Full Solutions
MATH101 April 2007

April 4, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

SOLUTION. Let $x = 2y + 1$ so that $dx = 2dy$ and hence
$\int (2y + 1)^5 \, dy = \int \frac{x^5}{2} \, dx = \frac{x^6}{12} + C = \frac{(2y+1)^6}{12} + C$

Question 1 (b)

SOLUTION. Evaluating directly, we have

$$\begin{aligned}
\int_{-1}^{0} (2x - e^x) \, dx &= x^2 - e^x \Big|_{-1}^{0} \\
&= 0 - e^0 - ((-1)^2 - e^{-1}) \\
&= -1 - 1 + e^{-1} \\
&= -2 + e^{-1}
\end{aligned}$$

Question 1 (c)

SOLUTION. Notice that our question is equivalent to
$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^4}{n^5} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \cdot \frac{i^4}{n^4}$
The values for the Riemann sum look like
$\Delta x = \dfrac{1}{n} = \dfrac{b-a}{n}$
$x_i = 0 + \dfrac{i}{n} = \dfrac{i}{n}$
and so
$a = 0$
and so with the above in the Δx, we can deduce that
$b = 1$
and our function should be
$f(x) = x^4$
Hence, we have
$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^4}{n^5} = \int_0^1 x^4 \, dx$
completing the question.

Question 1 (d)

SOLUTION. Notice that in our problem, we have
$\Delta x = \frac{4-0}{4} = 1 \qquad x_i = 0 + \frac{4i}{4} = i$
and so we have

$$\begin{aligned}
S_4 &= \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right) \\
S_4 &= \frac{1}{3}\left(f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)\right) \\
S_4 &= \frac{1}{3}\left(1 + \frac{4}{2} + \frac{2}{1+2^3} + \frac{4}{1+3^3} + \frac{1}{1+4^3}\right) \\
S_4 &= \frac{1}{3} + \frac{2}{3} + \frac{2}{27} + \frac{4}{84} + \frac{1}{195} \\
S_4 &= 1 + \frac{2}{27} + \frac{1}{21} + \frac{1}{195}
\end{aligned}$$

as required.

Question 1 (e)

SOLUTION. To solve this question, choose an x between $\pi/2$ and π. From here, the radius of a shell rotated about the y-axis is just $r = x$ and the height of this shell is $\frac{\sin x}{x} - 0 = \frac{\sin x}{x}$. The distance the height travels aroudn the y-axis is $2\pi r$ and hence, we have

$$\begin{aligned}
V &= \int_{\pi/2}^{\pi} 2\pi(x)(\sin(x)/x)\,dx \\
&= \int_{\pi/2}^{\pi} 2\pi\sin(x)\,dx \\
&= 2\pi(-\cos(x))\Big|_{\pi/2}^{\pi} \\
&= 2\pi(-\cos(\pi) + \cos(\pi/2)) \\
&= 2\pi(-(-1) + 0) \\
&= 2\pi
\end{aligned}$$

Question 1 (f)

SOLUTION. First, we need to show that the factor $x^2 + 9$ is irreducible. Looking at the discriminant, we see that it is $b^2 - 4ac = 0 - 4(1)(9) = -36 < 0$ and so this factor has no real roots. Thus, this is an irreducible quadratic. Hence, the partial fraction decomposition is
$\frac{10}{(x+1)^2(x^2+9)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+9}$

Question 1 (g)

SOLUTION. As in the hint, we have that the median m is the number such that

$$\frac{1}{2} = \int_m^{\infty} f(x)\,dx.$$

We are told that m=10. Hence, evaluating this integral gives

$$\begin{aligned}
\frac{1}{2} &= \int_{10}^{\infty} ke^{-kx}\,dx \\
&= \lim_{c\to\infty} \int_{10}^{c} ke^{-kx}\,dx \\
&= \lim_{c\to\infty} -e^{-kx}\Big|_{10}^{c} \\
&= \lim_{c\to\infty} -e^{-kc} + e^{-10k} \\
&= e^{-10k}
\end{aligned}$$

