Full Solutions
MATH200 December 2011

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Educational Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Exam Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Educational Resources.

Question 1 (a)

Easiness: 40/100

SOLUTION. The level curves of $f(x,y)$ satisfy $f(x,y)=C$ for arbitrary values of C such that a solution to $f(x,y)=C$ exists. Starting with that equation we have

$$e^{-x^2+4y^2}=C.$$

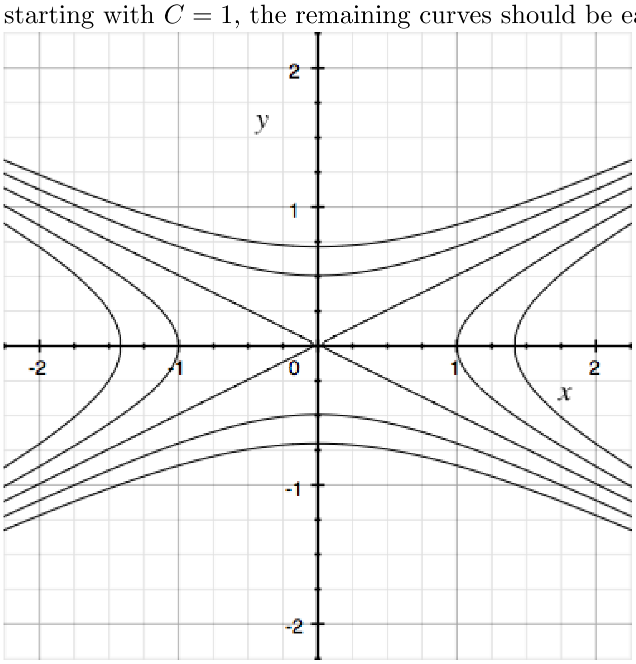
We notice that we can take the natural logarithm of both sides giving us

$$-x^2+4y^2=\ln(C),$$

where $\ln(C)$ is just an arbitrary constant as well! In other words, the level curves of $e^{-x^2+4y^2}$ look the same as those of $-x^2+4y^2=g(x,y)$. The only difference between the level curves in each situation is the values of C corresponding to each specific curve. The level curves of $g(x,y)$ are hyperbolae.
If we choose five different values of C

$$C=e^{-2},e^{-1},1,e^{1},e^{2}$$

and draw the resulting level curves, we get the figure below. (Hint: If you begin drawing the level curves starting with $C=1$, the remaining curves should be easier to draw)



Question 1 (b)

Easiness: 95/100

SOLUTION. The equation of the tangent plane is given by evaluating

$$\mathbf{n}\cdot\mathbf{v}=0$$

where $\mathbf{v}=[x-2,y-1,z-f(2,1)]^T$ and $\mathbf{n}$ is a normal vector to the surface $z=f(x,y)$ at $(x,y)=(2,1)$. The normal vector at the point (x,y) is given by

$$\mathbf{n}(x,y)=\begin{bmatrix} f_x(x,y)\\ f_y(x,y)\\ -1\end{bmatrix}=\begin{bmatrix} -2xe^{-x^2+4y^2}\\ 8ye^{-x^2+4y^2}\\ -1\end{bmatrix}\quad\rightarrow\quad \mathbf{n}(2,1)=\begin{bmatrix} -4\\ 8\\ -1\end{bmatrix}.$$

Evaluating $\mathbf{n} \cdot \mathbf{v} = 0$ gives

$$\mathbf{n} \cdot \mathbf{v} = -4(x-2) + 8(y-1) - 1(z-1) = 0.$$

We simplify the above equation (although it is not necessary) and get the tangent plane to $f(x, y)$ at $(2, 1)$:

$$-4x + 8y - z = -1.$$

Question 1 (c)

Easiness: 95/100

SOLUTION. From part (b), we found that the tangent plane to the surface $z = f(x, y)$ at the point $(2, 1)$ was

$$-4x + 8y - z = -1.$$

Since the point $(1.99, 1.01)$ is very close to $(2, 1)$, the tangent plane approximation gives a good estimate for the true value of $f(1.99, 1.01)$. Plugging in $(x, y) = (1.99, 1.01)$ into the tangent plane equation we get

$$\begin{aligned} -4(1.99) + 8(1.01) - z &= -1 \\ -7.96 + 8.08 - z &= -1 \\ 0.12 - z &= -1 \quad \rightarrow \quad z = 1.12 \end{aligned}$$

