Full Solutions
MATH221 December 2008

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How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

Solution. The column space of A is $R(A) = \text{span}\left(\left\{\begin{bmatrix}3\\1\\4\end{bmatrix}, \begin{bmatrix}0\\-2\\-2\end{bmatrix}, \begin{bmatrix}-2\\3\\1\end{bmatrix}\right\}\right)$

The set of vectors spanning the column space of A form a basis if it is linearly independent. We can check this by determining the determinante $\det(A)$ of A. If $\det(A) \neq 0$, then the columns of A are linearly independent.
$\det(A) = (3)(-2)(1) + (0)(3)(4) + (-2)(1)(-2) - (4)(-2)(-2) - (-2)(3)(3) - (1)(1)(0) = 0$
Hence, the columns of A are linear dependent. Indeed, we see that the third column can be expressed as $v_3 = (-2/3)v_1 + (-11/6)v_2$, while the set of the first two column vectors of A is linearly independent.
Therefore the basis of the column space of matrix A is $\left\{\begin{bmatrix}3\\1\\4\end{bmatrix}, \begin{bmatrix}0\\-2\\-2\end{bmatrix}\right\}$.

Question 2 (a)

Solution. The matrix is NOT invertible if its determinant is equal to 0. Expanding along the first row we obtain
$0 = \det(A) = (1)\begin{bmatrix}1 & 0\\-2 & h\end{bmatrix} + (-0)\begin{bmatrix}1 & 0\\0 & h\end{bmatrix} + (-2)\begin{bmatrix}1 & 1\\0 & -2\end{bmatrix} = h - 2(-2) = h + 4$
Therefore $h = -4$ makes the matrix A NOT invertible.

Question 3 (a)

Solution. Recall two key properties of the determinant of a matrix:

1. The determinant is unchanged if a multiple of one row is added to another.
2. Multiplying a row by a number c changes the determinant by a factor of c.

Start by performing two elementary row operations on A:

- Subtract the first row from the second row.
- Subtract the first row from the third row.

The effect of these operations is to transform A to the matrix
$B = \begin{pmatrix}1 & x & x^2\\0 & y-x & y^2-x^2\\0 & z-x & z^2-x^2\end{pmatrix}$.
By the above determinant property *1*, $\det(A) = \det(B)$.
The advantage gained in calculating $\det(B)$ is that its first column only contains a single non-zero entry, which is on the diagonal. Hence, when we calculate the determinant of B we can reduce it to calculating the determinant of a *2 x 2* submatrix,
$\det(B) = (1)\det\begin{pmatrix}y-x & y^2-x^2\\z-x & z^2-x^2\end{pmatrix} = \det\begin{pmatrix}y-x & (y-x)(y+x)\\z-x & (z-x)(z+x)\end{pmatrix}$
We can now use determinant property *2* to help us compute this *2 x 2* determinant,
$\det\begin{pmatrix}y-x & (y-x)(y+x)\\z-x & (z-x)(z+x)\end{pmatrix} = (z-x)(y-x)\det\begin{pmatrix}1 & (y+x)\\1 & (z+x)\end{pmatrix} = (z-x)(y-x)((z+x)-(y+x)) = (z-x)(y-x)(z-y)$.
Therefore, $\det(A) = (z-x)(y-x)(z-y)$.

Question 5 (a)

Solution. To calculate the eigenvalues of A we need to find values of λ such that $A - \lambda I$ has a non-trivial kernel. A good way to check this is to look for zeros of the determinant of the matrix above:

$0 = \det(A - \lambda I) = \det \begin{bmatrix} 5-\lambda & 0 \\ 2 & 1-\lambda \end{bmatrix} = (5-\lambda)(1-\lambda) - 2(0) = (5-\lambda)(1-\lambda).$

Luckily for us, the polynomial is already factorized, so that we can simply read off the eigenvalue $\lambda_1 = 1$ and $\lambda_2 = 5$.

