Full Solutions
MATH100 December 2014

April 5, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

SOLUTION. log is contiuous on $(0, \infty)$ and $\log(1) = 0$.

Question 1 (b)

SOLUTION. $\frac{1}{x} < 0$ for $x < 0$ and approaches $-\infty$ as $x \to 0$. Answer is C.

Question 1 (c)

SOLUTION. $f'(x) = 2x + 4 \cdot \frac{1}{2}x^{-1/2} = 2x + \frac{2}{\sqrt{x}}$, so the slope of the tangent line at $x = 4$ is $f'(4) = 8 + 1 = 9$. Answer is D.

Question 1 (d)

SOLUTION. $\frac{d}{dx}(\cos x) = -\sin x$. Answer is B.

Question 1 (e)

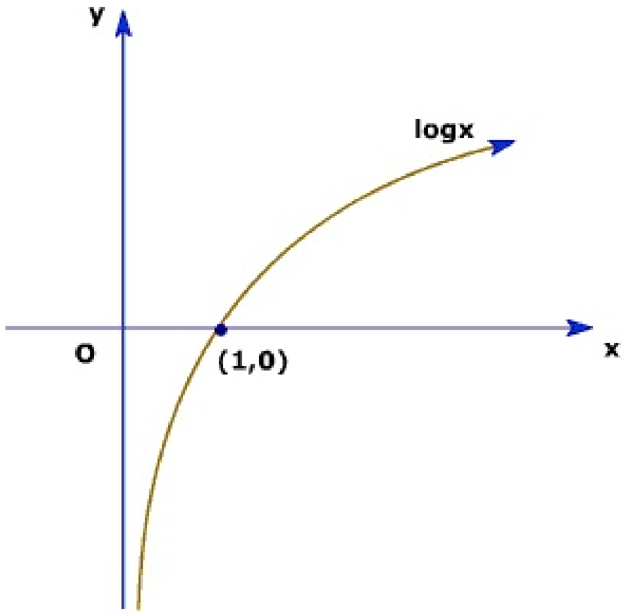
SOLUTION. If $x = \sin y$ then $1 = \cos y \frac{dy}{dx}$ and so $\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$
Answer is C.

Question 1 (f)

SOLUTION. $\sin(-\frac{\pi}{6}) = -\frac{1}{2}$ and $-\frac{\pi}{2} \leq -\frac{\pi}{6} \leq \frac{\pi}{2}$ and so $\arcsin(-\frac{1}{2}) = -\frac{\pi}{6}$. Answer is B.

Question 1 (g)

SOLUTION 1. From looking at the graph of $g(x) = \log(x)$ (picture below), we can see that all lines tangent to $g(x)$ lie above the function.



SOLUTION 2. $g'(x) = \frac{1}{x}$ and so $g''(x) = -\frac{1}{x}^2 < 0$ for all $x > 0$. Thus $g(x)$ is concave down for all $x > 0$ and so the line tangent to $g(x)$ at the point $(a, \log(a))$ lies above the function for all values of $a > 0$. Answer is A.

Question 1 (h)

SOLUTION. $f'(x) = \cos x = 0$ and so $f''(x) = -\sin x = 0$ at $x = 0$. $f''(x) > 0$ for x negative near 0 and $f''(x) < 0$ for x positive near 0. Thus the concavity of f changes at $x = 0$ and $f(x)$ has a point of inflection at $x = 0$. Answer is D.

Question 1 (i)

SOLUTION. $\tan(0) = 0$, so $g(x)$ is undefined at $x = 0$ and thus $g(x)$ has a discontinuity at $x = 0$. Answer is A.

Question 1 (j)

SOLUTION. $h'(x) = \frac{\sin x}{\cos x^2} = 0$ at $x = 0$. $h'(x) > 0$ for $0 < x < \frac{\pi}{2}$ and $h'(x) < 0$ for $-\frac{\pi}{2} < x < 0$, so $h(x)$ has a local minimum at $x = 0$. Answer is C.

