Full Solutions
MATH152 April 2011

April 4, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question A 01

Solution. To add vectors, we add component wise. Therefore,

$$\begin{aligned}
&\mathbf{x}+\mathbf{y}\\
&=[0,-3,2]+[1,1,1]\\
&=[0+1,-3+1,2+1]\\
&=[1,-2,3]
\end{aligned}$$

and so $\mathbf{x}+\mathbf{y}=[1,-2,3]$.

Question A 02

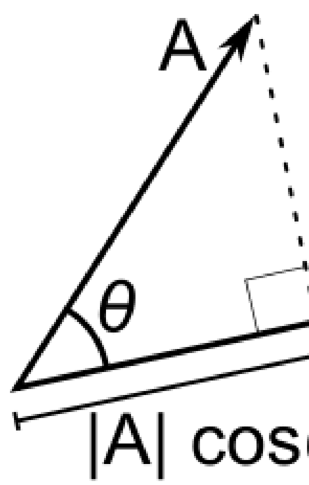
Solution. If we think of denoting the unit vectors in the x, y, z directions as $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ respectively then we can turn our vectors $\mathbf{x}$ and $\mathbf{y}$ into the following matrix,
$$\begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & -3 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

Firstly notice that we align the components of the vectors such that the x components are in the column with the heading $\hat{\mathbf{i}}$, the y components are in the column with the heading $\hat{\mathbf{j}}$, and the z components are in the column with the heading $\hat{\mathbf{k}}$. Secondly, this may look odd as a matrix since one of the rows is a vector. This matrix is artificial but if we treat it as if it were a standard matrix and computed its determinant, we would get

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & -3 & 2 \\ 1 & 1 & 1 \end{vmatrix} = -5\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}} = [-5,2,3]$$

which is precisely the definition of the cross product. Therefore $\mathbf{x}\times\mathbf{y}=[-5,2,3]$.

Question A 03



Solution. Consider connecting the two vectors into a right triangle similar the picture to the right
To get the component of $\mathbf{x}$ onto $\mathbf{y}$, we follow like in the picture and take,

$$\operatorname{proj}_{\mathbf{y}}\mathbf{x} = |\mathbf{x}|\cos(\theta)\hat{\mathbf{y}}$$

where θ is the angle between the vectors $\mathbf{x}$ and $\mathbf{y}$ and $\hat{\mathbf{y}}$ is a unit vector in the direction of $\mathbf{y}$. We can rewrite

$\cos(\theta)$ in terms of the dot (scalar) product as,

$$\cos(\theta) = \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}||\mathbf{y}|}.$$

Therefore, the projection of $\mathbf{x}$ onto $\mathbf{y}$ is

$$\mathrm{proj}_{\mathbf{y}}\mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{y}|}\hat{\mathbf{y}} = \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{y}|^2}\mathbf{y}$$

where we have recognized that

$$\hat{\mathbf{y}} = \frac{\mathbf{y}}{|\mathbf{y}|}.$$

Computing the dot product we get,

$$\mathbf{x} \cdot \mathbf{y} = 0 - 3 + 2 = -1$$

and for the magnitude of $\mathbf{y}$,

$$|\mathbf{y}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}.$$

Therefore we get that the projection of $\mathbf{x}$ onto $\mathbf{y}$ is,

$$\mathrm{proj}_{\mathbf{y}}\mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{y}|^2}\mathbf{y} = -\frac{1}{3}\mathbf{y} = \left[-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\right].$$

Question A 04

SOLUTION. With two equations (rows in the matrix), there can be at most 2 pivots and thus there is at least one free variable. A free variable implies that there are an infinite number of solutions provided that the right hand side vector vanishes wherever there is a row of zeros, otherwise a solution does not exist. Therefore, we either have at least one valid free variable which implies that there are an infinite number of solutions or the right hand side vector does not vanish whenever there is a row of zeros. Therefore we either have no solutions or an infinite number of solutions when a linear system has 2 equations with three unknowns. Therefore, the correct option is (d).

Advanced:
Recall that the rank (r) of a matrix defines the number of linearly independent rows or columns of a matrix or linear system and recall that for any matrix A size m by n that,

$$r \leq \min(n, m).$$

For this example we have m=2 and n=3 and so $r \leq 2$. The dimension of the nullspace is n-r and in this case it is at least 1 which implies that there are non-trivial solutions to"A$\mathbf{x}$ "$= \mathbf{0}$ and in fact there are an infinite number of them. A solution to"A**x** "= $\boldsymbol{b}$ exists as long as **"b"** is spanned by the column space of A. If this is true then"A**x** "= $\boldsymbol{b}$ has a particular solution along with the infinite set of solutions belonging to the nullspace and therefore overall there are an infinite number of solutions. However, if $\boldsymbol{b}$ is **not** spanned by the column space of A then there are no solutions to"A**x** "= $\boldsymbol{b}$. Therefore, we conclude that a linear system with 2 equations in 3 unknowns has (d) either no solutions or an infinite number of solutions.

Question A 05

SOLUTION. For any 2×2 matrix,
$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
we can write the inverse as,
$A^{-1} = \frac{1}{\text{Det}(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
with $\text{Det}(A) = ad - bc$. For our case, we have
$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$
and $\text{Det}(A) = 2(2) - 3(1) = 1$. Therefore,
$A^{-1} = \frac{1}{1} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$
is the inverse of the matrix.

Question A 06

SOLUTION. For the first line of code
```
A=[1 2 3;4 5 6;7 8 9];
```
we recall that when declaring a matrix in Matlab, square brackets [and] start and end the declaration. While inside a space indicates to move to the next column while a semicolon instructs to start a new row. Therefore we place the entries 1, 2, and 3 in the three columns of the first row; 4, 5, and 6 in the three columns of the second row and 7, 8, and 9 in the three columns of the last row. Therefore the matrix generated is

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

The semicolon at the end of the line indicates that output is suppressed and so the matrix will not be displayed. For the second line of code
```
A(2,:)
```
we recall that Matlab recognizes entries in a matrix first by row and column so a call A(i,j) would return an element in row i and column j. The colon instructs Matlab to take all the entries of a given field. Therefore A(:,j) instructs Matlab to output all entries in column j while A(i,:) instructs Matlab to output all of the entries in row i. Therefore A(2,:) instructs Matlab to output all of the entries from row 2. If we look at out matrix then the output would be
```
[4  5  6]
```
Notice the lack of semicolon on this output indicates that the output is **not** suppressed.

Question A 07

SOLUTION. From the hint we know that Matlab stores by row then by column. Therefore the first line of code
```
A=zeros(3,2);
```
would create a matrix of **zeros** with **3** rows and **2** columns,

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The next lines of code

```
A(1,1)=5;
A(2,2)=3;
A(3,1)=1;
```

would put a **5** in the **1**st row, **1**st column, a **3** in the **2**nd row, **2**nd column and a **1** in the **3**rd row, **1**st column and therefore,

$$A = \begin{bmatrix} 5 & 0 \\ 0 & 3 \\ 1 & 0 \end{bmatrix}$$

is our created matrix.

Question A 08

Solution. First we see the line

```
A=zeros(4,4);
```

from the hint we notice that Matlab stores by row then by column so this is an instruction to create a matrix filled with **zeros** that has **4** rows and **4** columns. Therefore after this call we have

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The next lines

```
for k=1:4
A(k,k)=-k;
end
```

initiates a for loop or a counting loop over a variable k. It says to start at k=1, perform whatever is inside the loop, increase k by 1 and then repeat until k=4. Here we see that the **k**th row and the **k**th column of A from $1 \leq \mathbf{k} \leq 4$ is filled with **-k**. Therefore, after this for loop we have

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}.$$

The next lines are

```
for k=1:4
A(1,k)=1;
end
```

which is another for loop. This time we always stay on row **1** but column **k** gets filled with **1** for $1 \leq \mathbf{k} \leq 4$. Notice since we still start at k=1 then the first entry we write to is A(1,1) which already has a -1 in it. Therefore, this will get overwritten with a 1. The matrix becomes

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}.$$

Since there are no more lines of code, this is the final matrix.

Question A 09

SOLUTION. From the current loop rule we know that each loop gets its own current which is an unknown for the system. We also need an unknown for the voltage drop that occurs across each current source. Since we have four loops we have **4** loop currents, i_1, i_2, i_3, i_4 and we have two current sources so we have **2** unknown voltage drops v_1 for source E1 and v_2 for current source E2. Therefore in total we have **6** unknowns for our linear system,

$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ v_1 \\ v_2 \end{bmatrix}.$$

Question A 10

SOLUTION. To be consistent with the labs, we will denote everything in terms of voltage drops so positive numbers are voltage drops and negative numbers are voltage gains.
In the third loop we move clockwise and start with the 9Ω resistor. The voltage across this resistor (and any resistor) is given by Ohm's Law V=IR so $V_{9\Omega} = 9i_3$ where we have a positive sign because we're moving with the current and so there is a voltage drop. Next we come to the 7Ω resistor which has both loop current 3 and loop current 1 flowing through it. Therefore there is a voltage drop contribution from loop current 3 by virtue that we're moving in the direction of that current but there is also a voltage gain from loop current 1 going in the opposite direction. Therefore we get that $V_{7\Omega} = 7(i_3 - i_1)$. Finally we reach the current source; since we are traversing it against the arrow we have a voltage drop and the value is one of our unknowns, v_2. The Kirchhoff rule states that the sum of voltage drops and gains in a loop must be zero and so we have that

$$0 = 9i_3 + 7(i_3 - i_1) + v_2$$

as the linear equation for loop 3.