To find the eigenvector $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ corresponding to the eigenvalue $\lambda_1 = 1$, we solve $(A - I)v = 0$. This equation is

$\begin{bmatrix} 4 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$

and thus $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector associated to $\lambda_1 = 1$.

Repeating the same steps for $\lambda_2 = 5$ gives us the eigenvector equation

$\begin{bmatrix} 0 & 0 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$

and thus $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_2 = 5$.

To summarize, 1 is an eigenvalue of A with corresponding eigenvector $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and 5 is an eigenvalue of A with corresponding eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Question 5 (b)

Solution. From part (a), we know that the eigenvalues for A are $\lambda_1 = 1$ and $\lambda_2 = 5$, with corresponding eigenvectors $v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, respectively.

Let $D = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$ denote the matrix of eigenvalues, and $P = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$ the matrix whose columns are the corresponding eigenvectors. Then A is diagonalized via $D = P^{-1}AP$, or, $A = PDP^{-1}$.

This decomposition allows for easy computations of powers of A, as for any integer k, we have $A^k = (PDP^{-1})^k = PD^kP^{-1}$, and $D^k = \begin{pmatrix} 1^k & 0 \\ 0 & 5^k \end{pmatrix}$. Therefore, we compute $P^{-1} = \begin{pmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{pmatrix}$, and multiply the three matrices to obtain

$A^k = PD^kP^{-1} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 5^k \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} 5^k & 0 \\ \frac{1}{2}(5^k - 1) & 1 \end{pmatrix}.$

Then for $k = 1000$, we have $A^{1000} = \begin{pmatrix} 5^{1000} & 0 \\ \frac{1}{2}(5^{1000} - 1) & 1 \end{pmatrix}.$

Question 1 (b)

Solution 1. To determine if b is in the column space of A, we have to make sure that b is a linear combination of the column space of A $A\vec{x} = b$

If there is a solution for $\vec{x}$, then we can write b as a linear combination of the columns of A We can do this by row reduce the matrix [A|b] $\begin{bmatrix} 3 & 0 & 1 \\ 1 & -2 & 1 \\ 4 & -2 & 1 \end{bmatrix} \xrightarrow[R2 \to R1 - 3R2]{R3 \to R3 - 4R2} \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 6 & -2 \\ 0 & 6 & -3 \end{bmatrix} \xrightarrow[R3 \to R2 - R3]{R2 \to R2/6} \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{-1}{3} \\ 0 & 0 & 1 \end{bmatrix}$

The last row of the reduced row form shows that there is no solution for this augmented matrix. Therefore, vector b $= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is not in the column space of matrix A $= \begin{pmatrix} 3 & 0 & -2 \\ 1 & -2 & 3 \\ 4 & -2 & 1 \end{pmatrix}$

Solution 2. The fastest way is to check if the matrix, whose first two columns are the basis of the column space of A and the third column is the vector b, has determinante 0. Thereby we check if these three vectors are linear dependent. If yes, then b lies in the column space of A.

$\det\left(\begin{pmatrix} 3 & 0 & 1 \\ 1 & -2 & 1 \\ 4 & -2 & 1 \end{pmatrix}\right) = 3*(-2)*1+0*1*4+1*1*(-2)-4*(-2)*1-(-2)*1*3+1*1*0 = 6 \neq 0$
Hence, the vectors are linear independent and b does not lies in the column space of A.

Question 2 (b)

Solution. This is **true**. A is not invertible when $\det(A) = 0$. We know that the characteristic polynomial is $p(\lambda) = \det(A - \lambda I)$. If $\lambda = 0$, then $p(\lambda) = \det(A)$. Therefore, $\det(A) \neq 0$ only if $p(\lambda) \neq 0$. However, we know that $p(0) = 0$ because the eigenvalue, λ is the root of the polynomial equation. $p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n)$. Therefore, if $\lambda = 0$, $p(\lambda) = \det(A) = 0$ and this shows that if $\lambda = 0$, then A is not invertible

