Full Solutions
MATH307 April 2006

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How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

SOLUTION. This matrix A in this question is given in the form of an LU decomposition with partial pivoting where P is just the matrix which swaps the rows of A. The matrix P also would not affect $R(A)$, $N(A)$, $R(A^T)$ where we need to look at the matrix U (echelon form) to find the bases.
To find a basis for the $R(A)$, we must identify the pivot columns in matrix U, which turns out to be the first and second column.
The pivots are highlighted in red.

$$U = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The basis for $R(A)$ is then the corresponding columns in A to the pivot columns in U.
The pivot columns of A are highlighted in red.

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & 1 & 8 & 8 \\ 0 & 1 & 4 & 2 \end{bmatrix}$$

$$R(A) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$$

To find $N(A)$, we can set $Ux = 0$ and solve for x. This would give us 2 equations and 2 free variables.

$$\begin{aligned} x_1 &= -2x_3 - 3x_4 \\ x_2 &= -4x_3 - 2x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned}$$

$$x = x_3 \begin{bmatrix} -2 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

And therefore

$$N(A) = \text{span}\left(\begin{bmatrix} -2 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right)$$

The basis of $R(A^T)$ can be taken from the rows of U that contain pivots.

$$R(A^T) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 4 \\ 2 \end{bmatrix} \right)$$

To find the basis for $N(A^T)$, we must row reduce A^T and solve for x in $A^T x = 0$
From row reducing A^T, you should acquire the matrix

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solving for x, you end up with 2 equations and 1 free variable.

$$\begin{aligned} x_1 &= 2x_3 \\ x_2 &= -x_3 \\ x_3 &= x_3 \end{aligned}$$

$$x = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

And hence

$$N(A^T) = \text{span}\left(\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right)$$

Question 1 (b)

SOLUTION. To find the condition on b_1, b_2, b_3, let us first solve the matrix problem $Ax = b$.
$\begin{bmatrix} 1 & 0 & 2 & 3 & | & b_1 \\ 2 & 1 & 8 & 8 & | & b_2 \\ 0 & 1 & 4 & 2 & | & b_3 \end{bmatrix}$
$P = \begin{bmatrix} 1 & 0 & 0 \\ \color{red}0 & \color{red}0 & \color{red}1 \\ \color{red}0 & \color{red}1 & \color{red}0 \end{bmatrix}$
$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \color{green}2 & \color{purple}1 & 1 \end{bmatrix}$
Since we are given the LU decomposition and partial pivoting of matrix A, we can simply follow the steps described by the matrix P (swapping the second and third row which is highlighted in *red*)and L (subtracting twice the first row (*green*) from the third row and subtracting once the second row from the third row(*purple*)), we would arrive at
$\begin{bmatrix} 1 & 0 & 2 & 3 & | & b_1 \\ 0 & 1 & 4 & 2 & | & b_3 \\ 0 & 0 & 0 & 0 & | & b_2 - b_3 - 2b_1 \end{bmatrix}$
In order for $Ax = b$ to have at least solution, the above matrix must be consistent, therefore the last row must contain all zeros.
The constraint on b_1, b_2, b_3 is the equation $-2b_1 + b_2 - b_3 = 0$.

Question 2 (a)

Solution. We want to find the ratio of $x_1(t)/x_2(t)$ as $t \to \infty$ for the difference equation $x(t+1) = Ax(t)$. The difference equation suggests that for every increment of 1 unit time, we want to multiply $x(t)$ by A.

The matrix A is already given in the diagonalized form of $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{-1}$

Given the initial condition of $x_0 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, we have the equation $x_k = A^k x_0$, which can also be expressed as the form

$$x_k = c_1(1)^k \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2(0.5)^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

where you can obtain c_1, c_2 by plugging in the initial condition x_0 and solving

$$x_0 = \begin{bmatrix} 4 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

As $k \to \infty$, the component with 0.5^k would reach 0 and you would be left with $x_k = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and therefore the ratio of $x_1(t)/x_2(t)$ would just be $1/2$.

Question 2 (b)

Solution. The solution for a differential equation of the form $\frac{\mathrm{d}x}{\mathrm{d}t} = Ax(t)$ and A is 2 by 2 and diagonalizable has the form
$x(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$ where λ are the eigenvalues of A and v are their respective eigenvectors.
c_1, c_2 can be obtained by plugging in the initial condition $x(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

Solving $\begin{bmatrix} 4 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ should give you $c_1 = 4, c_2 = -7$

therefore the solution is $x(t) = 4e^t \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 7e^{0.5t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Question 3 (a)

