Full Solutions
MATH152 April 2013

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How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question A 01

Solution. By the cross product formula,

$$\begin{aligned}
a \times b &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -3 & 3 \\ 2 & -3 & -2 \end{vmatrix} \\
&= \begin{vmatrix} -3 & 3 \\ -3 & -2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -3 & 3 \\ 2 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -3 & -3 \\ 2 & -3 \end{vmatrix} \mathbf{k} \\
&= 15\mathbf{i} - 0\mathbf{j} + 15\mathbf{k} \\
&= \begin{bmatrix} 15 \\ 0 \\ 15 \end{bmatrix}
\end{aligned}$$

Question A 02

Solution. We will use the formula,

$$\mathrm{Proj}_c a = \frac{a \cdot c}{|c|^2} c.$$

Note that

$$a \cdot c = -9 + 12 + 36 = 39$$

and

$$|c|^2 = c \cdot c = 9 + 16 + 144 = 169.$$

Hence,

$$\mathrm{Proj}_c a = \frac{a \cdot c}{|c|^2} c = \frac{39}{169} c = \frac{3}{13} \begin{bmatrix} 3 \\ -4 \\ 12 \end{bmatrix}.$$

Question A 03

Solution. a and d are orthogonal whenever their dot product is equal to zero. That is,

$$\begin{aligned}
0 = a \cdot d &= \begin{bmatrix} -3 \\ -3 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ k \\ 2k+1 \end{bmatrix} \\
&= (-3)(-1) + (-3)(k) + (3)(2k+1) \\
&= 3 - 3k + 6k + 3 \\
&= 6 + 3k \\
&= 3(2 + k).
\end{aligned}$$

Therefore, $k = -2$ is the value for which a and d are orthogonal.

Question A 04

SOLUTION. __NOTOC__ We reduce the matrix to row-echelon form. Subtracting the first row from the last row transforms

$$M = \begin{bmatrix} 1 & 7 & 3 \\ 0 & p & 5 \\ 1 & pq+7 & 8 \end{bmatrix}$$

into

$$\begin{bmatrix} 1 & 7 & 3 \\ 0 & p & 5 \\ 0 & pq & 5 \end{bmatrix}.$$

Now, subtracting the second row from the third row yields

$$\begin{bmatrix} 1 & 7 & 3 \\ 0 & p & 5 \\ 0 & pq-p & 0 \end{bmatrix}$$

This matrix either has rank 2 or rank 3. If *pq - p=p(q - 1) = 0*, then M has rank 2; otherwise, M has rank 3. In other words, M has rank 2 exactly when *p=0* or *q=1*.

Question A 05

SOLUTION. As calculated in A04, we take $q = 1, p = 0$,

$$M = \begin{pmatrix} 1 & 7 & 3 \\ 0 & 0 & 5 \\ 1 & 7 & 8 \end{pmatrix}.$$

Since M has rank 2, the columns in M must be linear dependent. We see that the second column is 7 times the first column. Hence, the second column is redundant.
The first and the third column of M are linearly independent since the third one has a 5 as second entry, where the first one has a 0. Therefore, there is no possible scalar such that columns 1 and 3 are multiples.

$$Mx = b$$

is only solvable, if b is a linear combination of the columns of M.
We want to find a vector, which is not a linear combination of the two linear independent columns. Once possible choice is to find a vector which orthogonal (and hence linear independent) to the first and third column by performing the cross product,

$$b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix} = \begin{pmatrix} -5 \\ -5 \\ 5 \end{pmatrix}.$$

Note that there are several choices that could be made and this method is just a quick and effective way for finding a suitable vector.

Question A 06

SOLUTION. Suppose we have three vectors over $\mathbb{R}^2$ say
$\begin{bmatrix} a \\ b \end{bmatrix}$ $\begin{bmatrix} c \\ d \end{bmatrix}$ $\begin{bmatrix} e \\ f \end{bmatrix}$
for fixed a,b,c,d,e,f. These vectors are linearly independent if we can write
$x \begin{bmatrix} a \\ b \end{bmatrix} + y \begin{bmatrix} c \\ d \end{bmatrix} + z \begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
and have x=y=z=0 be the only solution. We can write this as a system of linear equations given by:

$$\begin{aligned} xa + yc + ze &= 0 \\ xb + yd + zf &= 0 \end{aligned}$$

This is a system of two equations and three unknowns (given by x,y, and z). This always has a nontrivial solution (i.e. for any value of z, I can always find an x and y that solve this) and therefore x=y=z=0 is not the only solution. The answer then is that 3 vectors are always linearly dependent in $\mathbb{R}^2$

Question A 07

SOLUTION. **No** it is not possible for two planes to intersect in two linearly independent lines.
Two planes in $\mathbb{R}^3$ intersect in either a line, a plane (which is generated by two linearly independent vectors) or not at all. If they intersect in a plane or not at all, then their normal vectors are parallel. Otherwise they intersect in a line which cannot contain two parallel vectors.
For a more technical explanation, consider a point on both planes. Let the first plane be given by

$$n_1x + n_2y + n_3z = a$$

and the second plane by

$$m_1x + m_2y + m_3z = b.$$

The normal vectors are not parallel and so $< n_1, n_2, n_3 >$ and $< m_1, m_2, m_3 >$ are linearly independent. In a linear system, we can write this as

$$\begin{bmatrix} n_1 & n_2 & n_3 \\ m_1 & m_2 & m_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

Here we have two equations with three unknowns and this tells us that there is at least one free variable. Since the normal vectors are linearly independent, then there is only one free variable. Geometrically, one free variable corresponds to a line in space and so there is only one line which could contain the points of intersection of the two planes.

