Full Solutions
MATH101 April 2012

April 4, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

SOLUTION. Using the hint, we see that

$$\begin{aligned}\int_1^2 \frac{x^2+2}{x^2}\,dx &= \int_1^2 \left(\frac{x^2}{x^2}+\frac{2}{x^2}\right)dx\\ &= \int_1^2 (1+2x^{-2})\,dx\\ &= \left(x-2x^{-1}\right)\Big|_1^2\\ &= (2-2(2)^{-1})-(1-2(1)^{-1})\\ &= (2-1)-(-1)\\ &= 2\end{aligned}$$

Question 1 (b)

SOLUTION. Let $u = 1+\sin x$ so that $du = \cos x dx$. Further, notice that
$u(\pi/2) = 1+\sin(\pi/2) = 1+1 = 2$
and that
$u(0) = 1+\sin(0) = 1+0 = 1.$
This gives
$\int_0^{\pi/2} \frac{\cos x}{1+\sin x}\,dx = \int_1^2 \frac{du}{u} = (\ln|u|)|_1^2 = \ln(2)-\ln(1) = \ln(2)$

Question 1 (c)

SOLUTION. Since $x^{1/3}$ is an odd function and $\cos(x)$ is an even function, we know that their product is an odd function. More directly, notice that if

$$f(x) = x^{1/3}\cos(x)\,dx$$

then

$$f(-x) = (-x)^{1/3}\cos(-x) = -x^{1/3}\cos(x) = -f(x)$$

and so the function is odd. As the interval of integration is symmetric, we have that

$$\int_{-2012}^{2012} x^{1/3}\cos(x)\,dx = 0.$$

Question 1 (d)

SOLUTION. Using the hint, we have

$$
\begin{aligned}
f_{av} &= \frac{1}{\pi/k - 0}\int_0^{\pi/k} \sin(kx)\,dx \\
&= \frac{k}{\pi}\left(\frac{-1}{k}\cos(kx)\right)\Bigg|_0^{\pi/k} \\
&= \frac{-1}{\pi}\left(\cos(\pi) - \cos(0)\right) \\
&= \frac{-1}{\pi}(-1-1) \\
&= \frac{2}{\pi}
\end{aligned}
$$

Question 1 (e)

SOLUTION. Let $y = f(x)$. Then the problem becomes

$$\frac{dy}{dx} = xy$$

Separating and solving yields

$$
\begin{aligned}
\frac{dy}{dx} &= xy \\
\frac{dy}{y} &= xdx \\
\int \frac{dy}{y} &= \int xdx \\
\ln|y| &= \frac{x^2}{2} + C
\end{aligned}
$$

When $x = 0$ we know that $y = e$ and plugging this information in yields

$$1 = \ln(e) = \ln|e| = \frac{(0)^2}{2} + C = C$$

Continuing to simplify yields

$$
\begin{aligned}
\ln|y| &= \frac{x^2}{2} + C \\
\ln|y| &= \frac{x^2}{2} + 1 \\
|y| &= e^{x^2/2+1} \\
y &= \pm e^{x^2/2+1}
\end{aligned}
$$

Since y is a positive function, we know that the final answer is

$$y = e^{x^2/2+1}$$

as required.

Question 1 (f)

SOLUTION. Let $f(x) = 1$ and $g(x) = -e^x$. Notice that between *0* and *1*, we have that $g(x) \leq f(x)$ and so we have

$$\begin{aligned}
\bar{y} &= \frac{1}{2A}\int_0^1 (f(x)^2 - g(x)^2)\,dx \\
&= \frac{1}{2e}\int_0^1 (1^2 - (-e^x)^2)\,dx \\
&= \frac{1}{2e}\int_0^1 (1 - e^{2x})\,dx \\
&= \frac{1}{2e}\left(x - \frac{e^{2x}}{2}\right)\Big|_0^1 \\
&= \frac{1}{2e}\left(\left(1 - \frac{e^{2(1)}}{2}\right) - \left(0 - \frac{e^{2(0)}}{2}\right)\right) \\
&= \frac{1}{2e}\left(1 - \frac{e^2}{2} + \frac{1}{2}\right) \\
&= \frac{1}{2e}\left(\frac{3}{2} - \frac{e^2}{2}\right) \\
&= \frac{3-e^2}{4e}
\end{aligned}$$

completing the question.

