Full Solutions
MATH220 April 2005

April 4, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

Question 1

SOLUTION. The Intermediate Value Theorem states:

For any real numbers $a < b$, and for any function
$f : [a, b] \to \mathbb{R}$,
if f is continuous on the interval $[a,b\ ]$, then for every real number y between $f(a)$ and $f(b)$, there exists a real number c in the interval $[a,b\ ]$ such that $f(c)=y$.

We can usually spot implications in theorems by the use of the word *then.* So here we have that the statement A is

$$A$$

$=$ *f is a continuous function on the interval $[a,b\ ]$, with $a < b$.* and the conclusion of the theorem is statement B

$$B$$

$=$ *for all real numbers y between $f(a)$ and $f(b)$ there exists a real number c in the interval [a, b] such that $f(c) = y$.*
And so the Intermediate Value Theorem is simply the statement

$$A \Rightarrow B$$

And its negation is the statement

$$A \wedge \neg B$$

We'll need the negation of statement B for this:

$$\neg B$$

$=$ *there exists a real number y between $f(a)$ and $f(b)$ such that for all real number c in the interval [a, b] we have $f(c) \neq y$.*
And so the negation is

There exists a continuous function f on the interval $[a,b\ ]$, with $a < b$ such that there exists a real number y between f(a) and f(b) such that for all real number c in the interval [a, b] we have f(c) $\neq$ y".

Question 2

SOLUTION. The second statement should appear wrong fairly immediately since it talking about the existence of *the smallest real number* which cannot exist. Let's prove this:

The statement says that for *any* real number z it should be true that $y < z$, but if I pick z to be exactly that mysterious number y I should have $y < y$ which is clearly impossible, so the statement is false.

Now we can prove that the first statement is true:

That statement says that if any real number w is picked, then one can find a real number x that is larger than w. Well, sure enough, if one always choses $x = w + 1$, then clearly it is always going to be larger than the w that was picked in the first place.

Question 3 (a)

SOLUTION. Two sets A and B are said to have the same cardinality if there exists a bijection from one set to the other.
More precisely, that means there must exist at least one function $f\colon A \to B$ such that the function is injective and surjective; that is, for each element a of the set A there exists a unique element b in the set B such that $f(a) = b$ (that condition is the injectivity) AND for each element b of the B there exists a unique element a in the set A such that $f(a) = b$ (and this is the condition for surjectivity).

Question 3 (b)

SOLUTION. A set C is countable if it has the same cardinality as the set $\mathbb{N}$ of natural numbers. In other words, there exists a bijective function $f : C \to \mathbb{N}$. (Equivalently, there exists a bijective function $g : \mathbb{N} \to C$.)

Question 3 (c)

SOLUTION. Cantor's Diagonalization Argument is used to prove that the set of all real numbers is uncountable.
More precisely, it says that if you claim that the set of all real numbers is actually countable, that is, you have a bijection between all real number and the natural numbers, then he would be able to exhibit you a real number (using the diagonalization argument) that is **not** listed by your bijection. This means it is a contradiction to assume that such a bijection could exist and hence the real numbers are uncountable.

Question 4

SOLUTION. The first series

$$\sum_{k=1}^{\infty} \left(x - \frac{1}{2}\right)^k$$

converges for all values of x such that

$$-1 < x - \frac{1}{2} < 1$$

which we can rewrite as

$$-1 + \frac{1}{2} < x < 1 + \frac{1}{2}$$

that is

$$-\frac{1}{2} < x < \frac{3}{2}$$

The second series

$$\sum_{k=1}^{\infty}\left(x+\frac{1}{2}\right)^{k}$$

converges for all values of x such that

$$-1 < x + \frac{1}{2} < 1$$

which we can rewrite as

$$-1 - \frac{1}{2} < x < 1 - \frac{1}{2}$$

that is

$$-\frac{3}{2} < x < \frac{1}{2}$$

So values of x in the interval (-1/2,3/2) will make the first series converge and values in the interval (-3/2,1/2) will make the second series converge. Hence the values of x which make *exactly* one of the series convergent cannot be in the intersection of these two intervals. We can now conclude that that the set S is

$$S = \left(-\frac{3}{2}, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \frac{3}{2}\right)$$

Question 5

SOLUTION. We use induction.
In the base case when $n = 1$ we have $|a_1| \leq |a_1|$ so the statement is true.
Induction step: we assume that $|\sum_{i-1}^{n} a_i| \leq \sum_{i=1}^{n} |a_i|$ holds for some (fixed) value $n \in N$. Then it also holds for $n + 1$:

$$\begin{aligned}
|\sum_{i-1}^{n+1} a_i| &= |\sum_{i-1}^{n} a_i + a_{n+1}| \\
&\leq |\sum_{i-1}^{n} a_i| + |a_{n+1}| \\
&\leq \sum_{i=1}^{n} |a_i| + |a_{n+1}| \\
&= \sum_{i=1}^{n+1} |a_i|
\end{aligned}$$

Where the first inequality is true by the triangle inequality and the second by induction hypothesis. This finishes the induction step and so the result is true for all $n \in N$ by induction.

Question 6

SOLUTION. Let $x \in \mathbb{N}$.
Case 1

$$x = 3k$$

for some $k \in \mathbb{N}$.
In this case we write x as a sum of k 3's.
Case 2

$$x = 3k + 2$$

for some $k \in \mathbb{N}$ with $k \geq 1$.
In this case we write $x = 3k + 2 = 3(k - 1) + 5$ as the sum of $k - 1$ 3's and one 5.
Case 3

$$x = 3k + 1$$

for some $k \in \mathbb{N}$ with $k \geq 3$.
In this case we write $x = 3(k - 3) + 10$ as the sum of $k - 3$ 3's and two 5's.
This shows that all numbers other than 1,2,4,7 belong to T. It is easy to see that 1,2,4,7 are not in T.