Question A 08

SOLUTION 1. Since the matrix is in upper triangular form, we can see immediately that it is of full rank, as none of the diagonal entries are 0. Then the system has exactly 1 solution. Clearly, the zero vector is a solution (to any homogenous system), so the only solution is the trivial solution

$$x = 0.$$

SOLUTION 2. We can work out the solution coordinate by coordinate using the augmented matrix:

$$\left[\begin{array}{cccccc|c} 1 & 7 & 7 & \dots & 7 & & 0 \\ 0 & 2 & 7 & \dots & 7 & & 0 \\ 0 & 0 & 3 & \dots & 7 & & 0 \\ \vdots & \vdots & & \ddots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 13 & & 0 \end{array}\right]$$

The last row (13th) gives $13x_{13} = 0$, so $x_{13} = 0$. The 12th row gives $12x_{12} + 7x_{13} = 0$, but $x_{13} = 0$, so $x_{12} = 0$. Similarly, we calculate $x_1 = x_2 = \ldots = x_{10} = 0$, so the only solution is $x = 0$.

Question A 09

SOLUTION. Using the hint, we have that

$A^{-1} = \frac{1}{(1)(8)-(3)(3)} \begin{bmatrix} 8 & -3 \\ -3 & 1 \end{bmatrix} = -1 \begin{bmatrix} 8 & -3 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -8 & 3 \\ 3 & -1 \end{bmatrix}$

Question A 10

SOLUTION. Since A is **2**x2 matrix and B is a 2x**3** matrix, the resulting matrix AB will be **2**x**3**. Performing the matrix multiplication, we get

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 3 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ -3 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} (1)(1)+(3)(-3) & (1)(0)+(3)(1) & (1)(-1)+(3)(0) \\ (3)(1)+(8)(-3) & (3)(0)+(8)(1) & (3)(-1)+(8)(0) \end{bmatrix} \\ &= \begin{bmatrix} -8 & 3 & -1 \\ -21 & 8 & -3 \end{bmatrix} \end{aligned}$$

Question A 11

SOLUTION 1. Let $C = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$. Then multiplying out the required matrix gives

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = BC &= \begin{bmatrix} 1 & 0 & -1 \\ -3 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} \\ &= \begin{bmatrix} (1)(a)+(0)(b)+(-1)(c) & (1)(d)+(0)(e)+(-1)(f) \\ (-3)(a)+(1)(b)+(0)(c) & (-3)(d)+(1)(e)+(0)(f) \end{bmatrix} \\ &= \begin{bmatrix} a-c & d-f \\ -3a+b & -3d+e \end{bmatrix} \end{aligned}$$

This gives the equations

$$\begin{aligned} a - c &= 1 \\ d - f &= 0 \\ -3a + b &= 0 \\ -3d + e &= 1 \end{aligned}$$

Setting simple values like $a = 1$ and $e = 1$ gives us that $b = 3, c = 0, d = 0, f = 0$. This is a valid solution and thus such a matrix C exists and is given by

$C = \begin{bmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 0 \end{bmatrix}$

Solution 2. Proceeding as in hint 3, we find the inverse of

$\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

which is given by

$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

Thus, a matrix C that will work is

$\begin{bmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 0 \end{bmatrix}$

Since the zeroes at the bottom will cancel any terms in the last column of B once you perform the matrix multiplication BC.

Question A 12

Solution. We proceed as in the hint. Let

$D = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$

Then we have

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = DB &= \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ -3 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} (a)(1) + (b)(-3) & (a)(0) + (b)(1) & (a)(-1) + (b)(0) \\ (c)(1) + (d)(-3) & (c)(0) + (d)(1) & (c)(-1) + (d)(0) \\ (e)(1) + (f)(-3) & (e)(0) + (f)(1) & (e)(-1) + (f)(0) \end{bmatrix} \\ &= \begin{bmatrix} a - 3b & b & -a \\ c - 3d & d & -c \\ e - 3f & f & -e \end{bmatrix} \end{aligned}$$

Looking at the first row on the matrices on the left and right gives

$$\begin{aligned} a - 3b &= 1 \\ b &= 0 \\ -a &= 0 \end{aligned}$$

and this is inconsistent since the last two imply that both a and b are zero and this contradicts the first line. Thus no such matrix D exists.

Question A 13

SOLUTION 1. Notice that you can write the two given matrix-vector equations as a matrix-matrix equation:

$$M\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix},\quad M\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}\quad\rightarrow\quad M\begin{bmatrix}1&2\\1&1\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}=I_2.$$

Since M times the matrix of vectors is the 2-x-2 identity matrix, this means that

$$M=\begin{bmatrix}1&2\\1&1\end{bmatrix}^{-1}.$$

Using the 2-x-2 inverse matrix formula, we can write

$$M=\frac{1}{-1}\begin{bmatrix}1&-2\\-1&1\end{bmatrix}=\begin{bmatrix}-1&2\\1&-1\end{bmatrix}.$$

SOLUTION 2. Since M is a linear transformation,

$$M\begin{bmatrix}1\\0\end{bmatrix}=M\begin{bmatrix}2\\1\end{bmatrix}-M\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}-\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}-1\\1\end{bmatrix}$$

and

$$M\begin{bmatrix}0\\1\end{bmatrix}=2M\begin{bmatrix}1\\1\end{bmatrix}-M\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix}-\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}2\\-1\end{bmatrix}$$

Hence

$$M=\begin{bmatrix}-1&2\\1&-1\end{bmatrix}$$

Question A 14

SOLUTION. Euler's formula states that for a complex number given in argument form, we have

$$\cos(\theta)+i\sin(\theta)=e^{i\theta}$$

In the case of the complex number

$$z=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$$

we can either see that this corresponds to the point at distance 1 of the origin on the diagonal in the first quadrant of the complex plan and hence has an argument of 45° or of $\pi/4$; or you might directly remember the values of the sine and cosine of $\pi/4$. Either way, we obtain that

$$z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i = \cos(\pi/4) + i\sin(\pi/4) = e^{i\pi/4}$$

Taking the 2010 of z is much easier in this shape. We have

$$z^{2010} = (e^{i\pi/4})^{2010} = e^{i1005\pi/2}$$

In this notation we see that the argument of z^{2010} is $1005\pi/2$ which remains the same if we remove or add $4\pi/2$, so since $2005 = 4 \cdot 501 + 1$ we have that the angle $1005\pi/2 = \pi/2$ and so Using Euler's formula again yields

$$z^{2010} = e^{(1005)i\pi/2} = e^{i\pi/2}$$

Using Euler's formula the other way around we obtain that

$$z^{2010} = e^{i\pi/2} = \cos(\pi/2) + i\sin(\pi/2) = i$$

Since $\cos(\pi/2) = 0$ and $\sin(\pi/2) = 1$.