Question 1 (g)

SOLUTION. Notice that $e^{i/n}$ occurs in the summand. This suggests that $x_i = i/n$ and so we can take $a = 0$. Writing the sum as
$\sum_{i=1}^n \frac{ie^{i/n}}{n^2} = \sum_{i=1}^n \frac{1}{n} \cdot \frac{ie^{i/n}}{n}$
the last fraction suggests that a possible function is $f(x) = xe^x$ which can be seen by noting that we have $f(x_i) = \frac{ie^{i/n}}{n}$.
Also, we have
$\Delta x = \frac{1}{n} = \frac{b-a}{n}$
and since $a = 0$ we have that $b = 1$. Combining this gives
$\lim_{n\to\infty} \sum_{i=1}^n \frac{ie^{i/n}}{n^2} = \int_0^1 xe^x\,dx$
completing the question.

Question 1 (h)

SOLUTION 1. Notice that we have
$2.656565.... = 2 + \frac{65}{100} + \frac{65}{10000} + \frac{65}{1000000} = 2 + \sum_{n=1}^\infty \frac{65}{100^n}$
Now, recall that the sum of a geometric series is
$\sum_{n=1}^\infty ar^{n-1} = \frac{a}{1-r}$.
In our case, we have that $a = 65/100$ (the term when we plug in $n = 1$ into the sum) and we have that $r = 1/100$ and hence
$2 + \sum_{n=1}^\infty \frac{65}{100^n} = 2 + \frac{\frac{65}{100}}{1-\frac{1}{100}} = 2 + \frac{65}{99} = \frac{263}{99}$

completing the question.

SOLUTION 2. Let
$x = 2.65656565656565....$
Notice that
$100x = 265.656565656565...$
Subtracting the two equations gives
$99x = 263$
and so
$x = \frac{263}{99}$

Question 1 (i)

SOLUTION. Recall the Maclaurin (power) series of $\sin(x)$ is

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Note that the first two terms cancel exactly with the polynomial in the numerator. Thus,

$$\begin{aligned}\lim_{x\to 0} \frac{\sin(x) - x + x^3/6}{\sin(x^5)} &= \lim_{x\to 0} \frac{x^5/(5!) - x^7/(7!) + \cdots}{\sin(x^5)} \\ &= \lim_{x\to 0} \left(\frac{1}{5!} \frac{x^5}{\sin(x^5)} - \frac{x^2}{7!} \frac{x^5}{\sin(x^5)} + \dots \right)\end{aligned}$$

Now, as shown in Hint 2,

$$\lim_{x\to 0} \frac{y}{\sin(y)} = 1$$

we can also say that

$$\lim_{x\to 0} \frac{x^5}{\sin(x^5)} = 1$$

So the first term of the limit becomes $\frac{1}{5!}$. The remaining terms all have an $x^5/\sin(x^5)$ term which evaluates to 1, but also have an extra power of x which will make the entire term go to *0*. So the solution to the limit is $\frac{1}{5!}$.
This question demonstrates the strength of power series in more difficult calculus problems.

Question 1 (j)

SOLUTION. The Maclauren series of the integrand is found by:

$$\begin{aligned}x^4 e^{-x^2} &= x^4 \left(1 + \left(-x^2\right) + \left(-x^2\right)^2 / (2!) + \left(-x^2\right)^3 / (3!) + \cdots \right) \\ &= x^4 - x^6 + x^8/2 - x^{10}/6 + \dots\end{aligned}$$

