Full Solutions
MATH101 April 2013

April 4, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

SOLUTION. We proceed as suggested by the hint. First notice that
$\Delta x = (6-2)/4 = 1$.
Next, notice that

$$\begin{aligned} x_0 &= 2 + 0\Delta x = 2 \\ x_1 &= 2 + 1\Delta x = 3 \\ &\dots \\ x_4 &= 2 + 4\Delta x = 6 \end{aligned}$$

Thus, figuring out the $f(x_i)$ terms directly from the graph, we have
$T_4 = \frac{1}{2}[3 + (2)(8) + (2)(7) + (2)(6) + 4] = \frac{49}{2}$

Question 1 (b)

SOLUTION. We proceed as suggested by the hint. First notice that
$\Delta x = (6-2)/4 = 1$.
Next, notice that

$$\begin{aligned} x_0 &= 2 + 0\Delta x = 2 \\ x_1 &= 2 + 1\Delta x = 3 \\ &\dots \\ x_4 &= 2 + 4\Delta x = 6 \end{aligned}$$

Thus, figuring out the $f(x_i)$ terms directly from the graph, we have
$S_4 = \frac{1}{3}[3 + (4)(8) + (2)(7) + (4)(6) + 4] = \frac{77}{3}$

Question 1 (c)

SOLUTION. This is a rational function, so we try partial fractions. The integrand's denominator can be factored, then we apply partial fraction decomposition giving

$$\begin{aligned} \frac{dx}{x+x^2} &= \frac{1}{x(1+x)} \\ &= \frac{A}{x} + \frac{B}{1+x} \end{aligned}$$

Cross multiplying,

$$1 = A(1+x) + Bx$$

We can solve this several ways.
(1) Setting $x = 0$ we get $1 = A$, and setting $x = -1$ we get $1 = -B$, so $B = -1$.
(2) Comparing coefficients of the constant term, we get $1 = A$, and then comparing coefficients of the x term, we get $0 = A + B = 1 + B$, so $B = -1$.
(3) We get $A = 1$ and $B = -1$ by inspection.
Now we solve the integral:

$$\begin{aligned}\int_1^2 \frac{dx}{x+x^2} &= \int_1^2 \frac{dx}{x(1+x)} \\ &= \int_1^2 \left(\frac{1}{x} + \frac{-1}{1+x}\right) dx \\ &= (\ln|x| - \ln|1+x|)\Big|_1^2 \\ &= (\ln 2 - \ln 3) - (\ln 1 - \ln 2) \\ &= 2\ln 2 - \ln 3\end{aligned}$$

The answer $\ln \frac{4}{3}$ is equivalent.

Question 2 (a)

SOLUTION 1. First, note that

$$-\sin\left(\frac{1}{n}\right) \leq (-1)^n \sin\left(\frac{1}{n}\right) \leq \sin\left(\frac{1}{n}\right)$$

Further, since

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

and $\sin x$ is continuous at 0,

$$\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = \sin\left(\lim_{n\to\infty} \frac{1}{n}\right) = \sin(0) = 0$$

Similarly,

$$\lim_{n\to\infty} -\sin\left(\frac{1}{n}\right) = -0 = 0.$$

By the squeeze theorem,

$$\lim_{n\to\infty} (-1)^n \sin\left(\frac{1}{n}\right)$$

exists and equals 0 as well.

SOLUTION 2. The alternating series test says that

$$\sum_{n=1}^{\infty} (-1)^n a_n$$

converges if a_n is monotonically decreasing. In our case, $a_n = \sin\left(\frac{1}{n}\right)$, and since $\frac{1}{n}$ monotonically decreases as n increases, and *sin(x)* is a monotone function for x in *[0,1]* we find that indeed $a_n = \sin\left(\frac{1}{n}\right)$ is monotonically decreasing. Hence

$$\sum_{n=1}^{\infty} (-1)^n a_n$$

converges. But the question didn't ask if the series converges, but if the sequence $\{(-1)^n\ a_n\}$ converges. This, however, follow as well because if a series $\sum_{n=1}^{\infty} b_n$ converges, then the sequence $\{b_n\}$ must converge to zero, that is

$$0 = \lim_{n\to\infty} b_n = \lim_{n\to\infty} (-1)^n \sin\left(\frac{1}{n}\right)$$

Hence, $(-1)^n \sin\left(\frac{1}{n}\right)$ converges to *0*.

