Full Solutions
MATH110 April 2011

December 4, 2014

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Educational Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

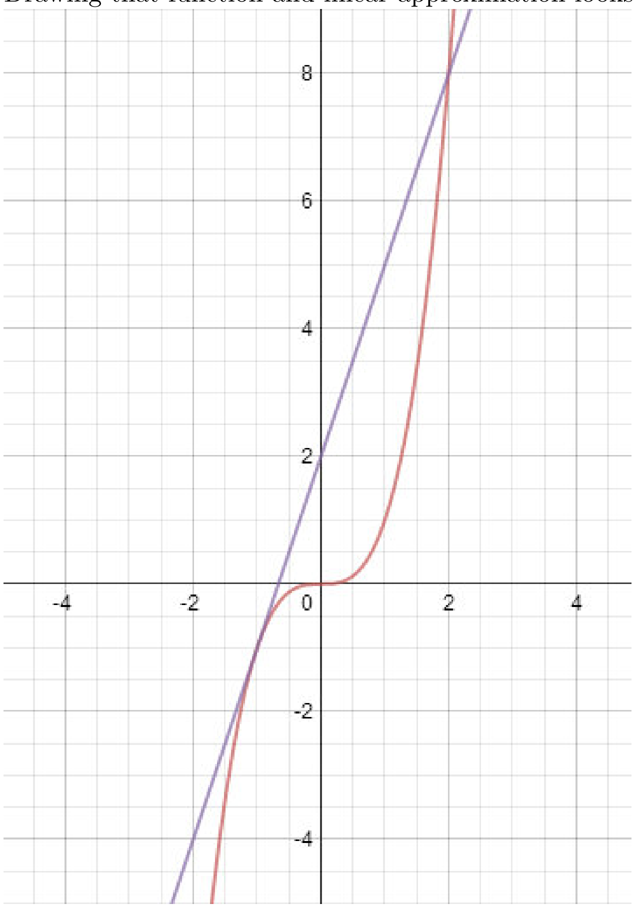
1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
    - Re-do related homework and webwork questions.
    - The Math Exam Resources offers mini video lectures on each topic.
    - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
    - Do as many exam simulations as possible.

    Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Educational Resources.

Question 1 (a)

SOLUTION. The statement is **false**, so a linear approximation L to a function f may satisfy $L(x) = f(x)$ for more than one value of x. To prove this, we need an example of such a function. In particular, we would like to find a function with a tangent line (the linear approxmiation) that intersects the function at another point.
One example would be the function $f(x) = x^3$, with the linear approximation at the point, say, $(-1, 1)$. Drawing that function and linear approximation looks like this:

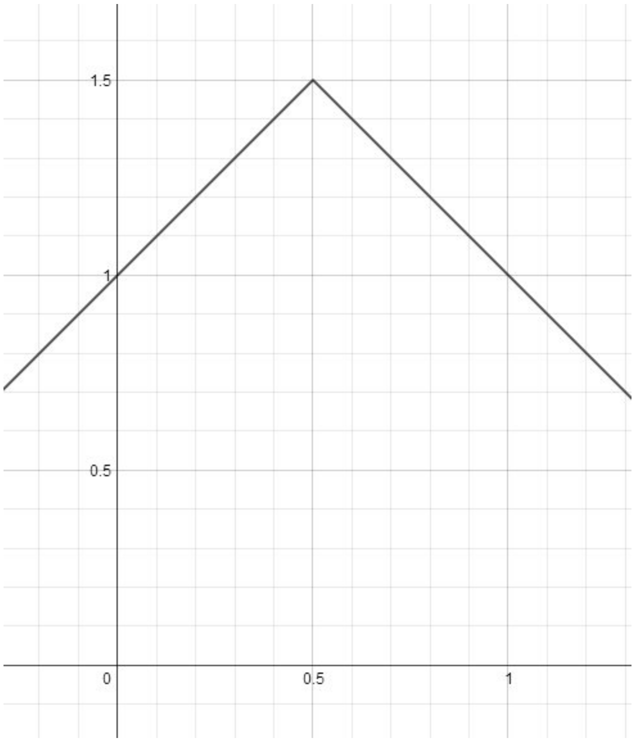


So we see that $L(x) = f(x)$ both at $x = -1$ and at $x = 2$. So the original statement is false.
If you want to make it even simpler, choose f to be linear, e.g. $f(x) = 17x+42$. Then the linear approximation L coincides with f at all points.