Question A 15

SOLUTION. (b) and (d) are the only correct statements independent of A and B.
(a) Recall that scaling a row is an elementary row operation which scales the determinant by the same factor, so if the determinant of A is non-zero, $\det A = \det B$ can fail to hold.
(b) All elementary row operations can be represented by invertible matrices, so since B is a product of these matrices, if B is invertible, then so is A. Alternatively, recall that elementary row operations can only scale the determinant of a matrix by a non-zero factor. Thus, if $\det B$ is non-zero (i.e. it is invertible) then $\det A$ is non-zero (so is invertible as well).
(c) This is false, elementary row operations do change eigenvectors. For a simple example, let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Subtracting the first row from the second row reduces B to A. However, $Av = v$, so v is an eigenvector of A, while

$$Bv = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

which is not a multiple of v. Hence v is not an eigenvector of B.
(d) If $\lambda = 0$ is an eigenvalue of A, then A is not invertible and the determinant of A is 0. As in (b), elementary row operations only scale the determinant by a constant non-zero factor. Therefore, the $\det(B) = c(\det(A)) = c(0) = 0$ and hence B is not invertible. Hence $\lambda = 0$ is also an eigenvalue of B.
Alternatively, applying an elementary row operation can be represented by multiplication of the original matrix by an elementary matrix (which is invertible). Suppose $B = EA$. Then if $Av = 0$, we have $Bv = E(Av) = E(0) = 0$, so 0 is an eigenvalue for B as well (in this case, v is also an eigenvector for this special eigenvalue).

Question A 16

SOLUTION. There is a small subtlety in this problem. Suppose first that $n = 1$. Then $Av = Bv$ becomes an

equation of real numbers, as *1x1* matrices are just one number, and a v is also just one number. This gives $(A-B)v=0$ and as $v \neq 0$ (the problem stated this was not zero) we have that $A=B$.
Now, for $n=2$, suppose that

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and let $B=-A$, Then

$$A - B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

Let

$$v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Then this vector is in the nullspace of both matrices, $Av = \mathit{0} = Bv$, but $A \neq B$ and hence the claim is not true for $n=2$.
Next, suppose that n is at least 3. We can generalize the above example nicely. Choose A to be the all zeroes matrix except with a 1 in the position $A_{1,1}$ (the top left corner) and let $B=-A$ so that it too has all zeroes except a -1 in the top left corner. Then $A-B$ has only a 2 in the top left corner and zeroes everywhere else. Lastly, let

$$v = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Then notice that $Av = 0 = Bv$ however $A \neq B$. Thus, the claim is false for all $n \geq 2$ but is true when $n=1$.

Question A 17

Solution. Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and c a real number. Then

$$\begin{aligned} T(cx) &= T\left(\begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix}\right) \\ &= \begin{bmatrix} -cx_3 \\ 1 \\ cx_1 + cx_2 + cx_3 \end{bmatrix} \end{aligned}$$

However,

$$\begin{aligned} cT(x) &= c\begin{bmatrix} -x_3 \\ 1 \\ x_1+x_2+x_3 \end{bmatrix} \\ &= \begin{bmatrix} -cx_3 \\ c \\ cx_1+cx_2+cx_3 \end{bmatrix} \\ &\neq \begin{bmatrix} -cx_3 \\ 1 \\ cx_1+cx_2+cx_3 \end{bmatrix} \\ &= T(cx) \end{aligned}$$

and so the map is not linear because the second rows do not match.
Alternatively, one could show that

$$\begin{aligned} T(x+y) &= T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) \\ &= T\left(\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{bmatrix}\right) \\ &= \begin{bmatrix} -(x_3+y_3) \\ 1 \\ x_1+y_1+x_2+y_2+x_3+y_3 \end{bmatrix} \\ &= \begin{bmatrix} -x_3-y_3 \\ 1 \\ x_1+y_1+x_2+y_2+x_3+y_3 \end{bmatrix} \end{aligned}$$

However

$$\begin{aligned} T(x)+T(y) &= T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) + T\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) \\ &= \begin{bmatrix} -x_3 \\ 1 \\ x_1+x_2+x_3 \end{bmatrix} + \begin{bmatrix} -y_3 \\ 1 \\ y_1+y_2+y_3 \end{bmatrix} \\ &= \begin{bmatrix} -x_3-y_3 \\ 2 \\ x_1+y_1+x_2+y_2+x_3+y_3 \end{bmatrix} \\ &\neq T(x+y) \end{aligned}$$

and once again, we see that the map is not linear. The moral of this question is that the one in the middle row is causing the map to not be linear.

Question A 18

SOLUTION. Using the hint, we see that

$$S\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -y+z \\ x+y+z \\ -x-2y \end{bmatrix} = \begin{bmatrix} 0 \\ x \\ -x \end{bmatrix} + \begin{bmatrix} -y \\ y \\ -2y \end{bmatrix} + \begin{bmatrix} z \\ z \\ 0 \end{bmatrix} = x\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Using these as our column vectors, we arrive at

$$S = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & 1 \\ -1 & -2 & 0 \end{bmatrix}$$

Question A 19

SOLUTION. Rearranging the equations gives
$x_1 + 2x_2 + 3x_3 = -4 \qquad \text{and} \qquad -2x_1 - 3x_2 - 4x_3 = 5$
So if we take
$A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -3 & -4 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad b = \begin{bmatrix} -4 \\ 5 \end{bmatrix}$
Then $Ax = b$ corresponds to the given problem.