Question 2 (b)

SOLUTION 1. This is a geometric series. The first term is given by $n = 2$:

$$a = \frac{3 \cdot 4^3}{8 \cdot 5^2}$$

The common ratio is given by the quotient of successive terms. For example, using the first two terms gives:

$$r = \frac{\frac{3\cdot 4^4}{8\cdot 5^3}}{\frac{3\cdot 4^3}{8\cdot 5^2}} = \frac{3 \cdot 4^4}{8 \cdot 5^3} \cdot \frac{8 \cdot 5^2}{3 \cdot 4^3} = \frac{4}{5}$$

Then using the formula

$$a + ar + ar^2 + \cdots = \frac{a}{1-r}$$

we get

$$\sum_{n=2}^{\infty} \frac{3 \cdot 4^{n+1}}{8 \cdot 5^n} = \frac{\frac{3\cdot 4^3}{8\cdot 5^2}}{1 - \frac{4}{5}} = \frac{24}{5}$$

SOLUTION 2. Simplifying the series helps to see the similarity to the geometric series:

$$\sum_{n=2}^{\infty} \frac{3 \cdot 4^{n+1}}{8 \cdot 5^n} = \frac{3 \cdot 4}{8} \sum_{n=2}^{\infty} \left(\frac{4}{5}\right)^n$$

Unlike the series above, the geometric series starts at $n = 0$, so we next rewrite our expression as

$$\begin{aligned}
\frac{3 \cdot 4}{8} \sum_{n=2}^{\infty} \left(\frac{4}{5}\right)^n &= \frac{3}{2} \left(\sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^n - \sum_{n=0}^{1} \left(\frac{4}{5}\right)^n \right) \\
&= \frac{3}{2} \left(\sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^n - \left(1 + \frac{4}{5}\right) \right)
\end{aligned}$$

Using the geometric series $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ with $r = 4/5$ we obtain

$$\frac{3}{2}\left(\sum_{n=0}^{\infty}\left(\frac{4}{5}\right)^n - \left(1+\frac{4}{5}\right)\right) = \frac{3}{2}\left(\frac{1}{1-4/5} - \left(1+\frac{4}{5}\right)\right)$$

Simplifying the expression above we arrive at the final answer $\sum_{n=2}^{\infty} \frac{3\cdot 4^{n+1}}{8\cdot 5^n} = \frac{24}{5}$

Question 2 (c)

SOLUTION. Recall the MacLaurin series for $\sin(x)$ is given by

$$\sin(x) = x - \frac{x^3}{3!} + \dots$$

so

$$\begin{aligned} x^2\sin(x^3) &= x^2\left(x^3 - \frac{x^9}{3!} + \dots\right) \\ &= x^5 - \frac{x^{11}}{3!} + \dots \end{aligned}$$

so

$$a = 1, b = -\frac{1}{6}$$

Question 3 (a)

SOLUTION. Using substitution,

$$\begin{aligned} u &= x^2 - x \\ du &= (2x-1)\,dx \end{aligned}$$

and the bounds change

$$\begin{aligned} x = 1 &\Rightarrow u = 0 \\ x = 3 &\Rightarrow u = 6 \end{aligned}$$

Then

$$\begin{aligned} \int_1^3 (2x-1)e^{x^2-x}\,dx &= \int_0^6 e^u\,du \\ &= e^u\Big|_0^6 \\ &= e^6 - 1 \end{aligned}$$

Question 3 (b)

SOLUTION. Recall for springs,

$$f(x) = kx$$

We are given that force is 10 N when x is 5 cm = 0.05 m, so

$$10 = k(0.05)$$

giving $k = 200$. Then, the work done in stretching the spring from its natural length to 50 cm = 0.5 m beyond its natural length is given by

$$\begin{aligned} W &= \int_0^{0.5} 200x \; dx \\ &= 200\frac{x^2}{2}\Big|_0^{0.5} \\ &= 25 \end{aligned}$$

So the work done in stretching the spring is 25J.