Question 1 (b)

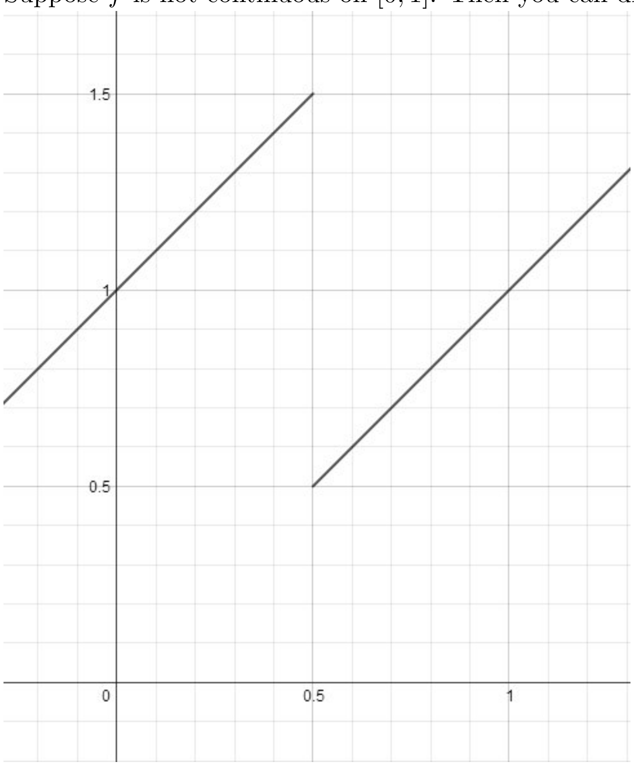
Easiness: 90/100

SOLUTION. This statement is **false**. It reads like the conclusion of Rolle's Theorem; however, Rolle's Theorem has two conditions: that f be continuous on a closed interval (in this case the interval $[0, 1]$) and differentiable on the same open interval (in this case $(0, 1)$). If we drop either of these two conditions, then the conclusion of Rolle's Theorem does not necessarily hold.
For example, suppose f is not differentiable on $(0, 1)$. Then we can draw the picture:



While this graph satisfies $f(0) = f(1) = 1$ there is no point where the derivative is zero, because the derivative is either 1, -1, or undefined.

Suppose f is not continuous on $[0, 1]$. Then you can draw a picture like this:



Where clearly the derivative is never zero, even though $f(0) = f(1) = 1$.

Either example (or a similar example) would suffice to prove the statement false.

Question 1 (c)

SOLUTION. An inflection point is a point where f'' is zero or undefined **and** the graph changes concavity. A critical point is a point c where either $f'(c) = 0$ or $f'(c)$ does not exist.
It is possible for a point to satisfy both definitions. Such a function has a point where both the first and second derivative are equal to zero. One such simple function is $f(x) = x^3$. It has a horizontal tangent line at $x = 0$ (making $x = 0$ a critical point) and also changes concavity at $x = 0$.
Thus the statement is **true**.

Question 2 (a)

SOLUTION. Simply trying to evaluate the limit as written gives the following:

$$\lim_{x\to\pi^+} \frac{\sin x}{x-\pi} = \frac{0}{0}$$

This is an indeterminate form and means that we can use L'Hopital's Rule. Differentiating the numerator and denominator of the function, we get the new limit:

$$\lim_{x\to\pi^+} \frac{\cos x}{1} = \lim_{x\to\pi^+} \cos x$$

Simply evaluating this limit gives

$$\lim_{x\to\pi^+} \cos x = -1$$

which is also the value of the original limit.