Question A 20

SOLUTION. Only (d) is correct.
Recall $(AB)^T = B^T A^T$ and $(AB)^{-1} = B^{-1}A^{-1}$ for invertible matrices. Then

$$(A(BC)^{-1}D)^T = (AC^{-1}B^{-1}D)^T = D^T(B^{-1})^T(C^{-1})^T A^T$$

Question A 21

SOLUTION. We will row reduce until we arrive at an upper triangular matrix, taking care to keep track of changes to the determinant due to elementary row operations. First, let's use the first row as a pivot

$$R2 \to R2 - R1,$$

$R3 \to R3 - 3R1$, and $R4 \to R4 + 2R1$. This doesn't change the determinant, and we end up with

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -3 & -1 \\ 0 & 6 & 9 & 3 \end{bmatrix}$$

Now use the second row as a pivot

$$R3 \to R3 + R2$$

and $R4 \to R4 - 6R2$. This doesn't change the determinant, and we end up with

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -3 & 3 \end{bmatrix}$$

Now use the third row as a pivot

$$R4 \to R4 - 3R3.$$

This doesn't change the determinant, and we end up with

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

Since this matrix is upper triangular, the determinant is the product of the diagonal entries, -6. The original matrix has the same determinant since we only added multiples of one row to another, which does not affect the determinant. Thus, $\det B = -6$.

Question A 22

SOLUTION. The probability of moving from location 3 to location 1 in one time step is $P_{1,3} = 0.1626$.

Question A 23

SOLUTION. The probability of moving from location i to location 2 in 10 time steps is $(P^{10})_{2,i}$. We know $(P^{10})_{2,1} = 0.3843$, $(P^{10})_{2,2} = 0.3753$, and $(P^{10})_{2,3} = 0.3786$, so to maximize the probability of getting to location 2, you should start from location 1.

Question A 24

SOLUTION. We are given that the walker is in location 3 when 10 days have passed and want to know the probability that the walker is in location 1 when 20 days have passed. This is identical to the situation if we are given that the walker is in location 3 when 0 days have passed and want to know the probability that the walker is in location 1 when 10 days have passed.
The probability of moving from location 3 to location 1 in 10 time steps is $(P^{10})_{1,3} = 0.2875$

Question A 25

SOLUTION. Notice that
$$\det(A - I\lambda) = \det\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) = -\lambda^3$$
and the roots of this polynomial are $\lambda = 0$. This is a triple root and so this is the only eigenvalue. Next, for any vector v, we have
$Av = 0 = 0v$
and hence every **nonzero** vector v is an eigenvector.

Question A 26

SOLUTION. Proceeding as in the hint, we have
$$\det(A - I\lambda) = \det\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\right) = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$$
and so the eigenvalues are given by $\lambda = \pm 1$

Question A 27

SOLUTION. The identity matrix has every non-zero vector as an eigenvector. So does the zero matrix.

Question A 28

SOLUTION. The hints should explain most of what's going on. Here is the output. For A=ones(4,4) we get

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Then after the loop executes for j=1, we get

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

(which is the same as the original A). After j=2, the matrix looks like

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

After j=3, the matrix looks like

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Lastly, when the loop hits its final value of j=4, the matrix looks like

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{bmatrix}$$

Finally, the code A(2,4) looks in row two column 4 for the value. The answer is 2.

Question A 29

SOLUTION. Following the hints, we can quickly see that

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 7 & 11 \end{bmatrix}$$

Keep in mind that the code b=3:4:11; gives us the second row that begins at 3, increments entries by 4 and then goes up to 11.

Question A 30

SOLUTION. The following code will work.

```
C=zeros(20);
for j=1:20;
C(j,j)=j;
end;
```

Question B 01 (a)

Solution. Notice that $(x, y) = (4, 3)$ is a point on the line. So the vector
$v = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$
is a matrix in the direction of L. To make it a unit vector, we divide the coordinates by the length of the vector. The length is given by
$\|v\| = \sqrt{(4)^2 + (3)^3} = \sqrt{25} = 5$
Thus, the unit vector is given by
$\hat{a} = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$

Question B 01 (b)

Solution 1. To reflect a vector about a line that goes through the origin, let $v = (v_x, v_y)$ be a vector in the direction of the line. The matrix which represents Ref_L is given by

$$\mathbf{A} = \frac{1}{\|v\|^2} \begin{bmatrix} v_x^2 - v_y^2 & 2v_xv_y \\ 2v_xv_y & v_y^2 - v_x^2 \end{bmatrix}$$

Since $v = (4, 3)$ is on the line, we have

$$\mathbf{A} = \frac{1}{4^2 + 3^2} \begin{bmatrix} 4^2 - 3^2 & 2(4)(3) \\ 2(4)(3) & 3^2 - 4^2 \end{bmatrix} = \frac{1}{25} \begin{bmatrix} 7 & 24 \\ 24 & -7 \end{bmatrix} = \begin{bmatrix} \frac{7}{25} & \frac{24}{25} \\ \frac{24}{25} & \frac{-7}{25} \end{bmatrix}$$

Solution 2. We can alternatively consider what a reflection does to vectors.
Any vector pointing along the line 3x=4y, <4,3>, for example, will be reflected onto itself.
Any vector perpendicular to <4,3> will point in the completely opposite direction after reflection. For example, <-3,4> will become <3,-4>.
Note that these two vectors are linearly independent. Thus if A denotes the reflection, $A \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and $A \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$, which we can write as a matrix equation

$$A \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix}.$$

Thus $A = \begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 7/25 & 24/25 \\ 24/25 & -7/25 \end{bmatrix}$.

Question B 01 (c)

Solution. Using our answer from (b),

$$\mathrm{Ref}_L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \frac{7}{25} & \frac{24}{25} \\ \frac{24}{25} & \frac{-7}{25} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7/25 \\ 24/25 \end{bmatrix}$$

Question B 01 (d)

Solution 1. From the statement of the problem we know that $\lambda = -1$ is an eigenvalue of the matrix A. The corresponding eigenvector $\mathbf{v}$ satisfies the following equation

$$(A - \lambda I)\mathbf{v} = (A + I)\mathbf{v} = \mathbf{0}$$

where I is the identity matrix. Solving for $\mathbf{v}$ gives:

$$(A+I)\mathbf{v} = \begin{bmatrix} \frac{7}{25}+1 & \frac{24}{25} \\ \frac{24}{25} & \frac{-7}{25}+1 \end{bmatrix}\mathbf{v} = \frac{2}{25}\begin{bmatrix} 16 & 12 \\ 12 & 9 \end{bmatrix}\mathbf{v} = \mathbf{0} \quad \to \quad \mathbf{v} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}.$$

Any non-zero constant multiple of $\mathbf{v}$ is also an acceptable answer.