Question A 11

SOLUTION. Notice that the current source E1 is on a branch shared by current loop i_1 and i_2. As these currents pass through E1, i_2 will face up while i_1 will face down. We know that the difference of these currents must result in a net current of 1A upwards (since the arrow points upwards). Therefore since i_2 is upwards then

$$i_2 - i_1 = 1.$$

Notice this is equivalent to considering down is positive in which case the net resulting current would be 1A up or -1A. In this case, choosing down as positive we'd have $i_1 - i_2 = -1$ which is exactly equivalent to above.

Full Solution for Those Interested:

If people are looking to practice there work with circuits, we can continue work from here and from A10 to solve for the unknown voltages and loop currents. Based on the same reasoning in A10 we can get that the Kirchhoff linear equation for loop 1 is

$$4i_1 + v_1 + 7(i_1 - i_3) - 3 = 0$$

where recall that positive numbers mean voltage drops (across the resistors and across the current source since it points upwards while our clockwise current points downwards there) and that negative numbers mean voltage gains (across the battery since we move from the negative to positive terminal). The 7Ω resistor takes the difference in currents from loop 1 and loop 3 since it shares those loop currents. Similarly we can get for loop 2

$$-v_1 + 6i_2 + 8(i_2 - i_4) = 0$$

where for this loop v_1 is a voltage gain because we are traversing upward, just like the arrow. We already have the third loop from A10 as

$$9i_3 + 7(i_3 - i_1) + v2 = 0$$

and so the fourth and final loop is

$$-v2 + 8(i_4 - i_2) + 5 + 10i_4$$

where here crossing the battery is a voltage **drop** because we are going from the positive to negative terminal. So far we only have 4 equations for 6 unknowns; the last two come from the current source. From above we already have that

$$i_2 - i_1 = 1$$

and through the current source E2 we have that

$$i_4 - i_3 = 2.$$

We can write this in a matrix problem as

$$\begin{bmatrix} 11 & 0 & -7 & 0 & 1 & 0 \\ 0 & 14 & 0 & -8 & -1 & 0 \\ -7 & 0 & 16 & 0 & 0 & 1 \\ 0 & -8 & 0 & 18 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ -5 \\ 1 \\ 2 \end{bmatrix}.$$

We can write this is an augmented matrix as

$$\left[\begin{array}{cccccc|c} 11 & 0 & -7 & 0 & 1 & 0 & 3 \\ 0 & 14 & 0 & -8 & -1 & 0 & 0 \\ -7 & 0 & 16 & 0 & 0 & 1 & 0 \\ 0 & -8 & 0 & 18 & 0 & -1 & -5 \\ -1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 & 2 \end{array}\right]$$

which we can row reduce to get

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & -0.5200 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0.4800 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1.2000 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0.8000 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0.3200 \\ 0 & 0 & 0 & 0 & 0 & 1 & 15.5600 \end{array}\right].$$

Therefore we see that our solution is

$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -0.5200 \\ 0.4800 \\ -1.2000 \\ 0.8000 \\ 0.3200 \\ 15.5600 \end{bmatrix}.$$

Do not worry about the presence of a negative sign, a negative loop current just means that the current (i_1 and i_3 in this case) actually flows counter-clockwise, not clockwise like we assumed.

Question A 12

SOLUTION 1. As rotation matrices are of the form

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

We must have that

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

This gives us that

$$\begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

or more simply

$-\sin(\theta) = -1$

and

$\cos(\theta) = 0$

Thus, we plug in these values into our matrix and we see that

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

will do the trick.

SOLUTION 2. As rotation matrices have the form

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

and a diagram shows that we are rotating the first vector (which is a unit vector on the positive y-axis) a total of $\frac{\pi}{2}$, we have that the rotation matrix that solves our problem is

$$\begin{bmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Question A 13

SOLUTION. As per the hint, we only have to consider the transformation of 4 special points. These points are the corners of the square. Linear transformations cannot distort the edges of a figure in such a way that they split an edge into a new corner. Therefore, the transformed figure must have the same amount of corners as the original figure. Because of this, it is easiest to just transform the corners and then connect them to make the new figure. Our transformation matrix is

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

and the four corners of the square are [0,0], [1,0], [0,1], and [1,1]. Firstly notice that the origin will always map to itself because no matrix can multiply [0,0] to produce anything but [0,0]. Therefore, we really only have to transform 3 points. The transformation of [1,0] is,

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

The transformation of [0,1] is,

$$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

and the transformation of [1,1] is

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

Therefore we have $[0, 0]$ transforms to$[0, 0]$, $[1, 0]$ transforms to $[1, 2]$, $[0, 1]$ transforms to$[2, 1]$ and $[1, 1]$ transforms to $[3, 3]$. The plot to the right400px|right|thumbnail shows the original points in blue while thetransformed points are in red. We complete the figure by connecting the corners.

Question A 14

SOLUTION. As per the hint, we only have to consider the transformation of 4 special points. These points are the corners of the square. Linear transformations cannot distort the edges of a figure in such a way that they split an edge into a new corner. Therefore, the transformed figure must have the same amount of corners as the original figure. Because of this, it is easiest to just transform the corners and then connect them to make the new figure. Our transformation matrix is

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

and the four corners of the square are [0,0], [1,0], [0,1], and [1,1]. Firstly notice that the origin will always map to itself because no matrix can multiply [0,0] to produce anything but [0,0]. Therefore, we really only have to transform 3 points. The transformation of [1,0] is,

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The transformation of [0,1] is,

$$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and the transformation of [1,1] is

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Therefore we have $[0,0]$ transforms to$[0,0]$, $[1,0]$ transforms to $[1,1]$, $[0,1]$ transforms to$[1,1]$ and $[1,1]$ transforms to $[2,2]$. The plot to the right400px|right|thumbnail shows the original points in blue while thetransformed points are in red. We complete the figure by connecting the corners.
Notice that in this case it may appear we have lost a vertex but really, as we computed above, two are just overlapping one another. We see that we have turned our square into a line, this is typical of a projection operator (recall that a projection operator maps vectors to a single vector). In this case, if we try to form a projection operator to the vector [1,1] we get (a multiple of) our linear transformation matrix A (try it yourself!)

Question A 15

Solution 1. We could start by finding both of the separate transformation matrices and then multiplying them together which would give us the general transformation matrix T. However since we only need T([1,0]), the transformation of the unit vector in the x-direction, it will suffice to just consider how that single vector is transformed and not deal with matrices at all.
We first rotate the vector [1,0] counter-clockwise by $\pi/6$ to get a new vector $\mathbf{v}_1$. After doing so, the new vector forms a right triangle with a vector in the x-direction and a vector in the y-direction. We can get its components by using trigonometry. The new vector will have length 1 (since it is a rotation of [1,0] which also has length 1) so its components are

$$\mathbf{v}_1 = \left[\cos\left(\frac{\pi}{6}\right), \sin\left(\frac{\pi}{6}\right)\right] = \left[\frac{\sqrt{3}}{2}, \frac{1}{2}\right].$$

We then must project this vector onto $\mathbf{u}$=[1,2]. Recall that a projection of a vector $\mathbf{v}$ onto a vector $\mathbf{u}$ is

$$\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{u}|^2}\mathbf{u} = \frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u}.$$

Using our vector $\mathbf{v}_1$ and projecting it onto $\mathbf{u}$ which has length $|\mathbf{u}| = \sqrt{5}$, we have that our new vector, $\mathbf{v_2}$, is

$$\begin{aligned} \mathbf{v_2} = \text{proj}_{\mathbf{u}}\mathbf{v}_1 &= \frac{\left[\frac{\sqrt{3}}{2}, \frac{1}{2}\right] \cdot [1,2]}{5}[1,2] \\ &= \frac{\frac{\sqrt{3}}{2}(1) + (\frac{1}{2})2}{5}[1,2] = \frac{\sqrt{3}+2}{10}[1,2]. \end{aligned}$$

Therefore, the transformation on the vector [1,0] is

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \frac{\sqrt{3}+2}{10}\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Solution 2. As practice for doing linear transformations, we will create the matrices anyway and show that they lead to the same conclusion as going step by step in Solution 1. First recall that a rotation matrix rotating counter-clockwise with an angle θ has the form

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

If you are having trouble remembering how to form a rotation matrix, see the solution to Question A28 on this exam for a more detailed explanation of these entries (they are components of the vectors generated by rotating the unit vectors).
Since in this example the angle rotating is $\pi/6$ then we get

$$A = \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}.$$

Next for the projection matrix, to get the entries in column k, we have to do a projection of the kth unit vector onto our desired vector **u**=[1,2]. We get that the first column is

$$\text{proj}_{\mathbf{u}}[1,0] = \frac{[1,0]\cdot[1,2]}{5}[1,2] = \frac{1}{5}[1,2].$$

The second column is

$$\text{proj}_{\mathbf{u}}[0,1] = \frac{[0,1]\cdot[1,2]}{5}[1,2] = \frac{2}{5}[1,2].$$

Therefore we get that the projection matrix is

$$B = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

The total transformation then is (remember that the rotation is applied first),

$$\begin{aligned} T = BA &= \frac{1}{10}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix} \\ &= \frac{1}{10}\begin{bmatrix} \sqrt{3}+2 & 2\sqrt{3}-1 \\ 2\sqrt{3}+4 & 4\sqrt{3}-2 \end{bmatrix} \end{aligned}$$

.
Therefore,

$$\begin{aligned} T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) &= \frac{1}{10}\begin{bmatrix} \sqrt{3}+2 & 2\sqrt{3}-1 \\ 2\sqrt{3}+4 & 4\sqrt{3}-2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{10}\begin{bmatrix} \sqrt{3}+2 \\ 2\sqrt{3}+4 \end{bmatrix} = \frac{\sqrt{3}+2}{10}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{aligned}$$

like we recovered in the first solution.

