Full Solutions
MATH110 April 2014

April 22, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
    - Re-do related homework and webwork questions.
    - The Math Education Resources offers mini video lectures on each topic.
    - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
    - Do as many exam simulations as possible.

    Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

SOLUTION. **False.** The statement would be true if the function is continuous. So many non-continuous functions can be found as counterexamples, such as:

$$f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

or

$$f(x) = \begin{cases} -|x| & x \neq 0 \\ -1 & x = 0 \end{cases}$$

Question 1 (b)

SOLUTION. **True.** Let a, b be two of the roots. Then f is continuous on $[a, b]$ since f is differentiable everywhere, and f is differentiable on (a, b). Rolle′s Theorem states that there must be a point with zero slope in the interval (a, b).

Question 1 (c)

SOLUTION. **True.** Look at $f''(x) = -\sin(x)$. There are infinitely many point of interest ($f''(x) = 0$) at $x = 0, \pm\pi, \pm 2\pi, \ldots$. Furthermore, at each point of interest, the second derivative changes sign. So each point of interest is an inflection point.

Question 1 (d)

SOLUTION. **False.** Only one side of the function needs to have an infinite limit. The other side could be defined. Counterexamples include:

$$f(x) = \begin{cases} \frac{1}{x-1} & x < 1 \\ 2 & x \geq 1 \end{cases}$$

or

$$f(x) = \begin{cases} 2 & x \leq 1 \\ \ln(x-1) & x > 1 \end{cases}$$

The statement is **False**.

Question 2 (a)

SOLUTION 1. Direct substitution gives “$\frac{0}{0}$“. So that means we should factor:

$$
\begin{aligned}
\lim_{x\to 2} \frac{x^4-8x}{x^2-4} &= \lim_{x\to 2} \frac{x(x^3-8)}{(x+2)(x-2)} \\
&= \lim_{x\to 2} \frac{x(x-2)(x^2+2x+4)}{(x-2)(x+2)} \\
&= \lim_{x\to 2} \frac{x(x^2+2x+4)}{x+2} \\
&= \lim_{x\to 2} \frac{(2)((2)^2+2(2)+4)}{(2)+2} = 6
\end{aligned}
$$

SOLUTION 2. After realizing the indeterminate form, use L′Hopital′s rule until we get the answer.

$$
\lim_{x\to 2} \frac{x^4-8x}{x^2-4} = \lim_{x\to 2} \frac{4x^3-8}{2x} = \frac{4(2)^3-8}{2(2)} = 6
$$

Question 2 (b)

SOLUTION 1. Direct substitution gives "$\frac{\infty}{\infty}$" so we can use L′Hopital′s rule again (and again):

$$
\lim_{x\to\infty} \frac{5x^2+\ln x}{2x^2+3x} = \lim_{x\to\infty} \frac{10x+\frac{1}{x}}{4x+3} = \lim_{x\to\infty} \frac{10-\frac{1}{x^2}}{4} = \frac{5}{2}
$$

SOLUTION 2. We can also divide by the highest power x^2 to get:

$$
\lim_{x\to\infty} \frac{5x^2+\ln x}{2x^2+3x} = \lim_{x\to\infty} \frac{10+\frac{\ln(x)}{x^2}}{4+\frac{3}{x^2}} = \frac{10}{4} = \frac{5}{2}
$$

To get the same result by noting that x^2 grows faster than $\ln(x)$.

Question 2 (c)

SOLUTION.

- Rolle′s Theorem: Let f be a continuous function on $[a,b]$, differentiable on (a,b), and satisfies $f(a) = f(b)$. Then there exists a point $c \in (a,b)$ such that $f'(c) = 0$.
- MVT: Let f be a continuous function on $[a,b]$ and differentiable on (a,b). Then there exists a point $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Question 3 (a)

SOLUTION. The domain is all the x values for which the function will return an answer. The only way this fails in this case is when the denominator is zero. Hence $(x-4)^2 = 0$ which means $x = 4$ is not in the domain. The domain is everything else

$$\mathbb{R}\backslash\{4\}.$$

Question 3 (b)

SOLUTION. Horizontal Asymptotes:

$$\lim_{x\to\infty} \frac{x^2}{(x-4)^2} = \lim_{x\to\infty} \frac{\frac{x^2}{x^2}}{\frac{(x-4)^2}{x^2}} = \lim_{x\to\infty} \frac{1}{\left(\frac{x-4}{x}\right)^2} = \lim_{x\to\infty} \frac{1}{\left(1-\frac{4}{x}\right)^2} = 1$$
$$\lim_{x\to-\infty} \frac{x^2}{(x-4)^2} = \lim_{x\to-\infty} \frac{\frac{x^2}{x^2}}{\frac{(x-4)^2}{x^2}} = \lim_{x\to-\infty} \frac{1}{\left(\frac{x-4}{x}\right)^2} = \lim_{x\to-\infty} \frac{1}{\left(1-\frac{4}{x}\right)^2} = 1$$

Vertical Asymptotes:

$$\lim_{x\to4-} \frac{x^2}{(x-4)^2} = \frac{4^2}{(0^-)^2} = \infty$$
$$\lim_{x\to4+} \frac{x^2}{(x-4)^2} = \frac{4^2}{(0^+)^2} = \infty$$

There is a positive and negative horizontal asymptote at $y = 1$ and a vertical asymptote at $x = 4$.