Solution. If P_2 is a vector space then the axioms must hold true
$\alpha_0 + \alpha_1 t + \alpha_2 t^2 + \beta_0 + \beta_1 t + \beta_2 t^2 = (\alpha_0 + \beta_0) + (\alpha_1 + \beta_1)t + (\alpha_2 + \beta_2)t^2$
$c(\alpha_0 + \alpha_1 t + \alpha_2 t^2) = c\alpha_0 + c\alpha_1 t + c\alpha_2 t^2$
We can also think of P_2 to be in $\mathbb{R}^3$
$\alpha_0 + \alpha_1 t + \alpha_2 t^2 \to \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix}$
such that commutative and associate properties are also satisfied.
To show that B is a basis, we have to show that it spans P_2 and are linearly independent
If we think of P_2 to be in $\mathbb{R}^3$ then we can immediately identify the basis for $\mathbb{R}^3$
Basis of $\mathbb{R}^3 : \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$
We can use this result and build the basis of P_2 such that
$B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t^2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ t \\ t^2 \end{bmatrix}.$
Therefore

$f(t) = \begin{bmatrix} 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \end{bmatrix} = 1 - 3t + 2t^2$

Question 3 (b)

SOLUTION. Note that differentiation is a linear operator
$f(t), g(t) \in P_2$ and some scalar c

$$\begin{aligned} T(f(t) + g(t)) &= (1+t)\frac{\mathrm{d}(g(t) + f(t))}{\mathrm{d}t} \\ &= (1+t)\frac{\mathrm{d}f(t))}{\mathrm{d}t} + (1+t)\frac{\mathrm{d}g(t))}{\mathrm{d}t} \\ &= T(g(t)) + T(f(t)) \end{aligned}$$

and

$$\begin{aligned} T(cf(t)) &= (1+t)\frac{\mathrm{d}cf(t)}{\mathrm{d}t} \\ &= (1+t)c\frac{\mathrm{d}f(t)}{\mathrm{d}t} \\ &= cT(f(t))) \end{aligned}$$

Thus as
$T(f(t) + g(t)) = T(g(t)) + T(f(t))$
and
$T(cf(t)) = cT(f(t)))$
we see that T is a linear transformation.

Question 3 (c)

SOLUTION. Given that the basis is $B = \begin{bmatrix} 1 \\ t \\ t^2 \end{bmatrix}$

If $f(t) = f \cdot B$

Then an arbitrary vector is $f = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix}$

$T(f) = (1+t)\frac{\mathrm{d}u}{\mathrm{d}x}$ The $(1+t)$ applies to all terms of $\frac{\mathrm{d}u}{\mathrm{d}x}$ can write the matrix A to be:
$A = (1+t)Df$
where D is the differentiation matrix
$D \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ 2\alpha_2 \\ 0 \end{bmatrix}$

Therefore $D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

Now we observe the linear trasnformation $(1+t)$ on $\begin{bmatrix} \alpha_1 \\ 2\alpha_2 \\ 0 \end{bmatrix}$

$(1+t)(\alpha_1 + 2\alpha_2 t) = (\alpha_1 + (\alpha_1 + 2\alpha_2)t + 2\alpha_2 t^2)$

$(1+t)\begin{bmatrix}\alpha_1\\2\alpha_2\\0\end{bmatrix} = \begin{bmatrix}\alpha_1\\2\alpha_2+\alpha_1\\2\alpha_2\end{bmatrix}$

therefore $(1+t)$ can be rewritten as $\begin{bmatrix}1&0&0\\1&1&0\\0&1&0\end{bmatrix}$

The transformation is therefore

$(1+t)D = \begin{bmatrix}1&0&0\\1&1&0\\0&1&0\end{bmatrix}\begin{bmatrix}0&1&0\\0&0&2\\0&0&0\end{bmatrix} = \begin{bmatrix}0&1&0\\0&1&2\\0&2&0\end{bmatrix}$

Question 4 (a)

SOLUTION. The eigenvalues are given by the formula

$det(A-\lambda * I) = \begin{vmatrix}a_{11}-\lambda & a_{12} & a_{13}\\ a_{21} & a_{22}-\lambda & a_{23}\\ a_{31} & a_{32} & a_{33}-\lambda\end{vmatrix} = 0$

Doing this gives the following polynomial

$-\lambda^3+\lambda^2+\frac{1}{36}\lambda-\frac{1}{36}=0$

The values we get are $\lambda = 1, \frac{1}{6}, \frac{-1}{6}$

These are the eigenvalues of Matrix A and it shows that 1 is an eigenvalue

Question 4 (b)

SOLUTION. $A * v = v$

Is the vector v corresponding to the eigenvalue 1

To solve this we modify the equation $A * v = \lambda v$ -> $A * v - \lambda v = (A - \lambda I) * v = 0$