Question A 16

Solution 1. The first plane

$$y - z = 1$$

has normal $\mathbf{n}_1$=[0,1,-1] (just the coefficients on x,y,z). The second plane

$$2x + y + z = 2$$

has normal $\mathbf{n}_2$=[2,1,1]. Let's call the direction our line has $\mathbf{v}$. Notice that since $\mathbf{v}$ is an intersection of the two planes then by definition it is on plane 1 and plane 2. Therefore

$$\mathbf{n}_1 \times \mathbf{v} = \mathbf{0} \quad \text{and} \quad \mathbf{n}_2 \times \mathbf{v} = \mathbf{0}.$$

Therefore we seek $\mathbf{v}$ so that it is orthogonal to both normal vectors. The cross product, by definition, provides this direction. Therefore,

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = [0, 1, -1] \times [2, 1, 1] = [2, -2, -2].$$

We want this to have length 1 and so we see that

$$|\mathbf{v}| = \sqrt{12} = 2\sqrt{3}.$$

Therefore we get that the unit norm vector, $\hat{\mathbf{v}}$ is

$$\hat{\mathbf{v}} = \frac{1}{\sqrt{3}}[1, -1, -1].$$

Solution 2. Since a vector on the line has to be on each plane, it has to be a solution of a linear system. We have the following equations

$$\begin{cases} \quad\ \ y - z = 1 \\ 2x + y + z = 2 \end{cases}$$

which is a linear system with 2 equations in 3 unknowns. We can write it in an augmented matrix as

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 1 \\ 2 & 1 & 1 & 2 \end{array}\right]$$

We will swap the first two rows and then row reduce to get

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1/2 \\ 0 & 1 & -1 & 1 \end{array}\right].$$

We notice there is a free variable and we let $x = t$. We then get

$$z = \frac{1}{2} - t$$
$$y = 1 + z = \frac{3}{2} - t$$

Therefore we get that the solution, $\mathbf{s}$, is

$$\mathbf{s} = \mathbf{s}_0 + t\mathbf{v} = \left[0, \frac{3}{2}, \frac{1}{2}\right] + t[1, -1, -1].$$

The direction part of the line, $\mathbf{v}$ is [1,-1,-1]. However to make it length 1 we must divide by its magnitude, $|\mathbf{v}| = \sqrt{3}$ to get,

$$\hat{\mathbf{v}} = \frac{1}{\sqrt{3}}[1, -1, -1]$$

just like in solution 1.

Question A 17

Solution. Using the hint, we have
$(4 - 2i)(-1 + 3i) = -4 - (-6) + (12 + 2)i = 2 + 14i$

Question A 18

Solution. Via the hint
$|-1 + 3i| = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$

Question A 19

Solution.

$$\frac{u}{z} = \frac{4 - 2i}{-1 + 3i} \cdot \frac{-1 - 3i}{-1 - 3i} = \frac{(4 - 2i)(-1 - 3i)}{1 + 9}$$

Continuing with the simplification, we have

$$\frac{u}{z} = \frac{-4 - 6 + (-12 + 2)i}{10} = \frac{-10 - 10i}{10} = -1 - i$$

Question A 20

Solution. The absolute value (or modulus) of a complex number $a + bi$ is

$$r = \sqrt{a^2 + b^2} = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

and the argument, θ, is the angle between the right side of the x-axis and the position of the complex number on the plane. In this case, the argument is $3\pi/4$. To see that, you can either just visually see that the point (-1,1) is clearly on the diagonal of the second quadrant of the plane, or using trigonometry have

$$\cos(\theta) = \frac{a}{r} = \frac{-1}{\sqrt{2}} \quad \text{and} \quad \sin(\theta) = \frac{b}{r} = \frac{1}{\sqrt{2}}$$

and from there notice that the corresponding angle is $3\pi/4$ (or in real life compute the angle using a calculator since you only know a very few values of sine and cosine).

So this allows us to write our complex number in polar form as

$$-1 + i = re^{\theta i} = \sqrt{2}e^{3\pi i/4}$$

Question A 21

SOLUTION. One would use the backslash operator'\', and store the answer into a vector called $\mathbf{x}$. The full command in your MATLAB console would look like

```
x = A\b;
```

Note that

```
x = inv(A)*b;
```

would also work, but requires to compute the inverse of A first. There is better ways of solving $Ax = b$ than that, and $x = A\backslash b$ is one of them. It does not use the inverse matrix (which is costly to compute).

Question A 22

SOLUTION. If the transition matrix is

$$P = \begin{bmatrix} 8/10 & 3/10 & 1/10 \\ 2/10 & 3/10 & 7/10 \\ 0 & 4/10 & 2/10 \end{bmatrix}$$

then the operation P$\mathbf{x}$=$\mathbf{b}$ gives us the probability of being in each of the three states $\mathbf{b}$ based on where we started, $\mathbf{x}$. We are told that we are equally likely to be in any of the three states so we have as our vector,

$$\mathbf{x} = [1/3, 1/3, 1/3].$$

To find out the probabilities of being in each state after one transition we compute P$\mathbf{x}$,

$$P\mathbf{x} = \begin{bmatrix} 8/10 & 3/10 & 1/10 \\ 2/10 & 3/10 & 7/10 \\ 0 & 4/10 & 2/10 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 2/5 \\ 1/5 \end{bmatrix}.$$

Therefore we get that **the probability of being in state 3 is 1/5**. Notice that the output probabilities sum to 1. This is a requirement since we only allow transitions from states 1 to 3, this summation to 1 simply says that no matter where we start we will end up somewhere in states 1 to 3.

Question A 23

SOLUTION. We are starting in state 2 so there is 100% probability of starting in state 2 and zero elsewhere, therefore our input vector is

$$\mathbf{x} = [0, 1, 0].$$

If we wanted to know the probabilities for moving one transition we would compute $\mathbf{t_1} = P\boldsymbol{x}$,

$$\mathbf{t}_1 = P\mathbf{x} = \begin{bmatrix} 8/10 & 3/10 & 1/10 \\ 2/10 & 3/10 & 7/10 \\ 0 & 4/10 & 2/10 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/10 \\ 3/10 \\ 4/10 \end{bmatrix}.$$

However we want to know the probabilities for a second transition. We can think of this as a first transition of our new vector $\mathbf{t_1}$. Therefore the second transition probabilities are $\mathbf{t_1} = P\ \mathbf{t_1}$,

$$\mathbf{t}_2 = P\mathbf{t}_1 = \begin{bmatrix} 8/10 & 3/10 & 1/10 \\ 2/10 & 3/10 & 7/10 \\ 0 & 4/10 & 2/10 \end{bmatrix} \begin{bmatrix} 3/10 \\ 3/10 \\ 4/10 \end{bmatrix} = \begin{bmatrix} 37/100 \\ 43/100 \\ 20/100 \end{bmatrix}.$$

Therefore, starting in state 2, it is **most likely that after two transitions, the person will still be in state 2** since 43/100 is the highest of the three probabilities.
Notice, we could also think of the problem as,

$$\mathbf{t}_2 = P\mathbf{t}_1 = P(P\mathbf{x}) = P^2\mathbf{x}$$

so that if we wanted the n^{th} transition probabilities we'd have $\mathbf{t_n} = P^n\ \mathbf{x}$.

