Full Solutions
MATH221 December 2007

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How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1

SOLUTION. To determine a basis for the null space of A, let $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ and suppose $Ax = 0$. We want to solve this equation for x.

To do so, we append the zero vector to A

$$\left(\begin{array}{cccc|c} 2 & -4 & 1 & t & 0 \\ 1 & -2 & 2 & t & 0 \\ 1 & -2 & 1 & 2t & 0 \\ 1 & -2 & 1 & t & 0 \end{array}\right)$$

and row reduce A in the standard way. Leaving aside these details (exercise), the end result is

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & t & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Therefore we deduce that x belongs to the null space of A if and only if

$$\begin{aligned} x_1 &= 2x_2 \\ x_3 &= 0 \\ tx_4 &= 0 \end{aligned}$$

.

Case 1: $t \neq 0$. Then $x_4 = 0$, so that $x = x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ where $x_2 \in \mathbb{R}$ is a free parameter. In this case we therefore conclude that $\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ is a basis for the null space of A.

Case 2: $t = 0$. Then there is no restriction on x_4, and we have $x = \begin{pmatrix} 2x_2 \\ x_2 \\ 0 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ where $x_2, x_4 \in \mathbb{R}$. In this case, we conclude that $\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for the null space of A.

Question 2

SOLUTION. We are going to use Gauss-Jordan elimination. Append the identity matrix to B,

$$\left(\begin{array}{cccc|cccc} 2 & 0 & -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

and proceed to row-reduce.

First, we multiply the first row by *1/2*.

Second, eliminate the *3* and *5* entries in the third row by subtracting *3* times the second row from the third, and subtracting *5* times the fourth row from the third.

After these steps, we obtain

$$\begin{pmatrix} 1 & 0 & -1 & 0 & | & 1/2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & | & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & | & 0 & -3 & 1 & -5 \\ 0 & 0 & 0 & 1 & | & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The last step is to add the third row to the first row. We find

$$\begin{pmatrix} 1 & 0 & 0 & 0 & | & 1/2 & -3 & 1 & -5 \\ 0 & 1 & 0 & 0 & | & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & | & 0 & -3 & 1 & -5 \\ 0 & 0 & 0 & 1 & | & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore, by Gauss-Jordan elimination, we have found

$$B^{-1} = \begin{pmatrix} 1/2 & -3 & 1 & -5 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & -5 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Question 3

SOLUTION. Letting $B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$, the task at hand is to solve $BAC = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ for A. Now B and C are invertible matrices (as they have have non-zero determinant), so we can solve for A by multiplying the above equation on the left by B^{-1}, and on the right by C^{-1}, giving

$$A = B^{-1} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} C^{-1}.$$

It is straightforward to compute $B^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$ and $C^{-1} = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$. Therefore

$$A = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}.$$

Question 4

SOLUTION. Let us assume that the vector v is contained in W. Then there are $a, b \in \mathbb{R}$ for which

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + b \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \\ -b \\ b-a \end{pmatrix}.$$

The second equation implies we must have b = *-2*. But this implies a - *1* = *1* and *-2* - a = *3*, i.e. that a must be equal to both *2* and *-5* simultaneously.
Since this is impossible, we have shown that v does not belong to W.

Question 5 (a)

SOLUTION. Recall that adding multiples of one row to another does not change the determinant. So to make our lives easier when calculating the determinant of B, we will simplify by row reduction as much as possible.
Subtracting the first row from the second, third, and fourth, results in the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 6 & 9 \\ 0 & 3 & 9 & t-1 \end{pmatrix}$$

whose determinant is the same as B. Now subtracting twice the second row from the third, and three times the second row from the fourth, gives

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 3 & t-10 \end{pmatrix}$$

whose determinant is still equal to that of B. A final step is to subtract $3/2$ times the third row from the fourth to have

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & t-10-9/2 \end{pmatrix}.$$

This matrix is upper-triangular, so we know how to calculate its determinant: it is simply the product of the diagonal entries. We therefore conclude that

$\det(B) = 1 \cdot 1 \cdot 2 \cdot (t - 10 - 9/2) = 2t - 29.$

Question 5 (b)

Solution. Recall that a (square) matrix is invertible if and only if its determinant is non-zero (why is this?). From part (a), we know that $\det(B) = 2t - 29$. Therefore, B is invertible if and only if $t \neq 29/2$.

Question 6 (a)

Solution. To see why $\mathcal{B}$ forms a basis for $\mathbb{R}^2$, we recall two facts:

- Orthogonal vectors are linearly independent.
- A linearly independent list of length n in a vector space of dimension n is a basis.

Since the two vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ comprising $\mathcal{B}$ are orthogonal, $\mathcal{B}$ is a linearly independent list of length 2 in $\mathbb{R}^2$.