Question 2 (a)

SOLUTION. This function is continuous at $t = 0$ so use direct substitution.

$$\begin{aligned}\lim_{t\to 0} \arcsin(\cos t) &= \arcsin(\cos 0) \\ &= \arcsin(1) \\ &= \pi/2\end{aligned}$$

Question 2 (b)

SOLUTION. As $x \to -3^+$ the numerator goes to -1 and the denominator goes to 0 but is positive. Hence the ratio goes to $-\infty$.

Question 2 (c)

SOLUTION. The function $\log x$ is defined on $(0, \infty)$.

- The term $\log x$ is defined when $x > 0$.
- In order for $\log(2 - \log x)$ to be defined we also need $2 - \log x > 0$ (ie. $\log x < 2$). Thus we need $x < e^2$.

Putting these two constraints together gives $0 < x < e^2$, so the domain of $f(x)$ is $(0, e^2)$.

Question 2 (d)

SOLUTION. We can assume $x < 0$ so $\sqrt{x^2} = -x$, as the square root is positive. Then

$$\begin{aligned}
\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1} - x} &= \lim_{x\to-\infty} \frac{x}{\sqrt{x^2}\sqrt{1+1/x^2} - x} \\
&= \lim_{x\to-\infty} \frac{x}{-x\sqrt{1+1/x^2} - x} \\
&= \lim_{x\to-\infty} \frac{x}{x(-\sqrt{1+1/x^2} - 1)} \\
&= \lim_{x\to-\infty} \frac{1}{-\sqrt{1+1/x^2} - 1} \\
&= -\frac{1}{2}
\end{aligned}$$

Question 2 (e)

SOLUTION. This follows by the Squeeze Theorem since $-1 \le \cos(x) \le 1$, so $-\frac{1}{x} \le \frac{\cos(x^2)}{x} \le \frac{1}{x}$ and then taking the limit as $x \to \infty$, we get $\lim_{x\to+\infty} \frac{\cos(x^2)}{x} = 0$ and so the limit is $e^0 = 1$

Question 3 (a)

SOLUTION. Product and chain rules:

$$\begin{aligned}
f'(x) &= e^x \cdot \frac{d}{dt}\cos(\pi x) + \cos(\pi) \cdot \frac{d}{dt} e^x \\
&= -e^x \sin \pi x \cdot \pi + e^x \cos(\pi x) \\
&= e^x(\cos(\pi x) - \pi \sin(\pi x))
\end{aligned}$$

Question 3 (b)

SOLUTION.

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\sin(\log x)} \cdot \frac{d}{dx}\sin(\log x) \\ &= \frac{1}{\sin(\log x)} \cdot \cos(\log x) \cdot \frac{1}{x} \\ &= \frac{\cos(\log x)}{x\sin(\log x)}\end{aligned}$$

Question 3 (c)

SOLUTION. Convert the base to base e.

$$\begin{aligned} y &= e^{\left((\log(x))^2\right)} \\ \frac{dy}{dx} &= 2\log x \cdot \frac{1}{x} \cdot e^{(\log(x))^2} \\ &= 2\log(x) \cdot \frac{1}{x} \cdot x^{\log(x)} \\ &= 2\log(x) \cdot x^{\log(x)-1}\end{aligned}$$

Question 3 (d)

SOLUTION. We apply L'Hôpital's Rule since we deal with an indeterminate form "0/0". L'Hôpital's Rule states $\lim_{t\to a} \frac{f(t)}{g(t)} = \lim_{t\to a} \frac{f'(t)}{g'(t)}$ Using $f(t) = \sqrt{t+1} - e^t$ and $g(t) = t$, we obtain $\lim_{t\to 0} \frac{\frac{1}{2\sqrt{t+a}} - e^t}{1} = -\frac{1}{2}$

Question 3 (e)

SOLUTION. We have $V = \frac{4\pi r^3}{3}$ and so $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$. Thus $\frac{dr}{dt} = \frac{\frac{dV}{dt}}{4\pi r^2} = \frac{100}{4\pi\cdot(20)^2} = \frac{1}{16\pi}$ cm/s

Question 3 (f)

SOLUTION. The function arcsin has domain $[-1,1]$ and range $[\pi/2, \pi/2]$, so our answer must lie in this range.