In this case λ=1 so $(A-I)*v = 0$. We also want this to be non-trivial $(v \neq 0)$. we now take the reduced row echelon form a $(A-I)$ $rref(A-I) = rref\left(\begin{bmatrix}-\frac{1}{2} & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{2} & -\frac{2}{3} & \frac{1}{2}\\ 0 & -\frac{1}{2} & -\frac{5}{6}\end{bmatrix}\right) = \begin{bmatrix}1&0&-\frac{7}{3}\\0&1&-\frac{5}{2}\\0&0&0\end{bmatrix}$

This gives us an eigenvector $v = \begin{bmatrix}\frac{7}{3}\\\frac{5}{2}\\1\end{bmatrix}$

For this application we want the components of the vectors to add to 1. To do this we divide each component by the sum of all the components.

Once we do this we get our final vector $v = \begin{bmatrix}\frac{2}{5}\\\frac{3}{7}\\\frac{6}{35}\end{bmatrix}$

Question 4 (c)

SOLUTION. $AS = \Lambda S$ where Λ is the matrix with the eigenvalues on the diagonal and S is a matrix with eigenvectors as the columns. In this case S will be invertible so we can modify the equation to $S^{-1}AS = \Lambda$ So the matrix S is a 3x3 matrix with columns that equal the eigenvectors. Using the same method to find the eigenvectors as in part (b) we get the following eigenvectors

$\lambda_1 = 1$ has eigenvector $v = \begin{bmatrix}\frac{2}{5}\\\frac{3}{7}\\\frac{6}{35}\end{bmatrix}$

$\lambda_2 = \frac{1}{6}$ has eigenvector $v = \begin{bmatrix}-\frac{1}{2}\\0\\\frac{1}{2}\end{bmatrix}$

$\lambda_3 = -\frac{1}{6}$ has eigenvector $v = \begin{bmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

Therefore we have the matrix S $S = \begin{bmatrix} \frac{2}{5} & -\frac{1}{2} & 0 \\ \frac{3}{7} & 0 & -\frac{1}{2} \\ \frac{6}{35} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Question 4 (d)

SOLUTION. For this problem we want to find $x(t)$. We know that $x(0) = \begin{bmatrix} 0 \\ 0 \\ 350 \end{bmatrix}$. And $x_{(1)} = Ax_{(0)}$. We can generalize to the form $x_{(n)} = Ax_{(n-1)}$ This can further be generalized to $x_{(n)} = A^n x_{(0)}$

From part c we found that $S^{-1}AS = \Lambda$. We can rearrange this to $A = S\Lambda S^{-1}$. So $A^n = (S\Lambda S^{-1})^n = S\Lambda^n S^{-1}$ because S and S^{-1} will cancel each other out.

This means we can easily find $\lim_{n\to\infty} x_n = A^n x_0 = S\Lambda^n S^{-1} x_0$ Recall that two of our eigenvalues were less than zero so when we take this limit they will go to zero.

$\lim_{n\to\infty} x_n = \begin{bmatrix} \frac{2}{5} & -\frac{1}{2} & 0 \\ \frac{3}{7} & 0 & -\frac{1}{2} \\ \frac{6}{35} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{5} & -\frac{1}{2} & 0 \\ \frac{3}{7} & 0 & -\frac{1}{2} \\ \frac{6}{35} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 350 \end{bmatrix}$

Once we do this we find the answer $x_n = \begin{bmatrix} 140 \\ 150 \\ 60 \end{bmatrix}$

Question 5 (a)

SOLUTION. If we find the row reduced echelon form we get that

$rref(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

Take the pivot columns to be our column space

$R(A) = \{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \}$

We can then find the orthonormal basis

let $r_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $r_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

let u_1, u_2 be the orthonormal vectors we wish to find

$u_1 = \frac{r_1}{\|r_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$u_2 = \frac{r_2 - proj_{u_1}(r_1)}{\|r_2 - proj_{e_1}(r_1)\|} = \frac{1}{\sqrt{\frac{3}{2}}} \begin{bmatrix} -0.5 \\ 0.5 \\ 1 \end{bmatrix}$

$A = QR$

where Q is the matrix formed by the orthonormal basis and R is upper triangular with position diagonal entries

$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2\sqrt{3}} \\ 0 & \sqrt{\frac{2}{3}} \end{bmatrix} R$

Solving yields

$R = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{\frac{3}{2}} \end{bmatrix}$

Question 5 (b)

SOLUTION. u_3 is orthogonal to the R(A) since it is a part of the orthonormal basis for $\mathbb{R}^3$
Recall that $R(A)^\perp = N(A^T)$
Since the dim of $N(A^T) = 1$, the basis of $N(A^T)$ is u_3
We can find u_3 either by finding $N(A^T)$ or using gram-schmidt process or with cross products
We solve it by finding $N(A^T)$
$A^T u_3 = 0$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$a = c$
$b = -a$