Question 3 (b)

Solution. We want to make use of the result of 3a. The given matrix A is not in the right form to use the formula of 3a. However by multiplying the first row by *1/3*, and multiplying the third row by *1/10* will give us the desired form with $x = 1$, $y = 5$, and $z = 7$. More precisely,
$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 5 & 25 \\ 1 & 7 & 49 \end{pmatrix} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/10 \end{pmatrix} A.$
Recall that multiplying a row of a matrix by a number c changes the determinant by a factor of c. For this question, this means that
$\left(\frac{1}{3}\right)\left(\frac{1}{10}\right)\det(A) = \det\begin{pmatrix} 1 & 1 & 1 \\ 1 & 5 & 25 \\ 1 & 7 & 49 \end{pmatrix}.$
By 3a, $\det\begin{pmatrix} 1 & 1 & 1 \\ 1 & 5 & 25 \\ 1 & 7 & 49 \end{pmatrix} = (7-1)(5-1)(7-5) = 48.$
So $\det(A) = 30(48) = 1440$.

Question 4 (a)

Solution. The vector u can be written as $u = v - \frac{1}{2}w$, so indeed $u \in \text{span}\{v, w\} = W$.
To show that u and w form an orthogonal basis for W, we first check (or see by observation) that their inner-product is zero, $\langle u, w\rangle = 2(0) + 1(0) + 2(0) + 1(0) = 0$, so they are orthogonal.
Now recall that in an inner-product space, orthogonal vectors are linearly independent, so u and w are linearly independent vectors in W.
Moreover, u and w span W, precisely because $v = u + \frac{1}{2}w$, so every vector in W is an element in $\text{span}\{u, w\}$.
Therefore u and w form a basis for W.

Question 4 (b)

Solution. The vector in W which is closest to b will be the"orthogonal projection of b onto W". From part (a), we know that the vectors $u = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ and $w = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \end{pmatrix}$ form an orthogonal basis for W. Let us make an *orthonormal* basis by normalizing u and w to

$$e_1 = \frac{u}{\|u\|} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} \quad \text{and} \quad e_2 = \frac{w}{\|w\|} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}.$$

Then the orthogonal projection of b onto W is

$$P_W(b) = \langle b, e_1\rangle e_1 + \langle b, e_2\rangle e_2 = \frac{1}{\sqrt{2}}(1\cdot 1 + 3\cdot 1)e_1 + \frac{1}{\sqrt{2}}(2\cdot 1 + 4\cdot 1)e_2 = 2\sqrt{2}e_1 + 3\sqrt{2}e_2 = \begin{pmatrix}2\\3\\2\\3\end{pmatrix}.$$

.

Thus $\begin{pmatrix}2\\3\\2\\3\end{pmatrix}$ is the vector in W closest to the vector $b = \begin{pmatrix}1\\2\\3\\4\end{pmatrix}$.

Question 6 (a)

SOLUTION. To begin with, we assume that people live in either Alberta or BC. Thus, each year 10 We represent a transition matrix as follows: T=[T_{ij}], where T_{ij} is the We let i = Alberta =1 and j = BC = 2, i.e. T_{11} =T_{12}= T_{21} = and finally T_{22} Thus, we have $T = \begin{bmatrix}0.90 & 0.10\\0.15 & 0.85\end{bmatrix}$.

Question 6 (b)

SOLUTION. The population distribution after n number of years from the initial year is $P(n) = P_o * T^n$, where $P = \begin{bmatrix}P_{Alberta} & P_{BC}\end{bmatrix}$ and $P_o = \begin{bmatrix}InitialP_{Alberta} & initialP_{BC}\end{bmatrix} = \begin{bmatrix}100,000 & 100,000\end{bmatrix}$

At n, $P(n) = P_o * T^n = \begin{bmatrix}100,000 & 100,00\end{bmatrix}\begin{bmatrix}(0.90) & (0.10)\\(0.15) & (0.85)\end{bmatrix}^n$

We take the limit as n->infinity.