Question A 24

SOLUTION. From the hint we see that a probability equilibrium vector gives us the probabilities of a final state from any initial state. This means that once we transition to a certain point, further transitions do not impact the probability any more. Let's assume we have reached this special equilibrium vector $\mathbf{v}$ then

$$P\mathbf{v} = \mathbf{v}$$

which says exactly what we stated; attempting to transition $\mathbf{v}$ does nothing to it. Therefore the special vector $\mathbf{v}$, will satisfy

$$(P - I)\mathbf{v} = \mathbf{0}$$

$$\begin{bmatrix} -2/10 & 3/10 & 1/10 \\ 2/10 & -7/10 & 7/10 \\ 0 & 4/10 & -8/10 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which we can solve (using Gaussian elimination) to get, for a free parameter, t,

$$\mathbf{v} = \left[\frac{7}{2}t, 2t, t\right].$$

Now recall that the input vector $\mathbf{v}$ is a set of probabilities and so the entries of $\mathbf{v}$ must sum to 1. Therefore we have that,

$$v_1 + v_2 + v_3 = \frac{7}{2}t + 2t + t = \frac{13}{2}t = 1$$

and therefore, t=2/13 so that our equilibrium vector $\mathbf{v}$ is

$$\mathbf{v} = \left[\frac{7}{13}, \frac{4}{13}, \frac{2}{13}\right].$$

Note *(This is not necessary to answer the question):* Our result means that starting from any initial state, we will end up in state 1 with a probability of 7/13, state 2 with a probability of 4/13 and state 3 with a probability of 2/13. If we try this with our initial state from A 23, [0,1,0] we notice that after 8 transitions,

$$P^8[0,1,0] = [0.5309904100, 0.3114556300, 0.1575539600]$$

where our equilibrium vector is

$$\mathbf{v} = [0.5384615385, 0.3076923077, 0.1538461538]$$

which is already a fairly good agreement!
Note further: It may have crossed some minds that the equilibrium vector is always the eigenvector for the transition matrix corresponding to an eigenvalue of 1, whose entries sum to 1. For probability matrices where the columns sum to 1 there will always be such a vector.

Question A 25

SOLUTION. We want to find the eigenvalues of A, i.e. we seek λ such that,
$A\mathbf{x} = \lambda\mathbf{x}$.
Equivalently we can write this as,
$$\underbrace{\begin{bmatrix} 1-\lambda & 0 & p \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{bmatrix}}_{M} \mathbf{x} = 0.$$
In order for a non-trivial ($\mathbf{x} \neq \mathbf{0}$) solution we require the determinant of M to be zero. Computing this determinant and setting it to zero we get,
$$\begin{vmatrix} 1-\lambda & 0 & p \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = (\lambda-1)((\lambda-1)^2 - p) = 0.$$
Immediately we can conclude that $\lambda = 1$ is an eigenvalue regardless of p. The other two eigenvalues are $\lambda_{\pm} = 1 \pm \sqrt{p}$.
In order for these to be **distinct**, we require that $p \neq 0$. In order for them to be **real**, we require that $p > 0$. Notice $p > 0$ satisfies both of these conditions. Also, as long as $p > 0$ then $\lambda_{\pm}$ will be distinct from the third eigenvalue $\lambda = 1$. Therefore we conclude that if $p > 0$ then the eigenvalues of A will be both **real** and **distinct**.

Question A 26

SOLUTION. We compute the determinant of our matrix

$$\begin{aligned} \det(B) &= (a^2c + a^2b + a^2b - (a^3 + a^2c + ab^2)) \\ &= 2a^2b - a^3 - ab^2 \end{aligned}$$

In order for the matrix to be invertible, we need the determinant to be nonzero. For that, we need to factor our determinant

$$\begin{aligned} \det(B) &= 2a^2b - a^3 - ab^2 \\ &= -a(a^2 - 2ab + b^2) \\ &= -a(a-b)^2 \end{aligned}$$

So we obtain that

$$\det(B) = -a(a-b)^2$$

and so, for the matrix to be invertible, that is, for the matrix not to have a determinant of zero, we require that $a \neq 0$ and $a \neq b$.

Question A 27

SOLUTION. Recall that the definition of an inverse of A denoted A^{-1} is

$$AA^{-1} = A^{-1}A = I$$

where I is the identity. Now assume we want to solve $A\mathbf{x} = 0$ for some $\mathbf{x}$. If A is invertible then by the above property we can left multiply by $A^{-1}>$ to get

$$\mathbf{x} = A^{-1}0 = 0$$

so we conclude that if A is invertible then $A\mathbf{x} = 0$ only has $\mathbf{x} = 0$, the trivial solution and that **option c is correct**.

Now assume we wanted to find eigenvalues $A\mathbf{v} = \lambda\mathbf{v}$. If we wanted to find zero eigenvalues then we'd need to solve $A\mathbf{v} = 0$ but as we saw above, the only solution to this if A is invertible is $\mathbf{v} = 0$ which is not a valid eigenvector. Therefore, if A is invertible then it does not have any zero eigenvalues and **option d is correct**.

A square matrix A of size $n \times n$ always has n eigenvalues. If the matrix has rank r then there are always r non-zero eigenvalues and $n - r$ zero eigenvalues. Since in this case we have shown that there are no zero eigenvalues then $n - r = 0$ or $n = r$, thus the matrix A has rank n and **option a is correct**.

Recall that rank has to do with the number of pivots in a row reduced matrix. If A has full rank (rank n) then there will be n pivots which means two things. We have shown that all invertible matrices have rank n. Firstly this means that the row reduced form of A is the identity matrix (since it has n pivots) so **option b is correct** and secondly it means that there are no rows of zeros which means that $\det(A) \neq 0$, so **option e is correct**.

Therefore, from starting with the definition of an inverse we were able to deduce that **all** of the options are correct.

Question A 28

SOLUTION 1. By the position of this question inside the exam, and given that it is a multiple choice question, it is very likely that a conceptual answer is sufficient.

Conceptual Answer:

With a good understanding of vector geometry, we can quickly see which operations are equivalent to the identity matrix. Recall that the identity matrix means that the vector you finish with is the same as the vector you begin with. We will go through each scenario and argue whether the net transformation recovers the original vector.

(a)$\text{Refl}_\beta\text{Refl}_\beta$:

Recall that a reflection is angle preserving. For example, if we have a vector $\mathbf{u}$ that we want to reflect over some vector $\mathbf{v}$ and there is a counter-clockwise angle θ from $\mathbf{u}$ to $\mathbf{v}$ then the reflection will produce a new vector $\mathbf{x}$ that is a counter-clockwise angle θ from $\mathbf{v}$, i.e. 2θ counter-clockwise from $\mathbf{u}$. Therefore, in this case we have a vector that we are instructed to reflect over a line at an angle β from the x-axis. We are then instructed to take the new vector and reflect it once again over the same line. In other words we are asking for a "mirror-image of a mirror-image" which is the original vector. Therefore, we **do** get that this operation is equivalent to an identity matrix.

(b)$\text{Rot}_\alpha\text{Rot}_{-\alpha}$:

In this example we take a vector and for the first operation, rotate it an angle α. We then take the new vector and rotate it an angle $-\alpha$. This is equivalent to spinning something clockwise and then counter-clockwise by the same amount. Doing so would result in the original vector and therefore we **do** get that this operation is equivalent to an identity matrix.

(c)$\text{Proj}_\theta\text{Proj}_\theta$:
For this question we need to understand what a projection operation does. If we want to project a vector **u** onto a vector **v** then the net result will be to determine all the components of **u** so that it lies parallel to **v** or points in the same direction. Therefore, in this case if our vector **v** makes an angle θ with the x-axis and we want to project some vector **u** onto it then after the first projection operation we will have a vector, **w** that lies on top of **v**. Any future attempts to project this new vector onto **v** will result in an identity because there is nothing left to make parallel to **v**. Therefore applying a second projection operator will just map **w** to itself. Therefore we have that we started with a vector **u** and ended with a vector **w** which for arbitrary **u** is **not** equivalent to an identity matrix operation. In fact the only way it could be an identity matrix is if the original vector were **v** itself so that the projection does nothing to the vector.
(d)$\text{Proj}_\theta\text{Proj}_{\pi/2-\theta}$:
Recall from part (c) that projection is an operation that traps vectors to the original vector being projected on. This means that after one operation, further projections onto the same vector will result in no changes. Therefore in this example after the first operation we have a resultant vector in the direction of a vector at an angle $\pi/2 - \theta$ with the x-axis. After the second projection, our resultant vector is in the direction of a vector making an angle θ with the x-axis. Therefore, unless the original vector made an angle θ with the x-axis then this resultant vector will not be the same as the original. Therefore, for any arbitrary starting vector, this will **not** be an identity matrix operation.
(e)$\text{Rot}_\alpha\text{Refl}_{\alpha/2}$:
In this example we consider taking an initial vector, reflecting it by an angle of $\alpha/2$ and then rotating it counter-clockwise by an angle α. Consider if the original angle is clockwise of the reflector at $\alpha/2$. In this case if we reflect it so that it is now counter-clockwise to the reflector and rotate it even further counter-clockwise then there is no way of returning to the original vector. In order for it to be an identity operator, the operation has has to return the original vector for any arbitrary initial vector. We have shown with this class of examples that the operation will not produce the original vector ever and so it can't hold for any arbitrary initial vector. Therefore this is **not** an identity matrix operation.
Therefore, we conclude that only **(a)** and **(b)** are equivalent to identity matrix operations.