Therefore, integrating term-by-term gives:

$$I = \int_0^1 x^4 e^{-x^2} dx = 1/5 - 1/7 + 1/18 - 1/66 + \dots$$

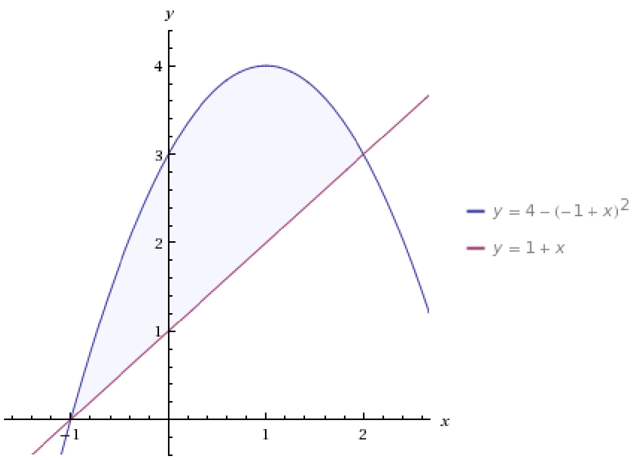
This confirms the given approximation, $a = 1/5 - 1/7 + 1/18$.
Moreover, this shows that the integral can be represented by an alternating series in the form

$$\sum_{n=1}^{k} (-1)^{n-1} a_n$$

Since the positive numbers a_n are evidently monotonically decreasing, the *Alternating Series Estimation Theorem* then says that an error bound of a partial sum can be given by the next term a_{k+1}. In this case, the statement translates to the bound:

$$|I - a| \leq 1/66$$

Question 2 (a)



SOLUTION.
Since the two curves $y_1 = f(x) = 4 - (x-1)^2$ and $y_2 = g(x) = x + 1$ intersects when

$$\begin{aligned} 0 &= f(x) - g(x) \\ &= 4 - (x-1)^2 - (x+1) \\ &= 2 + x - x^2 \\ &= -(x-2)(x+1) \end{aligned}$$

Thus, the curves intersect at *x=-1* and *x=2*. Moreover, from the sketch, we can see that *f(x) > g(x)* in the interval *[-1, 2]*.
Therefore, the area A bounded by the curves is given by:

$$\begin{aligned} A &= \int_{-1}^{2} f(x) - g(x) dx \\ &= \int_{-1}^{2} (2 + x - x^2) dx \\ &= \left[2x + x^2/2 - x^3/3\right]_{-1}^{2} \\ &= (4 + 2 - 8/3) - (-2 + 1/2 + 1/3) \\ &= 9/2 \end{aligned}$$

Question 2 (b)

SOLUTION. We use the method of washers.
For any x in *(-1,2)*, the larger and smaller radii R and r of the washer is given by

$$R(x) = 5 - (x + 1) = 4 - x$$

$$r(x) = 5 - (4 - (x - 1)^2) = 1 + (x - 1)^2$$

(The simplifications given above are not really necessary)
Notice that the larger radius R is given by the distance from the axis of revolution *y=5* to the lower curve, because it is further away.
Therefore, the area of the washer is

$$A(x) = (R^2 - r^2)\pi = \left[(4 - x)^2 - \left(1 + (x - 1)^2\right)^2\right]\pi$$

Now we can write down the integral that represents the volume of the solid of revolution:

$$\begin{aligned} V &= \int_{-1}^{2} A(x)dx \\ &= \pi\left[\int_{-1}^{2} (4 - x)^2 - \left(1 + (x - 1)^2\right)^2 dx\right] \end{aligned}$$

According to the question statement, we may stop here.
If you are interested to find the value of the integral, read on.
To solve this integral, we split it in two parts and write

$$V = \pi\left[\int_{-1}^{2} (4 - x)^2 dx - \int_{-1}^{2} \left(1 + (x - 1)^2\right)^2 dx\right]$$