We evaluate lim n->infinity T^n by first finding matrices D and Q such that $T = Q * D * Q^{-1}$. So, $T^n = Q*D^n*Q^{-1}$, where D is a diagonal matrix, whose diagonal entries are eigenvalues of T, and their corresponding eigenvectors are the columns of the matrix Q.

Finding Eigenvalues

$$(A - I\lambda) = \begin{bmatrix}0.90-\lambda & 0.10\\0.15 & 0.85-\lambda\end{bmatrix} \Rightarrow (0.9-\lambda)(0.85-\lambda) - (0.1 * 0.15) =$$

$\lambda^2 - 1.7\lambda - 0.735 = (\lambda - 1)(\lambda - 0.75)) \Rightarrow \lambda_1 = 1, \lambda_2 = 0.75$

Finding Eigenvectors

$$A\vec{v} = \lambda\vec{v} \Rightarrow (A - I\lambda)\vec{v} = \vec{0}$$

$(A - I\lambda_1)\vec{v_1} = \begin{bmatrix}-0.1 & 0.1\\0.15 & -0.15\end{bmatrix}\vec{v_1} = \vec{0} \Rightarrow v_{11} = v_{21} \Rightarrow \vec{v_1} = \begin{bmatrix}1\\1\end{bmatrix}$

$(A - I\lambda_2)\vec{v_2} = \begin{bmatrix}0.15 & 0.1\\0.15 & 0.1\end{bmatrix}\vec{v_2} = \vec{0} \Rightarrow -1.5v_{12} = v_{22} \Rightarrow \vec{v_2} = \begin{bmatrix}1\\-1.5\end{bmatrix}$

So we have $D = \begin{bmatrix}\lambda_1 & 0\\0 & \lambda_2\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 0.75\end{bmatrix}$ and $Q = \begin{bmatrix}\vec{v_1} & \vec{v_2}\end{bmatrix} = \begin{bmatrix}1 & 1\\1 & -1.5\end{bmatrix}$.

Since $Q * Q^{-1} = I, T^n = (Q * D * Q^{-1})^n = (Q * D * Q^{-1}) * (Q * D * Q^{-1}) * ... * (Q * D * Q^{-1}) * (Q * D * Q^{-1})$ $= Q * D * (Q^{-1} * Q) * D * (Q^{-1} * Q) * D * ... * (Q^{-1} * Q) * D * Q^{-1} = Q * D^n * Q^{-1}$.

Notice that $D^n = \begin{bmatrix} 1^n & 0 \\ 0 & (0.75)^n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ as n goes to infinity

Hence, $P(n) = P_o * Q * D^n * Q^{-1}$ as n goes to infinity $= \begin{bmatrix} 100,000 & 100,000 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1.5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.6 & 0.4 \\ 0.4 & -0.4 \end{bmatrix}$

$= \begin{bmatrix} 120,000 & 80,000 \end{bmatrix}$

Hence, the limiting distribution distribution is $P_{Alberta}^{limit}$=120,000people and P_{BC}^{limit}=80,000 people.

Question 7 (a)

Solution. To find the characteristic polyomial, we need to compute $\det(A - xI)$. We have

$(A - xI) = \begin{bmatrix} 1-x & 0 & -1 \\ 2 & 3-x & -1 \\ 0 & 6 & -x \end{bmatrix}$

and so the characteristic polynomial is

$$\begin{aligned} \det(A - xI) &= (1-x)[(3-x)(-x) - (6)(-1)] - 0 + (-1)[(2)(6) - 0(3-x)] \\ &= (1-x)[-3x + x^2 + 6] + (-1)(12 - 0) \\ &= (1-x)[x^2 - 3x + 6] - 12 \\ &= -x^3 + 4x^2 - 9x - 6. \end{aligned}$$

Question 7 (b)

Solution. The statement is true: If A and B are similar, then there is an invertible matrix P for which $A = P^{-1}BP$. Then we can write

$$\begin{aligned} xI - A &= xI - P^{-1}BP \\ &= P^{-1}(xI - B)P. \end{aligned}$$

Therefore we compute that

$\det(xI - A) = \det(P^{-1}(xI - B)P) = \det(P)^{-1}\det(xI - B)\det(P) = \det(xI - B)$.