Solving for k yields

$$e^{10k} = 2$$

and thus

$$k = \frac{\ln(2)}{10}$$

Question 1 (h)

SOLUTION. We first compute the derivative of $y = 1 + (2/3)x^{3/2}$ and see that

$$\frac{dy}{dx} = \sqrt{x}$$

Plugging this into the arc length formula gives

$$\int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 \sqrt{1 + \sqrt{x}^2}\,dx = \int_0^1 \sqrt{1+x}\,dx$$

Now, let $u = 1 + x$ so that $du = dx$ and $u(0) = 1$ and $u(1) = 2$. Then

$$\begin{aligned}
\int_0^1 \sqrt{1+x}\,dx &= \int_1^2 \sqrt{u}\,du \\
&= \frac{2u^{3/2}}{3}\Big|_1^2 \\
&= \frac{2}{3}\left(2^{3/2} - 1\right) \\
&= \frac{2}{3}\left(2\sqrt{2} - 1\right)
\end{aligned}$$

completing the question.

Question 1 (i)

SOLUTION. First, we remove the improper integral to get
$\int_e^\infty \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty}\int_e^b \frac{1}{x(\ln x)^2}$.
Then we proceed by substitution. Let $u = \ln x$ so that $du = \frac{1}{x}dx$. The endpoints become $u(e) = \ln e = 1$ and $u(b) = \ln b$. Hence, we have

$$\begin{aligned}\int_e^\infty \frac{1}{x(\ln x)^2}\,dx &= \lim_{b\to\infty}\int_e^b \frac{1}{x(\ln x)^2}\,dx \\ &= \lim_{b\to\infty}\int_1^{\ln b}\frac{1}{(u)^2}\,du \\ &= \lim_{b\to\infty} -u^{-1}|_1^{\ln b} \\ &= \lim_{b\to\infty} -\frac{1}{\ln b}+\frac{1}{1} \\ &= 1\end{aligned}$$

completing the question.

Question 1 (j)

SOLUTION. We apply t^3 for 'x in the Maclaurin series for *cos(x)* to get
$\cos(t^3) = 1 - \frac{t^6}{2!} + \frac{t^{12}}{4!} - ... = \sum_{n=0}^\infty (-1)^n \frac{t^{6n}}{(2n)!}$
Now we multiply by t to get
$t\cos(t^3) = t - \frac{t^7}{2!} + \frac{t^{13}}{4!} - ... = \sum_{n=0}^\infty (-1)^n \frac{t^{6n+1}}{(2n)!}$
Now we integrate to get
$\int t\cos(t^3) = C + \frac{t^2}{2} - \frac{t^8}{8\cdot 2!} + \frac{t^{14}}{14\cdot 4!} - ... = \sum_{n=0}^\infty (-1)^n \frac{t^{6n+2}}{(6n+2)(2n)!}$
Applying the endpoints gives
$\int_0^x t\cos(t^3) = \frac{x^2}{2} - \frac{x^8}{8\cdot 2!} + \frac{x^{14}}{14\cdot 4!} - ... = \sum_{n=0}^\infty (-1)^n \frac{x^{6n+2}}{(6n+2)(2n)!}$
and so the first three nonzero terms are
$\frac{x^2}{2}, -\frac{x^8}{8\cdot 2!}, \frac{x^{14}}{14\cdot 4!}$
completing the question.

Question 1 (k)

SOLUTION. First, we flip the bounds of integration
$f(x) = \int_{e^x}^0 \cos^3 t\,dt = -\int_0^{e^x}\cos^3 t\,dt$
Taking the derivative using the fundamental theorem of calculus part 1 and the chain rule, we have
$f'(x) = -\frac{d}{dx}\int_0^{e^x}\cos^3 t\,dt = -e^x\cos^3(e^x)$
completing the question.