We refrain from expanding the brackets but make a change of variables *u=4-x, du=-dx* and *v=x-1, dv=dx* for the first and second integral respectively. Not forgetting to update the integration limits, we arrive at

$$\begin{aligned} V &= \pi\left[\int_{-1}^{2} (4 - x)^2 dx - \int_{-1}^{2} \left(1 + (x - 1)^2\right)^2 dx\right] \\ &= \pi\left[\int_{5}^{2} u^2(-du) - \int_{-2}^{1} \left(1 + v^2\right)^2 dv\right] \\ &= \pi\left[\int_{2}^{5} u^2 du - \int_{-2}^{1} \left(1 + 2v^2 + v^4\right) dv\right] \\ &= \pi\left(\left[u^3/3\right]_2^5 - \left[v + 2v^3/3 + v^5/5\right]_{-2}^{1}\right) \\ &= \pi\left((125 - 8)/3 - [3 + 2(1 + 8)/3 + (1 + 32)/5]\right) \\ &= \pi\left(39 - 3 - 6 - 6 - 3/5\right) \\ &= \pi\left(24 - 3/5\right) = 117\pi/5 \end{aligned}$$

Question 3 (a)

SOLUTION. The *bad* part of the integrand is $\ln x$. The hints says we'd like to differentiate it (to make it 1/x). The relevant integration technique that involves differentiation is then *integration by parts*. First we identify $u = \ln x$ and $dv = 1/x^{101}dx$, then, $v = -1/(100x^{100})$ by anti-differentiation, and $du = (1/x)dx$ by diffentiation. It seems we are not getting an integral that is harder then before, so let's move on and apply integration by parts:

$$\begin{aligned}\int \frac{\ln x}{x^{101}}dx &= \int udv = uv - \int vdu \\ &= -\frac{1}{100}\frac{\ln x}{x^{100}} - \int \left(-\frac{1}{100}\frac{1}{x^{100}}\right)\left(\frac{1}{x}dx\right) \\ &= -\frac{1}{100}\frac{\ln x}{x^{100}} + \frac{1}{100}\int \frac{1}{x^{101}}dx \\ &= \frac{1}{100}\left(-\frac{\ln x}{x^{100}} - \frac{1}{100}\frac{1}{x^{100}}\right) + C \\ &= -\frac{1}{100x^{100}}\left(\ln x + \frac{1}{100}\right) + C\end{aligned}$$

Question 3 (b)

SOLUTION. We can write the integral as $\int_0^3 (x+1)\sqrt{9-x^2}dx = \int_0^3 x\sqrt{9-x^2}dx + \int_0^3 \sqrt{9-x^2}dx$.
Now we compute: for $\int_0^3 x\sqrt{9-x^2}dx$ we use the substitution $u = 9-x^2$, $du = -2xdx$ to get: $\int_0^3 x\sqrt{9-x^2}dx = -\int_9^0 \frac{1}{2}\sqrt{u}du = -\frac{1}{3}u^{3/2}|_9^0 = -(0-\frac{1}{3}9^{3/2}) = 9$
For $\int_0^3 \sqrt{9-x^2}dx$ we note that this corresponds to one fourth of the area of a circle of radius 3, so $\int_0^3 \sqrt{9-x^2}dx = \frac{1}{4}9\pi$
Hence the final result is $\int_0^3 (x+1)\sqrt{9-x^2}dx = 9 + \frac{9\pi}{4}$

Question 3 (c)

SOLUTION. We want to write the integrand using partial fractions:

$$\begin{aligned}\frac{4x+8}{(x-2)(x^2+4)} &= \frac{A}{x-2} + \frac{Bx+C}{x^2+4} \\ &= \frac{A(x^2+4)+(Bx+C)(x-2)}{(x-2)(x^2+4)} \\ &= \frac{Ax^2+4A+Bx^2+Cx-2Bx-2C}{(x-2)(x^2+4)} \\ &= \frac{(A+B)x^2+(-2B+C)x+4A-2C}{(x-2)(x^2+4)}\end{aligned}$$

Comparing coefficients we get the equations:

$$
\begin{aligned}
0 &= A + B \\
4 &= -2B + C \\
8 &= 4A - 2C
\end{aligned}
$$

Now we solve

$$A = -B,$$

which gives $4 = 2A + C$, adding twice this equation to the third one we get $16 = 8A$ so $A = 2$. This means $B = -2$ and $C = 0$.
So we have

$$
\begin{aligned}
\int \frac{4x+8}{(x-2)(x^2+4)}dx &= 2\int \frac{1}{x-2}dx - \int \frac{2x}{x^2+4}dx \\
&= 2\ln|x-2| - \ln|x^2+4| + c
\end{aligned}
$$