This equality means precisely that the characteristic polynomials for A and B are identical. Since the eigenvalues of a matrix are the zeros of its characteristic polynomial, and the characteristic polynomials for A and B are identical, we conclude that the eigenvalues of A and B are the same.

Question 8 (a)

Solution. We know that $Av = \lambda v$, thus $A(Bv) = ABv = BAv = B(Av) = B(\lambda v) = \lambda(Bv)$.

This shows that Bv is the eigenvector of A with the eigenvalue λ, such that $A(Bv) = \lambda(Bv)$.

Since B is invertible and v is not the zero vector, we know that Bv is not a zero vector. We can conclude that Bv is an eigenvector of A with the same eigenvalue.

Question 8 (b)

Solution. The statement is true. In order to prove T is a linear transformation, we need to show that for any $c \in \mathbb{R}$ and vectors $v_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, v_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$, the properties

- $T(cv_1) = cT(v_1)$, and
- $T(v_1 + v_2) = T(v_1) + T(v_2)$

are satisfied .
Let us check these properties:

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T(c\begin{pmatrix} x_1\\ y_1 \end{pmatrix})=T\begin{pmatrix} cx_1\\ cy_1 \end{pmatrix}=\begin{pmatrix} cx_1+cy_1\\ cx_1-cy_1 \end{pmatrix}=\begin{pmatrix} c(x_1+y_1)\\ c(x_1-y_1) \end{pmatrix}=c\begin{pmatrix} x_1+y_1\\ x_1-y_1 \end{pmatrix}=cT\begin{pmatrix} x_1\\ y_1 \end{pmatrix} ,

-

\begin{align} T(\begin{pmatrix} x_1\\ y_1 \end{pmatrix}+\begin{pmatrix} x_2\\ y_2 \end{pmatrix})&= T\begin{pmatrix} x_1+x_2\\ y_1+y_2 \end{pmatrix}\\ &=\begin{pmatrix} x_1+x_2+y_1+y_2\\ x_1+x_2-y_1-y_2 \end{pmatrix}\\ &= \begin{pmatrix} x_1+y_1\\ x_1-y_1 \end{pmatrix}+\begin{pmatrix} x_2+y_2\\ x_2-y_2 \end{pmatrix}\\ &= T\begin{pmatrix} x_1\\ y_1 \end{pmatrix}+T\begin{pmatrix} x_2\\ y_2 \end{pmatrix}. \end{align}
Therefore, T is a linear transformation.

Question 9 (a)

Solution. In general, if $\{e_1, \dots, e_n\}$ is a basis for $\mathbb{R}^n$, and T is a linear transformation mapping $\mathbb{R}^n \to \mathbb{R}^n$, we know that for each j, $T(e_j)$ is a linear combination of the basis vectors, i.e.
$T(e_j) = \sum_{i=1}^n a_{ij} e_i, \quad a_{ij} \in \mathbb{R}$.
Recall T is completely determined by its action on the basis vectors. So the coordinates a_{ij} completely characterize the matrix T, in the sense that if we know all the a_{ij}, then we know exactly the linear transformation T. Therefore, the matrix whose entry in the ith row and jth column is a_{ij} is called"the matrix of T relative to the basis" $\{e_1, \dots, e_n\}$.
For this problem, we are given that $T(b_1) = 2 \cdot b_1 + 1 \cdot b_2$ and $T(b_2) = 0 \cdot b_1 + 1 \cdot b_2$. Therefore, writing
$T(b_j) = \sum_{i=1}^2 a_{ij} b_i \quad (j = 1, 2)$
we identify $a_{11} = 2, a_{21} = 1, a_{12} = 0, a_{22} = 1$. Hence the matrix of T relative to the given basis B is
$[T]_B = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$.