Question 2 (b)

SOLUTION. We can evaluate this limit directly. We start by evaluating what happens to $\sin x$ and $\cos x$ as $x \to 0^+$. Because this is a directional limit from the right, we will have $\sin x \to 0^+$ (on a sine graph, as you approach $x = 0$ from the right, the y value is approaching zero *from above*, hence approaching 0^+) and $\cos x \to 1^-$ (similarly, approaching $x = 0$ from the right on a cosine graph means approaching $y = 1$ from below, hence 1^-). Replacing $\sin x$ with y and $\cos x$ with z, we can thus rewrite the limit as a fraction of the limits:

$$\lim_{x\to 0^+} \frac{\ln(\sin x)}{\ln(\cos x)} = \frac{\lim_{y\to 0^+} \ln(y)}{\lim_{z\to 1^-} \ln(z)}$$

Again, the fact that this is a directional limit means that for the first limit, as $y \to 0^+$, $\ln(y) \to -\infty$ and for the second limit, as $z \to 1^-$, $\ln(z) \to 0^-$. So we have:

$$\frac{\lim_{y\to 0^+} \ln(y)}{\lim_{z\to 1^-} \ln(z)} = \frac{-\infty}{0^-}$$

Recall that when a function goes to zero from the left in the denominator, this means that we eventually divide by a very small negative value (for example, -1/100000). Using fraction rules, this very small value in the denominator is the same as multiplying by a very large negative value (in my example, -100000) in the numerator. Thus going to zero in the denominator corresponds to multiplying the numerator by a very large number. Since our numerator is already going to negative infinity, the fact that we have 0^- in the denominator will make the value of the limit greater, and turn it back from negative to positive. Thus the value of this limit is $+\infty$,

$$\lim_{x\to 0^+} \frac{\ln(\sin x)}{\ln(\cos x)} = +\infty$$

Question 3 (a)

Easiness: 100/100

SOLUTION. `Suppose a function` f `is continuous on the interval` *[a,b]*.
`Suppose further that` f `is differentiable on the interval (a,b).`
`Then there exists some c in the interval (a,b) satisfying`

$f'(c) = \frac{f(b)-f(a)}{b-a}$.

Question 3 (b)

Easiness: 62/100

SOLUTION. Because the function $\sin x$ is continuous on the interval $[0, b]$ and is differentiable on the interval $(0, b)$, we can apply the Mean Value Theorem. Thus we know that there is some c in the interval $(0, b)$ so that:

$$\begin{aligned} \cos(c) &= \frac{\sin(b) - \sin(0)}{b - 0} \\ &= \frac{\sin(b)}{b} \end{aligned}$$

Question 3 (c)

Easiness: 37/100

SOLUTION. In the previous question, we showed that for some c in the interval $(0, b)$

$$\cos(c) = \frac{\sin(b)}{b}$$

Now we know that for all possible values of c, $\cos(c)$ is bounded above by 1, or

$$\cos(c) \leq 1$$

Substituting what we know from part (b), this is the same as

$$\frac{\sin(b)}{b} \leq 1$$

Because $b > 0$ we can multiply both sides by b to get

$$\sin(b) \leq b$$

Which completes our proof.

Question 4 (a)

SOLUTION. Implicitly differentiating $x^4 + y^4 = 2$ gives

$$4x^3 + 4y^3 \cdot y' = 0$$

And then we solve for y':

$$\begin{aligned} 4y^3 \cdot y' &= -4x^3 \\ y' &= \frac{-x^3}{y^3} \end{aligned}$$

Question 4 (b)

SOLUTION. To find y'' we simply need to differentiate the expression $y' = -x^3/y^3$ from part (a). Using implicit differentiation and the quotient rule, we have:

$$\begin{aligned} y'' &= \frac{-3x^2(y^3) - (-x^3)3y^2y'}{(y^3)^2} \\ &= \frac{3x^2y^2(-y + xy')}{y^6} \\ &= \frac{3x^2(-y + xy')}{y^4} \end{aligned}$$

Question 4 (c)