Question 3 (d)

SOLUTION. The integral is equal to the sum of the two standard Type I integrals:

$$\int_{-\infty}^{0} \frac{x}{x^2+1}dx + \int_{0}^{\infty} \frac{x}{x^2+1}dx.$$

By the substitution $u{=}x^2{+}1$, $du = 2dx$, we see that

$$
\begin{aligned}
\int \frac{x}{x^2+1}dx &= \frac{1}{2}\int \frac{1}{u}du \\
&= \frac{1}{2}\ln|u| + C \\
&= \frac{1}{2}\ln|x^2+1| + C
\end{aligned}
$$

So, we calculate

$$
\begin{aligned}
\int_{0}^{\infty} \frac{x}{x^2+1}dx &= \lim_{a\to\infty} \frac{1}{2}\ln(x^2+1)\big|_0^a \\
&= \lim_{a\to\infty} (\frac{1}{2}\ln(a^2+1) - \ln(1)) \\
&= \infty
\end{aligned}
$$

Since this part diverges the whole integral diverges also.

Question 4 (a)

SOLUTION. We will compare our series with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$. So we look at the limit:

$$\lim_{n\to\infty} \frac{\left(\frac{n^2-\sin(n)}{n^6+n^2}\right)}{\left(\frac{1}{n^4}\right)} = \lim_{n\to\infty} \frac{n^6+n^4\sin(n)}{n^6+n^2}$$
$$= \lim_{n\to\infty} \frac{1-\frac{\sin(n)}{n^2}}{1+\frac{1}{n^4}}$$
$$= \frac{1}{1} = 1$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges, by limit comparison our series also converges.

Question 4 (b)

SOLUTION. We want to use the Ratio test, so we look at the limit:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{(2n+2)!}{((n+1)^2+1)((n+1)!)^2}}{\frac{(2n)!}{(n^2+1)(n!)^2}}$$
$$= \lim_{n\to\infty} \frac{(2n+2)!(n^2+1)(n!)^2}{(2n)!((n+1)^2+1)((n+1)!)^2}$$
$$= \lim_{n\to\infty} \frac{(2n+2)!}{(2n)!} \cdot \frac{(n^2+1)}{(n+1)^2+1} \cdot \left(\frac{n!}{(n+1)!}\right)^2$$
$$= \lim_{n\to\infty} (2n+2)(2n+1) \cdot \frac{n^2+1}{n^2+2n+2} \cdot \frac{1}{(n+1)^2}$$
$$= \lim_{n\to\infty} \frac{2n+2}{n+1} \cdot \frac{2n+1}{n+1} \cdot \frac{n^2+1}{n^2+2n+2} = 2\cdot 2\cdot 1 = 4$$

Since this limit is > 1 we know that this series diverges.

Question 4 (c)

SOLUTION. We will use the integral test. Note that

$$\int_2^\infty \frac{dx}{x\ln(x)^{101}} = \lim_{a\to\infty} \int_2^a \frac{dx}{x\ln(x)^{101}}$$

We can calculate $\int_2^a \frac{dx}{x\ln(x)^{101}}$ by substitution with $u = \ln x$ (then du=(1/x)dx) to get

$$\int_2^a \frac{dx}{x\ln(x)^{101}} = \int_{\ln(2)}^{\ln(a)} u^{-101} du$$
$$= -\frac{1}{100u^{100}}\Bigg|_{\ln(2)}^{\ln(a)}$$
$$= -\frac{1}{100}\left(\frac{1}{(\ln(a))^{100}} - \frac{1}{(\ln(2))^{100}}\right)$$

Hence

$$\int_2^\infty \frac{dx}{x\ln(x)^{101}} = \lim_{a\to\infty} \frac{1}{100}\left(\frac{1}{\ln(2)^{100}} - \frac{1}{\ln(a)^{100}}\right) = \frac{1}{100\ln(2)^{100}}.$$

This implies that the series converges absolutely by the integral test.