Therefore, by the two bulleted points above, $\mathcal{B}$ is a basis for $\mathbb{R}^2$.

Question 6 (b)

Solution. The *coordinates* of $\begin{pmatrix} 7 \\ 10 \end{pmatrix}$ *with respect to the basis* $\mathcal{B} = \{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\}$ are the coefficients $a, b \in \mathbb{R}$ for which $\begin{pmatrix} 7 \\ 10 \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

That is, we want to solve

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 7 \\ 10 \end{pmatrix}$$

for $\begin{pmatrix} a \\ b \end{pmatrix}$.

By inverting the matrix $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, we find

$$\begin{pmatrix} a \\ b \end{pmatrix} = -\tfrac{1}{2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 7 \\ 10 \end{pmatrix} = \begin{pmatrix} 17/2 \\ -3/2 \end{pmatrix}.$$

Question 6 (c)

Solution. Let $\overline{\mathcal{B}} = \{e_1, e_2\}$ denote the standard basis of $\mathbb{R}^2$, where $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and denote the given basis by $\mathcal{B} = \{b_1, b_2\}$, where $b_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $b_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Let us briefly recall how given $[T]_{\mathcal{B}}$ (the matrix of T with respect to $\mathcal{B}$), we can find $[T]_{\overline{\mathcal{B}}}$ (the matrix of T with respect to $\overline{\mathcal{B}}$.).

We first find the linear relation between the basis $\mathcal{B}$ and $\overline{\mathcal{B}}$:

$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$.

Letting $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ the above equality gives us the following relation for the coordinates of a vector $v \in \mathbb{R}^2$ with respect to the two bases:

$[v]_{\mathcal{B}} = P[v]_{\overline{\mathcal{B}}}$.

Now, by definition, we have $[Tv]_{\mathcal{B}} = [T]_{\mathcal{B}}[v]_{\mathcal{B}}$ and $[Tv]_{\overline{\mathcal{B}}} = [T]_{\overline{\mathcal{B}}}[v]_{\overline{\mathcal{B}}}$.

Therefore, we can write the following string of equalities:

$P[T]_{\overline{\mathcal{B}}}[v]_{\overline{\mathcal{B}}} = P[Tv]_{\overline{\mathcal{B}}} = [Tv]_{\mathcal{B}} = [T]_{\mathcal{B}}[v]_{\mathcal{B}} = [T]_{\mathcal{B}}P[v]_{\overline{\mathcal{B}}}$.

Multiplying by P^{-1} in the above gives us $[T]_{\overline{\mathcal{B}}}[v]_{\overline{\mathcal{B}}} = P^{-1}[T]_{\mathcal{B}}P[v]_{\overline{\mathcal{B}}}$. As this equality is true for every v, we conclude that

$[T]_{\overline{\mathcal{B}}} = P^{-1}[T]_{\mathcal{B}}P$.

For this question, we are given $[T]_{\mathcal{B}} = \begin{pmatrix} 2 & -3 \\ 0 & 2 \end{pmatrix}$, and we can compute $P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Putting these together, we find that

$[T]_{\overline{\mathcal{B}}} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1/2 & 3/2 \\ -3/2 & 7/2 \end{pmatrix}$.

Question 7 (a)

Solution. Recall that 1 is an eigenvalue of A if and only if $\det(A - I) = 0$. Therefore it suffices to show that $\det(A - I) = 0$. We have

$A - I = \begin{pmatrix} -1 & -4 & -6 \\ -1 & -1 & -3 \\ 1 & 2 & 4 \end{pmatrix}$

We would prefer to avoid doing lots of computation by finding the determinant right away here. Instead, recall that adding multiples of one row to another does not change the determinant. Therefore, adding the second row to the third, and then subtracting the first row from the second, yields the matrix

$\begin{pmatrix} -1 & -4 & -6 \\ 0 & 3 & 3 \\ 0 & 1 & 1 \end{pmatrix}$

which has the same determinant as A - I. But note that two rows are just multiples of each other. Therefore the determinant of the above matrix is zero, and hence $\det(A - I) = 0$.