Therefore,

$$f(1.99, 1.01) \approx 1.12.$$

Question 2 (a)

SOLUTION. The variable z is a function of both x, y which, in turn, are both functions of t. Thus, any change in z in response to a change in t will be the result of how x and y change with t. We can write the derivative of z with respect to t using the chain rule:

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

We evaluate the partial derivatives of x, y with respect to t:

$$\frac{\partial x}{\partial t} = -r\sin(t), \quad \frac{\partial y}{\partial t} = r\cos(t).$$

And thus,

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x} \cdot (-r\sin(t)) + \frac{\partial f}{\partial y} \cdot r\cos(t).$$

Question 2 (b)

Easiness: 3/100

SOLUTION. To evaluate this derivative we merely need to recognize that we apply the first derivative operator to our answer from part (a):

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial t}\right).$$

We need to apply the chain rule again as in part (a):

$$\begin{aligned}
\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial t}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial t}\right)\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial t}\right)\frac{\partial y}{\partial t} \\
&= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial t}\right)\cdot(-r\sin(t)) + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial t}\right)\cdot r\cos(t) \\
&= -y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial t}\right) + x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial t}\right).
\end{aligned}$$

where we have use the fact that $x = r\ cos(t)$, $y = r\ sin(t)$ to write the last equation above. From part (a), we found that

$$\begin{aligned}
\frac{\partial z}{\partial t} &= \frac{\partial f}{\partial x}\cdot(-r\sin(t)) + \frac{\partial f}{\partial y}\cdot r\cos(t) \\
&= -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}.
\end{aligned}$$

Using this, we continue to evaluate the partial derivatives above

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial t}\right) &= -y\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial y} + x\frac{\partial^2 f}{\partial y\partial x} \\
\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial t}\right) &= -\frac{\partial f}{\partial x} - y\frac{\partial^2 f}{\partial y\partial x} + x\frac{\partial^2 f}{\partial y^2}
\end{aligned}$$

finally giving us

$$\begin{aligned}
\frac{\partial^2 z}{\partial t^2} &= -y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial t}\right) + x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial t}\right) \\
&= -y\left(-y\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial y} + x\frac{\partial^2 f}{\partial y\partial x}\right) + x\left(-\frac{\partial f}{\partial x} - y\frac{\partial^2 f}{\partial y\partial x} + x\frac{\partial^2 f}{\partial y^2}\right) \\
&= y^2\frac{\partial^2 f}{\partial x^2} + x^2\frac{\partial^2 f}{\partial y^2} - 2xy\frac{\partial^2 f}{\partial y\partial x} - y\frac{\partial f}{\partial y} - x\frac{\partial f}{\partial x}
\end{aligned}$$

Writing the result above in terms of r, t gives the final answer:

$$\frac{\partial^2 z}{\partial t^2} = r^2\sin^2(t)\frac{\partial^2 f}{\partial x^2} + r^2\cos^2(t)\frac{\partial^2 f}{\partial y^2} - 2r^2\cos(t)\sin(t)\frac{\partial^2 f}{\partial y\partial x} - r\sin(t)\frac{\partial f}{\partial y} - r\cos(t)\frac{\partial f}{\partial x}$$

Question 3 (a) **Easiness: 23/100**

SOLUTION. We let the vector $\mathbf{v}$ be the velocity vector. If the bee travels along a path in $\mathbb{R}^3$ given by $[x(t), y(t), z(t)]^T$, then $\mathbf{v}$ is given by

$$\mathbf{v} = \left[\tfrac{d}{dt}x(t), \tfrac{d}{dt}y(t), \tfrac{d}{dt}z(t)\right]^T$$