Solution 2. Any vector perpendicular to the line of reflection will flip it's direction, and hence be an eigenvector with eigenvalue *-1*. To find the vector that is perpendicular to the line we rewrite *3x=4y* in normal form

$$\langle 3, -4\rangle \cdot \langle x, y\rangle = 0$$

In other words, all point on the line *3x=4y* are perpendicular to $\langle 3, -4\rangle$. Hence $\langle 3, -4\rangle$ must be an eigenvector with eigenvalue *-1*.

Question B 02 (a)

Solution. Let

$$v_{AB} = B - A = \begin{bmatrix} 1 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}$$

and let

$$v_{AC} = C - A = \begin{bmatrix} 1 & 1 & 4 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \end{bmatrix}$$

Then, the area of a triangle is given by

$$\text{Area} = 1/2\|v_{AB} \times v_{AC}\|$$

Computing the cross product gives

$$v_{AB} \times v_{AC} = \begin{bmatrix} -2 & -1 & 1 \end{bmatrix}$$

Taking the norm and dividing by 2 yields

$$\text{Area} = 1/2\|v_{AB} \times v_{AC}\| = 1/2\|\begin{bmatrix} -2 & -1 & 1 \end{bmatrix}\| = \frac{\sqrt{(-2)^2 + (-1)^2 + 1^2}}{2} = \frac{\sqrt{6}}{2}$$

Question B 02 (b)

Solution. Let
$\vec{AB} = B - A = \begin{bmatrix} 1 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}$
and let

$\vec{BC} = C - B = \begin{bmatrix}1 & 1 & 4\end{bmatrix} - \begin{bmatrix}1 & 0 & 3\end{bmatrix} = \begin{bmatrix}0 & 1 & 1\end{bmatrix}$
as in part (a). The vector are orthogonal if their dot product is zero. Calculating this dot product yields
$\vec{AB} \cdot \vec{BC} = (1)(0) + (-1)(1) + (1)(1) = 0$
Thus, vectors $\vec{BA}$ and $\vec{CB}$ are orthogonal and so the triangle is right angled.

Question B 02 (c)

Solution. Since the points A, B, and C all lie in the plane P, we have the following equations are satisfied:

$$\begin{aligned} b + 2c &= d \\ a + 3c &= d \\ a + b + 4c &= d \end{aligned}$$

. Writing these equations as a linear system gives

$$\begin{bmatrix} 0 & 1 & 2 & -1 \\ 1 & 0 & 3 & -1 \\ 1 & 1 & 4 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \mathbf{0}$$

Row reducing this system such that we have a pivot in each row we get:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \mathbf{0} \quad \rightarrow \quad a = -2d,\, b = -d,\, c = d$$

So we can see that we have one free parameter. In other words, any specific values of a,b, and c are determined by our choice of d. If we choose d = 1, then a = -2, b = -1, c = 1. Hence, the plane defined by the points A,B and C is $-2x - y + z = 1$.
(Note that any constant multiple the planar equation is also a correct answer: e.g. $4x + 2y - 2z = -2$.)

Question B 02 (d)

Solution 1. Let $D = (x, y, z)$. First we compute the vectors as stated in hint one. These are given by

$$\begin{aligned} \vec{AD} &= (x, y - 1, z - 2) \\ \vec{CD} &= (x - 1, y - 1, z - 4) \\ \vec{AB} &= (1, -1, 1) \\ \vec{CB} &= (0, -1, -1) \end{aligned}$$

Taking dot products of these values as stated in hint 2 we have

$$\begin{aligned} 0 &= \vec{AD} \cdot \vec{CD} = x(x - 1) + (y - 1)^2 + (z - 2)(z - 4) \\ 0 &= \vec{CD} \cdot \vec{CB} = (1 - y) + (4 - z) = -y - z + 5 \\ 0 &= \vec{AD} \cdot \vec{AB} = x - (y - 1) + (z - 2) = x - y + z - 1 \\ 0 &= \vec{CB} \cdot \vec{AB} = (0)(1) + (-1)(-1) + (1)(-1) \end{aligned}$$

It turns out we will not use the first equation. Rearranging the middle two equations above gives

$$
\begin{aligned}
5 &= y + z \\
1 - x &= -y + z
\end{aligned}
$$

Lastly, recall that the plane P is given by $4x + 2y - 2z = -2$. Using this with the two equations above gives
$-2 = 4x + 2y - 2z = 4x - 2(-y + z) = 4x - 2(1 - x) = 6x - 2$
and solving yields that $x = 0$. Using this, the system of two equations above reduces to

$$
\begin{aligned}
5 &= y + z \\
1 &= -y + z
\end{aligned}
$$

Summing these yields $6 = 2z$ and so $z = 3$ and substituting this above gives $y = 2$. Thus the solution is

$$D = (x, y, z) = (0, 2, 3)$$

Solution 2. From part (b) we have that the triangle ABC is a right-angled triangle and so if we reflect along the hypotenuse connecting AC we will have a rectangle. We can achieve this by adding the vector BC to the vector A. We have

$$BC =< 1, 1, 4 > - < 1, 0, 3 >=< 0, 1, 1 >$$

and so

$$D = A + BC =< 0, 1, 2 > + < 0, 1, 1 >=< 0, 2, 3 > .$$

Alternatively we could add the vector BA to the vector C:

$$BA =< 0, 1, 2 > - < 1, 0, 3 >=< -1, 1, -1 >$$

and

$$D = C + BA =< 1, 1, 4 > + < -1, 1, -1 >=< 0, 2, 3 > .$$

This point is on the plane because the vectors BA and BC span the plane and this point is written as a linear combination of these vectors.