Question 3 (c)

SOLUTION. Let

$$u(x) = x^2 \qquad u'(x) = 2x$$
$$v(x) = (x-4)^2 \qquad v'(x) = 2(x-4)$$

Then we can apply the quotient rule.

$$\begin{aligned} f(x) &= \frac{x^2}{(x-4)^2} \\ f'(x) &= \frac{(x-4)^2 \cdot (2x) - 2(x-4) \cdot (x^2)}{(x-4)^4} \\ &= -\frac{8x}{(x-4)^3} \end{aligned}$$

Question 3 (d)

SOLUTION. We can use the first derivative of the previous part.
$f'(x) = -\frac{8x}{(x-4)^3}$
We first identify the critical points as $x = 0$ only. Then we can partition the domain and test points in each interval.

Interval	$f'(x)$	INC/DEC
$(-\infty, 0)$	$f'(-1) < 0$	DEC
$(0, 4)$	$f'(2) > 0$	INC
$(4, \infty)$	$f'(10) < 0$	DEC

The final answer is that $f(x)$ is increasing on $(0, 4)$ and decreasing on $(-\infty, 0)$ and $(4, \infty)$.

Question 3 (e)

SOLUTION. Recall that
$f'(x) = -\frac{8x}{(x-4)^3}$
Let

$$u(x) = -8x \qquad u'(x) = -8$$
$$v(x) = (x-4)^3 \qquad v'(x) = 3(x-4)^2$$

Then we can apply the quotient rule.

$$\begin{aligned} f'(x) &= \frac{-8x}{(x-4)^3} \\ f''(x) &= \frac{(x-4)^3 \cdot (-8) - (-8x) \cdot 3(x-4)^2}{(x-4)^6} \\ &= \frac{16(x+2)}{(x-4)^4} \end{aligned}$$

Question 3 (f)

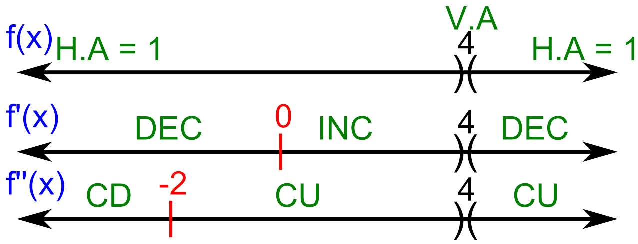
SOLUTION. The second derivative is
$f''(x) = \frac{16(x+2)}{(x-4)^4}$.
The point of interest in this case is $x = -2$. So we need to partition the domain and test values within the intervals.

Interval	$f''(x)$	CU/CD
$(-\infty, -2)$	$f''(-10) < 0$	CD
$(-2, 4)$	$f''(0) > 0$	CU
$(4, \infty)$	$f''(10) > 0$	CU

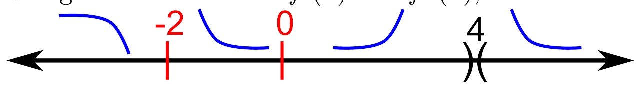
The function is concave up on $(-2, 4)$ and $(4, \infty)$ and concave down on $(-\infty, -2)$.

Question 3 (g)

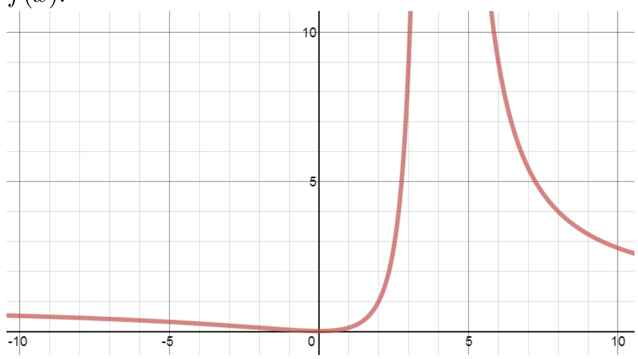
SOLUTION. We want to combine all of the information from the previous parts. One way we can visualise the information is to write the information over three number lines for $f(x)$, $f'(x)$, and $f''(x)$.