We are told that the bee is travelling along a path defined by the intersection of the two surfaces

$$3z + x^2 + y^2 = 2$$

and $z = x^2 - y^2$. Therefore, we can differentiate each relationship with respect to t to get a pair of equations describing the velocity of the bee at any point along the curve:

$$3\frac{dz}{dt} + 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0, \quad \frac{dz}{dt} = 2x\frac{dx}{dt} - 2y\frac{dy}{dt}.$$

We are interested in analyzing the velocity at (1,1,0), so plug this point into the above equations:

$$3\frac{dz}{dt} + 2\frac{dx}{dt} + 2\frac{dy}{dt} = 0, \quad \frac{dz}{dt} - 2\frac{dx}{dt} + 2\frac{dy}{dt} = 0.$$

Solving the two equations to get $\frac{dx}{dt}$, $\frac{dy}{dt}$ in terms of $\frac{dz}{dt}$ gives:

$$\frac{dx}{dt} = -\frac{1}{2}\frac{dz}{dt}, \quad \frac{dy}{dt} = -\frac{dz}{dt}$$

So at the point (1,1,0), we can write the velocity vector in terms of $z'(t)$:

$$\mathbf{v} = \left(\frac{d}{dt}z(t)\right)\left[-\tfrac{1}{2}, -1, 1\right]^T.$$

We know that the speed of the bee is 6 at the point (1,1,0), so we use the equation

$$|\mathbf{v}| = \left(\sqrt{\left(\frac{d}{dt}z(t)\right)^2}\right)\left(\sqrt{\left(-\frac{1}{2}\right)^2 + (-1)^2 + (1)^2}\right) = 6$$

Solving for $\frac{dz}{dt}$ gives

$$\left(\sqrt{\left(\frac{d}{dt}z(t)\right)^2}\right) = \frac{6}{\sqrt{9/4}} = 4 \quad \rightarrow \quad \frac{d}{dt}z(t) = \pm 4.$$

Taking the positive solution, (since the bee is travelling in the direction of increasing z, we can write the velocity vector

$$\mathbf{v} = \left[-2, -4, 4\right]^T$$

Question 3 (b)

Easiness: 90/100

SOLUTION. To evaluate the change in temperature, T, experienced by the bee with respect to time at time 2, we need to evaluate the full derivative of $T = T(x(t), y(t), z(t), t)$ with respect to t and substitute in the value $t = 2$. Writing the expression for the full time derivative gives:

$$\begin{aligned}\frac{dT}{dt} &= \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt} \\ &= \frac{\partial T}{\partial t} + \nabla T \cdot \mathbf{v}\end{aligned}$$

where $\mathbf{v}$ is the velocity vector of the bee. Evaluating the terms in the above equation we find:

$$\frac{\partial T}{\partial t} = 2y, \quad \nabla T = [y-3, x+2t, 1]^T.$$

Note that we don't need to re-evaluate $\mathbf{v}$ since we can just use our results from part (a) here. At time $t = 2$, the bee is at $[1, 1, 0]$, so the above terms satisfy,

$$\frac{\partial T}{\partial t} = 2, \quad \nabla T = [-2, 5, 1]^T$$

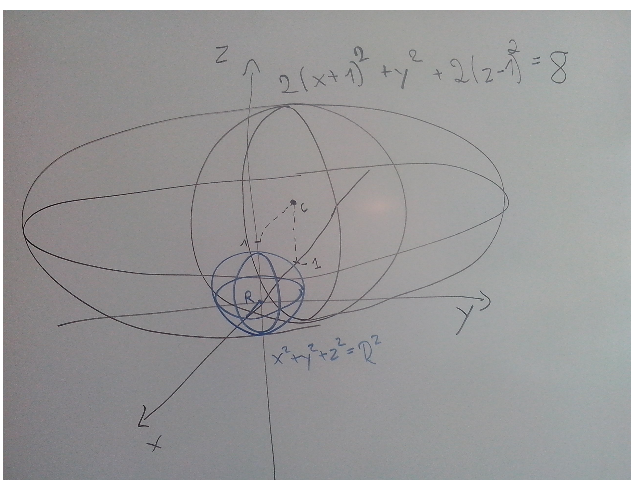
at $t = 2$. Using this and the fact that $\mathbf{v} = [-2, -4, 4]$, we evaluate the change in temperature:

$$\frac{dT}{dt} = 2 + [-2, 5, 1]^T \cdot [-2, -4, 4]^T = 2 + 4 - 20 + 4 = -10.$$

Therefore, the bee is experiencing a change in temperature of -10 temperature units per second at t = 2.