- We know that $\sin(x) = \sin(x - 2\pi)$; so $\sin(9\pi/11) = \sin(31\pi/11)$.
- Also we know that $\sin(x) = \sin(\pi - x)$ and so, $\sin(9\pi/11) = \sin(2\pi/11)$.

Since $-\pi/2 \leq 2\pi/11 \leq \pi/2$, we conclude that

$$\begin{aligned}\arcsin\left(\sin(\frac{31\pi}{11})\right) &= \arcsin\left(\sin(\frac{9\pi}{11})\right) \\ &= \frac{2\pi}{11}\end{aligned}$$

Question 4 (a)

SOLUTION. If we try substitution we get "0/0" so use L'Hôpital's Rule $\lim_{x\to 0} \frac{\cos x - e^{x^2}}{x^2} = \lim_{x\to 0} \frac{-\sin x - 2xe^{x^2}}{2x}$ Still get "0/0", so use L'Hôpital's Rule again

$$\begin{aligned} &= \lim_{x\to 0} \frac{-\cos x - (4x^2e^{x^2} + 2e^{x^2})}{2} \\ &= -\frac{3}{2} \end{aligned}$$

Question 4 (b)

SOLUTION. Use the quotient rule.

$$\begin{aligned} g'(x) &= \frac{x^2 f'(x) - 2xf(x)}{x^4} \\ g'(1) &= \frac{f'(1) - 2f(1)}{1} \\ 4 &= 3 - 2f(1) \\ f(1) &= -\frac{1}{2} \end{aligned}$$

Question 4 (c)

SOLUTION. The general form of a linear approximation at $x = a$ is $L(x) = f(a) + (x - a)f'(a)$ We need $f(x)$ and its derivative, both at $x = 0$:

$$\begin{aligned} f(x) &= \frac{e^x - e^{-x}}{e^x + e^{-x}} \\ f(0) &= 0 \\ f'(x) &= \frac{(e^x - e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2} \\ &= \frac{(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 - e^{-2x})}{(e^x + e^{-x})^2} \\ &= \frac{4}{(e^x + e^{-x})^2} \\ f'(0) &= 1 \end{aligned}$$

Hence $L(x) = x$ and $L(1/10) = 1/10$.

Question 5 (a)

SOLUTION. We have $f'(x) = a\cos(ax) - 2x + 2$.We need to impose the condition that $f'(0) = 0$ which yields

$$
\begin{aligned}
a \cdot \cos(a \cdot 0) - 2 \cdot 0 + 2 &= 0 && \text{and so} \\
a + 2 &= 0
\end{aligned}
$$

Hence $a = -2$.

Question 5 (b)

SOLUTION. We find the critical points of $f(x)$ on $[0, 3]$ by solving $f'(x) = 0$ which yields $3x^2 - 6 = 0$. Hence $x = \pm\sqrt{2}$ and only $x = \sqrt{2}$ lies in the interval we want. Now check the ends of the interval and the critical number:

$$
\begin{aligned}
f(0) &= 4 \\
f(3) &= 27 - 18 + 4 = 13 \\
f(\sqrt{2}) &= 2\sqrt{2} - 6\sqrt{2} + 4 \\
&= 4 - 4\sqrt{2}
\end{aligned}
$$

Since $\sqrt{2} > 1$, the last of these is negative, while the other two are positive. So the global minimum is $4 - 4\sqrt{2}$ with coordinates $(x, y) = (\sqrt{2}, 4 - 4\sqrt{2})$.

Question 5 (c)

SOLUTION. We have

$$f'(x) = 3x^{\frac{1}{2}} - \frac{5}{x^2}$$

and so

$$f(x) = 2x^{\frac{3}{2}} + \frac{5}{x} + C$$

for some constant C

We can find C using the information that $f(1) = -4$. Hence

$$f(x) = 2 \cdot x^{\frac{3}{2}} + \frac{5}{x} + C$$

means that

$$
\begin{aligned}
f(1) &= -4 = 2 + 5 + C \\
-11 &= C
\end{aligned}
$$

which finally gives $f(x) = 2x^{\frac{3}{2}} + \frac{5}{x} - 11$.