Question B 03 (a)

Solution. Elementary row operations don't change the solution of a homogeneous equation, hence the solutions of $Ux = 0$ also solve $Ax = 0$.
From the first row on matrix U, we have $\mathbf{x_1} = 7\mathbf{x_3}$.
From the second row on matrix U, we have $\mathbf{x_2} = -4\mathbf{x_3}$.
Hence the general solution to is $\begin{bmatrix} \mathbf{x_1} \\ \mathbf{x_2} \\ \mathbf{x_3} \end{bmatrix} = a \begin{bmatrix} 7 \\ -4 \\ 1 \end{bmatrix}$, where a is any real number.

Question B 03 (b)

SOLUTION 1. No, this is not possible. Elementary row operations change the right hand side and hence, in general, solutions of $Ax = b$ are not related to solutions of $Ux = b$. Only in the special case of a homogeneous system, when $b = 0$, is this possible.

An easy way to see this is by a counter example. We will look at two matrices A_1 and A_2 that can both be row reduced to U, but have different solutions to $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Let us choose $A_1 = U$ and $A_2 = \begin{bmatrix} 1 & 0 & -7 \\ 1 & 1 & -3 \end{bmatrix}$

A_2 was obtained by adding the first row of U to the second row of U. Hence, by subtracting row 1 from row 2 in A_2 gives U, which shows that A_2 can be row reduced to U.

Solving $A_1\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ we get

$$\left[\begin{array}{ccc|c} 1 & 0 & -7 & 1 \\ 0 & 1 & 4 & 0 \end{array}\right]$$

with solution

$$\mathbf{x}_1 = a \begin{bmatrix} 7 \\ -4 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

While solving $A_2\mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ we get

$$\left[\begin{array}{ccc|c} 1 & 0 & -7 & 1 \\ 1 & 1 & -3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -7 & 1 \\ 0 & 1 & 4 & -1 \end{array}\right]$$

with solution

$$\mathbf{x}_2 = b \begin{bmatrix} 7 \\ -4 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

As we see, regardless of a and b we always find $\mathbf{x}_1 \neq \mathbf{x}_2$. Thus using the given information, we cannot tell which solution set is correct.

SOLUTION 2. Another way to view this question is to notice that elementary row operations combine to give us that

$A = EU$

where E is an invertible matrix. Thus solving

$Ax = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Reduces to solving

$Ux = E^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

This last solution varies with the matrix E. Notice that if the vector on the right were the zero vector, then this matrix would not matter (as it didn't in part (a)) but here it can make a huge difference. A concrete counter example is given in solution 1.

Question B 03 (c)

SOLUTION 1. Rewriting the given equation gives

$$
\begin{aligned}
A\mathbf{x} &= A\begin{bmatrix}0\\1\\-2\end{bmatrix}\\
0 &= A\mathbf{x} - A\begin{bmatrix}0\\1\\-2\end{bmatrix}\\
&= A\underbrace{\left(\mathbf{x} - \begin{bmatrix}0\\1\\-2\end{bmatrix}\right)}_{=\mathbf{y}}
\end{aligned}
$$

From part (a) we know that the solution to this homogeneous equation is given by

$$\mathbf{y} = a\begin{bmatrix}7\\-4\\1\end{bmatrix}$$

Solving for $\boldsymbol{x}$ we obtain

$$\mathbf{x} = \mathbf{y} + \begin{bmatrix}0\\1\\-2\end{bmatrix} = a\begin{bmatrix}7\\-4\\1\end{bmatrix} + \begin{bmatrix}0\\1\\-2\end{bmatrix}$$

SOLUTION 2. Since U can be obtained from A by elementary row operations, there is an invertible matrix E such that $A = EU$. Hence

$$
\begin{aligned}
Ax &= A\begin{bmatrix}0\\1\\-2\end{bmatrix}\\
EUx &= EU\begin{bmatrix}0\\1\\-2\end{bmatrix}\\
Ux &= U\begin{bmatrix}0\\1\\-2\end{bmatrix}\\
\begin{bmatrix}1 & 0 & -7\\0 & 1 & 4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} &= \begin{bmatrix}14\\-7\end{bmatrix}
\end{aligned}
$$

We see that x_3 is a free parameter. Further, $x_1 - 7x_3 = 14$ implies $x_1 = 7x_3+14$ and $x_2 + 4x_3 = -7$ implies $x_2 = -4x_3 - 7$. Therefore, the solution is given by

$$\begin{bmatrix}7x_3+14\\-4x_3-7\\x_3\end{bmatrix} = x_3\begin{bmatrix}7\\-4\\1\end{bmatrix} + \begin{bmatrix}14\\-7\\0\end{bmatrix}$$

where x_3 is any constant.
Note that the set of solutions found here is the same as the set of solutions found in solution 1, see by setting $a = x_3 + 2$.

Question B 04 (a)

SOLUTION. Proceeding as usual, we look for non-zero solutions to

$$0 = (A - 3I)v = \begin{bmatrix} -10 & 0 & 1 \\ 0 & 0 & 0 \\ -2 & 0 & -7 \end{bmatrix} v$$

The free variable is v_2, while there is a system of two equations and two unknowns for v_1 and v_3. The (simplest and only) solution to this smaller system is $v_1 = 0 = v_3$. Hence, any vector of the form

$$v = \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix}$$

with $a \neq 0$ is an eigenvector of A with eigenvalue 3.