Question 4

SOLUTION. If the sphere with radius in question R is enclosed by the ellipse, and the radius is maximal, then the surface of the ellipse and the sphere just touch in one point.



The tangent planes to the surface $2(x+1)^2 + y^2 + 2(z-1)^2 = 8$ and the sphere with radius R are parallel at the points where the surface and the sphere touch. Hence the normal vectors to the tangent planes are also parallel.

Regarding the surface equation and the sphere equation as level sets of functions $f, g : \mathbb{R}^3 \to \mathbb{R}$, then the normal vectors to these tangent planes are given by the gradients of the functions f and g.
The sphere equation with the origin as center and radius R is $x^2 + y^2 + z^2 = R^2$. Then $f(x, y, z) = 2(x+1)^2 + y^2 + 2(z-1)^2$ and $g(x, y, z) = x^2 + y^2 + z^2$. We are looking for x, y, z, λ such that the gradients are parallel,

$$\nabla f = \lambda \nabla g$$
$$\begin{pmatrix} 4(x+1) \\ 2y \\ 4(z-1) \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

The second line yields that $\lambda = 1$ or $y = 0$.

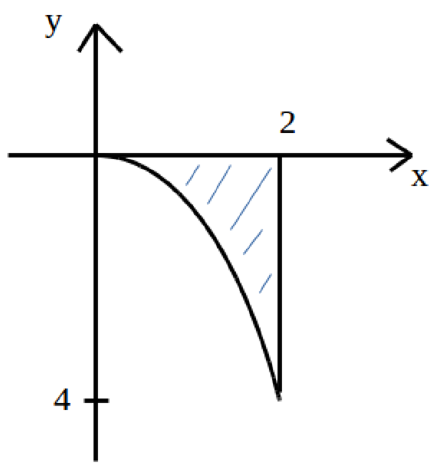
- $\lambda = 1$ leads to $x = -2, z = 2$ and using the surface equation we obtain the points $p_1 = (-2, 2, 2)$, $p_2 = (-2, -2, 2)$
- $y = 0$ leads to $x = -z$ and using the surface equation we obtain the points $p_3 = (\sqrt{2} - 1, 0, -\sqrt{2} + 1)$, $p_4 = (-\sqrt{2} - 1, 0, \sqrt{2} + 1)$.

These four points are candidates for where the ellipse and the sphere touch. Since we are searching for the radius R such that the sqhere is enclosed in the ellipse, we need the point p_i with the shortest distance to the origin.

$$||p_1|| = \sqrt{12}$$
$$||p_2|| = \sqrt{12}$$
$$||p_3|| = \sqrt{6 - 4\sqrt{2}}$$
$$||p_4|| = \sqrt{6 + 4\sqrt{2}}$$

Since $\sqrt{6 - 4\sqrt{2}}$ is the smallest of these numbers, this is the radius R of the sqhere.

Question 5 (a)



SOLUTION. (i)

(ii) Since $\cos(x^3)$ is hard to integrate with respect to x, we change the order of integration. Using the sketch of (i) and that $x = \sqrt{-y}$ is equivalent to $y = -x^2$ (and $x \geq 0$) we find that

$$\begin{aligned}
\int_{-4}^{0}\int_{\sqrt{-y}}^{2} \cos(x^3)\ \mathrm{d}x\mathrm{d}y &= \int_{0}^{2}\int_{-x^2}^{0} \cos(x^3)\ \mathrm{d}y\mathrm{d}x \\
&= \int_{0}^{2} \cos(x^3)[y]_{-x^2}^{0}\ \mathrm{d}x \\
&= \int_{0}^{2} \cos(x^3)x^2\ \mathrm{d}x \\
&= [\frac{1}{3}\sin(x^3)]_0^2 = \frac{1}{3}\sin(8) - \frac{1}{3}\sin(0) \\
&= \frac{1}{3}\sin(8)
\end{aligned}$$