Question 9 (b)

Solution. Let $B' = \{e_1, e_2\}$ denote the standard basis for $\mathbb{R}^2$. We determined in part (a) that the matrix of T relative to the basis $B = \{b_1, b_2\}$ was $[T]_B = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$, and we want to determine $[T]_{B'}$, the matrix of T relative to the basis B'. We can begin by writing
$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$.
Let $P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.The above equality gives us a relation $[v]_{B'} = P[v]_B$ (check this!) for the coordinates of a vector $v \in \mathbb{R}^2$ with respect to each basis (recall that if $v = a_1 b_1 + a_2 b_2$, where $a_1, a_2 \in \mathbb{R}$, then the coordinates of v with respect to the basis B is $[v]_B = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$).
Now, we have by definition, $[Tv]_B = [T]_B [v]_B$ and $[Tv]_{B'} = [T]_{B'} [v]_{B'}$.
Therefore, we can write

$$\begin{aligned} P^{-1}[T]_{B'}[v]_{B'} = P^{-1}[Tv]_{B'} &= [Tv]_B \\ &= [T]_B [v]_B = [T]_B P^{-1} [v]_{B'}, \end{aligned}$$

and so multiplying by P, we find $[T]_{B'}[v]_{B'} = P[T]_B P^{-1}[v]_{B'}$, from which we conclude that $[T]_{B'} = P[T]_B P^{-1}$. To determine $[T]_{B'}$, we just need compute

$$P^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

and hence

$$[T]_{B'} = P^{-1}[T]_B P = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ 6 & -2 \end{pmatrix}.$$

Question 10 (a)

SOLUTION. The statement is true. To prove that $Av_1, \ldots, Av_n$ is a basis for $\mathbb{R}^n$, we actually only need to verify that $Av_1, \ldots, Av_n$ are linearly independent vectors. (Why? Recall that for a finite-dimensional vector space V, a linearly independent list of vectors with length equal to the dimension of V is a spanning list, and hence a basis. In our case $V = \mathbb{R}^n$ and $\dim V = n$.)

To prove linear independence, assume

$$c_1 A v_1 + \ldots + c_n A v_n = 0,$$

where $c_1, \ldots, c_n \in \mathbb{R}$. We want to show that $c_1 = \ldots = c_n = 0$. To begin, write the above equation as

$$A(c_1 v_1 + \ldots + c_n v_n) = 0.$$

We are given that A is invertible, so apply A^{-1} to the above equation to find

$$c_1 v_1 + \ldots + c_n v_n = A^{-1}A(c_1 v_1 + \ldots + c_n v_n) = A^{-1}0 = 0.$$

Now the vectors $v_1, \ldots, v_n$ are linearly independent because they are a basis. This means that the only solution to $c_1 v_1 + \ldots + c_n v_n = 0$ is $c_1 = \ldots = c_n = 0$, which is exactly what we wanted to show.
Therefore $Av_1, \ldots, Av_n$ is a linearly independent list of vectors with length $n = \dim\mathbb{R}^n$, and thus a basis.

Question 10 (b)

SOLUTION. The statement is **false**.

To construct a counterexample, we need to find a *2 x 2* matrix which has characteristic polynomial $(\lambda - 2)^2$ but has only **one** linearly independent eigenvector. The first matrix that comes to mind which has characteristic polynomial $(\lambda - 2)^2$ is the scalar multiple of the identity matrix,

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

However this clearly isn't a counterexample because it is a diagonal matrix. Can we modify it so that it isn't a diagonal matrix, but still has the same characteristic polynomial? A first simple modification is to investigate the diagonalizability of

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

Our requirement that A has only one linearly independent eigenvector means there can be only one linearly independent solution to

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which is clearly the case, as $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the only linearly independent eigenvector. Therefore,

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

is a counterexample to the statement.

Good Luck for your exams!