Question 3 (c)

SOLUTION. By the Fundamental Theorem of Calculus,

$$\frac{d}{dx}\int_0^x f(t)\; dt = f(x),$$

so differentiating both sides gives

$$x\pi\cos(\pi x) + \sin(\pi x) = f(x)$$

At $x = 4$, we have $f(4) = 4\pi$

Question 4 (a)

SOLUTION. To get our bounds of integration, we need to know where the intersection points are. Also, from the graph, the top function changes, so we need to find out exactly where this occurs, which is another point of intersection. The intersections can be found by solving

$$\begin{aligned} 4 + \pi\sin x &= 4 + 2\pi - 2x \\ x &= \pi - \frac{\pi}{2}\sin x \end{aligned}$$

This isn't easy to solve, but the graph shows three intersection points which can be estimated then checked. It looks like the three intersection points are $x = \pi/2, \pi, 3\pi/2$. At $x = \pi/2$,

$$\pi - \frac{\pi}{2}\sin\frac{\pi}{2} = \pi - \frac{\pi}{2}\cdot 1 = \frac{\pi}{2}$$

so this is a solution. At $x = \pi$,

$$\pi - \frac{\pi}{2}\sin\pi = \pi - \frac{\pi}{2}\cdot 0 = \pi$$

so this is a solution. At $x = 3\pi/2$,

$$\pi - \frac{\pi}{2}\sin\frac{3\pi}{2} = \pi - \frac{\pi}{2}\cdot(-1) = \frac{3\pi}{2}$$

so this is a solution. Then, the intersection points are indeed $x = \pi/2, \pi, 3\pi/2$. Notice the top function on $[\pi/2, \pi]$ is $4 + \pi\sin x$, whereas the top function on $[\pi, 3\pi/2]$ is $4 + 2\pi - 2x$.
We have

$$\begin{aligned}
A &= \int_{\pi/2}^{\pi} [(4+\pi\sin x) - (4 + 2\pi - 2x)]\ dx \\
&\quad + \int_{\pi}^{3\pi/2} [(4 + 2\pi - 2x) - (4 + \pi\sin x)]\ dx \\
&= (-\pi\cos x - 2\pi x + x^2)\big|_{\pi/2}^{\pi} + (\pi\cos x + 2\pi x - x^2)\big|_{\pi}^{3\pi/2} \\
&= 2(\pi - \pi^2/4)
\end{aligned}$$

Question 4 (b)

SOLUTION. To begin with, to get our bounds of integration and where the intersection points are, we follow same steps as in part (a) to find that the curves intersect when $x = \pi/2, \pi, 3\pi/2$. Further, the top function on $[\pi/2, \pi]$ is $4 + \pi\sin x$, whereas the top function on $[\pi, 3\pi/2]$ is $4 + 2\pi - 2x$.
Since we are revolving around the horizontal line $y = -1$, the radius of the discs change for every value of x. Hence we will integrate over all x values to get the volume of revolution. To find the radius of the discs as a function of x, note that we need to add 1 to the function value since the radius is given by

$$\text{radius} = \text{function} - \text{axis} = f(x) - (-1) = f(x) + 1$$

where $f(x)$ is the function. The volume of revolution is given by $\pi\displaystyle\int_a^b (\text{outer radius}^2 - \text{inner radius}^2)\ dx$, so

$$\begin{aligned}
V &= \pi\int_{\pi/2}^{\pi} [(4 + \pi\sin x + 1)^2 - (4 + 2\pi - 2x + 1)^2]\ dx \\
&\quad + \pi\int_{\pi}^{3\pi/2} [(4 + 2\pi - 2x + 1)^2 - (4 + \pi\sin x + 1)^2]\ dx
\end{aligned}$$

Question 5 (a)

SOLUTION. The widths of the rectangles in a n-rectangle Riemann sum is given by
$\Delta x = \dfrac{b-a}{n} = \dfrac{4-2}{n} = \dfrac{2}{n}$,
and the right-hand endpoints are given by

$x_i = a + i\Delta x = 2 + i\frac{2}{n}$.
Then using the given formulas,

$$\begin{aligned}
\int_2^4 x^2 \, dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(2 + i\frac{2}{n}\right)^2 \cdot \frac{2}{n} \\
&= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left(4 + \frac{8}{n}i + \frac{4}{n^2}i^2\right) \\
&= \lim_{n\to\infty} \frac{8}{n} \left(\sum_{i=1}^{n} 1 + \frac{2}{n}\sum_{i=1}^{n} i + \frac{1}{n^2}\sum_{i=1}^{n} i^2\right) \\
&= \lim_{n\to\infty} \frac{8}{n} \left(n + \frac{2}{n}\frac{n(n+1)}{2} + \frac{1}{n^2}\frac{n(n+1)(2n+1)}{6}\right) \\
&= 8\left(1 + 1 + \frac{1}{3}\right) \\
&= \frac{56}{3}
\end{aligned}$$