Question 2 (a)

SOLUTION. A picture is included below.
First, we determine the points of intersection. Setting the two curves equal to each other, we have

$$4 - x^2 = (x-2)^2 = x^2 - 4x + 4$$

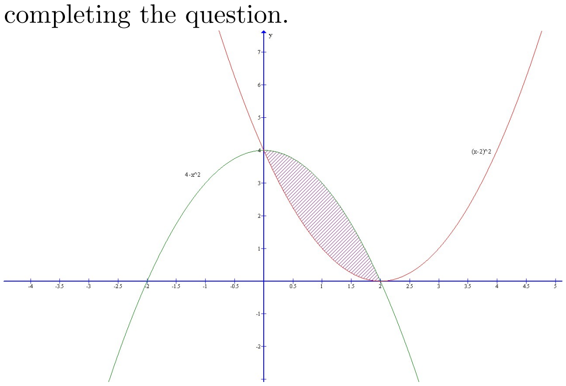
Bringing the terms to the right hand side, we have

$$0 = 2x^2 - 4x = 2x(x-2)$$

and so the points of intersection are $x = 0$ and $x = 2$. Notice either form the picture or testing a point between 0 and 2 (say $x = 1$) and plugging it into both equations, we see that the curve $4 - x^2$ is bigger between 0 and 2. Hence, our area is

$$\begin{aligned}
\int_0^2 (4 - x^2 - (x-2)^2)\, dx &= \int_0^2 (4 - x^2 - x^2 + 4x - 4)\, dx \\
&= \int_0^2 (-2x^2 + 4x)\, dx \\
&= \frac{-2x^3}{3} + 2x^2 \Big|_0^2 \\
&= \frac{-2(2)^3}{3} + 2(2)^2 \\
&= \frac{8}{3}
\end{aligned}$$

completing the question.



Question 2 (b)

SOLUTION. The questions asks for when $f_{av} = 0$. Plugging this into the formula in the hint gives

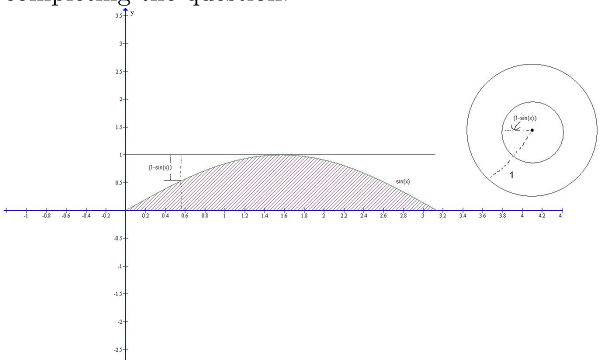
$$\begin{aligned}
f_{av} &= \frac{1}{b-0} \int_0^b (3x^2 - 6x + 2)\, dx \\
&= \frac{1}{b}\, x^3 - 3x^2 + 2x \Big|_0^b \\
&= \frac{1}{b}(b^3 - 3b^2 + 2b) \\
&= b^2 - 3b + 2 \\
&= (b-2)(b-1)
\end{aligned}$$

and so we have that $b = 1$ or $b = 2$ completing the question.

Question 2 (c)

SOLUTION. A picture is included below.

Imagine yourself standing on the x-axis at x=-1 looking towards the rotated object. This top down view (the circles in the picture) show that what we are doing is taking a cylinder of radius 1 and height π, and subtracting from it the shape that we get from removing the inner shape of radius $(1 - \sin(x))^2$. This gives the volume to be
$V = \int_0^\pi \pi(1^2 - (1 - \sin(x))^2)\, dx = \pi \int_0^\pi (2\sin(x) - \sin^2(x))\, dx$
completing the question.