SOLUTION 2. **Computational Solution:**
This question can of course also be solved computationally in terms of transformation matrices if one were so inclined. We will present this solution now as an exercise to those wishing to improve their understanding of linear transformations and also to appease the worries of those who struggle with understanding the conceptual framework in the previous solutions.
For the purposes of short form we will denote the following matrices

- A_α-the matrix corresponding to the transformation Rot_α.
- B_β-the matrix corresponding to the transformation Refl_β.
- C_θ-the matrix corresponding to the transformation Proj_θ.

Recall that to obtain transformation matrices we take as the columns, the transformations on the basis vectors. Therefore, the first column would represent the transformation on $[1, 0]$ the basis vector in the x-direction.
Rotation Matrix:
Consider rotating the vector $[1, 0]$ by a counter-clockwise angle α. Since [1,0] has magnitude 1 then the new vector **u** can be written in component form as

$$\mathbf{u} = \langle |u| \cos(\alpha), |u| \sin(\alpha)\rangle = \langle \cos(\alpha), \sin(\alpha)\rangle$$

and therefore this is the first column of A_α. We choose counter-clockwise angles to be consistent with measuring angles from the positive x-axis towards the positive y-axis. If we rotate the y-basis [0,1] by the same angle we'd get a vector **v** (which also has magnitude 1),

$$\mathbf{v} = \langle -\sin(\alpha), \cos(\alpha)\rangle.$$

Note the negative sign in the x-component comes from the counter-clockwise rotation of [0,1] into the negative-x, positive-y quadrant. Therefore we see that the rotation matrix can be written as,

$$A_\alpha = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}.$$

Reflection Matrix: To formulate the reflection matrix we will think of it in terms of rotation. We want to reflect the basis vectors around a vector called the reflector, that makes an angle β with the x-axis. Recall that reflection is angle preserving so whatever angle the reflector makes with the original vector it must maintain that same angle with the new vector but in the opposite direction. For example if an initial vector is $\pi/6$ clockwise from the reflector than the new vector will be $\pi/6$ counter-clockwise from the reflector. The basis vector [1,0] is, by definition of the reflector, at a clockwise angle of β from the reflector. Therefore the new vector must be an angle of β counter-clockwise from the reflector or 2β from the positive x-axis. Therefore we can consider reflecting [1,0] the same as rotating it by 2β. Therefore the new vector $\mathbf{u}$ is

$$\mathbf{u} = \langle \cos(2\beta), \sin(2\beta) \rangle$$

where once again the magnitude is 1 because the vector [1,0] has magnitude 1. Now the vector [0,1] makes an angle of $\pi/2 - \beta$ clockwise from the y-axis with the reflector. Therefore the new vector $\mathbf{v}$ must be an angle of $2(\pi/2 - \beta)$ clockwise from the y-axis. However, we require that we measure angles from the positive x-axis and thus $\mathbf{v}$ makes an angle of

$$\frac{\pi}{2} - 2\left(\frac{\pi}{2} - \beta\right) = 2\beta - \frac{\pi}{2}$$

with the positive x-axis. Therefore we can think of $\mathbf{v}$ as a rotation from the x-axis of $2\beta - \pi/2$. Now recall that

$$\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$$
$$\sin\left(x - \frac{\pi}{2}\right) = -\cos(x).$$

Therefore we can write,

$$\mathbf{v} = \langle \sin(2\beta), -\cos(2\beta) \rangle$$

so that the reflection matrix B_β can be written as

$$B_\beta = \begin{bmatrix} \cos(2\beta) & \sin(2\beta) \\ \sin(2\beta) & -\cos(2\beta) \end{bmatrix}.$$

Projection: For the projection operator, we want to have an initial vector get projected onto a vector that makes an angle θ with the x-axis. Therefore, if we know immediately that the resultant vector $\mathbf{u}$ can be written as,

$$\mathbf{u} = |\mathbf{u}| \langle \cos(\theta), \sin(\theta) \rangle.$$

We also need to recall that the formula for projecting some vector $\mathbf{v}$ onto another vector $\mathbf{w}$ is

$$\mathrm{proj}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{w} \cdot \mathbf{v}}{|\mathbf{w}|^2}\mathbf{w}.$$

Now, if we call the basis vector [1,0], $\mathbf{i}$, then we have,

$$\begin{aligned}
\text{proj}_{\mathbf{u}}\mathbf{i} &= \frac{\mathbf{i}\cdot\mathbf{u}}{|\mathbf{u}|^2}\mathbf{u} \\
&= \frac{\langle 1,0\rangle \cdot \langle |\mathbf{u}\cos(\theta), |\mathbf{u}|\sin(\theta)\rangle}{|\mathbf{u}|^2}|\mathbf{u}|\langle\cos(\theta),\sin(\theta)\rangle \\
&= \langle\cos(\theta)^2, \cos(\theta)\sin(\theta)\rangle.
\end{aligned}$$

Similarly, for the other basis vector, [0,1], which we call $\mathbf{j}$, we get,

$$\begin{aligned}
\text{proj}_{\mathbf{u}}\mathbf{j} &= \frac{\mathbf{j}\cdot\mathbf{u}}{|\mathbf{u}|^2}\mathbf{u} \\
&= \frac{\langle 0,1\rangle \cdot \langle |\mathbf{u}\cos(\theta), |\mathbf{u}|\sin(\theta)\rangle}{|\mathbf{u}|^2}|\mathbf{u}|\langle\cos(\theta),\sin(\theta)\rangle \\
&= \langle\cos(\theta)\sin(\theta), \sin(\theta)^2\rangle.
\end{aligned}$$

Therefore, the projection matrix is,

$$C_\theta = \begin{bmatrix} \cos(\theta)^2 & \cos(\theta)\sin(\theta) \\ \cos(\theta)\sin(\theta) & \sin(\theta)^2 \end{bmatrix}.$$

Now that we have the transformation matrices, we can answer the questions.
(a)$\mathbf{Refl}_\beta\mathbf{Refl}_\beta$:
This is equivalent to

$$\begin{aligned}
B_\beta B_\beta &= \begin{bmatrix} \cos(2\beta) & \sin(2\beta) \\ \sin(2\beta) & -\cos(2\beta) \end{bmatrix} \begin{bmatrix} \cos(2\beta) & \sin(2\beta) \\ \sin(2\beta) & -\cos(2\beta) \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

where we have required the identity $\cos(\theta)^2 + \sin(\theta)^2 = 1$. Notice, this **is** indeed an identity matrix.
(b)$\mathbf{Rot}_\alpha\mathbf{Rot}_{-\alpha}$:
This is equivalent to

$$\begin{aligned}
A_\alpha A_{-\alpha} &= \begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) \\ \sin(-\alpha) & \cos(-\alpha) \end{bmatrix} \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

where we have taken advantage of the odd and even properties of $\sin(\theta)$ and $\cos(\theta)$. Notice this **is** and identity matrix.
(c)$\mathbf{Proj}_\theta\mathbf{Proj}_\theta$:
This is equivalent to

$$\begin{aligned}
C_\theta C_\theta &= \begin{bmatrix} \cos(\theta)^2 & \cos(\theta)\sin(\theta) \\ \cos(\theta)\sin(\theta) & \sin(\theta)^2 \end{bmatrix} \begin{bmatrix} \cos(\theta)^2 & \cos(\theta)\sin(\theta) \\ \cos(\theta)\sin(\theta) & \sin(\theta)^2 \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta)^2 & \cos(\theta)\sin(\theta) \\ \cos(\theta)\sin(\theta) & \sin(\theta)^2 \end{bmatrix}
\end{aligned}$$

which is **not** an identity matrix.

(d)$\mathbf{Proj}_\theta\mathbf{Proj}_{\pi/2-\theta}$:
This is equivalent to

$$\begin{aligned} C_\theta C_{\pi/2-\theta} &= \begin{bmatrix} \cos(\theta)^2 & \cos(\theta)\sin(\theta) \\ \cos(\theta)\sin(\theta) & \sin(\theta)^2 \end{bmatrix} \begin{bmatrix} \sin(\theta)^2 & \cos(\theta)\sin(\theta) \\ \cos(\theta)\sin(\theta) & \cos(\theta)^2 \end{bmatrix} \\ &= 2\cos(\theta)\sin(\theta) \begin{bmatrix} \cos(\theta)\sin(\theta) & \cos(\theta)^2 \\ \sin(\theta)^2 & \cos(\theta)\sin(\theta) \end{bmatrix} \end{aligned}$$

where we have used trig identities to simplify the argument $\pi/2 - \theta$. This is **not** an identity matrix.
(e)$\mathbf{Rot}_\alpha\mathbf{Refl}_{\alpha/2}$:
This is equivalent to

$$\begin{aligned} A_\alpha B_{\alpha/2} &= \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{bmatrix} \\ &= \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix} \end{aligned}$$

which is **not** an identity matrix. Therefore, we conclude, like before, that only **(a)** and **(b)** are equivalent to an identity matrix.