SOLUTION. It will be complicated to simply plug our expressions for y' and y'' into the formula for curvature. Instead it might be wiser to calculate the actual value of y' and y'' at the point $(1, 1)$.
For y' we have

$$y' = \frac{-(1)^3}{(1)^3} = -1$$

And for y'' we have

$$y'' = \frac{3(1)^2(-(1) + (1)(-1))}{(1)^4} = -6$$

Plugging these values of $y' = -1$ and $y'' = 6$ into the formula for K we get:

$$K = \frac{-6}{(1 + (-1)^2)^{3/2}} = \frac{-6}{2^{3/2}} = \frac{-3}{\sqrt{2}}.$$

Question 5

Easiness: 5/100

SOLUTION. We know that the amount of ^{90}Sr remaining at a time t is modeled by the equation $S(t) = S_0 e^{kt}$, where S_0 is the initial quantity and k is a rate of decay. Our first task is to find k. Using the information that the half-life of ^{90}Sr is 29 years, we know that when $t = 29$, $S(29) = S_0/2$. Plugging this into our equation of exponential decay, we get:

$$\begin{aligned} S_0 e^{k\cdot 29} &= \frac{S_0}{2} \\ e^{k\cdot 29} &= \frac{1}{2} \\ k\cdot 29 &= \ln\left(\frac{1}{2}\right) \\ k &= \frac{\ln\left(\frac{1}{2}\right)}{29} \end{aligned}$$

Note that $k < 0$ as expected: the amount of ^{90}Sr declines over time. Putting k back into the formula $S(t) = S_0 e^{kt}$, we get

$$S(t) = S_0 e^{(\ln(1/2)/29)t}$$

You can leave this expression as is, or simplify it to get

$$S(t) = S_0 \left(1/2\right)^{t/29}$$

To find the percentage of ^{90}Sr remaining today, we simply plug in the amount of time that has elapsed since the initial explosion in 1986; in 2011, that is 25 years. So

$$S(25) = S_0(1/2)^{25/29}$$

The percentage of S_0 remaining is thus

$$\frac{S(25)}{S(0)} = \frac{S_0(1/2)^{25/29}}{S_0} = (1/2)^{25/29}$$

Question 6

Easiness: 50/100

SOLUTION. We are given the volume of a cone.

$$V = \frac{1}{3}\pi r^2 h$$

If we differentiate this expression as it is, we will end up with an expression relating dV/dt to both variables of r and h. To avoid this, we need to write the volume formula in terms of just r or h, by writing one variable in terms of another. The question states that the length of the stalagmite is always five times its radius, so we can write $h = 5r$, or, alternatively, $r = h/5$.
(i) Finding an expression in terms of h: Using $r = h/5$, we plug it into our volume formula to get

$$\begin{aligned} V &= \frac{1}{3}\pi\left(\frac{h}{5}\right)^2 h \\ &= \frac{1}{3}\pi\frac{h^3}{25} \end{aligned}$$

Differentiating yields

$$\begin{aligned}\frac{dV}{dt} &= \frac{1}{3}\pi\frac{3h^2}{25}\frac{dh}{dt}\\ &= \frac{\pi h^2}{25}\frac{dh}{dt}\end{aligned}$$

And because we know that $dh/dt = 0.13$ mm/year, we finally reach an expression relating the rate of change in volume to the height h

$$\frac{dV}{dt} = \frac{\pi h^2}{25}0.13$$

(ii) Finding an expression in terms of r: To find an expression in terms of r we simply use the relationship $h = 5r$ to get

$$\begin{aligned}V &= \frac{1}{3}\pi r^2(5r)\\ &= \frac{5}{3}\pi r^3\end{aligned}$$

Differentiating yields

$$\begin{aligned}\frac{dV}{dt} &= \frac{5}{3}\pi 3r^2\frac{dr}{dt}\\ &= 5\pi r^2\frac{dr}{dt}\end{aligned}$$

To find an expression for dV/dt solely in terms of r, we need to find dr/dt. We can do this by differentiating the equation $h = 5r$ to get

$$\frac{dh}{dt} = 5\frac{dr}{dt}$$

$$\frac{0.13}{5} = \frac{dr}{dt}$$

Plugging this into our formula for dV/dt gives

$$\begin{aligned}\frac{dV}{dt} &= 5\pi r^2\frac{0.13}{5}\\ &= \pi 0.13r^2\end{aligned}$$

in terms of r, as desired.