Using information from $f'(x)$ and $f''(x)$, we can sketch the shape of the graph in each section



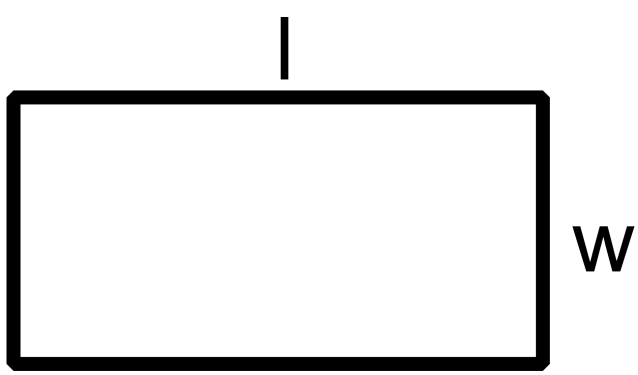
Finally, we can join up the pieces and adjust the vertical shift so that it lines up with the information from $f(x)$.



Question 4

SOLUTION. We proceed using the standard template for solving optimisation questions. The exact steps might vary between course and section, but the overall steps will be similar.

1. A diagram.



So let's define the rectangular pasture to have length l and width w. We want to minimise cost C.
Total cost of fencing will be $C = 10w + 10w + 10l + 15l = 20w + 25l$. We also need to have the area be $500 = lw$.
Combining the two equations we get

$$C(l) = 25l + 20\left(\frac{500}{l}\right) = 25l + \frac{10000}{l}.$$

In particular, the domain of the function C is $l \in (0, \infty)$.
Differentiate, we get

$$C'(l) = 25 - \frac{10000}{l^2}.$$

In the domain, the critical point is the solution of $25l^2 = 10000$ which is $l = \pm 20$. We disregard the negative solution not in the domain. So that means $l = 20$. To see that it's a global min,

Interval	$C'(l)$	INC/DEC
$(0, 20)$	$C'(10) < 0$	DEC
$(20, \infty)$	$C'(100) > 0$	INC

So that means $l = 20$ is a local min. Since it is decreasing to the left, $C(20)$ must be lower than all other values to the left. By the same argument, it must be lower than all the points to the right as well. So this is indeed a global min.
The lowest cost for the fence can be reached by making the length of $20m$ and width $25m$.

Question 5

Solution. We can proceed by using the standard template for solving optimisation questions. The exact steps might vary between course and section, but the overall steps will be similar.
1. Let's define the coordinates of $P = (x, y)$.
2. The area of the triangle is therefore

$$A = \frac{1}{2}(1+x)\sqrt{1-x^2}$$

3. The area equation is already in one variable. The domain is $x \in [-1, 1]$. In that we can have x between -1 and 1 (the diameter of the circle) and we can include the endpoints since this will give an area of 0.
4. Differentiate:

$$\begin{aligned} A'(x) &= \frac{1}{2}\left[(1+x)\cdot\frac{-2x}{2\sqrt{1-x^2}} + 1\cdot\sqrt{1-x^2}\right] \\ &= \frac{1}{2}\left[\frac{-x-x^2+(1-x^2)}{\sqrt{1-x^2}}\right] \\ &= \frac{1}{2}\left[\frac{1-x-2x^2}{\sqrt{1-x^2}}\right] \end{aligned}$$

In this case, the critical points are when the numerator is equal to zero (denominator is zero when $x = \pm 1$).

$$\begin{aligned} x &= \frac{1 \pm \sqrt{1-4(-2)(1)}}{-4} \\ &= \frac{1 \pm \sqrt{9}}{-4} = \frac{1}{2}, -1 \end{aligned}$$

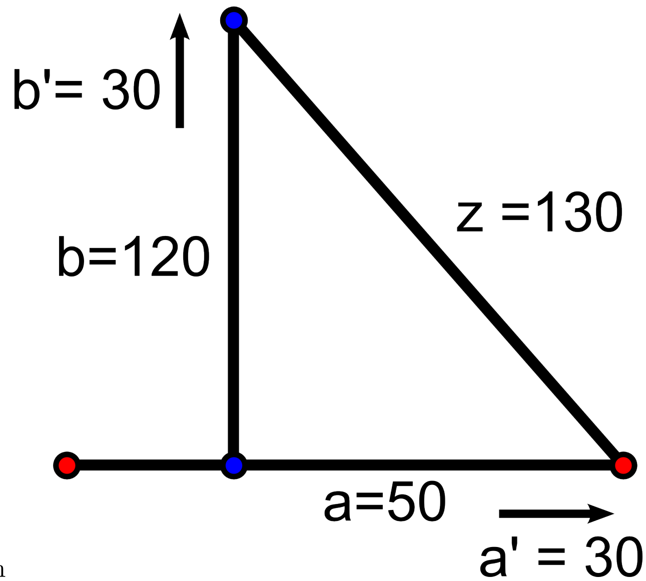
We can then use the closed interval method:

$$\begin{aligned} A(1) &= 0 \\ A(-1) &= 0 \\ A\left(\frac{1}{2}\right) &= \frac{3\sqrt{3}}{4} > 0 \end{aligned}$$

5. The maximum area happens at point $P = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

Question 6

Solution. This is a related rates problem. So we go about it using the standard related rates approach.