Question A 29

Solution. As mentioned in the hint, if we can find a counter example to the statement for any n, then the statement is false since, if true, it must hold for **ALL** n.
Consider 2×2 matrices with all entries zero except for the alternating diagonals. Specifically, we mean let

$$A = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}.$$

We choose these matrices because we know that the determinant of a diagonal matrix is just the product of its diagonal elements. Of course, since these are 2×2 matrices we could easily compute any determinant, but these matrices also satisfy a condition of the statement which is that det(A)=det(B)=0. We can then compute det(A+B),

$$\det(A + B) = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$$

which is **not** zero for arbitrary a and b with $a \neq 0$, $b \neq 0$. Therefore we have found a whole class of 2×2 matrices that satisfy det(A)=det(B)=0 but which do not satisfy det(A+B)=0. Since the statement claims to be true for **ALL** $n \times n$ matrices we can use our counter example to verify the statement is **false**.
Note that only one counterexample would be enough, so your solution is just as good, if you choose numbers for a and b, e.g. $a = b = 1$.

Question A 30

Solution 1. The normal of the plane is given by,

$$\mathbf{n} = \mathbf{v} \times \mathbf{w}$$

and therefore if a vector is in the plane it must be orthogonal to this normal vector. Recall that vectors are orthogonal if the scalar (dot) product is zero. Therefore we seek to check if,

$$(\mathbf{v} \times (\mathbf{v} \times \mathbf{w})) \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{v} \times \mathbf{n}) \cdot \mathbf{n} = 0.$$

To do this we use the triple scalar triple product which for any three vectors **a**, **b**, and **c** states,

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}).$$

Therefore we see that,

$$\mathbf{n} \cdot (\mathbf{v} \times \mathbf{n}) = \mathbf{v} \cdot (\mathbf{n} \times \mathbf{n}) = 0$$

by the virtue that parallel vectors have a zero cross-product. Therefore we see that the vector $\mathbf{v} \times (\mathbf{v} \times \mathbf{w})$ is orthogonal to the normal vector of the plane and thus lies in the plane itself. Therefore, the statement is **true**.

SOLUTION 2. From hint 2, we recall that since the plane is in $\mathbb{R}^2$, the two linearly independent vectors inside are enough to span the entire space. Therefore, any vector, **y**, inside P can be written as a linearly dependent combination of **v** and **w**,

$$\mathbf{y} = d_1\mathbf{v} + d_2\mathbf{w}$$

for some scalars, d_1 and d_2. Recall that for any vectors, **a**, **b**, and **c**, we can define the vector triple product as

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$

We wish to see if the vector $\mathbf{v} \times (\mathbf{v} \times \mathbf{w})$ is inside the plane P. Using the vector triple product,

$$\mathbf{v} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{v} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{v})\mathbf{w}.$$

But this is a linear combination of the vectors **v** and **w**, and hence is inside the plane P. To see this more clearly, define

$$\begin{aligned} d_1 &= (\mathbf{v} \cdot \mathbf{w}) \\ d_2 &= -(\mathbf{v} \cdot \mathbf{v}) \end{aligned}$$

then we have that

$$\mathbf{v} \times (\mathbf{v} \times \mathbf{w}) = d_1\mathbf{v} + d_2\mathbf{w}$$

and therefore the vector $\mathbf{v} \times (\mathbf{v} \times \mathbf{w})$ is inside the plane P, making the statement **true**.

Question B 01 (a)

SOLUTION. We have measured the total **output** in tons. In a linear system sense, this would be the vector **b**. How do we relate this output to the **input** or **unknowns**, which in this case is the total amount of yield per hectare for each wheat type. This would be the vector **x** in A**x**=**b**. If we consider a unit analysis we have to multiply **x** which is in tons per hectare by something that results in tons. Therefore, the entries of the matrix A must be hectares. We know that for the first field, a third of the area goes to each type of wheat.

Since each field has a total area of 1 hectare, we know that the area taken up by each wheat in the first field is 1/3 hectare. Therefore we know that

$$\frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 = 12$$

since the total output is 12 tons. Similarly, for the second field we have that a half hectare is for type two and a half for type 3. Therefore we get a second equation

$$0x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 = 10.$$

Finally for the last field we have that a half hectare is for type 1 wheat and a half hectare for type two. Therefore,

$$\frac{1}{2}x_1 + \frac{1}{2}x_2 + 0x_3 = 16.$$

Notice we have three linear equations and thus we have a linear system. As mentioned above, we store the areas in the matrix A, the wheat type unknowns are our **x** and the resulting output is our **b**. Therefore we have

$$A\mathbf{x} = \mathbf{b}$$

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 10 \\ 16 \end{bmatrix}$$

Question B 01 (b)

SOLUTION. Recall from part (a) that we have our linear system of the form

$$A\mathbf{x} = \mathbf{b}$$

is:

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 10 \\ 16 \end{bmatrix}$$

Therefore, we form an augmented matrix by adding the output vector **b** as a new column in A. Therefore the augmented matrix is

$$\left[\begin{array}{ccc|c} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 12 \\ 0 & \frac{1}{2} & \frac{1}{2} & 10 \\ \frac{1}{2} & \frac{1}{2} & 0 & 16 \end{array}\right]$$

Question B 01 (c)

SOLUTION. We take our augmented matrix from part (b) and we put it into reduced row-echelon form. Recall that this procedure ensures that every entry above and below a pivot is a zero. We could also just do a row-echelon form which ensure that only numbers below pivots are zero. However, for the purpose of example we will highlight the full reduced row-echelon form step by step.
We start with the augmented matrix

$$\left[\begin{array}{ccc|c} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 12 \\ 0 & \frac{1}{2} & \frac{1}{2} & 10 \\ \frac{1}{2} & \frac{1}{2} & 0 & 16 \end{array}\right]$$

We see that the first row can contain a pivot in its first column. We set that pivot to 1 by multiplying the first row by 3 (recall that elementary row operations do not change the solution to the linear system).

$$\left[\begin{array}{ccc|c} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 12 \\ 0 & \frac{1}{2} & \frac{1}{2} & 10 \\ \frac{1}{2} & \frac{1}{2} & 0 & 16 \end{array}\right] \overrightarrow{3R1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 36 \\ 0 & \frac{1}{2} & \frac{1}{2} & 10 \\ \frac{1}{2} & \frac{1}{2} & 0 & 16 \end{array}\right]$$

We let R_i indicate an operation on row i. We now have to clear out the rest of the entries in column 1 to be zero. We see that column one on row two is already zero and so we can move on to row three. To remove the entry in column 1 of row 3, we can subtract off 1/2 of row 1. Therefore

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 36 \\ 0 & \frac{1}{2} & \frac{1}{2} & 10 \\ \frac{1}{2} & \frac{1}{2} & 0 & 16 \end{array}\right] \overrightarrow{R3-1/2R1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 36 \\ 0 & \frac{1}{2} & \frac{1}{2} & 10 \\ 0 & 0 & -1/2 & -2 \end{array}\right]$$

We now move on to the second column and we seek to place a pivot in the second row. We want the pivot to have the value of 1 and so we multiply the whole row by 2. Therefore

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 36 \\ 0 & \frac{1}{2} & \frac{1}{2} & 10 \\ 0 & 0 & -1/2 & -2 \end{array}\right] \overrightarrow{2R2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 36 \\ 0 & 1 & 1 & 20 \\ 0 & 0 & -1/2 & -2 \end{array}\right]$$

To make the entries above and below this new pivot zero we must subtract row 2 from row 1 and we leave row three alone since it is already zero.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 36 \\ 0 & 1 & 1 & 20 \\ 0 & 0 & -1/2 & -2 \end{array}\right] \overrightarrow{R1-R2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 16 \\ 0 & 1 & 1 & 20 \\ 0 & 0 & -1/2 & -2 \end{array}\right]$$

Finally for the last pivot in the third row and third column we have to multiply row three by -2. We will then have to subtract row three from row two so that the third column entry in the second row vanishes. We do not need to perform any operations on the first row since there is already a zero in the third column.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 16 \\ 0 & 1 & 1 & 20 \\ 0 & 0 & -1/2 & -2 \end{array}\right] \overrightarrow{-2R3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 16 \\ 0 & 1 & 1 & 20 \\ 0 & 0 & 1 & 4 \end{array}\right] \overrightarrow{R2 - R3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 16 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Having all entries zero except for the pivots we have our matrix in reduced row-echelon form. The advantage to putting the matrix in this form is that we can immediately determine the vector $\mathbf{x}$. We have that,

$$\begin{aligned} x_1 &= 16 \\ x_2 &= 16 \\ x_3 &= 4 \end{aligned}$$

Therefore we conclude that the yield of the wheat is 16 tons/hectare for type 1, 16 tons/hectare for type 2 and 4 tons/hectare for type 3. Notice that if we were so inclined we could put our solution in for $\mathbf{x}$ in the original problem an we do indeed recover the right output vector $\mathbf{b}$. If instead you opted to just put the matrix in row-echelon form, you would get

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 36 \\ 0 & 1 & 1 & 20 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

which of course leads to the same conclusion but some arithmetic is involved. This type of situation is often the trade off in deciding which process to choose. Doing a row-echelon form is often quicker but because it only guarantees entries **below** pivots are zero, then arithmetic will almost always be required to get the solution. Conversely, the reduced row-echelon form takes longer to obtain but then the unknowns are clearly presented.