Question 5 (b)

SOLUTION. Note that we do not need to know the density, since it cancels out later. First, we calculate the area by adding the areas of the rectangle and semicircle:

$$A = 6 \cdot 2 + \frac{1}{2}\pi \cdot 3^2 = 12 + \frac{9}{2}\pi$$

We identify the endpoints of the integral as $[-3, 3]$, the top function as $\sqrt{3^2 - x^2}$ and the bottom function as -2. Then

$$\begin{aligned}
\overline{x} &= \frac{1}{A}\int_{-3}^{3} x(\sqrt{3^2 - x^2} + 2)\, dx \\
&= 0
\end{aligned}$$

by (odd) symmetry. Alternatively, you could argue from a physical point of view that the x-coordinate of the centre of mass must be 0. By (even) symmetry,

$$\begin{aligned}
\overline{y} &= \frac{1}{A}\int_{-3}^{3}\frac{1}{2}[(3^2 - x^2) - 2^2)]\ dx \\
&= \frac{1}{2A}\int_{-3}^{3}(5 - x^2)\ dx \\
&= \frac{1}{2A}\cdot 2\int_{0}^{3}(5 - x^2)\ dx \\
&= \frac{1}{A}\int_{0}^{3}(5 - x^2)\ dx \\
&= \frac{1}{A}\left(5x - \frac{x^3}{3}\right)\Big|_0^3\ dx \\
&= \frac{1}{A}(15 - 9) \\
&= \frac{6}{12 + \frac{9}{2}\pi} \\
&= \frac{12}{24 + 9\pi}
\end{aligned}$$

Question 6 (a)

SOLUTION. From now on, we will use $\arctan x$ instead of $\tan^{-1} x$. By integration by parts, set $u = \arctan x$ and $dv = dx$. Then $du = \dfrac{1}{1 + x^2}\ dx$ and $v = x$. Then

$$\begin{aligned}
\int_0^1 \arctan x\ dx &= x\arctan x\big|_0^1 - \int_0^1 \frac{x}{1 + x^2}\ dx \\
&= \arctan(1) - \frac{1}{2}\ln|1 + x^2|\big|_0^1 \\
&= \frac{\pi}{4} - \frac{1}{2}\ln 2
\end{aligned}$$

Question 6 (b)

SOLUTION 1. Let's complete the square in the denominator.

$$x^2 - 2x + 5 = (x - 1)^2 + 4$$

This suggests using the substitution $u = x - 1$, so that $du = dx$ and $2x - 1 = 2(x - 1) + 1 = 2u + 1$. Then

$$\begin{aligned}
\int \frac{2x - 1}{x^2 - 2x + 5}\ dx &= \int \frac{2u + 1}{u^2 + 4}\, du \\
&= \int \frac{2u}{u^2 + 4}\ du + \int \frac{1}{u^2 + 4}\, du
\end{aligned}$$

For the first integral, we use the substitution $v = u^2 + 4$, so that $dv = 2u\ du$.

$$\begin{aligned}\int \frac{2u}{u^2+4}\,du &= \int \frac{1}{v}\,dv\\ &= \ln|v| + C_1\\ &= \ln|u^2+4| + C_1\\ &= \ln|(x-1)^2+4| + C_1\\ &= \ln|x^2-2x+5| + C_1\end{aligned}$$

For the second integal, this looks like an arctan integral, but we need to scale. We use the substitution $2w = u$, so that $2\,dw = du$ and $u^2+4 = 4[(u/2)^2+1] = 4(w^2+1)$. Then

$$\begin{aligned}\int \frac{1}{u^2+4}\,du &= \int \frac{1}{4(w^2+1)} 2\,dw\\ &= \frac{1}{2}\int \frac{1}{w^2+1}\,dw\\ &= \frac{1}{2}\arctan w + C_2,\\ &= \frac{1}{2}\arctan\left(\frac{u}{2}\right) + C_2\\ &= \frac{1}{2}\arctan\left(\frac{x-1}{2}\right) + C_2\end{aligned}$$

Altogether,

$$\int \frac{2x-1}{x^2-2x+5}\,dx = \ln|x^2-2x+5| + \frac{1}{2}\arctan\left(\frac{x-1}{2}\right) + C_3$$

with the arbitrary constant of integration C_3.