Question B 04 (b)

SOLUTION 1. Let's compute the characteristic polynomial

$$\begin{aligned}
\det(A - \lambda I) &= \det\left(\begin{bmatrix} -7 & 0 & 1 \\ 0 & 3 & 0 \\ -2 & 0 & -4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right) \\
&= \det\left(\begin{bmatrix} -7-\lambda & 0 & 1 \\ 0 & 3-\lambda & 0 \\ -2 & 0 & -4-\lambda \end{bmatrix} \right) \\
&= (-7-\lambda)(3-\lambda)(-4-\lambda) - (1(3-\lambda)(-2)) \\
&= (3-\lambda)((-7-\lambda)(-4-\lambda) + 2) \\
&= (3-\lambda)(28 + 11\lambda + \lambda^2 + 2) \\
&= (3-\lambda)(30 + 11\lambda + \lambda^2) \\
&= (3-\lambda)(\lambda+5)(\lambda+6)
\end{aligned}$$

Solving for the roots gives the eigenvalues $\lambda = -6, -5, 3$

SOLUTION 2. A party trick is to use the fact that the product of eigenvalues equals the determinant

$$\lambda_1 \lambda_2 \lambda_3 = \det(A)$$

and that the sum of the eigenvalues equals the sum of the diagonal entries of A

$$\lambda_1 + \lambda_2 + \lambda_3 = -7 + 3 - 4 = -8$$

The determinant of A is quickly calculated as

$$\det(A) = (-7)(3)(-4) + 0 + 0 - (-2)(3)(1) - 0 - 0 = 90$$

Since $\lambda_1 = 3$ is given, we get the two equations

$$\lambda_2\lambda_3 = 30, \quad \text{and} \quad \lambda_2 + \lambda_3 = -11$$

Hence $\lambda_2 = -11 - \lambda_3$ and thus

$$\begin{aligned} (-11 - \lambda_3)\lambda_3 &= 30 \\ \lambda_3^2 + 11\lambda_3 + 30 &= 0 \\ (\lambda_3 + 5)(\lambda_3 + 6) &= 0 \end{aligned}$$

Hence the eigenvalues are $\lambda = 3, -5, -6$.

Question B 04 (c)

SOLUTION. We compute the matrices $A - \lambda I$ and check its nullspace

$$\begin{bmatrix} -7 & 0 & 1 \\ 0 & 3 & 0 \\ -2 & 0 & -4 \end{bmatrix} - \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 8 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

and from this we can see that

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

is an eigenvector associated to $\lambda = -5$ (to get all eigenvectors, multiply this vector by any non-zero number). Repeating this with $\lambda = -6$

$$\begin{bmatrix} -7 & 0 & 1 \\ 0 & 3 & 0 \\ -2 & 0 & -4 \end{bmatrix} - \begin{bmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 9 & 0 \\ -2 & 0 & 2 \end{bmatrix}$$

and from this we can see that

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

is an eigenvector associated to $\lambda = -6$ (to get all eigenvectors, multiply this vector by any non-zero number).

Question B 05 (a)

SOLUTION. The first and second entry of $P^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ give the probability that, after k steps, the random walker is in location 1 and location 2, respectively, when the random walker starts in location 1. Therefore we calculate

$$P^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & 3/4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

which tells us that the probability of being in state 1 is 1/2.

Question B 05 (b)

SOLUTION. We begin by computing the eigenvalues.

$$\begin{aligned}
\det(P - \lambda I) &= \det\left(\begin{bmatrix} 0 & 1/2 \\ 1 & 1/2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} -\lambda & 1/2 \\ 1 & 1/2 - \lambda \end{bmatrix}\right) \\
&= (-\lambda)(1/2 - \lambda) - 1/2 \\
&= \lambda^2 - \frac{1}{2}\lambda - \frac{1}{2} \\
&= \frac{1}{2}(2\lambda^2 - \lambda - 1) \\
&= \frac{1}{2}(2\lambda + 1)(\lambda - 1)
\end{aligned}$$

and thus the roots (and hence eigenvalues) are $\lambda = -1/2, 1$. To compute the eigenvectors, we look at the nullspace of $P - \lambda I$ for each eigenvalue. When $\lambda = -1/2$, we have

$$\begin{bmatrix} 0 & 1/2 \\ 1 & 1/2 \end{bmatrix} - \begin{bmatrix} -1/2 & 0 \\ 0 & -1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 1 \end{bmatrix}$$

and a vector (hence an eigenvector) in the kernel of this matrix is given by $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$
Similarly, for $\lambda = 1$, we have

$$\begin{bmatrix} 0 & 1/2 \\ 1 & 1/2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1/2 \\ 1 & -1/2 \end{bmatrix}$$

and a vector (hence an eigenvector) in the kernel of this matrix is given by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$
Adjoin these eigenvectors to make a matrix

$$M = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

Then, our theory of diagonalizability gives us that
$D = M^{-1}PM \qquad$ or, equivalently, $\qquad P = MDM^{-1}$
where

$$D = \begin{bmatrix} -1/2 & 0 \\ 0 & 1 \end{bmatrix}$$

is the diagonal matrix consisting of the eigenvalues. Next, notice that

$$P^{20} = (MDM^{-1})^{20} = \underbrace{(MDM^{-1})(MDM^{-1})...(MDM^{-1})}_{20 \text{ times}} = MD^{20}M^{-1}$$

Finally, our work will pay off: Taking the 20th power of a diagonal matrix is easy and convenient. Having to inverting a 2x2 matrix is a small price to pay for this convenience.

$$\begin{aligned}
P^{20} = MD^{20}M^{-1} &= \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}^{20} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} (1/2)^{20} & 0 \\ 0 & (1)^{20} \end{bmatrix} \left(\frac{1}{3} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \right) \\
&= \frac{1}{3} \begin{bmatrix} -1/(2^{20}) & 1 \\ 1/(2^{20}) & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \\
&= \frac{1}{3} \begin{bmatrix} -1/(2^{19}) + 1 & 1/(2^{20}) + 1 \\ 1/(2^{19}) + 2 & -1/(2^{20}) + 2 \end{bmatrix}
\end{aligned}$$

Applying the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to this matrix gives

$P^{20} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{-1/(2^{19})+1}{3} \\ \frac{1/(2^{19})+2}{3} \end{bmatrix}$

and this completes the problem.