Question 6 (a)

SOLUTION.

$$
\begin{aligned}
g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{x+h}{2+3x+3h} - \frac{x}{2+3x}\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{(x+h)(2+3x)-x(2+3x+3h)}{(2+3x)(2+3x+3h)}\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{2x+2h+3x^2+3xh-2x-3x^2-3xh}{(2+3x)(2+3x+3h)}\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{2h}{(2+3x)(2+3x+3h)}\right] \\
&= \lim_{h\to 0} \frac{2}{(2+3x)(2+3x+3h)} \\
&= \frac{2}{(2+3x)^2}
\end{aligned}
$$

Question 6 (b)

SOLUTION. Multiply by the conjugate and then simplify:

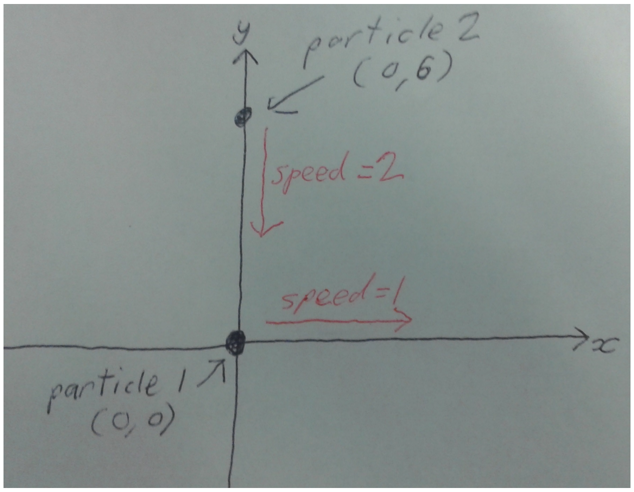
$$
\begin{aligned}
\sqrt{x^2+2x}-\sqrt{x^2-2x} &= (\sqrt{x^2+2x}-\sqrt{x^2-2x})\cdot\frac{\sqrt{x^2+2x}+\sqrt{x^2-2x}}{\sqrt{x^2+2x}+\sqrt{x^2-2x}} \\
&= \frac{x^2+2x-(x^2-2x)}{\sqrt{x^2+2x}+\sqrt{x^2-2x}} \\
&= \frac{4x}{\sqrt{x^2+2x}+\sqrt{x^2-2x}} \\
&= \frac{4x}{x(\sqrt{1+2/x}+\sqrt{1-2/x})} \\
&= \frac{4}{\sqrt{1+2/x}+\sqrt{1-2/x}}
\end{aligned}
$$

assuming $x > 0$. Hence

$$
\begin{aligned}
\lim_{x\to+\infty}(\sqrt{x^2+2x}-\sqrt{x^2-2x}) &= \lim_{x\to+\infty}\frac{4}{\sqrt{1+2/x}+\sqrt{1-2/x}} \\
&= \frac{4}{2} = 2
\end{aligned}
$$

Question 7 (a)

SOLUTION. Drawn is the starting position and velocity of the particles.



Let t be time in seconds. The position of the first particle is given by $(0, y)$ where $y = 6 - 2t$ The position of the second particle is $(x, 0)$ with $x = t$ The squared-distance between the particles is therefore

$$\begin{aligned} d^2 &= (\text{difference in } x)^2 + (\text{difference in } y)^2 \\ &= t^2 + (6-2t)^2 \\ &= 5t^2 - 24t + 36 \\ d &= \sqrt{5t^2 - 24t + 36} \end{aligned}$$

Question 7 (b)

Solution 1. So we need to minimise the distance d for $t \in [0, 3]$. We instead minimise d^2, which is equivalent, using the closed interval method:

$$d(0)^2 = 36 \qquad\qquad d(3)^2 = 45 - 72 + 36 = 9$$

The critical numbers are then given by $\frac{d}{dt}(d^2) = 10t - 24 = 0$ So the only critical number is $t = 12/5$. Then

$$\begin{aligned} d(12/5)^2 &= \frac{144}{5} - 24 \cdot \frac{12}{5} + 36 \\ &= \frac{144 - 2 \times 144 + 180}{5} \\ &= \frac{36}{5} < 9 \end{aligned}$$

By the closed interval method, the minimum distance is reached when $t = \frac{12}{5}$ seconds.

