noticing that the original differential equation has the restriction that K cannot be 0, we may divide by K as well to reduce the problem to

$$\frac{1}{2} = M = 1 \mp \sqrt{1 - \tfrac{2H}{K}}$$

Isolating for the square root and squaring both sides gives

$$\frac{1}{4} = M = 1 - \tfrac{2H}{K}$$

Solving for H gives
$H = \dfrac{3}{8}K$

Question 8 (a)

SOLUTION 1. We have via the fundamental theorem of calculus that
$I_a = \frac{d}{dx}\int_x^7 \cos(\sin(t))\,dt = -\frac{d}{dx}\int_7^x \cos(\sin(t))\,dt = -\cos(\sin(x))$

SOLUTION 2. Recall the Fundamental Theorem of Calculus in its general form:
$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\;dt = h'(x)f(h(x)) - g'(x)f(g(x))$
Then

$$\begin{aligned} I_a &= \frac{d}{dx}\int_x^7 \cos(\sin(t))\,dt \\ &= \left(\frac{d}{dx}7\right)\cos(\sin(7)) - \left(\frac{d}{dx}x\right)\cos(\sin(x)) \\ &= -\cos(\sin(x)) \end{aligned}$$

Question 8 (b)

SOLUTION 1. Since integrals give you areas, which are just constants, the derivative of a constant is 0.

SOLUTION 2. For a more convoluted solution, notice that via the FTC, we have

$$\begin{aligned} I_b &= \frac{d}{dx}\int_1^3 e^{4^t}\;dt \\ &= \frac{d}{dx}\int_1^x e^{4^t}\;dt + \frac{d}{dx}\int_x^3 e^{4^t}\;dt \\ &= \frac{d}{dx}\int_1^x e^{4^t}\;dt - \frac{d}{dx}\int_3^x e^{4^t}\;dt \\ &= e^{4^x} - e^{4^x} \\ &= 0 \end{aligned}$$

.

SOLUTION 3. Another convoluted solution: recall the Fundamental Theorem of Calculus in its general form:
$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\;dt = h'(x)f(h(x)) - g'(x)f(g(x))$
Then
$I_a = \frac{d}{dx}\int_1^3 e^{4^t}\;dt = \left(\frac{d}{dx}3\right)e^{4^3} - \left(\frac{d}{dx}1\right)e^{4^1} = 0$

Question 8 (c)

SOLUTION 1. Plugging in 0 into the integral yields
$\int_0^0 \frac{t^2}{1+t^6}\,dt = 0$
and plugging in 0 into x^3 also yields 0 hence we may apply L'Hopital's rule to see that

$\lim_{x\to 0}\frac{1}{x^3}\int_0^x \frac{t^2}{1+t^6}\,dt = \lim_{x\to 0}\frac{\frac{x^2}{1+x^6}}{3x^2} = \lim_{x\to 0}\frac{1}{3(1+x^6)} = \frac{1}{3}$
Notice that in the first equality, we also used the fundamental theorem of calculus.

Solution 2. Let's try to evaluate the inside integral
$\int_0^x \frac{t^2}{1+t^6}\,dt.$
Let $u = t^3$ so that $du = 3t^2dt$, $u(0) = (0)^3 = 0$ and $u(x) = x^3$. Plugging in yields
$\int_0^x \frac{t^2}{1+t^6}\,dt = \int_0^{x^3} \frac{\frac{1}{3}du}{1+u^2} = \frac{1}{3}\arctan(u)\big|_0^{x^3} = \frac{\arctan(x^3)}{3}.$
With the last equaltiy holding since plugging in 0 gives $\frac{\arctan(0)}{3} = 0.$
Now our goal is to evaluate the limit
$\lim_{x\to 0}\frac{\arctan(x^3)}{3x^3}.$
Plugging in zero into the numerator and denominator gives an indeterminant form 0/0 and so we may apply L'Hopital's rule to see that
$\lim_{x\to 0}\frac{\arctan(x^3)}{3x^3} = \lim_{x\to 0}\frac{\frac{3x^2}{1+x^6}}{3(3x^2)} = \lim_{x\to 0}\frac{1}{3(1+x^6)} = \frac{1}{3}$
completing the problem.

Solution 3. Proceed as in solution 2 to see that the integral evaluates to $\frac{\arctan(x^3)}{3}$. Now we seek to evaluate
$\lim_{x\to 0}\frac{\arctan(x^3)}{3x^3}$
Taking the Taylor expansion of the function yields

$$\begin{aligned}\frac{\arctan(x^3)}{x^3} &= \frac{1}{3x^3}\sum_{n=0}^{\infty}(-1)^n\frac{(x^3)^{2n+1}}{2n+1}\\ &= \frac{1}{3x^3}\sum_{n=0}^{\infty}(-1)^n\frac{x^{6n+3}}{2n+1}\\ &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{6n}}{3(2n+1)}\end{aligned}$$

As all terms bigger than when n equals 0 contain a positive power of x, taking the limit of the above as x tends to 0 leaves only the constant term which is
$\frac{(-1)^0x^{6(0)}}{3(2(0)+1)} = \frac{1}{3}$