Question B 02 (a)

SOLUTION. Recall that a solution to

$$\mathbf{y}' = A\mathbf{y}$$

with eigenvalues λ_1, λ_2 and eigenvectors $\mathbf{k}_1, \mathbf{k}_2$ is

$$\mathbf{y}(t) = c_1\mathbf{k}_1 \exp(\lambda_1 t) + c_2\mathbf{k}_2 \exp(\lambda_2 t).$$

Therefore, for this specific example, the solution is,

$$\mathbf{y}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \exp(-t) + c_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} \exp(-2t).$$

Question B 02 (b)

SOLUTION. Recall from part (a) that we figured out that the general solution was

$$\mathbf{y}(t) = c_1 \begin{bmatrix}1\\2\end{bmatrix} \exp(-t) + c_2 \begin{bmatrix}2\\3\end{bmatrix} \exp(-2t).$$

With the initial condition

$$\mathbf{y}(0) = \begin{bmatrix}1\\1\end{bmatrix},$$

we can find the constants c_1 and c_2 for this particular problem. Plugging in the initial condition we get,

$$\mathbf{y}(0) = c_1 \begin{bmatrix}1\\2\end{bmatrix} \exp(0) + c_2 \begin{bmatrix}2\\3\end{bmatrix} \exp(0) = \begin{bmatrix}c_1 + 2c_2\\2c_1 + 3c_2\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}.$$

Therefore we conclude that,

$$\begin{cases} c_1 + 2c_2 = 1 \\ 2c_1 + 3c_2 = 1 \end{cases}$$

which has solution, $c_1 = -1$ and $c_2 = 1$. Therefore the solution to this problem with initial condition

$$\mathbf{y}(0) = \begin{bmatrix}1\\1\end{bmatrix}$$

is

$$\mathbf{y}(t) = -\begin{bmatrix}1\\2\end{bmatrix} \exp(-t) + \begin{bmatrix}2\\3\end{bmatrix} \exp(-2t).$$

Question B 03 (a)

SOLUTION. From the hint we recall that an eigenvalue, eigenvector pair satisfies,

$$A\mathbf{x} = \lambda\mathbf{x}.$$

We have in our problem that [1,1] transforms to [2,2], which in matrix form is,

$$A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix} = 2\begin{bmatrix}1\\1\end{bmatrix}.$$

Therefore we see that [1,1] is an eigenvector of A with eigenvalue 2. For the other case we have that [1,-1] gets transformed to [0,0]. Therefore,

$$A\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} = 0\begin{bmatrix}1\\-1\end{bmatrix}$$

and so we see that [1,-1] is also an eigenvector with eigenvalue 0. Recall that an $n \times n$ matrix has at most n eigenvalues. Therefore since in this case, n=2, we have found all the eigenvalues of the system. We conclude that A has eigenvalues and eigenvectors,

$$\lambda_1 = 2, \quad \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\lambda_2 = 0, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Question B 03 (b)

Solution. Recall that a matrix is invertible if and only if its determinant, det(A) $\neq 0$. Also recall that the determinant of a matrix is the product of its eigenvalues. In part(a), we deduced that the eigenvalues were $\lambda_1 = 2$ and $\lambda_2 = 0$. Therefore we have that,

$$\det(A) = \lambda_1 \lambda_2 = 2(0) = 0.$$

Therefore we conclude that det(A)=0 and thus that the matrix A is **not** invertible.

Question B 03 (c)

Solution. Let's write our matrix A as

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with yet to be determined entries. In the problem we know that

$$A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and we expand this out to get

$$a - b = 0,$$
$$c - d = 0.$$

Therefore, *a*=*b* and *c*=*d*. We also have that

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

which we can expand out to get

$$a + b = 2a = 2,$$
$$c + d = 2c = 2.$$

Therefore we conclude that *a*=*b*=*c*=*d*=*1*. Therefore our matrix A is

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Notice that we can indeed confirm that this matrix has eigenvalues 2 and 0 and is not invertible.

Question B 04 (a)

SOLUTION. Following the hint, we seek to show the properties of a linear transformation,

$$\begin{aligned} T(\mathbf{x}+\mathbf{y}) &= T(\mathbf{x}) + T(\mathbf{y}), \\ T(c\mathbf{x}) &= cT(\mathbf{x}). \end{aligned}$$

For the first property,

$$\begin{aligned} &T(\mathbf{x}+\mathbf{y}) \\ &= (\mathbf{x}+\mathbf{y}) \times \mathbf{a} \\ &= (\mathbf{x} \times \mathbf{a}) + (\mathbf{y} \times \mathbf{a}) \\ &= T(\mathbf{x}) + T(\mathbf{y}), \end{aligned}$$

where we have used that the cross product is distributive over addition. The second property holds by the compatibility of scalar multiplication with the cross product,

$$\begin{aligned} &T(c\mathbf{x}) \\ &= (c\mathbf{x}) \times \mathbf{a} \\ &= c(\mathbf{x} \times \mathbf{a}) \\ &= cT(\mathbf{x}). \end{aligned}$$

Therefore we have concluded that T is a linear transformation.

Question B 04 (b)

SOLUTION. The columns of a linear transformation matrix, A, are populated by the transformed vectors of each of the basis vectors. In this case the first column of A will be $[1,0,0] \times \mathbf{a}$, the second, $[0,1,0] \times \mathbf{a}$, and the third, $[0,0,1] \times \mathbf{a}$. By computing these cross products, we get,

$$\begin{aligned} [1,0,0] \times \mathbf{a} = [1,2,-1] = [0,1,2] \\ [0,1,0] \times \mathbf{a} = [1,2,-1] = [-1,0,-1] \\ [0,0,1] \times \mathbf{a} = [1,2,-1] = [-2,1,0] \end{aligned}$$

Therefore the linear transformation matrix A which corresponds to T is,

$$A = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}.$$

Notice if we wanted some sort of confirmation of our answer we can pick a random vector, say $\boldsymbol{x}$=[3,4,7], compute $\mathbf{x} \times \mathbf{a}$ and compute $A\boldsymbol{x}$ and make sure they are the same thing.

$$[3, 4, 7] \times [1, 2, -1] = [-18, 10, 2]$$

and

$$A \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -18 \\ 10 \\ 2 \end{bmatrix}$$

which is indeed the same result.

Question B 04 (c)

SOLUTION. Recall from part(b) that we computed the transformation matrix A as

$$A = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}.$$

To compute the determinant we can use co-factor expansion or any special tricks one may have for computing 3×3 determinants.

$$\det(A) = \begin{vmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \\ 2 & -1 & 0 \end{vmatrix} = -1 \begin{vmatrix} -1 & -2 \\ -1 & 0 \end{vmatrix} + 2 \begin{vmatrix} -1 & -2 \\ 0 & 1 \end{vmatrix} = 0.$$

Therefore we conclude that *det(A)=0*.
Is there anyway we could have expected this before hand?
Firstly, notice that the next part of the question part(d) asks for *any* nonzero vector that has a zero cross product. This implies that A has a null space and thus that *det(A)* is zero. However, this reason is somewhat cheating since it involves analyzing the wording of the problem.
A more independent reasoning would be that if the determinant weren't zero, we would have an empty null space (or a unique solution to $A\boldsymbol{x}=\boldsymbol{0}$). However, we know from the properties of cross-products that any vector in a cross product with itself is zero. Therefore we know that $\mathbf{a} \times \mathbf{a} = \mathbf{0}$, and hence $T(\boldsymbol{a}) = A\boldsymbol{a} = 0$, i.e. $\mathbf{a}$ is in the null space of A. Hence *det(A)* must be zero.
In fact, any vector parallel to $\mathbf{a}$ will also have a zero cross product since

$$\mathbf{a} \times (k\mathbf{a}) = k(\mathbf{a} \times \mathbf{a}) = \mathbf{0}.$$

Question B 04 (d)

SOLUTION. In part(b), we have that

$$A = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}$$

and in part(c) we concluded that det(A)=0. Therefore A has a null space and there are many solutions such that A$\mathbf{x}$=$\mathbf{0}$. We could perform some sort of Gaussian elimination and determine the entire null-space but

the question asks for **any** vector and so we are permitted to just guess. In fact we saw in part(c) that any vector parallel to **a** had a zero cross product. Therefore $T(\mathbf{a}) = \mathbf{a} \times \mathbf{a} = \mathbf{0}$.
"Note that, since $T(\boldsymbol{a}) = A\boldsymbol{a}$ we can also verify this using the matrix representation:

$$\begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore a vector such that T($\mathbf{x}$)=$\mathbf{0}$ is $\mathbf{x}$=$\mathbf{a}$ (or any vector of the form $k\boldsymbol{a}$, due to the linearity of T).