Question 5 (b)

Solution. We use the change of variables from cartesian coordinates to polar coordinates

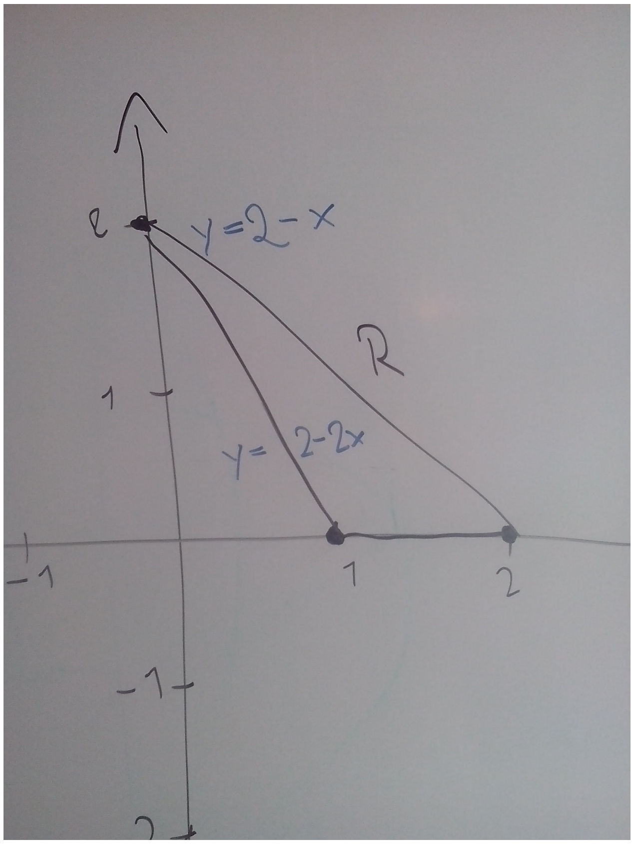
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos(\phi) \\ r\sin(\phi) \end{pmatrix}$$

The region of integration in polar coordinates is $D = \{(r,\phi)|\ r \in [0,\sqrt{2}],\ \phi \in [0,\frac{\pi}{4}]\}$. The function $y\sqrt{x^2+y^2}$ becomes $r^2\sin(\phi)$. The volume factor for polar coordinates is r. Hence,

$$\begin{aligned}
\iint_D y\sqrt{x^2+y^2}\ \mathrm{d}A &= \int_0^{\sqrt{2}}\int_0^{\frac{\pi}{4}} r^2\sin(\phi)r\ \mathrm{d}\phi\mathrm{d}r \\
&= \int_0^{\sqrt{2}} r^3[-\cos(\phi)]_0^{\frac{\pi}{4}}\ \mathrm{d}r \\
&= \left[\frac{1}{4}r^4\right]_0^{\sqrt{2}} (-\cos(\frac{\pi}{4}) + \cos(0)) \\
&= 1 - \frac{1}{\sqrt{2}}
\end{aligned}$$

Question 6

Solution. Since the region of integration is a triangle with vertices $(1,0),\ (2,0),\ (0,2)$ the region is bounded by the function $y = 2-x$ and $y = 2-2x$, see sketch



We want to integrate with respect to x first, hence, solving for x, the integration boundaries become

$$\begin{gathered}\tfrac{1}{2}(2-y) \leq x \leq 2-y \\ 0 \leq y \leq 2\end{gathered}$$

and the region is $R = \{(x,y) \in \mathbb{R}^2 |\ 1 - \dfrac{y}{2} \leq x \leq 2-y,\ 0 \leq y \leq 2\}$
To calculate $\overline{y} = \dfrac{\iint_R y\rho(x,y)\ \mathrm{d}A}{\iint_R y\rho(x,y)\ \mathrm{d}A}$ which is the y coordinate of the center of mass, we calculate at first the integral in the numerator.