Question 7 (b)

Solution. We already verified that 1 is an eigenvalue of A in part (a). To determine other possible eigenvalues, we solve $\det(A - \lambda I) = 0$ for λ. We have

$A - \lambda I = \begin{pmatrix} -\lambda & -4 & -6 \\ -1 & -\lambda & -3 \\ 1 & 2 & 5-\lambda \end{pmatrix}$

Let us make use of the fact that adding multiples of one row to another does not change the determinant to avoid lots of computation. Adding the second row of $A - \lambda I$ to the third, while subtracting λ times the second row from the first, yields the matrix

$B = \begin{pmatrix} 0 & \lambda^2 - 4 & 3\lambda - 6 \\ -1 & -\lambda & -3 \\ 0 & 2-\lambda & 2-\lambda \end{pmatrix} = \begin{pmatrix} 0 & (\lambda+2)(\lambda-2) & 3(\lambda-2) \\ -1 & -\lambda & -3 \\ 0 & 2-\lambda & 2-\lambda \end{pmatrix}$

which has the same determinant has $A - \lambda I$. By expanding down the first column, the determinant is

$\det(A - \lambda I) = \det B = (\lambda+2)(\lambda-2)(2-\lambda) - 3(\lambda-2)(2-\lambda) = (\lambda-2)^2(1-\lambda)$

Therefore, the other eigenvalue of A is 2.

Question 7 (c)

SOLUTION. From part (b), we know that the eigenvalues of A are 1 and 2. We also saw in part (b) that the eigenvalue 2 is a repeated root of the characteristic polynomial for A, while the eigenvalue 1 is not.
By definition, the dimension of each respective eigenspace is *at least 1*. As well, recall that the dimension of each eigenspace is *at most the multiplicity of the eigenvalue as a root of the characteristic polynomial.*

- As the eigenvalue 1 is not a repeated root of the characteristic polynomial for A, the dimension of its eigenspace must be *at least 1*, and *at most 1*, i.e. it is equal to 1.

- The eigenvalue 2 is a twice repeated root of the characteristic polynomial for A; therefore the dimension of its eigenspace may be either 1 or 2. We will need to investigate further to determine which one it is.

To calculate the dimension of the eigenspace corresponding to the eigenvalue 2, we want to determine $\dim \operatorname{null}(A-2I)$, that is, the dimension of the kernel of A - *2I*. We have
$A - 2I = \begin{pmatrix} -2 & -4 & -6 \\ -1 & -2 & -3 \\ 1 & 2 & 3 \end{pmatrix}$.
All three rows of A - *2I* are multiples of each other, so $\operatorname{rank}(A-2I) = 1$. Therefore, by the rank-nullity theorem,
$\dim \operatorname{null}(A-2I) = \dim(\mathbb{R}^3) - \dim \operatorname{rank}(A-2I) = 3 - 1 = 2$.
In conclusion,

- For the eigenvalue 1, the dimension of its eigenspace is 1.
- For the eigenvalue 2, the dimension of its eigenspace is 2.

Question 8 (a)

SOLUTION. Since T is a reflection, it does not change the length of any vector. Therefore, 1 and -1 are the only possible eigenvalues for T.
The eigenvalue 1 corresponds to a vector which remains invariant under T, and therefore these vectors must lie on the line of reflection $y = \frac{1}{2}x$. In particular, we see that $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector for T corresponding to the eigenvalue 1, as it lies on the line $y = \frac{1}{2}x$.
The eigenvalue -1 corresponds to a 180 degree reversal in orientation; these vectors must therefore lie on the line *perpendicular* to the line of reflection, that is, any vector lying on the line $y = -2x$. Since $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ lies on the line $y = -2x$, we conclude that it is an eigenvector corresponding to the eigenvalue -1.
We therefore have two (linearly independent) eigenvectors for T, $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$, which form a basis for $\mathbb{R}^2$.

Question 8 (b)

SOLUTION. There is no basis of $\mathbb{R}^2$ consisting of eigenvectors of T. This is because T has no eigenvectors. Why? If v were an eigenvector of T, then $T(v) = \lambda v$ for some $\lambda \in \mathbb{R}$. As T is a rotation, it does not change the length of v, so the only possible values for λ are ± 1. This implies *T(v)* lies either parallel to v (if $\lambda = 1$) or anti-parallel (if $\lambda = -1$). But this is impossible, as *T(v)* is a clockwise rotation by 5 degrees.
Therefore T has no eigenvectors.

Question 8 (c)

SOLUTION. Note that every non-zero vector in $\mathbb{R}^2$ is an eigenvector for T (corresponding to the eigenvalue 5).

In particular, the standard basis $\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\}$ is a basis of $\mathbb{R}^2$ consisting of eigenvectors of T.

Question 8 (d)

Solution. No such basis exists.
Note that 1 is the only eigenvalue for T, as the characteristic polynomial is $(\lambda - 1)^2$. To determine the eigenvectors corresponding to the eigenvalue 1, we solve the corresponding equation
$$\begin{pmatrix}0 & 0\\2/3 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$
Thus $\begin{pmatrix}x\\y\end{pmatrix}$ is an eigenvector for T if and only if $x = 0$. This means that there is only **one** linearly independent eigenvector, namely $\begin{pmatrix}0\\1\end{pmatrix}$.
As a basis of $\mathbb{R}^2$ consists of two linearly independent vectors, there exists no basis consisting of eigenvectors of T.