Solution 2. Note that this is already in reduced form using partial fraction decomposition. We see that the numerator is *almost* the derivative of the denominator. We can modify it to be exactly the derivative by subtracting 1, then adding 1 to make up for it. Then using a simple substitution in the first integral and completing the square in the second,

$$\begin{aligned}\int \frac{2x-1}{x^2-2x+5}\,dx &= \int \frac{2x-2}{x^2-2x+5}\,dx + \int \frac{1}{x^2-2x+5}\,dx\\ &= \ln|x^2-2x+5| + \int \frac{1}{(x-1)^2+2^2}\,dx\\ &= \ln|x^2-2x+5| + \frac{1}{2}\arctan\left(\frac{x-1}{2}\right) + C\end{aligned}$$

Question 7 (a)

Solution. The question asks us to confirm that

$$\int_0^{\pi/4} \cos^4\theta\,d\theta = (8+3\pi)/32$$

So we will calculate the integral on the left hand side, and then compare our answer to the given right hand side.
The key identity is the double angle formula for cosine,

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$$

where the second equality came from $\sin^2 x + \cos^2 x = 1$. This gives

$$\cos^2 x = \frac{1}{2}(\cos 2x + 1)$$

and hence

$$\cos^4 x = \left(\frac{1}{2}(\cos 2x + 1)\right)^2$$

Then with $x = \theta$,

$$\begin{aligned}\int_0^{\pi/4} \cos^4\theta \; d\theta &= \int_0^{\pi/4} \left(\frac{1}{2}(\cos 2\theta + 1)\right)^2 \; d\theta \\ &= \frac{1}{4}\int_0^{\pi/4} (\cos^2 2\theta + 2\cos 2\theta + 1) \; d\theta\end{aligned}$$

"Note that this is **not** a substitution in the integral. All we do is rewrite $cos^4\ \theta$ using the double angle formula."
We notice that we have a term that looks like $\cos^2$ (something) again. The identity above holds for any argument and so we could write

$$\cos^2 \clubsuit = \frac{1}{2}(\cos 2\clubsuit + 1).$$

We have $\cos^2(2\theta)$ and so we can replace $\clubsuit$ with 2θ

$$\begin{aligned}\int_0^{\pi/4} \cos^4\theta \; d\theta &= \frac{1}{4}\int_0^{\pi/4} (\cos^2 2\theta + 2\cos 2\theta + 1) \; d\theta \\ &= \frac{1}{4}\int_0^{\pi/4} \left[\frac{1}{2}(\cos(2(2\theta)) + 1) + 2\cos 2\theta + 1\right] \; d\theta\end{aligned}$$

To make things cleaner, we'll factor out 1/2 from every summand in the integral:

$$\begin{aligned}\int_0^{\pi/4} \cos^4\theta \; d\theta &= \frac{1}{4}\int_0^{\pi/4} \left[\frac{1}{2}(\cos 4\theta + 1) + 2\cos 2\theta + 1\right] \; d\theta \\ &= \frac{1}{8}\int_0^{\pi/4} [(\cos 4\theta + 1) + 4\cos 2\theta + 2] \; d\theta \\ &= \frac{1}{8}\int_0^{\pi/4} [\cos 4\theta + 4\cos 2\theta + 3] \; d\theta\end{aligned}$$

Now, we can integrate each term, remembering the scaling factor:

$$\begin{aligned}
\int_0^{\pi/4} \cos^4\theta \, d\theta &= \frac{1}{8}\int_0^{\pi/4} [\cos 4\theta + 4\cos 2\theta + 3] \, d\theta \\
&= \frac{1}{8}\left[\frac{1}{4}\sin 4\theta + 2\sin 2\theta + 3\theta\right]\Big|_0^{\pi/4} \\
&= \frac{1}{8}\left[\frac{1}{4}\sin\pi + 2\sin\frac{\pi}{2} + 3\frac{\pi}{4} - \frac{1}{4}\sin 0 - 2\sin 0 - 3\cdot 0\right] \\
&= \frac{1}{8}\left[2 + \frac{3\pi}{4}\right] \\
&= \frac{1}{4} + \frac{3\pi}{32} \\
&= (8+3\pi)/32
\end{aligned}$$

as required.