Question 2 (d)

Solution. First we compute the area.
$A = \int_0^1 e^{-x}\, dx = -e^{-x}\big|_0^1 = -e^{-1} + e^0 = -\frac{1}{e} + 1 = \frac{e-1}{e}$
Next, to compute the x-coordinate, we want to compute
$\bar{x} = \frac{1}{A}\int_a^b x f(x)\, dx = \frac{e}{e-1}\int_0^1 xe^{-x}\, dx$
To do this, we use integration by parts. Let

$$\begin{aligned} u &= x & v &= -e^{-x} \\ du &= dx & dv &= e^{-x} \end{aligned}$$

So that

$$\begin{aligned} \bar{x} &= \frac{e}{e-1}\int_0^1 xe^{-x}\, dx \\ &= \frac{e}{e-1}\left(-xe^{-x}\big|_0^1 + \int_0^1 e^{-x}\, dx\right) \\ &= \frac{e}{e-1}\left(-e^{-1} - e^{-x}\big|_0^1\right) \\ &= \frac{e}{e-1}\left(-e^{-1} - e^{-1} + e^0\right) \\ &= \frac{e}{e-1}\left(1 - \frac{2}{e}\right) \\ &= \frac{e}{e-1}\left(\frac{e-2}{e}\right) \\ &= \frac{e-2}{e-1} \end{aligned}$$

completing the question.

Question 3 (a)

Solution 1. Let $u = x^2$ so that $du = 2xdx$. Then

$$\begin{aligned}\int \frac{x}{\sqrt{1-x^4}}\,dx &= \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} \\ &= \frac{-1}{2}\arccos(u) + C \\ &= \frac{-\arccos(x^2)}{2} + C\end{aligned}$$

completing the question.

Solution 2. If you forgot the derivative of $\arccos(x)$, you can use trig substitutions to derive it. The first step though remains the same: Let $u = x^2$ so that $du = 2xdx$. Then

$$\int \frac{x}{\sqrt{1-x^4}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}}$$

Now, let $u = \sin\theta$ so $du = \cos\theta d\theta$. Then

$$\frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\int \frac{\cos\theta d\theta}{\sqrt{1-\sin^2\theta}}$$

Using the trig identity $\sin^2\theta + \cos^2\theta = 1$, we have

$$\begin{aligned}\frac{1}{2}\int \frac{\cos\theta d\theta}{\sqrt{1-\sin^2\theta}} &= \frac{1}{2}\int \frac{\cos\theta d\theta}{\sqrt{\cos^2\theta}} \\ &= \frac{1}{2}\int d\theta \\ &= \frac{\theta}{2} + C \\ &= \frac{\arcsin(u)}{2} + C \\ &= \frac{\arcsin(x^2)}{2} + C\end{aligned}$$

completing the question.
Notice that this answer is different from the answer in the previous part. However, since

$$\arccos(x) + \arcsin(x) = \frac{\pi}{2}$$

the answers are actually the same up to a constant!

Question 3 (b)

Solution. We proceed by partial fractions. First, we solve for the constants in
$\frac{2x+3}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} = \frac{A(x+1)+B}{(x+1)^2}$
As the denominators above are equal, the numerators must also be equal and so
$2x + 3 = A(x + 1) + B$
Now, plugging in $x = -1$ into the above gives
$2(-1) + 3 = A(-1 + 1) + B$

and simplifying gives $B = 1$. Plugging in $x = 0$ into the above yields
$3 = 2(0) + 3 = A(0 + 1) + B = A + B = A + 1$
and so $A = 2$. Therefore, we have

$$\begin{aligned}\int_0^1 \frac{2x+3}{(x+1)^2}\,dx &= \int_0^1 \frac{2}{x+1}\,dx + \int_0^1 \frac{1}{(x+1)^2}\,dx \\ &= 2\ln|x+1|\big|_0^1 - (x+1)^{-1}\big|_0^1 \\ &= 2\ln|1+1| - 2\ln|0+1| - ((1+1)^{-1} - (0+1)^{-1}) \\ &= 2\ln(2) - \frac{1}{2} + 1 \\ &= 2\ln(2) + \frac{1}{2}\end{aligned}$$

completing the question.