1. Draw a diagram
2. Variables: Define the distance between the two ships to be z. We want to find $z'(t)$. Define the distance between the ships and the origin to be a and b respectively. Set the original location of ship B to be at the origin. After 4 hrs, ship A is 50 km east of the origin and ship B is 120 km north of the origin.
3. Equations: The Pythagorean theorem will be useful here. The distance between them is $z = \sqrt{50^2 + 120^2} = 130km$. Since A is going away from the origin and B is also going away from the origin, we have $a' = 20$ and $b' = 30$.
4. Differentiating the distance formula
$$z^2 = a^2 + b^2,$$
we have
$$zz' = aa' + bb',$$
substitute to get
$$130z' = 50 \cdot 20 + 120 \cdot 30.$$
5. Rearrange to get $z' = \frac{460}{13} \approx 35.38km/hr$. The distance between the ships are increasing at $\frac{460}{13} \approx 35.38km/hr$.

Question 7 (a)

SOLUTION. Check that both sides agree:

$$\text{LHS} = 10(0) = 0$$
$$\text{RHS} = e^{2(0)} - e^{-2(0)} = 1 - 1 = 0$$

So both sides agree and the point $(0, 0)$ is on the curve.

Question 7 (b)

SOLUTION. Since we know the point is on the curve, we can just differentiate both sides with respect to t

$$\frac{\mathrm{d}}{\mathrm{d}t}10x = \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{2y} - e^{-2y}\right)$$
$$10x' = 2y'\left(e^{2y} + e^{-2y}\right)$$

Substituting $x = 0, y = 0, x'(0) = 4$ gives:

$$10(4) = 2y'\left(e^{2(0)} + e^{-2(0)}\right)$$
$$40 = 4y'$$

Rearranging gives the change in y-coordinate to be $10m/s$.

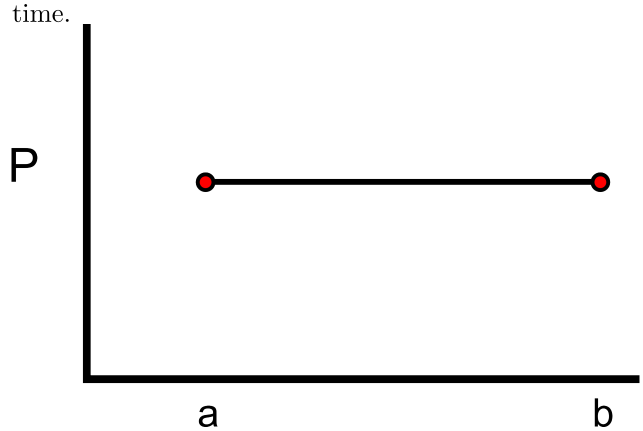
Question 8 (a)

SOLUTION. We make the assumption that ρ is the only parameter that changes with time. So we get:

$$P(t) = 1 - e^{-\pi\rho(t)D^2}$$
$$P'(t) = \pi D^2 \rho'(t) e^{-\pi\rho(t)D^2}$$

Question 8 (b)

SOLUTION. Since we have $P'(t) = \pi D^2 \rho'(t) e^{-\pi\rho(t)D^2}$, in the case where $\rho'(t) = 0$, we get $P'(t) = 0$. Hence the probability of interaction is not changing over time. Hence $P(t)$ will be the constant function during that time.



Question 9 (a)

SOLUTION. For this linear approximation, we want to pick $f(x) = \ln(x)$ and $a = 1$. Then

$$f(x) = \ln(x) \qquad\qquad f(1) = 0$$
$$f'(x) = \frac{1}{x} \qquad\qquad f'(1) = 1$$

So that means our linear approximation is:

$$L(x) = f(a) + f'(a)(x - a)$$
$$L(x) = 0 + 1(x - 1) = x - 1$$

Since we are interested in $x = 0.9$, we get

$$L(0.9) = (0.9 - 1) = -0.1.$$

So $\ln(0.9) \approx -0.1$.

Question 9 (c)

SOLUTION. Take the constant or linear function. We have $f(x) = mx + b$. Then we just apply the linear approximation by finding $f(a) = ma + b$ and $f'(a) = m$. Now we substitute.

$$\begin{aligned} L(x) &= f(a) + f'(a)(x - a) \\ &= (ma + b) + m(x - a) \\ &= ma + b + mx - ma \\ &= mx + b. \end{aligned}$$

We get the original function back.

Good Luck for your exams!