Question 7 (b)

SOLUTION. We want to be able to use part (a), so we need some way of converting the polynomial into something involving trigonometric functions. In particular, recall that
$\tan^2\theta + 1 = \sec^2\theta$
so let's try the substitution $x = \tan\theta$, so $dx = \sec^2\theta \, d\theta$, and the boundaries of integration change according to $x = -1 = \tan\theta \Rightarrow \theta = -\pi/4$ and $x = 1 = \tan\theta \Rightarrow \theta = \pi/4$. Then, using symmetry at the end, as the resulting function is even, we see that

$$\begin{aligned}
\int_{-1}^{1} \frac{dx}{(x^2+1)^3} &= \int_{-\pi/4}^{\pi/4} \frac{\sec^2\theta}{(\tan^2\theta+1)^3} \, d\theta \\
&= \int_{-\pi/4}^{\pi/4} \frac{\sec^2\theta}{(\sec^2\theta)^3} \, d\theta \\
&= \int_{-\pi/4}^{\pi/4} \frac{\sec^2\theta}{\sec^6\theta} \, d\theta \\
&= \int_{-\pi/4}^{\pi/4} \frac{1}{\sec^4\theta} \, d\theta \\
&= \int_{-\pi/4}^{\pi/4} \cos^4\theta \, d\theta \\
&= 2\int_0^{\pi/4} \cos^4\theta \, d\theta \\
&= 2\left(\frac{8+3\pi}{32}\right) \\
&= \frac{8+3\pi}{16}
\end{aligned}$$

Question 8

SOLUTION. Let $y(t)$ be the mass of sugar in the tank at time t (in minutes), with units of kg. The rate of change of $y(t)$ is the difference between the incoming rate of salt and the outgoing rate of salt. The rate is simply the concentration multiplied by the flow rate; for incoming, we know the concentration is 0.01 kg/L and the flow rate is 20 L/min; for the outgoing, the concentration is $y/1000$ and the flow rate is still 20 L/min.

$$\begin{aligned}\frac{dy}{dt} &= (\text{rate in}) - (\text{rate out}) \\ &= 0.01 \cdot 20 - \frac{y}{1000} \cdot 20 \\ &= \frac{10-y}{50}.\end{aligned}$$

If $y \neq 10$, then

$$\begin{aligned}\frac{1}{10-y}\frac{dy}{dt} &= \frac{1}{50} \\ \int \frac{1}{10-y}\frac{dy}{dt}\, dt &= \int \frac{1}{50}\, dt \\ -\ln|10-y| &= \frac{1}{50}t + C\end{aligned}$$

We have an initial condition of $y(0) = 0$, so

$$-\ln 10 = C.$$

(Incidentally, this initial condition implies that $y \neq 10$, since $y = 10$ is a steady-state solution, where the mass never changes from 10.)
Now we solve

$$\begin{aligned}-\ln|10-y| &= \frac{1}{50}t - \ln 10 \\ |10-y| &= 10e^{-t/50}\end{aligned}$$

Since $y(t)$ is continuous, $10 - y(0) > 0$, and the right-hand side of the equation above is always positive, we conclude $10 - y(t) > 0$ for all t. Then

$$\begin{aligned}10 - y &= 10e^{-t/50} \\ y &= 10 - 10e^{-t/50}\end{aligned}$$

The amount of sugar in the tank after 1 hour, or 60 minutes, is

$$y(60) = 10 - 10e^{-6/5}$$

Question 9 (a)

Solution. This is the series

$$\sum_{n=1}^{\infty} \frac{1}{2n-1}$$

Notice

$$\frac{1}{2n-1} > \frac{1}{2n} > 0$$

for all n, and

$$\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges by p-series (it's just a multiple of the harmonic series). Then by the comparison test, the series

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \cdots$$

diverges.

Question 9 (b)

SOLUTION. We use the ratio test:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{2(n+1)+1}{2^{2(n+1)+1}} \cdot \frac{2^{2n+1}}{2n+1} \right| &= \lim_{n\to\infty} \frac{2n+3}{2n+1} \cdot \frac{2^{2n+1}}{2^{2n+3}} \\
&= \lim_{n\to\infty} \frac{2+\frac{3}{n}}{2+\frac{1}{n}} \cdot \frac{1}{2^2} \\
&= \frac{1}{4} \\
&< 1
\end{aligned}$$

So by the ratio test, the series converges.