Question B 05 (c)

SOLUTION. As in part (b), we deduce using the diagonalized matrix that

$$\begin{aligned}
P^{n} = MD^{n}M^{-1} &= \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}^{n} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1/2^n & 0 \\ 0 & 1 \end{bmatrix} \left(\frac{1}{3} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \right) \\
&= \frac{1}{3} \begin{bmatrix} -1/2^n & 1 \\ 1/2^n & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \\
&= \frac{1}{3} \begin{bmatrix} -1/2^{n-1} + 1 & 1/2^n + 1 \\ 1/2^{n-1} + 2 & -1/2^n + 2 \end{bmatrix}
\end{aligned}$$

Applying the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to this matrix gives

$P^{n} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{-1/2^{n-1}+1}{3} \\ \frac{1/2^{n-1}+2}{3} \end{bmatrix}$

Taking the limit as n tends to infinity gives

$\lim_{n\to\infty} P^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$

and this completes the problem.

Question B 06 (a)

SOLUTION. This equation is equivalent to $\mathbf{y}'(\mathbf{t}) = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \mathbf{y}(\mathbf{t})$

Question B 06 (b)

SOLUTION. To begin, we compute the eigenvalues of the matrix from part (a) (call this matrix M). To start, compute the characteristic polynomial.

$$\begin{aligned}
\det(M-\lambda I) &= \det\left(\begin{bmatrix}3 & -2\\ 4 & -1\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\\
&= \det\left(\begin{bmatrix}3-\lambda & -2\\ 4 & -1-\lambda\end{bmatrix}\right)\\
&= (3-\lambda)(-1-\lambda)-(-2)(4)\\
&= -3-2\lambda+\lambda^2+8\\
&= \lambda^2-2\lambda+5
\end{aligned}$$

This polynomial has two complex roots given by

$$\begin{aligned}
\lambda &= \frac{-b\pm\sqrt{b^2-4ac}}{2a}\\
&= \frac{-(-2)\pm\sqrt{(-2)^2-4(1)(5)}}{2(1)}\\
&= \frac{2\pm\sqrt{-16}}{2}\\
&= \frac{2\pm 4i}{2}\\
&= 1\pm 2i
\end{aligned}$$

Using $\lambda_a = 1+2i$, we have the eigenvector which can be computed by
$\begin{bmatrix}3-(1+2i) & -2\\ 4 & -1-(1+2i)\end{bmatrix} = \begin{bmatrix}1-i & -1\\ 2 & -1-i\end{bmatrix}$
Multiplying the first row with *1+i* we obtain $\begin{bmatrix}2 & -1-i\\ 2 & -1-i\end{bmatrix} = \begin{bmatrix}2 & -(1+i)\\ 0 & 0\end{bmatrix}$
The eigenvector v_a is such that $2v_{a,1}-(1+i)v_{a,2}=0$. Choosing $v_{a,1}=1$ we have $v_{a,2}=\frac{2}{i+1}=\frac{2(i-1)}{(i+1)(i-1)}=1-i$, hence the eigenvector we find is
$v_a = \begin{bmatrix}1\\ 1-i\end{bmatrix}$
Important: Please keep in mind that this is one possible eigenvector and there are others that differ from this by a scalar multiple. Since the eigenvalues λ_a, λ_b are complex conjugate, the eigenvector v_b is the complex conjugate of v_a, which is
$v_b = \begin{bmatrix}1\\ 1+i\end{bmatrix}$
Now we calculate by using Eulers formula, $e^{x+iy}=e^x(\cos(y)+i\sin(y))$,

$$\begin{aligned}
v_a e^{\lambda_a t} &= \begin{bmatrix}1\\ 1-i\end{bmatrix} e^{(1+2i)t}\\
&= \begin{bmatrix}1\\ 1-i\end{bmatrix} e^t(\cos(2t)+i\sin(2t))\\
&= \begin{bmatrix}e^t\cos(2t)+ie^t\sin(2t)\\ e^t(\cos(2t)+\sin(2t))+ie^t(\sin(2t)-\cos(2t))\end{bmatrix}\\
&= e^t\begin{bmatrix}\cos(2t)\\ \cos(2t)+\sin(2t)\end{bmatrix} + ie^t\begin{bmatrix}\sin(2t)\\ \sin(2t)-\cos(2t)\end{bmatrix}
\end{aligned}$$

Hence, the **real** solution of y is given by
$y(t) = C_1 e^t \begin{bmatrix} \cos(2t) \\ \cos(2t)+\sin(2t) \end{bmatrix} + C_2 e^t \begin{bmatrix} \sin(2t) \\ \sin(2t)-\cos(2t) \end{bmatrix}$
for constants C_1, C_2.
"**Note:** For the general (complex) solution you would **not** drop the part $v_b e^{\lambda_b t}$ and the solution would be $y(t) = B_1 v_a e^{\lambda_a t} + B_2 v_b e^{\lambda_b t}$
for constants B_1, B_2. But for the real solution, it is enough to only consider $v_a e^{\lambda_a t}$."

Question B 06 (c)

SOLUTION. Using the solution from part (b),
$y(t) = C_1 e^t \begin{bmatrix} \cos(2t) \\ \cos(2t)+\sin(2t) \end{bmatrix} + C_2 e^t \begin{bmatrix} \sin(2t) \\ \sin(2t)-\cos(2t) \end{bmatrix}$
we have for $t = 0$
$y(0) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix}$
Because the given intial values are $y(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, we write
$\begin{bmatrix} -1 \\ 1 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix}$
This is

$$\begin{aligned} -1 &= C_1 \\ 1 &= C_1 - C_2 \end{aligned}$$

and $C_1 = -1$ and $C_2 = -2$. Hence, the solution is

$$\begin{aligned} y(t) &= -e^t \begin{bmatrix} \cos(2t) \\ \cos(2t)+\sin(2t) \end{bmatrix} - 2e^t \begin{bmatrix} \sin(2t) \\ \sin(2t)-\cos(2t) \end{bmatrix} \\ &= e^t \begin{bmatrix} -2\sin(2t)-\cos(2t) \\ -3\sin(2t)+\cos(2t) \end{bmatrix} \end{aligned}$$

Good Luck for your exams!