Question 7 (a)

Easiness: 100/100

SOLUTION. We know that our line passes through the point (1,1) and has slope m. Using the point-slope formula of line, we get

$$y - 1 = m(x - 1).$$

Solving for y gives us $y = mx - m + 1$

Question 7 (b)

SOLUTION. Our line has a y-intercept when $x = 0$. Plugging this into our equation of the line, we get

$$y = m(0) - m + 1 = -m + 1.$$

So our y-intercept is $(0, -m + 1)$.
Similarly, our x-intercept occurs when $y = 0$. Setting our equation equal to zero and solving for x, we get

$$0 = mx - m + 1$$

We see that if *m=0*, then there is no x-intercept (the line is horizontal). Otherwise

$$x = \frac{m-1}{m}$$

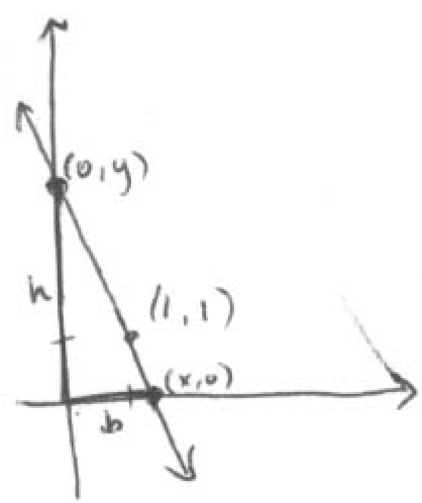
and our x-intercept is $\left(\frac{m-1}{m}, 0\right)$.

Question 7 (c)

SOLUTION. The question is asking us to minimize the area of a triangle. So the function that we will be minimizing is

$$A = \frac{1}{2}bh$$

where b is the triangle's base and h is its height. In order to optimize (in this case, minimize), we need to reduce this formula to a function of one variable.
If we draw a picture of a line with negative slope that passes through (1,1), we will see the triangle formed by the axes and the line and can label its base and height.



So the base of the triangle is equal to the distance between 0 and the x-intercept of the line (which is simply the value of the x-intercept) and similarly, the height is equal to the y-intercept. We calculated the x and y intercept in the previous part of the question, so now we can simply plug them into our area formula for b and h.

$$\begin{aligned} A &= \frac{1}{2}\left(\frac{m-1}{m}\right)(-m+1) \\ &= \frac{-m^2+2m-1}{2m} \end{aligned}$$

We now take the derivative of A...

$$\begin{aligned} A' &= \frac{(-2m+2)(2m)-(-m^2+2m-1)(2)}{4m^2} \\ &= \frac{-2(m-1)(m)+(m-1)^2}{2m^2} \\ &= \frac{(m-1)(-2m+m-1)}{2m^2} \\ &= \frac{(m-1)(-m-1)}{2m^2} \end{aligned}$$

...and find its critical points. The derivative is undefined when $m=0$. Setting it equal to zero:

$$0 = \frac{(m-1)(-m-1)}{2m^2}$$

$$0 = (m-1)(-m-1)$$

yields additional critical points of $m=-1$, and $m=1$. However, as stated in the question, $m<0$, so the only valid critical point is $m=-1$. To check that this is a minimum, you could use the first or second derivative test.

Question 8 (a)

Solution. The function $\ln(x)$ is defined for all $x>0$. Therefore, in order for $\ln(4-x^2)$ to be defined, we need $4-x^2>0$. If we factor the left side of the inequality, we get

$$(2-x)(2+x) > 0$$

For what values of x is this inequality true?
If $x=2$ or -2, then the left side of the inequality is zero, which is not allowed. So 2 and -2 cannot be part of our domain. Similarly, if $x>2$ or $x<-2$, then the left side of the inequality will be negative, contradicting the inequality. So these values are also excluded from our domain. If we choose $-2<x<2$ we find that the left hand side of the inequality is positive, satisfying the inequality. Thus our domain is $-2<x<2$, which written in interval notation is $(-2,2)$.