Question 10 (a)

SOLUTION. Since

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$$

we have

$$\frac{1}{1+x^3} = \frac{1}{1-(-x^3)} = \sum_{n=0}^{\infty} (-x^3)^n = \sum_{n=0}^{\infty} (-1)^n x^{3n}$$

Integrating term by term,

$$\begin{aligned}
\int \frac{1}{1+x^3}\, dx &= \int \left(\sum_{n=0}^{\infty} (-1)^n x^{3n} \right) dx \\
&= \sum_{n=0}^{\infty} \int (-1)^n x^{3n}\, dx \\
&= \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{3n+1} x^{3n+1} \right) + C
\end{aligned}$$

Question 10 (b)

SOLUTION. By part (a),

$$\int_0^{1/4} \frac{1}{1+x^3}\, dx = \left[\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{3n+1} x^{3n+1}\right) + C\right]\Bigg|_0^{1/4}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{3n+1}\left(\frac{1}{4}\right)^{3n+1}$$

This is an alternating sum with

$$b_n = \frac{1}{3n+1}\left(\frac{1}{4}\right)^{3n+1}.$$

Clearly $b_{n+1} \leq b_n$ and $\lim_{n\to\infty} b_n = 0$. Then by Alternating Series Remainder Theorem, the error in the approximation is bounded as follows:

$$|\text{error}| \leq b_{n+1}$$

Then we are guaranteed the error is less than 10^{-5} if $b_{n+1} \leq 10^{-5}$, that is, we must solve

$$\frac{1}{3(n+1)+1}\left(\frac{1}{4}\right)^{3(n+1)+1} \leq 10^{-5} = \frac{1}{10^5}.$$

For $n+1 = 1$,

$$\frac{1}{3+1}\left(\frac{1}{4}\right)^{3+1} = \frac{1}{4^5} > \frac{1}{10^5}$$

so we do not have enough terms yet. For $n+1 = 2$,

$$\frac{1}{3(2)+1}\left(\frac{1}{4}\right)^{3(2)+1} \leq \frac{1}{7}\frac{1}{16000} = \frac{1}{112000} < \frac{1}{10^5}$$

where we have used $4^7 = 2^{14} = 2^{10}\cdot 16 = 1024\cdot 16 \geq 1000\cdot 16 = 16000$ to approximate. Thus, $n+1 = 2$ is sufficient. Since the series starts at 0, we need 2 terms (the $n=0$ and $n=1$ term, specifically).

Question 11

SOLUTION. We begin by calculating the radius of convergence, using the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x+2)^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{(-1)^n(x+2)^n}\right| = \lim_{n\to\infty}\sqrt{\frac{n}{n+1}}\cdot|x+2|$$
$$= \lim_{n\to\infty}\sqrt{\frac{1}{1+\frac{1}{n}}}\cdot|x+2|$$
$$= |x+2|$$

By the Ratio Test, the series converges if $|x+2| < 1$ and diverges if $|x+2| > 1$. Now we need to check the endpoints.
When $x+2 = -1$, that is, $x = -3$, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

which diverges by the p-series test, since $1/2 < 1$.
When $x+2 = 1$, that is, $x = -1$, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

This is an alternating series with

$$b_n = \frac{1}{\sqrt{n}}$$

Clearly, $b_{n+1} \leq b_n$ and $\lim\limits_{n\to\infty} b_n = 0$. By the Alternating Series Test,

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

converges. Therefore, the interval of convergence is $(-3, -1]$.

Question 12 (a)

Solution. Since this integral can be solved directly, we try the comparison test instead to prove divergence. Notice that for large x, the integrand behaves like $1/x$. Since $\int_2^{\infty} 1/x \, dx$ diverges, it will be our goal to show that our integrand is larger than a constant times $1/x$.
First, since $\sin x \geq -1$, we have

$$\frac{x + \sin x}{1 + x^2} \geq \frac{x-1}{1+x^2}$$

Now, for $x \geq 2$, we have

$$\frac{x-1}{1+x^2} \geq \frac{1}{3x}$$

To prove this, notice if $x \geq 2$, then

$$\begin{aligned} x(2x-3) &\geq 2 \cdot 1 \geq 1 \\ 3x^2 - 3x &\geq x^2 + 1 \\ \frac{x-1}{1+x^2} &\geq \frac{1}{3x} \end{aligned}$$