Solution 2. So we need to minimise the distance d for $t \in [0, 3]$. We do this using the closed interval method:

$$d(0) = \sqrt{36} = 6 \qquad\qquad d(3) = \sqrt{45 - 72 + 36} = \sqrt{9} = 3$$

The critical numbers are then given by $\frac{d}{dt}(d) = \frac{1}{2}(5t^2 - 24t + 36)^{-1/2}(10t - 24) = \frac{10t-24}{2\sqrt{5t^2-24t+36}} = 0$ So the only critical number is when $10t - 24 = 0$. This is at $t = 12/5$. Then

$$\begin{aligned} d(12/5) &= \sqrt{\frac{144}{5} - 24 \cdot \frac{12}{5} + 36} \\ &= \sqrt{\frac{144 - 2 \times 144 + 180}{5}} \\ &= \sqrt{\frac{36}{5}} < \sqrt{\frac{36}{4}} = 3 \end{aligned}$$

By the closed interval method the minimum distance is reached when $t = \frac{12}{5}$ seconds.

Question 8 (a)

SOLUTION. The domain of $\log(x)$ is $x > 0$, while the domain of $\sqrt{x}$ is $x \geq 0$. So for their ratio to be defined, we need $x > 0$ and the domain of f is $(0, \infty)$.

Question 8 (b)

SOLUTION. Since the function is defined only for $x > 0$ there is no y-intercept. Solving $f(x) = 0$ gives $x = 1$ so the only intercept is an x-intercept at $(1, 0)$.

Question 8 (c)

SOLUTION. Since the function is only defined for $x > 0$, we need only consider $\lim\limits_{x\to+\infty} f(x)$. Since $\lim_{x\to+\infty} \frac{\log x}{\sqrt{x}} = \frac{\infty}{\infty}$ we use L'Hospital's rule to obtain

$$\begin{aligned} \lim_{x\to+\infty} \frac{\log x}{\sqrt{x}} &= \lim_{x\to+\infty} \frac{x^{-1}}{x^{-1/2}/2} \\ &= \lim_{x\to+\infty} \frac{2}{\sqrt{x}} = 0 \end{aligned}$$

So the only horizontal asymptote is $y = 0$.

Question 8 (d)

SOLUTION. Since the function is defined for all $x > 0$ the only possible vertical asymptote is as $x \to 0^+$. Now, as $x \to 0^+$ the numerator becomes large and negative, while the denominator becomes small and positive. Hence $\lim_{x\to 0^+} \frac{\log x}{\sqrt{x}} = -\infty$ This asymptote at $x = 0$ is the only vertical asymptote.

Question 8 (e)

SOLUTION. First find the derivative

$$\begin{aligned} f'(x) &= \frac{\sqrt{x} \cdot \frac{1}{x} - \frac{1}{2\sqrt{x}} \log(x)}{x} \\ &= \frac{2 - \log(x)}{2x^{3/2}} \end{aligned}$$

So the derivative exists everywhere on the domain. Any local maximum or minimum occur when $f'(x) = 0$. This is zero when $2 - \log(x) = 0$, that is, when $x = e^2$.

$$(0, e^2) : f'(x) = \frac{pos}{pos} = pos \qquad \text{increasing}$$
$$(e^2, \infty) : f'(x) = \frac{neg}{pos} = neg \qquad \text{decreasing}$$

Hence $x = e^2$ gives the only local maximum at the coordinates $(e^2, f(e^2)) = (e^2, \frac{2}{e})$ and there is no local minimum.