Question 5

SOLUTION. We will use the ratio test:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(x-1)^{n+1}}{2^{n+1}(n+3)}}{\frac{(x-1)^n}{2^n(n+2)}}\right| \\
&= \lim_{n\to\infty}\frac{|x-1|(n+2)}{2\cdot(n+3)} \\
&= |\frac{x-1}{2}|
\end{aligned}$$

We want this limit to be < 1 so that the series converges absolutely. This means $|\frac{x-1}{2}| < 1$ which means that the radius of convergence is 2 and that $-1 < x < 3$. Now we have to test the endpoints.
At $x = 3$ we get:
$\sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{2^n(n+2)} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+2)}$
which converges by alternate series test, if we sum two consecutive terms we get:
$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} < \frac{1}{n^2}$
and the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges.
At $x = -1$ we get
$\sum_{n=0}^{\infty}\frac{(-1)^n(-2)^n}{2^n(n+2)} = \sum_{n=0}^{\infty}\frac{1}{(n+2)}$
which diverges because it is the tail of the harmonic series.
So, the interval of convergence is $(-1, 3]$.

Question 6

SOLUTION. Denote y as the distance from the top of the bowl to the water level and r the radius of the water surface.
Then, consider a right-angled triangle with a water surface radius r as base, then the height is y, and the hypoteneuse is 3, which is the radius of the bowl. (The top of the triangle is exactly at the center of the the sphere, half of which becomes the bowl).
Then $r^2 + y^2 = 3^2 = 9$ implies $r^2 = 9 - y^2$, so the area of the water surface is given as a function of y by

$$A(y) = r^2\pi = \pi(9 - y^2)$$

A thin slab of water thus has volume $A(y)\triangle y$ and mass

$$M(y) = 1000A(y)\triangle y = 1000\pi(9 - y^2)\triangle y$$

This slab of water travels a distance of *y+2* to be pumped out (*+2* because of the 2m vertical outlet) and requires a force of Mg
Therefore, summing the work done for each slab, and letting $\triangle y \to 0$, we get

$$\begin{aligned}
W &= \int_0^3 1000\pi g(2+y)(9-y^2)dy \\
&= 1000\pi g\left[\int_0^3 (18 + 9y - 2y^2 - y^3)dy\right] \\
&= 1000\pi g\left[18y + 9y^2/2 - 2y^3/3 - y^4/4\right]_0^3 \\
&= 1000\pi g\left(18\cdot 3 + 9(3^2)/2 - 2(3^3)/3 - 3^4/4\right)
\end{aligned}$$

Question 7 (a)

SOLUTION. The formula for the trapezoidal approximation of the integral is as follows:

$$T_4 = \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4))$$

The *4* of T_4 means that our interval of integration has been divided into four pieces, each of length *(b-a)/4=1/4*. So $\Delta x = 1/4$. We also know that x_0 and x_4 are the endpoints of the interval of integration. So we have

$$\begin{aligned} x_0 &= 1 \\ x_1 &= 5/4 \\ x_2 &= 6/4 \\ x_3 &= 7/4 \\ x_4 &= 2 \end{aligned}$$

We simply plug this all into the formula given above to get:

$$T_4 = \frac{1}{8}\left(\frac{1}{1} + 2\frac{1}{5/4} + 2\frac{1}{6/4} + 2\frac{1}{7/4} + \frac{1}{2})\right)$$

Question 7 (b)

SOLUTION. The Simpson's rule approximation for $n = 4$ is given by:

$$S_4 = \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right)$$

In our situation, $\Delta x = \frac{1}{4}$, $f(x) = \frac{1}{x}$, $x_0 = 1$ and $x_4 = 2$. Thus we can calculate S_4 as:
$S_4 = \frac{1/4}{3}\left(\frac{1}{1} + 4\frac{1}{5/4} + 2\frac{1}{6/4} + 4\frac{1}{7/4} + \frac{1}{2}\right)$

Question 7 (c)