where in the second line we added x^2 to both sides, and in the third line we divided by $(3x)(1+x^2) > 0$.
Now, since

$$\int_2^\infty \frac{1}{3x}\,dx = \frac{1}{3}\int_2^\infty \frac{1}{x}\,dx$$

diverges by p-test, we have that

$$\begin{aligned}\int_2^\infty \frac{x+\sin x}{1+x^2}\,dx &\geq \int_2^\infty \frac{x-1}{1+x^2}\,dx\\ &\geq \int_2^\infty \frac{1}{3x}\,dx\\ &= \frac{1}{3}\int_2^\infty \frac{1}{x}\,dx > \infty\end{aligned}$$

and hence the original integral diverges by the Comparison Test, as required.

Question 12 (b)

SOLUTION. If $f(x) = \dfrac{x+\sin x}{1+x^2}$, then its clear it is positive, tends to 0, and agrees with the summands on the natural numbers. So we should prove it is never eventually decreasing. Taking the derivative,

$$f'(x) = \frac{(1+\cos x)(1+x^2) - 2x(x+\sin x)}{(1+x^2)^2}$$

We want to show that f is never eventually decreasing. For this, it suffices to find a sequence x_n of values that grow with no limit, and for which *f '(x_n)* is positive. How would be go about finding such a sequence? Let us look at the enumerator more carefully. Our best bet to make *f '(x_n)* positive is to choose x_n such that the positive term is maximal. This is achieved when *cos(x_n) = 1.* In this case, *sin(x_n) = 0* and we find that the enumerator becomes *2(1+$x_n{}^2$)-2$x_n{}^2$ = 2 > 0.* Hence, we choose $x_n = 2n\pi$ for any integer n, and double check that indeed the numerator becomes:

$$\begin{aligned}&(1+\cos(2n\pi))(1+(2n\pi)^2) - 2(2n\pi)((2n\pi)+\sin(2n\pi))\\ &= 2(1+4n^2\pi^2) - 8n^2\pi^2\\ &= 2 > 0\end{aligned}$$

The denominator is always positive, so $f'(x_n) > 0$ for these values of x_n, so that $f(x_n)$ is increasing at these values of x_n. Since $2n\pi$ grows with no limit, $f(x)$ is never eventually decreasing, and hence the integral test fails.

Question 12 (c)

SOLUTION 1. Limit comparison test finishes this problem fairly quickly. By inspection, the summand looks like $1/n$ for large n. Let's prove this rigorously.

$$\lim_{n\to\infty} \frac{n+\sin n}{1+n^2}\cdot\frac{n}{1} = \lim_{n\to\infty}\frac{1+\frac{\sin n}{n}}{\frac{1}{n^2}+1} = 1 > 0$$

and finite. Here, we used $-1/n \leq (\sin n)/n \leq 1/n$ and both the left and right sides limit to 0, so squeeze theorem gives $\lim_{n\to\infty}(\sin n)/n = 0$. Since $\sum_{n=2}^{\infty}\frac{1}{n}$ diverges by p-series, we conclude $\sum_{n=2}^{\infty}\frac{n+\sin n}{1+n^2}$ diverges as well by the limit comparison test.

SOLUTION 2. We can prove this using the comparison test, in the same way as part (a), with n replacing x and sums instead of integrals.
Notice that for large x, the summands behaves like $1/n$, so this will be our goal. First, since $\sin n \geq -1$, we have

$$\frac{n + \sin n}{1 + n^2} \geq \frac{n - 1}{1 + n^2}$$

Now, for $n \geq 2$, we have

$$\frac{n - 1}{1 + n^2} \geq \frac{1}{3n}$$

To prove this, notice if $n \geq 2$, then

$$\begin{aligned} n(2n - 3) &\geq 2 \cdot 1 \geq 1 \\ 3n^2 - 3x &\geq n^2 + 1 \\ \frac{n-1}{1+n^2} &\geq \frac{1}{3n} \end{aligned}$$

where in the second line we added n^2 to both sides, and in the third line we divided by $(3n)(1 + n^2) > 0$.
Now, since

$$\sum_{n=2}^{\infty} \frac{1}{3n} = \frac{1}{3} \sum_{n=2}^{\infty} \frac{1}{n}$$

diverges by p-series, we have that $\displaystyle\sum_{n=2}^{\infty} \frac{n + \sin n}{1 + n^2}$ diverges by the comparison test.

Good Luck for your exams!