Question 8 (f)

Solution. The second derivative is defined everywhere on the domain. The only zero occurs at

$$3\log(x) = 8$$
$$x = e^{8/3}$$

So we examine:

$$(0, e^{8/3}) : f''(x) = \frac{neg}{pos} = neg \qquad \text{concave down}$$
$$(e^{8/3}, \infty) : f''(x) = \frac{pos}{pos} = pos \qquad \text{concave up}$$

So, the only inflection point is at $(e^{\frac{8}{3}}, f(e^{\frac{8}{3}})) = (e^{\frac{8}{3}}, \frac{8}{3e^{4/3}})$.

Question 9 (a)

Solution. Set $q = 0$ then the equation becomes $x^4 + 4x = 0$. Factoring gives $x^3(x + 4) = 0$. Hence $x = -4, x = 0$ are two solutions. Thus $q = 0$ satisfies the equation having at least two solutions.

Question 9 (b)

Solution.

- Let $f(x) = x^4 + 4x + q$ and suppose that the equation has 3 solutions $a < b < c$. ie $f(a) = f(b) = f(c) = 0$.
- Now since f is a polynomial it is continuous and differentiable everywhere, so we can apply the Mean Value Theorem or Rolle′s theorem.
- By Rolle′s theorem there must be points p, q with p between a and b and q between b and c so that $f'(p) = f'(q) = 0$ with $p < q$.
- But $f'(x) = 4x^3 + 4 = 4(x^3 + 1)$ which only has a single real zero at $x = -1$.
- Hence the derivative cannot have 2 distinct zeros, so we have reached a contradiction and so f cannot have 3 solutions.

Question 10

Solution. In order to be continuous at $x = 0$ we need $\lim_{x\to 0^-} f(x) = f(0) = \lim_{x\to 0^+} f(x)$

- $f(0) = 0$

- $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x = 0$

So it remains to determine which a gives $\lim_{x\to 0^+} x^a \sin\left(\frac{1}{x}\right) = 0$

- If $a = 0$ then $\lim_{x\to 0^+} x^a \sin\left(\frac{1}{x}\right) = \lim_{x\to 0^+} \sin\left(\frac{1}{x}\right)$ which simply oscillates and so does not exist.
- If $a < 0$ then x^a diverges to $+\infty$ as $x \to 0^+$ and so $f(x)$ does not converge (it gets larger and larger and oscillates wildly).
- Finally if $a > 0$ then we can use the Squeeze Theorem. Since $-1 \le \sin\left(\frac{1}{x}\right) \le 1 - x^a \le f(x) \le x^a$ And as $x \to 0^+$ we know that $\lim_{x\to 0+} -x^a = \lim_{x\to 0+} x^a = 0$

So by the Squeeze Theorem, $f(x) \to 0$ as $x \to 0^+$. Thus $f(x)$ is continuous at $x = 0$ when $a > 0$.

Question 11 (a)

Solution. Since the function is differentiable, we can use the closed interval method.

$$f(1) = e - e^{-1}$$
$$f(-1) = -(e - e^{-1})$$

Differentiating gives $\frac{d}{dx} f(x) = e^x + e^{-x}$ which is nowhere near zero since $e^x > 0$ for all x Thus the maximum value is $e - e^{-1}$ and the minimum value is $-(e - e^{-1})$.

Question 11 (b)

Solution. We need the third derivative of $g(x)$:

$$\begin{aligned} g(x) &= e^x + e^{-x} \\ g'(x) &= e^x - e^{-x} \\ g''(x) &= e^x + e^{-x} \\ g'''(x) &= e^x - e^{-x} = f(x) \end{aligned}$$

Then the absolute error is given by $R_2(x) = \frac{1}{3!} f(c) x^3$ for c between 0 and x. Hence

$$R_2(1) = \frac{f(c)}{6}$$

for c between 0 and 1. From the previous part we know this is maximised when $c = 1$ and gives $R_2(1) \le \frac{e-e^{-1}}{6}$. And since $2 < e < 3$ we have $\frac{1}{3} < \frac{1}{e} < \frac{1}{2}$ and so $R_2(1) \le \frac{e-e^{-1}}{6} \le \frac{3-\frac{1}{3}}{6} = \frac{8}{18} = \frac{4}{9}$

Good Luck for your exams!