Question 3 (c)

SOLUTION. First, we complete the square in our integral to get

$$\int \frac{dx}{\sqrt{x^2+2x+5}} = \int \frac{dx}{\sqrt{x^2+2x+1+4}} = \int \frac{dx}{\sqrt{(x+1)^2+4}}$$

Now, let $x + 1 = 2\tan\theta$ so that $dx = 2\sec^2\theta$. This gives

$$\int \frac{dx}{\sqrt{(x+1)^2+4}} = \int \frac{2\sec^2\theta d\theta}{\sqrt{4\tan^2\theta+4}}$$

Using the trig identity $\tan^2\theta + 1 = \sec^2\theta$, we have

$$\int \frac{2\sec^2\theta d\theta}{\sqrt{4\tan^2\theta+4}} = \int \frac{2\sec^2\theta d\theta}{\sqrt{4\sec^2\theta}} = \int \sec\theta d\theta$$

This last integral has a clever trick. Multiply top and bottom by

$$\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}$$

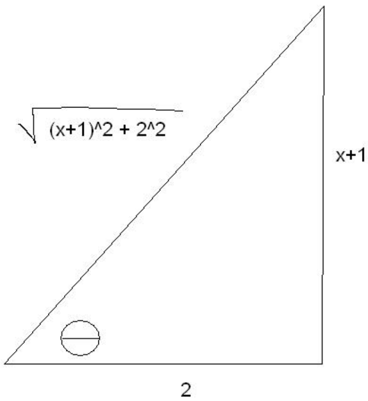
to get

$$\begin{aligned}\int \sec\theta d\theta &= \int \sec\theta \cdot \frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta} d\theta \\ &= \int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta} d\theta\end{aligned}$$

Let $u = \sec\theta + \tan\theta$ so $du = \sec^2\theta + \sec\theta\tan\theta$ and so the above integral is

$$\begin{aligned}\int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta} d\theta &= \int \frac{du}{u} \\ &= \ln|u| \\ &= \ln|\sec\theta+\tan\theta|\end{aligned}$$

To get x back in the expression we need to draw our triangle.



(Here we used Pythagorean theorem to get the hypotenuse.) From the picture, we see that

$$\cos\theta = \frac{2}{\sqrt{(x+1)^2+4}}$$

and so

$$\sec\theta = \frac{\sqrt{(x+1)^2+4}}{2}$$

Thus,

$$\int \frac{dx}{\sqrt{x^2+2x+5}} = \ln|\sec\theta + \tan\theta| = \ln\left|\frac{\sqrt{(x+1)^2+4}}{2} + \frac{x+1}{2}\right|$$

completing the question.

Question 3 (d)

SOLUTION. Notice that

$$\begin{aligned}\int(\cos^3 x)(\sin^4 x)\,dx &= \int(\cos^2 x)(\sin^4 x)\cos x\,dx \\ &= \int(1-\sin^2 x))(\sin^4 x)\cos x\,dx\end{aligned}$$

So let $u = \sin x$ so $du = (\cos x)dx$. Thus,

$$\begin{aligned}\int(\cos^3 x)(\sin^4 x)\,dx &= \int(1-\sin^2 x))(\sin^4 x)\cos x\,dx\\ &= \int(1-u^2))u^4\,du\\ &= \int u^4-u^6\,du\\ &= \frac{u^5}{5}-\frac{u^7}{7}+C\\ &= \frac{\sin^5 x}{5}-\frac{\sin^7 x}{7}+C\end{aligned}$$

completing the question.

Question 4 (a)

SOLUTION. Bringing the y' term over we have
$4y''=-y'$
Integrating both sides yields
$4y'=-y+C$
Now, bring the y term to the left and dividing by $\frac{1}{4}$, we have
$y'+\frac{y}{4}=\frac{C}{4}$
Now, multiplying both sides by $e^{x/4}$ yields
$e^{x/4}y'+e^{x/4}\frac{y}{4}=e^{x/4}\frac{C}{4}$
The left hand side is now a product rule derivative, namely
$(e^{x/4}y)'=e^{x/4}\frac{C}{4}$
Integrating both sides yields
$e^{x/4}y=\int e^{x/4}\frac{C}{4}\,dx=e^{x/4}C+D$
Dividing both sides by $e^{x/4}$ gives
$y=C+De^{-x/4}$
completing the question.