$$\begin{aligned}\iint_R y\rho(x,y)\ \mathrm{d}A &= \int_0^2 \int_{1-\frac{y}{2}}^{2-y} y^3\ \mathrm{d}x\mathrm{d}y \\ &= \int_0^2 y^3(2-y-1+\frac{y}{2})\ \mathrm{d}y \\ &= \int_0^2 (y^3 - \frac{y^4}{2})\ \mathrm{d}y \\ &= \frac{2^4}{4} - \frac{2^5}{10} = \frac{4}{5}\end{aligned}$$

The integral in the denominator is very similar

$$\iint_R \rho(x,y)\ \mathrm{d}A = \int_0^2 \int_{1-\frac{y}{2}}^{2-y} y^2\ \mathrm{d}x\mathrm{d}y = \int_0^2 (y^2 - \frac{y^3}{2})\ \mathrm{d}y = \frac{2}{3}$$

Hence
$\overline{y} = \frac{\frac{4}{5}}{\frac{2}{3}} = \frac{6}{5}$

Question 7

SOLUTION. The plane $z + y = 1$ intesects $z = 0$ at the line $y = 1, z = 0$. Since the region is is the first octant, we know that $x, y, z \geq 0$.
Using this information and the sketch in the hint, we find the region E as $E = \{(x, y, z) \in \mathbb{R}^3 |\ 0 \leq z \leq 1 - y,\ x^2 \leq y \leq 1,\ 0 \leq x \leq 1\}$
Hence the integral is

$$\begin{aligned}
\iiint_E x \ \mathrm{d}V &= \int_0^1 \int_{x^2}^1 \int_0^{1-y} x \ \mathrm{d}z\mathrm{d}y\mathrm{d}x \\
&= \int_0^1 \int_{x^2}^1 x(1 - y) \ \mathrm{d}y\mathrm{d}x \\
&= \int_0^1 x \left[y - \frac{y^2}{2} \right]_{x^2}^1 \mathrm{d}x \\
&= \int_0^1 \frac{x}{2} - x^3 + \frac{x^5}{2} \ \mathrm{d}x \\
&= \left[\frac{x^2}{4} - \frac{x^4}{4} + \frac{x^6}{12} \right]_0^1 = \frac{1}{12}
\end{aligned}$$

Question 8 (a)

SOLUTION. The volume of the snowman is the volume of the body V_b, the volume of the head V_h minus the volume of the intesection V_i.
First we calculate V_i, which is the volume of the intersection of the two balls. The intersection is bounded below from the surface $x^2 + y^2 + (z - 4)^2 = 4$ and above from $x^2 + y^2 + z^2 = 12$. Since this shape is radially symmetric to the z-axes, we change to cylindrical coordinates. This means $x^2 + y^2 = r^2$ and $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos(\phi) \\ r\sin(\phi) \\ z \end{pmatrix}$.
We find the intersection of the surfaces $r^2 + (z - 4)^2 = 4$ and $r^2 + z^2 = 12$ to find the upper bound for the radius r:
$4 = r^2 + (z - 4)^2 = r^2 + z^2 - 8z + 16 = 12 - 8z + 16$
which yields $z = 3$ and $r = \sqrt{3}$.
With these bounds we write V_i as
$V_i = \{(r, \phi, z) |\ 4 - \sqrt{4 - r^2} \leq z \leq \sqrt{12 - r^2}, 0 \leq r \leq \sqrt{3}, 0 \leq \phi \leq 2\pi\}$.
Recall that the volume factor for cylindrical coordinates is r.

$$
\begin{aligned}
\iiint_{V_i} 1 \, \mathrm{d}V &= \int_0^{\sqrt{3}} \int_{4-\sqrt{4-r^2}}^{\sqrt{12-r^2}} \int_0^{2\pi} r \mathrm{d}\phi \mathrm{d}z \mathrm{d}r \\
&= \int_0^{\sqrt{3}} 2\pi r(\sqrt{12-r^2} - 4 + \sqrt{4-r^2}) \, \mathrm{d}r \\
&= \pi \left[-\frac{2}{3}(12-r^2)^{\frac{3}{2}} - 4r^2 - \frac{2}{3}(4-r^2)^{\frac{3}{2}} \right]_0^{\sqrt{3}} \\
&= \pi \left[-\frac{2}{3}9^{\frac{3}{2}} - 12 - \frac{2}{3} + \frac{2}{3}12^{\frac{3}{2}} + 0 + \frac{2}{3}4^{\frac{3}{2}} \right] = \pi \frac{4}{3}(12\sqrt{3} - 19)
\end{aligned}
$$