Question 8 (b)

Solution. To find vertical asymptotes, we will take the limit of the function at $x=2$ and $x=-2$. Note that we are taking one-sided limits because the function is not defined on both sides of the point.

$$\lim_{x\to -2^+} \ln(4-x^2)$$

$$\lim_{x\to 2^-} \ln(4-x^2)$$

As x goes to -2 (from the right) or 2 (from the left), the expression $4-x^2 \to 0^+$. So using $y = 4-x^2$ we can rewrite both of these limits as

$$\lim_{y\to 0^+} \ln(y)$$

which is simply $-\infty$. So the function has vertical asymptotes at both $x=-2$ and $x=2$.
Since the function is only defined in the interval $(-2,2)$, which is bound on both sides, the function does not have horizontal asymptotes.

Question 8 (c)

SOLUTION. At a function's x-intercept, its y-value is equal to zero. Thus to find our x-intercepts, we need to set the function equal to 0 and solve for x.

$$0 = \ln(4-x^2)$$

We raise both sides by e to cancel the logarithm and get:

$$\begin{aligned} e^0 &= e^{\ln(4-x^2)} \\ 1 &= 4-x^2 \\ x^2 &= 3 \\ \pm\sqrt{3} &= x \end{aligned}$$

So our two x-intercepts are at $x = \pm\sqrt{3}$.
The y-intercept of a function occurs when $x=0$. Therefore the y-intercept is $y = \ln 4$.

Question 8 (d)

SOLUTION. To determine where f is increasing or decreasing, we look at the sign of the first derivative. The first derivative, calculated using the chain rule, is:

$$f'(x) = \frac{-2x}{4-x^2} = \frac{2x}{x^2-4}$$

It has critical points at $x = 0, -2, 2$. The values *-2* and *2* are not part of the domain, but $x =$ *0* is. We first note that the denominator is always negative on the domain $(-2,2)$. Hence, the sign of the derivative is always the opposite sign of the numerator. For $x <$ *0* we see that $f'(x) > 0$, while for $x >$ *0* we see that $f'(x) < 0$.
Hence, f is increasing on *(-2,0)* and decreasing on *(0,2)*.
"Note. This also means that f has a local maximum at $x = 0$.

Question 8 (e)

SOLUTION. We find the second derivative of f by using the quotient rule on the first derivative:

$$\begin{aligned} f''(x) &= \frac{2(x^2-4) - 2x(2x)}{(x^2-4)^2} \\ &= \frac{-2x^2-8}{(x^2-4)^2} \end{aligned}$$

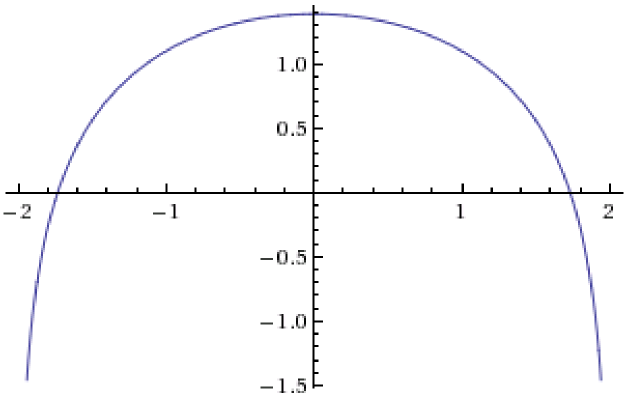
We find possible inflection points by setting $f''(x)$ equal to zero and also by determining where it is undefined. In this case, f'' is never zero and is undefined when $x = -2, 2$. Because these points are not included in our domain, f has no inflection points, i.e. f does not change its concavity. Therefore, it suffices to find the concavity at any point in the domain, e.g. at $x = 0$:

$$f''(0) = \frac{-8}{16} < 0$$

and hence f is concave down on its whole domain.