$N(A^T) = \{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \}$

Normalize the vector

$u_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Alternatively, if you did not recognize that $u_3 \in N(A^T)$ we can still compute u_3 with cross product

$u_3 = u_1 \times u_2 = \begin{vmatrix} i & j & k \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{\sqrt{2}}{2\sqrt{3}} & \frac{\sqrt{2}}{2\sqrt{3}} & \sqrt{\frac{2}{3}} \end{vmatrix}$

$= \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$

And we note our answer from computing the nullspace versus the cross product is a multiple of -1 but either one can be used as u_3 for the orthonormal basis of $\mathbb{R}^3$

Question 5 (c)

SOLUTION. To find $b^\perp$ we take the project of b onto the null space u_3 from part b
$proj_{u_3} = (u_3 \cdot b)u_3$

$= \frac{1}{3}(\begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}) \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

We can find $b^\parallel$ by vector addition
$b - b^\perp = b^\parallel$

$= \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$

$b^\parallel = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$

$b^\perp = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Question 5 (d)

SOLUTION. From part c, we note b is not in R(A) therefore it is not solvable.
We will proceed by solving the least squares solution by QR decomposition
A=QR from part a
$Ax = b$
$QRx = b$
since Q has columns of orthonormal vectors, Q is orthogonal $Q^TQ = I$
$Rx = Q^Tb$
$x = R^{-1}Q^Tb$

$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2\sqrt{3}} \\ 0 & \sqrt{\frac{2}{3}} \end{bmatrix}$

$Q^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{\sqrt{2}}{2\sqrt{3}} & \frac{\sqrt{2}}{2\sqrt{3}} & \sqrt{\frac{2}{3}} \end{bmatrix}$

$R = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{\frac{3}{2}} \end{bmatrix}$

$R^{-1} = \frac{1}{det(R)}adj(R) = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} & -\frac{1}{2} \\ 0 & \sqrt{\frac{3}{2}} \end{bmatrix}$

$x = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} & -\frac{1}{\sqrt{2}} \\ 0 & \sqrt{\frac{3}{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{\sqrt{2}}{2\sqrt{3}} & \frac{\sqrt{2}}{2\sqrt{3}} & \sqrt{\frac{2}{3}} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

$\frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} & -\frac{1}{\sqrt{2}} \\ 0 & \sqrt{\frac{3}{2}} \end{bmatrix} \begin{bmatrix} \frac{5}{\sqrt{2}} \\ \frac{9\sqrt{2}}{2\sqrt{3}} \end{bmatrix}$

The solution to our least square problem is

$x = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Question 6 (a)

SOLUTION. True
Given A as an $n \times m$ and B as an $m \times q$ matrix (to be defined)
then AB is an $n \times q$ matrix. A can have at the most n pivot columns, i.e. rank(A) = n and AB can have at the most n pivot columns, hence rank(AB) $\leq$ rank(A).
Likewise, since R(AB) is also contained in the the subspace R(A) , the easiest way to see this is if given vector y if
$\vec{y} = (AB)\vec{x}$
then
$\vec{y} = A(B\vec{x})$
hence the rank of AB is contained in the rank of A

Question 6 (b)

SOLUTION. The answer is True
Given $A\vec{x} = \vec{b}$ then $\vec{b} \in R(A)$ (i.e. is a vector in the range of A, or a vector b is in the span of A).
And $\vec{y} \in N(A^T)$, meaning that y is in the nullspace of A^T
From the orthogonality of subspaces, we now that $N(A^T) = R(A)^\perp$
Hence, $\vec{y}^T\vec{b} = 0$ or $< y, b >= 0$

Question 6 (c)

SOLUTION. False.
Proof by counter example.
In the case that U = W, then it is not necessary that $U \perp W$ but instead parallel, for example in the $\mathbb{R}^2$ case let $a = [1, 0]$, part of subspace U and $b = [0, 1]$ an element of subspace V, it is clear that $a \cdot b = 0$, hence $U \perp W$, now let $c = [2, 0]$, part of subspace W. $b \cdot c =$ but $c \cdot a \neq 0$, hence U is not perpendicular to W.

Question 6 (d)

SOLUTION. True
Given A is an $n \times n$ matrix and $\vec{v}$ and $\vec{u}$ are elements in $\mathbb{R}^n$. Say that v is an eigenvector for A^T and w is an eigenvector for A. So we have that

Question 7 (a)

SOLUTION. No content found.

Question 7 (b)

SOLUTION. No content found.

Question 7 (c)

SOLUTION. No content found.

Question 7 (d)

SOLUTION. No content found.

Good Luck for your exams!