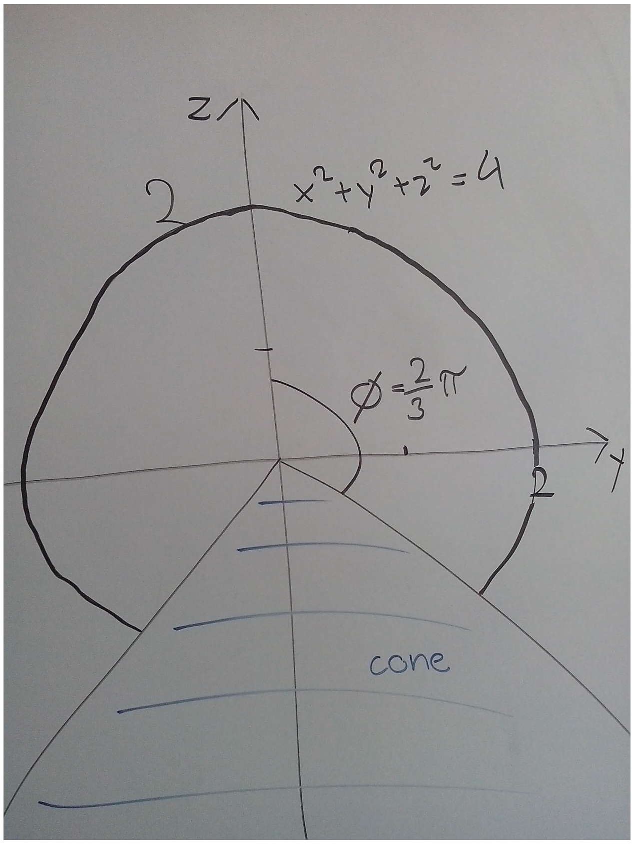
The ball V_b which makes the body has volume $V_b = \frac{4}{3}\pi\sqrt{12}^3 = 32\sqrt{3}\pi$

The ball V_h which makes the head has volume $V_h = \frac{4}{3}\pi 2^3 = \frac{32}{3}\pi$

Hence, the volume of the snowman is
$V_b + V_h - V_i = \pi \left(36 + 16\sqrt{3}\right)$

Question 8 (b) i

SOLUTION. Region (1) is part of head or body?
In the integral (1), spherical coordinates are used, where the radius covers the interval $\rho \in [0, 2]$, the horizontal angle covers $\theta \in [0, 2\pi]$ which is the whole circle and the vertical angle covers $\phi \in [0, 2/3\pi]$. Hence, the region of integration is a ball with radius 2, where a cone is cut out from below.
Since the head of the snowman is a ball with radius 2, this region must be part of the head.
Finding the equation of the enclosing sphere
Since the region of integration is part of the ball with radius 2, whose centre is in the origin, it is enclosed by the sphere $x^2 + y^2 + z^2 = 2^2 = 4$.
Finding the equation of the enclosing cone
It is left to find the equation of the cone which is cut out from the region of integration. The cone is defined through the angle ϕ, as seen in the following figure, which shows a cross-section of the region of integration through the zy-plane.



To find the equation of the cone, we must rewrite the expression $\phi = \frac{2}{3}\pi$ into cartesian coordinates. The relationship between spherical and cartesian coordinates is

$$\begin{aligned} x &= \rho\cos(\theta)\sin(\phi) \\ y &= \rho\sin(\theta)\sin(\phi) \\ z &= \rho\cos(\phi) \end{aligned}$$

where $\rho = \sqrt{x^2+y^2+z^2}$. Pluging in $\phi = \frac{2}{3}\pi$ gives for z

$$\begin{aligned} z &= \rho\cos\left(\frac{2}{3}\pi\right) = -\frac{1}{2}\rho = -\frac{1}{2}\sqrt{x^2+y^2+z^2} \\ \Rightarrow\ z^2 &= \frac{1}{4}(x^2+y^2+z^2) \\ \Rightarrow\ 4z^2 &= x^2+y^2+z^2 \\ \Rightarrow\ 3z^2 &= x^2+y^2 \end{aligned}$$

which is the required cone equation.