Question B 05 (a)

Solution. We want to solve something of the form

$$A\mathbf{x} = \lambda\mathbf{x}(A - \lambda I)\mathbf{x} \qquad = \mathbf{0}$$

where I is the identity matrix. In this case we have $\lambda_1 = 1$ so we have that

$$A - \lambda_1 I = A - I = \begin{bmatrix} 2 & 1 & -3 \\ -1 & -1 & 2 \\ 1 & 0 & -1 \end{bmatrix}.$$

To solve (A-I)$\mathbf{x}$ we write $\mathbf{x}$=[a,b,c] and solve

$$\begin{bmatrix} 2 & 1 & -3 \\ -1 & -1 & 2 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which is equivalent to solving

$$\begin{aligned} 2a + b - 3c &= 0, \\ -a - b + 2c &= 0, \\ a - c &= 0. \end{aligned}$$

This has solution b=c=a for any arbitrary a. Therefore we take a=1, without loss of generality, and our eigenvetor is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Question B 05 (b)

Solution 1. Recall that our matrix is

$$A = \begin{bmatrix} 3 & 1 & -3 \\ -1 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

with one eigenvalue being, $\lambda_1 = 1$. Also recall that the product of the eigenvalues is equal to the determinant of the matrix and the sum is equal to the trace. We have that tr(A)=3 and det(A)=2. Therefore,

$$\begin{aligned} \det(A) &= \lambda_1\lambda_2\lambda_3 = \lambda_2\lambda_3 = 2, \\ \text{tr}(A) &= \lambda_1 + \lambda_2 + \lambda_3 = 1 + \lambda_2 + \lambda_3 = 3. \end{aligned}$$

Therefore we have that

$$\begin{aligned} \lambda_2\lambda_3 &= 2, \\ \lambda_2 + \lambda_3 &= 2. \end{aligned}$$

We can rearrange for λ_3 to get $\lambda_3^2 - 2\lambda_3 + 2 = 0$ which has solution

$$\lambda_3 = 1 \pm i$$

where $i = \sqrt{-1}$. If we were to plug this in to get λ_2 we'd get

$$\lambda_2 = 1 \mp i.$$

This is completely unsurprising since we know that complex eigenvalues occur in complex conjugate pairs. Therefore we take one of them for λ_2 and one for λ_3 to get that the three eigenvalues are

$$\begin{aligned} \lambda_1 &= 1, \\ \lambda_2 &= 1 + i, \\ \lambda_3 &= 1 - i. \end{aligned}$$

SOLUTION 2. If you are unfamiliar of the relationship to determinant and trace to eigenvalues then we can compute the remaining terms through the characteristic polynomial. For

$$A = \begin{bmatrix} 3 & 1 & -3 \\ -1 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

we wish to solve (A-λI)**x**=**0**. For this to have a non-trivial solution we require *det(A-λI)=0*. Therefore we require

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 & -3 \\ -1 & 0-\lambda & 2 \\ 1 & 0 & 0-\lambda \end{vmatrix} = \lambda^3 - 3\lambda^2 + 4\lambda - 2 = 0.$$

We already know that $\lambda_1 = 1$ is a solution, so we can long divide the polynomial by $(\lambda - 1)$ to get an equation for the remaining eigenvalues. If we do this long division we get,

$$\begin{array}{r} \lambda^2 - 2\lambda + 2 \\ \lambda - 1\overline{)\lambda^3 - 3\lambda^2 + 4\lambda - 2} \end{array}$$

and so the other eigenvalues, λ_2 and λ_2 must solve the equation, $\lambda^2 - 2\lambda + 2$ which is the same equation we found in Solution 1. Therefore we know that the solution is

$$\lambda = 1 \pm i$$

giving the three eigenvalues as

$$\begin{aligned}\lambda_1 &= 1,\\ \lambda_2 &= 1+i,\\ \lambda_3 &= 1-i.\end{aligned}$$

Question B 05 (c)

SOLUTION. In part (b) we found that the other two eigenvalues λ_2 and λ_3 were complex conjugates, $1 \pm i$. Therefore, the eigenvectors for λ_2 and λ_3 will also occur in complex conjugate pairs. It is then sufficient to just find one of the vectors and then conjugate it to find the other. We will solve the vector for $\lambda_2 = 1+i$. This means we are seeking a vector for which (A-λ_2I)$\mathbf{x}$=$\mathbf{0}$.

$$A - \lambda_2 I = A - (1+i)I = \begin{bmatrix} 2-i & 1 & -3 \\ -1 & -1-i & 2 \\ 1 & 0 & -1-i \end{bmatrix}.$$

If we denote the vector $\mathbf{x}$ as [a,b,c] then we need to solve

$$\begin{bmatrix} 2-i & 1 & -3 \\ -1 & -1-i & 2 \\ 1 & 0 & -1-i \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which is equivalent to solving

$$\begin{aligned} (2-i)a + b - 3c &= 0,\\ -a + (-1-i)b + 2c &= 0,\\ a + (-1-i)c &= 0. \end{aligned}$$

If we solve this we get a=(1+i)c and b=-ic for arbitrary c. We can set c=1 without loss of generality to get that the eigenvector for λ_2 is

$$\mathbf{x}_2 = \begin{bmatrix} 1+i \\ -i \\ 1 \end{bmatrix}.$$

This means automatically that the eigenvector for λ_3 is just the conjugate of the eigenvector for λ_2. Therefore,

$$\mathbf{x}_3 = \overline{\mathbf{x}_2} = \begin{bmatrix} 1-i \\ i \\ 1 \end{bmatrix}.$$

If you still have trouble computing eigenvectors, it is a worthwhile exercise to actually compute $\mathbf{x}_3$ using the method above and comparing with this result.

Question B 06 (a)

SOLUTION. For a vector $\mathbf{x}$, with components x_1 and x_2, the unique solution is

$$\begin{aligned} x_1 &= 2, \\ x_2 &= 3. \end{aligned}$$

Since there is a unique solution all pivots must be 1. Since we have two equations in two unknowns we have 2 rows and 2 columns. Therefore the augmented matrix would look like

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right]$$

Question B 06 (b)

SOLUTION. We have 2 equations in 3 unknowns so the original matrix is 2×3. Such a system always has infinitely many solutions and hence at least one free variable. Denote the components of the solution vector $\mathbf{x}$ as x_1, x_2, and x_3. The solution we are given suggests $x_3 = t$ as the free variable. Further

$$\begin{aligned} x_1 &= 1 + t \\ x_2 &= 2 \\ x_3 &= t \end{aligned}$$

Therefore $x_1 = 1 + x_3$ or $x_1 - x_3 = 1$. This means we can write our augmented matrix as

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 2 \end{array}\right]$$

Question B 06 (c)

SOLUTION. We have 3 equations in 3 unknowns so we have a 3×3 matrix. In order for a solution to not be found we require an impossible statement to occur. One of the most common ways is to have a line which claims $0 = a$ for some non-zero number a. One such augmented matrix could be

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 7 \end{array}\right].$$

Notice in this example, if the solution vector $\mathbf{x}$ has components x_1, x_2, and x_3 then we are trying to claim that $0x_3 = 7$ which is never true. Therefore this system has no solution.

Question B 06 (d)

SOLUTION. We have 3 equations in 2 unknowns. We know that if we have n unknowns then we require n equations for a unique solution. Therefore, in this case we only need two of the equations to uniquely determine the components of a solution vector $\mathbf{x}$. The only way it will satisfy all three equations is if the third equation happens to be equivalent to one of the other two equations. Since we want no solution, we will require the three equations to produce contradicting information. As an example, take the augmented matrix,

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 1 & 5 \end{array}\right].$$

For a solution vector $\mathbf{x}$, with components x_1 and x_2, the matrix claims

$$\begin{aligned} x_1 &= 2, \\ x_2 &= 3, \\ x_2 &= 5. \end{aligned}$$

Two of the equations are contradicting each other and therefore there is no solution. This matrix is not in reduced row-echelon form because of the last row. If we put it in that form,

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{array}\right]$$

we see the other type of indicator for no solution that occurred in part (c).

Question B 06 (e)

Solution. We want to continue with part (d) but now find all matrices that produce no solution to 3 equations in 2 unknowns. We start with an augmented matrix in the form

$$\left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b \\ c & d & e \end{array}\right]$$

for some numbers a, b, c, d, e. Notice we have put the matrix in reduced row-echelon form up to the first two rows. We did this because we know that if there were only two equations then we could find a unique solution depending the values of a and b. The problem now is for those same a, b that would work in the 2×2 case we want to find a condition on e that creates no solution. Notice we can row-reduce that last row as follows

$$\left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b \\ c & d & e \end{array}\right] R3 - c\overrightarrow{R1} - cR2 \left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & e - ca - db \end{array}\right]$$

where R_i indicates an operation on row i. Notice **if** $e = ca + db$ then we have a row of zeros in the matrix. This implies for a vector $\mathbf{x}$ with components x_1 and x_2 that

$$\begin{aligned} x_1 &= a \\ x_2 &= b \\ 0x_2 &= 0 \end{aligned}$$

which would always hold true. Therefore, in this case there would be a solution. As discussed in part (d) this means that the third equation was secretly a linear combination of the first two equations and thus didn't really add another constraint to the problem. Since we are seeking there to be **no solution** we see then that we must require,

$$e \neq ac + bd.$$

Good Luck for your exams!