Solution 4. We start with the Taylor expansion for $\frac{1}{1-x}$
$\frac{1}{1-x} = \sum_{n=0}^{\infty}x^n$
Plugging in $x = -t^6$ yields
$\frac{1}{1+t^6} = \sum_{n=0}^{\infty}(-t^6)^n = \sum_{n=0}^{\infty}(-1)^nt^{6n}$
Multiplying by t^2 gives
$\frac{t^2}{1+t^6} = \sum_{n=0}^{\infty}(-1)^nt^{6n+2}$
Now integrating from 0 to x gives
$\int_0^x \frac{t^2}{1+t^6} = \sum_{n=0}^{\infty}(-1)^n\int_0^x t^{6n+2} = \sum_{n=0}^{\infty}\frac{(-1)^nx^{6n+3}}{6n+3}$

Dividing by x^3 gives

$$\frac{1}{x^3}\int_0^x \frac{t^2}{1+t^6} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{6n}}{6n+3}$$

As all terms bigger than when n equals 0 contain a positive power of x, taking the limit of the above as x tends to 0 leaves only the constant term which is

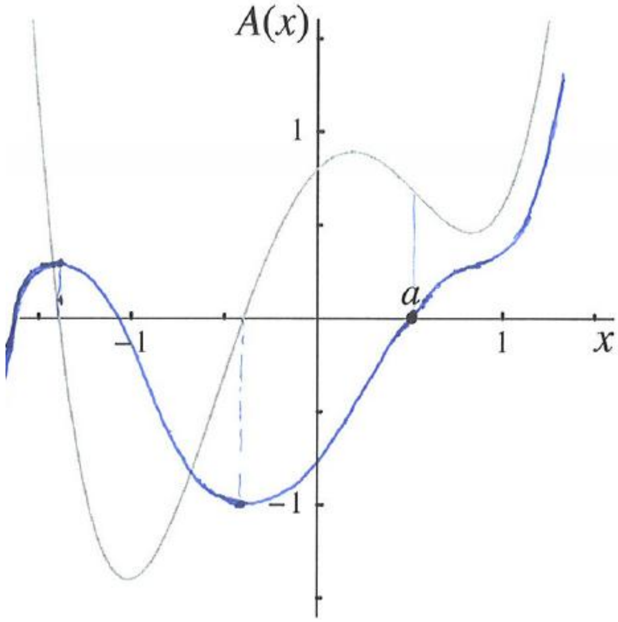
$$\frac{(-1)^0 x^{6(0)}}{6(0)+3} = \frac{1}{3}$$

which is the desired answer.

Question 9

Solution. Attached is a picture. Notice that there is a local minimum at $x = -0.4$ and a local maximum at $x = -1.4$. Also note that plugging in a into the integral gives

$$A(a) = \int_a^a f(s)\,ds = 0.$$



Question 10

Solution 1. We mechanically plug in the derivatives to the Taylor series formula given by

$$F(x) = \sum_{n=0}^{\infty}\frac{F^{(n)}(0)x^n}{n!}$$

where

$$F(x) = \int_0^x \ln(1+s^2)\,ds.$$

Now, we have

$$F(0) = \int_0^0 \ln(1+s^2)\,ds = 0.$$

Using the FTC, we get

$F'(0) = \frac{d}{dx}\Big|_{x=0} \int_0^x \ln(1+s^2)\, ds = \ln(1+x^2)\big|_{x=0} = \ln(1) = 0.$
The next derivative is (via the chain rule)
$F''(0) = \frac{d}{dx}\Big|_{x=0} \ln(1+x^2) = \frac{2x}{1+x^2}\Big|_{x=0} = 0.$
The next derivative is (via the quotient rule)
$F'''(0) = \frac{d}{dx}\Big|_{x=0} \frac{2x}{1+x^2} = \frac{2(1+x^2)-(2x)^2}{(1+x^2)^2}\Big|_{x=0} = 2.$
The last derivative is

$$\begin{aligned} F''''(0) &= \frac{d}{dx}\Big|_{x=0} \frac{2(1+x^2)-(2x)^2}{(1+x^2)^2} \\ &= \frac{d}{dx}\Big|_{x=0} \frac{2-2x^2}{(1+x^2)^2} \\ &= \frac{-4x(1+x^2)^2 - 2(1-x^2)\cdot 2(1+x^2)(2x)}{(1+x^2)^4}\Big|_{x=0} \\ &= 0 \end{aligned}$$

.
Thus, the fourth Taylor polynomial is
$T_4(x) = \frac{x^3}{3}$
as required.

Solution 2. Starting with
$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$
We see that
$\ln(1+s^2) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{s^{2n}}{n}.$
Integrating yields
$\int_0^x \ln(1+s^2) = \sum_{n=1}^{\infty} (-1)^{n+1} \int_0^x \frac{s^{2n}}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{n(2n+1)}$
Plugging in the first few values yields
$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{n(2n+1)} = \frac{x^3}{3} - \frac{x^5}{10} + ...$
Since the question is seeking $T_4(x)$, we have that
$T_4(x) = \frac{x^3}{3}$
as required.

Good Luck for your exams!