SOLUTION. We will use the hint in the statement of the problem: if we can find an upper bound K on $|f^{(4)}|$, then we know that $|I - S_4| \leq \frac{K(b-a)^5}{180 \cdot 4^4}$.
First we find an upper bound on $f^{(4)}$.

$$\begin{aligned} f'(x) &= -x^{-2} \\ f''(x) &= 2x^{-3} \\ f^{(3)} &= -6x^{-4} \\ f^{(4)} &= 24x^{-5} \end{aligned}$$

On our interval of integration, *[1,2]*, $|f^{(4)}(x)| \leq \frac{24}{1^5} = 24$. So our upper bound on $|f^{(4)}|$ is $K = 24$.

Now we simply plug this into the error formula given above to get:
$|I - S_4| \leq \frac{24(2-1)^5}{180 \cdot 4^4} = \frac{24}{180 \cdot 4^4} = \frac{1}{1920}$.

Question 8 (a)

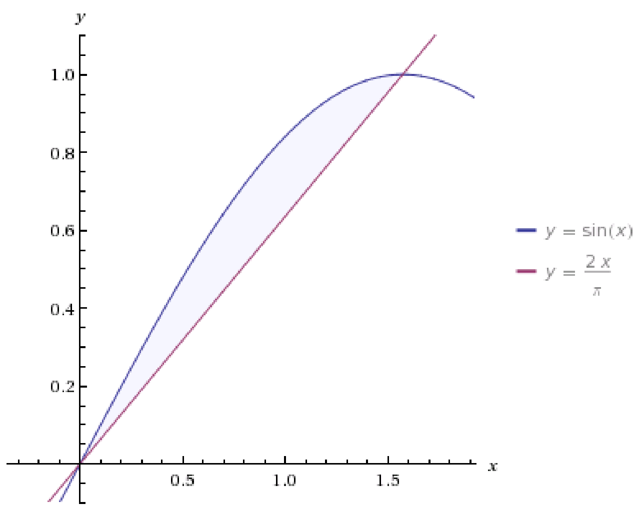
SOLUTION. Recall that the Maclaurin series of $\ln(1+x)$ is

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$$

with radius of convergence *R=1*.
If we put *x=1/2*, then we obtain the given series and hence we find

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^n} = \ln(1+1/2) = \ln(3/2)$$

Question 8 (b)



SOLUTION.
Following the hint, we want to find a straight line that is below $y = \sin(x)$ with minimal error.
To do this, we require the straight line to pass through $y = \sin(x)$ at both $x = 0$ and $x = \pi/2$, which are the endpoints of the domain of integration.
Since $y = \sin(0) = 0$ at $x = 0$, we look for a straight line that passes through the origin with formula $y = f(x) = cx$. Then we put $x = \pi/2$, $y = \sin(\pi/2) = 1$ to find $1 = c\pi/2$, so $c = 2/\pi$.
Note that the slope of the line $c = 2/\pi < 1$, while the initial slope of $y = \sin(x)$ is given by $\cos(0) = 1$, so the linear function $y = (2/\pi)x$ is really below $y = \sin(x)$ in the interval $[0, \pi/2]$. Hence, we can deduce the following inequalities:

$$\begin{aligned} \frac{2}{\pi}x &\leq \sin x \\ \Rightarrow -a \sin x &\leq -\frac{2a}{\pi}x \\ \Rightarrow e^{-a \sin x} &\leq e^{-\frac{2a}{\pi}x} \end{aligned}$$

because $a > 0$ and e^x is a strictly increasing function. So by comparison, we obtain

$$\begin{aligned}\int_0^{\pi/2} e^{-a\sin x}dx &\leq \int_0^{\pi/2} e^{-\frac{2a}{\pi}x}dx\\ &= -\frac{\pi}{2a}\left[e^{-\frac{2a}{\pi}x}\right]_0^{\pi/2}\\ &= -\frac{\pi}{2a}\left(e^{-a}-1\right)\\ &= \frac{\pi}{2a}\left(1-e^{-a}\right) < \frac{\pi}{2a}\end{aligned}$$

because $e^{-a} > 0$ for any a.

Good Luck for your exams!