Question 7

SOLUTION. Using the ratio test seems appropriate here. If we denote the general term of the series by

$$a_k = \frac{k(k+1)}{2^k} \quad \text{for all } k \geq 1$$

Then the ratio test asks us to look at

$$\begin{aligned}\frac{a_{k+1}}{a_k} &= \frac{(k+1)(k+2)2^k}{k(k+1)2^{k+1}} \\ &= \frac{k+2}{2k}\end{aligned}$$

(Note: we removed the absolute value since the terms are clearly all positive). And so

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{k+2}{2k} = \frac{1}{2}$$

Since this is a number less than one, the ratio test guarantees that the series converges absolutely.

Question 8

SOLUTION. We can show that the sequence b_n is Cauchy. Let $\epsilon > 0$, then if n and m are larger than $1/(2\epsilon)$ (that's the value of N, you can see from the argument below how we might have guessed it), then if $m \geq n$, we have that

$$|b_m - b_n| < \frac{1}{m+n} \leq \frac{1}{2n} < \epsilon$$

And since Cauchy sequences of real numbers are always convergent, we know that the limit of this sequence exists.

Question 9

SOLUTION. Let $\varepsilon > 0$, we want to show that for some positive integer N_ε, we have that

$$\left|\left(-\frac{2}{3}\right)^n\right| < \varepsilon \quad \text{if } n \geq N_\varepsilon$$

First, we can get rid of the absolute value since

$$\left|\left(-\frac{2}{3}\right)^n\right| = \left(\frac{2}{3}\right)^n$$

and so we wonder if for some value of n large enough, we will have

$$\left(\frac{2}{3}\right)^n < \varepsilon$$

for the value of ε that we fixed (and that we think of as being a very small number). Since we are interested in the value of n, we take the logarithm on each side and obtain

$$\underbrace{\ln\left[\left(\frac{2}{3}\right)^n\right]}_{=n\ln(2/3)} < \ln(\varepsilon)$$

Now since 2/3 is a number smaller than 1, its logarithm will be a negative number, so when we divide in the above by $\ln(2/3)$ we have to flip the inequality sign, we obtain

$$n > \frac{\ln(\varepsilon)}{\ln(2/3)}$$

Note that since ε is also a very small number, its logarithm is also negative and hence the fraction on the right-hand side is a positive number (as expected). We can thus choose N_ε to be the nearest integer larger than the fraction, which we can write as

$$N_\varepsilon = \left\lceil \frac{\ln(\varepsilon)}{\ln(2/3)} \right\rceil$$

which concludes our proof.

Note

It is not a bad idea to see that ε proofs allow you to actually compute something about how quickly the sequence will converge to its limit. In this case for example, if we choose $\varepsilon = 0.001$ then the above proof gives us the value

$$N_{0.001} = \left\lceil \frac{\ln(0.001)}{\ln(2/3)} \right\rceil = \lceil 17.0366 \rceil = 18$$

So for any $n \geq 18$, the value of $(-2/3)^n$ should be less than 0.001, indeed we have

$$\left|\left(-\frac{2}{3}\right)^{17}\right| = 0.00101 \quad \text{and} \quad \left|\left(-\frac{2}{3}\right)^{18}\right| = 0.0006766$$

Question 10

SOLUTION. Since f is bounded there exists a constant M such that

$$|f(n)| < M \quad \text{for all } n \in \mathbb{N}$$

We will show that $\{f(n)a_n\}$ converges to zero. Let $\epsilon > 0$. Since $\{a_n\}$ converges to zero there exists a positive integer N_ε such that

$$|a_n| < \frac{\epsilon}{M} \quad \text{for all } n > N_\epsilon$$

Hence

$$|f(n)a_n| = |f(n)||a_n| < M\frac{\epsilon}{M} = \epsilon \quad \text{for all } n > N_\epsilon$$

This shows that $\{f(n)a_n\}$ converges to zero.

Question 11

SOLUTION. The condition

$$x^2 < 5$$

is equivalent to the conditions

$$-\sqrt{5} < x < \sqrt{5}$$

This means, we can rewrite the set M as being

$$M = \mathbb{Q} \cap (-\sqrt{5}, \sqrt{5})$$

The infimum of a set being the largest real number that is smaller or equal to all the numbers of the set, it seems to be a good idea to think that - is a good candidate for the infimum of the set M.
Since - is clearly smaller than all the numbers of the set M, we only need to show that it is the largest real number with that property.
Assume that there is a number m with that property as well, that is strictly larger than -, that is

$$m \leq x \quad \text{for all } x \in M \quad \text{and} \quad m > -\sqrt{5}$$

We'll simply show that such a m cannot exist and hence - is the infimum of the set M.
Indeed, if m is strictly larger than - than it means that the interval

$$(-\sqrt{5}, m)$$

is not empty. Since non-empty interval of real numbers contain infinitely many numbers, we can find two distinct numbers in that interval x_1 and x_2, that is

$$-\sqrt{5} < x_1 < x_2 < m$$

Now a property of the rational numbers is that you can always find a rational number between any two distinct real numbers. That means there must exist a rational number q between x_1 and x_2, so we have

$$-\sqrt{5} < x_1 < q < x_2 < m < \sqrt{5}$$

but in particular that means that q is in the interval (-,) and since it is rational, q is an element of the set M. But by construction, $q < m$ which contradicts the assumption that m could be a lower bound of the set M. This means that such a number m cannot exist and hence the infimum of the set M has to be -.

Good Luck for your exams!