Question 8 (b) ii

SOLUTION. Integral (2) is written in cylindrical coordinates, where the limit of the variables of integration is $\theta \in [0, 2\pi]$ for the horizontal angle, $r \in [0, \sqrt{3}]$ for the radius $r = \sqrt{x^2+y^2}$ and $z \in \left[\sqrt{3}r, 4 - \frac{r}{\sqrt{3}}\right]$ for the vertical z axis.

Finding the equation of the enclosing cone from below
The variables r, θ are in dependent and the angle covers the whole horizontal circle. The variable z is depending or r. Hence, the region of integration is bounded from below by the equation

$$\begin{aligned} z &= \sqrt{3}r = \sqrt{3}\sqrt{x^2+y^2} \\ z^2 &= 3x^2+3y^2 \end{aligned}$$

which is a cone-equation.
Finding the equation of the enclosing cone from above
Further, the region of integration is bounded from above by the equation

$$\begin{aligned} z &= 4 - \frac{r}{\sqrt{3}} = 4 - \frac{\sqrt{x^2+y^2}}{\sqrt{3}} \\ 3(z-4)^2 &= x^2+y^2 \end{aligned}$$

which is also a cone-equation.
Region of integration is part of body of head?
The lower cone $z^2 = 3x^2+3y^2$ contains the point $(0,0,0)$ which is the center of the sphere $x^2+y^2+z^2 = 12$, the body of the snowman. Hence, a region above this cone is part of the body of the snowman.
The upper cone $3(z-4)^2 = x^2+y^2$ contains the point $(0,0,4)$ which is the center of the sphere $x^2+y^2+(z-4)^2 = 4$, the head of the snowman. Hence, a region below this cone is part of the head of the snowman.
So, the region of integration contains part of the head and the body of the snowman.

Question 8 (b) iii

SOLUTION. Region of integration is part of head or body?
In the integral (3), spherical coordinates are used, where the radius covers the interval $\rho \in [0, 2\sqrt{3}] = [0, \sqrt{12}]$, the horizontal angle covers $\theta \in [0, 2\pi]$ which is the whole circle and the vertical angle covers $\phi \in \left[\frac{\pi}{6}, \pi\right]$.
Hence, the region of integration is a ball with radius $\sqrt{12}$, where a cone is cut out from above. Since the body of the snowman is a ball with radius $\sqrt{12}$,this region must be part of the body .
Finding the equation of the enclosing sphere
Since the region of integration is part of the ball with radius $\sqrt{12}$, whose centre is in the origin, it is enclosed by the sphere
$x^2+y^2+z^2 = 12$
Finding the equation of the enclosing cone
It is left to find the equation of the cone which is cut out from the region of integration. The cone is defined through the angle ϕ.
To find the equation of the cone, we must rewrite the expression $\phi = \frac{1}{6}\pi$ into cartesian coordinates. The relationship between spherical and cartesian coordinates is

$$\begin{aligned} x &= \rho\cos(\theta)\sin(\phi) \\ y &= \rho\sin(\theta)\sin(\phi) \\ z &= \rho\cos(\phi) \end{aligned}$$

where $\rho = \sqrt{x^2+y^2+z^2}$. Pluging in $\phi = \frac{1}{6}\pi$ gives for z

$$\begin{aligned} z &= \rho\cos\left(\frac{1}{6}\pi\right) = \frac{\sqrt{3}}{2}\rho = \frac{\sqrt{3}}{2}\sqrt{x^2+y^2+z^2} \\ \Rightarrow\ z^2 &= \frac{3}{4}(x^2+y^2+z^2) \\ \Rightarrow\ 4z^2 &= 3x^2+3y^2+3z^2 \\ \Rightarrow\ z^2 &= 3x^2+3y^2 \end{aligned}$$

which is the required cone equation.

Good Luck for your exams!