Question 9 (a)

Solution. No content found.

Question 9 (b)

Solution. No content found.

Question 9 (c)

Solution. No content found.

Question 9 (d)

Solution. No content found.

Question Section 101 10 (a)

Solution. No content found.

Question Section 101 10 (b)

Solution. No content found.

Question Section 102 10 (a)

Solution. No content found.

Question Section 102 10 (b)

Solution. No content found.

Question Section 102 10 (c)

SOLUTION. No content found.

Question Section 103 10 (a)

SOLUTION. Let us first write the system in matrix notation:
$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$.
Notice that $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}^2 \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, and $\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}^3 \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$...
Therefore, we *hypothesize* that for every $n \in \mathbb{N}$,
$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$.
This explicit formula can (and should) be established rigorously by induction (we leave this as an exercise).
Now we want to compute $\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}^n$. Recall that matrix powers are easy to compute via matrix diagonalization.

Therefore, we find the eigenvalues and eigenvectors for $A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$.

The eigenvalues are determined from $\det(A - \lambda I) = (1-\lambda)(2-\lambda) - 6 = (\lambda - 4)(\lambda + 1) = 0$, so the eigenvalues of A are *4* and *-1*.
To find an eigenvector associated with the eigenvalue *4*, we solve $(A - 4I)x = 0$, i.e. the equation
$\begin{pmatrix} -3 & 3 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

from which it is clear that $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector.

To find an eigenvector associated with the eigenvalue *-1*, we solve
$\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

from which we find $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ is an eigenvector.

Now we form the matrix $P = \begin{pmatrix} 1 & -3 \\ 1 & 2 \end{pmatrix}$ whose columns are the eigenvectors of A, and the diagonal matrix $\Lambda = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$ with the corresponding eigenvalues. We know that $A = P\Lambda P^{-1}$ and hence $A^n = P\Lambda^n P^{-1} = P \begin{pmatrix} 4^n & 0 \\ 0 & (-1)^n \end{pmatrix} P^{-1}$.

The final step is to compute $P^{-1} = \frac{1}{5} \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}$, and then A^n:
$A^n = \frac{1}{5} \begin{pmatrix} 1 & -3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 4^n & 0 \\ 0 & (-1)^n \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} = \frac{4^n}{5} \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} + \frac{(-1)^n}{5} \begin{pmatrix} 3 & -3 \\ -2 & 2 \end{pmatrix}$.
Check that for $n = 1$ we recover the matrix A.
Finally, the question asks us to find x_n and y_n explicitly, with the given initial condition. Having computed A^n, we can now compute
$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \left[\frac{4^n}{5} \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} + \frac{(-1)^n}{5} \begin{pmatrix} 3 & -3 \\ -2 & 2 \end{pmatrix} \right] \begin{pmatrix} 5 \\ 10 \end{pmatrix}$.
Upon multiplying and simplifying the above equation, we find that

$$x_n = 2(4)^{n+1} + 3(-1)^{n+1}$$
$$y_n = 2[4^{n+1} + (-1)^n]$$

.

Question Section 103 10 (b)

SOLUTION. From part (a), we know that $x_n = 2(4)^{n+1} + 3(-1)^{n+1}$ for every integer n. Therefore,
$\frac{x_{n+1}}{x_n} = \frac{2(4)^{n+2}+3(-1)^{n+2}}{2(4)^{n+1}+3(-1)^{n+1}} = \frac{4+\frac{3}{4}(-1/4)^n}{1+\frac{3}{2}(-1/4)^{n+1}}$
where to obtain the last equality, we divided through by $2(4)^{n+1}$. As $\lim_{n\to\infty}(-1/4)^n = 0$,, we conclude that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 4$.

Question Section 103 10 (c)

SOLUTION. From part (a), we found that

$$x_n = 2(4)^{n+1} + 3(-1)^{n+1}$$
$$y_n = 2[4^{n+1} + (-1)^n]$$

.
Therefore,
$\frac{x_n}{y_n} = \frac{2(4)^{n+1}+3(-1)^{n+1}}{2[4^{n+1}+(-1)^n]} = \frac{1+\frac{3}{2}(-1/4)^{n+1}}{1+\frac{1}{4}(-1/4)^n}$,
where we have obtained the last equality by dividing top and bottom by $2(4)^{n+1}$.
As $\lim_{n\to\infty} \left(\frac{-1}{4}\right)^n = 0$, we conclude $\lim_{n\to\infty} \frac{x_n}{y_n} = 1$.

Good Luck for your exams!