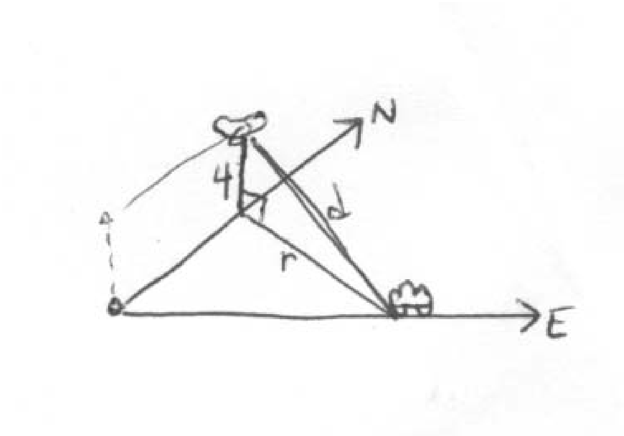
Question 8 (f)

SOLUTION. Using the information from parts (a) - (e), we draw the graph as follows.
First, we know that the function is undefined outside of $(-2, 2)$ so we will shade out that portion of the graph.
Now we add in the two x-intercepts at $-\sqrt{3}$ and $\sqrt{3}$. We also include the information about the asymptotes - both $x = -2$ and $x = 2$ were asymptotes, with the function going to $-\infty$.
Finally, incorporating the information from (d) and (e) about increase/decrease and concavity, and the maximum at $x = 0$, we connect the dots and finish the drawing.



Question 9

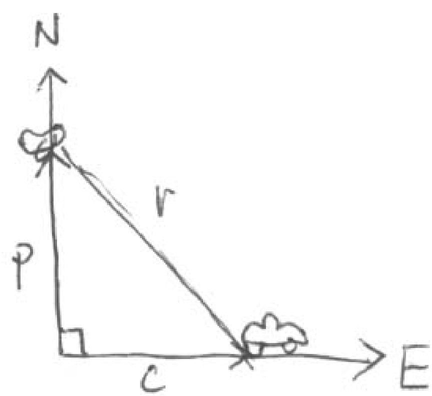
SOLUTION. First I will find a formula describing the distance between the car and the plane.
I can draw a triangle whose sides are as follows: the plane's height above the ground, the distance between the plane and the car (denoted d), and the distance between the position of the car and the plane if both were on the ground (r). The triangle looks like this:



Because my triangle is a right triangle, I can relate its sides using the Pythagorean theorem. So I have

$$d^2 = 4^2 + r^2$$

If I look down on the plane and car and ignore the height difference, I can find the distance between their positions on the ground (denoted r before) by using another right triangle, drawn here



Where p is the distance the plane has traveled and c is the distance the car has traveled. These are all related in the formula

$$r^2 = p^2 + c^2$$

I can then plug this into my distance formula found above to get an equation in terms of the car's distance (c) and the plane's distance (p), which relates to the information I was given in the statement of the problem.

$$d^2 = 4^2 + p^2 + c^2$$

I can now differentiate with respect to time to get:

$$2d\frac{dd}{dt} = 2p\frac{dp}{dt} + 2c\frac{dc}{dt}$$

or

$$\frac{dd}{dt} = \left(p\frac{dp}{dt} + c\frac{dc}{dt}\right)/d$$

Now, we already have that $\frac{dp}{dt} = 600$ km/h and $\frac{dc}{dt} = 100$ km/h. To solve for $\frac{dd}{dt}$ we need p, c and d. However, we know that the plane and car have been traveling for one hour, and have therefore gone 600 km and 100 km respectively. These are the values of p and c. To find d, we plug $p = \mathit{600}$ and $c = \mathit{100}$ into our original distance formula above to get $d = \sqrt{16 + 600^2 + 100^2}$. Plugging these all into our equation that describes dd/dt we get:

$$\frac{dd}{dt} = \frac{600 \cdot 600 + 100 \cdot 100}{\sqrt{16 + 600^2 + 100^2}} \approx 608.26\,\text{km/h}$$

(As mentioned in the problem, you don't need to compute the answer, I just added it to give you a sense of the value.)

Good Luck for your exams!