Question 4 (b)

SOLUTION. **Particular Solution**
We proceed as in the hints. First, we find a solution to
$y''-4y'+5y=5x^2-3x-2$
by noticing that since the right hand side is a quadratic polynomial and the left hand side is a second order differential equation, we have that a solution is given by
$y=Ax^2+Bx+C$
and we solve for A,B and C. Direct substitution yields

$$\begin{aligned}5x^2-3x-2 &= y''-4y'+5y\\ &= 2A-4(2Ax+B)+5(Ax^2+Bx+C)\\ &= 2A-8Ax-4B+5Ax^2+5Bx+5C\\ &= 5Ax^2+(5B-8A)x+5C-4B+2A\end{aligned}$$

Isolating to one side yields
$(5-5A)x^2+(8A-5B-3)x+4B-2A-5C-2=0$
Since this must hold for all x values, we have that each of the individual coefficients must be zero. This gives

$$
\begin{aligned}
5-5A&=0\\
8A-5B-3&=0\\
4B-2A-5C-2&=0
\end{aligned}
$$

Solving gives A=1, B=1 and C= 0. Hence, a solution to our differential equation is
$y=x^2+x$

Homogeneous solution

The story is not over yet. We could add the particular solution to solutions found via
$y''-4y'+5y=0$
Adding these solutions together would give the full set of solutions. To solve the homogeneous part notice that the corresponding polynomial given by
$r^2-4r+5=0$
has two imaginary roots given by (using the quadratic formula)
$r=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(5)}}{2}=\frac{4\pm\sqrt{-4}}{2}=2\pm i$
Hence, to this new differential equation, we have the solutions
$y=C_1e^{2x}\cos(x)+C_2e^{2x}\sin(x)$
Combining solutions, we have that
$y=C_1e^{2x}\cos(x)+C_2e^{2x}\sin(x)+x^2+x.$

Initial Conditions

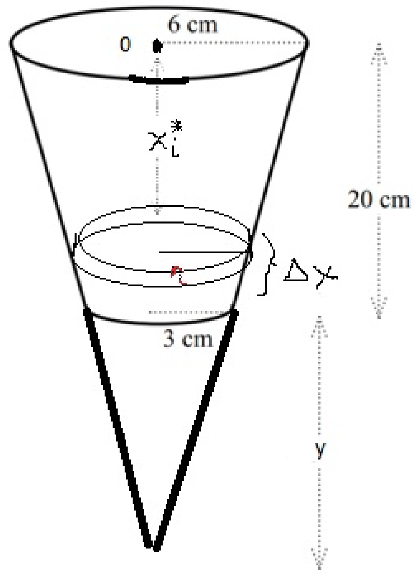
Now we would be done except, this is an initial value problem so we need to solve for the constants. The initial conditions are $y(0)=0$ and $y'(0)=2$. Hence, plugging in these values, we get that
$0=y(0)=C_1e^{2(0)}\cos(0)+C_2e^{2(0)}\sin(0)+(0)^2+(0)=C_1$
and that (taking the derivative via the product rule)

$$
\begin{aligned}
y'&=C_1(2e^{2x}\cos(x)-\sin(x)e^{2x})+C_2(2e^{2x}\sin(x)+\cos(x)e^{2x})+2x+1\\
2=y'(0)&=C_1(2e^{2(0)}\cos(0)-\sin(0)e^{2(0)})+C_2(2e^{2(0)}\sin(0)+\cos(0)e^{2(0)})+2(0)+1\\
2=y'(0)&=0(2-0)+C_2(0+1)+2(0)+1\\
1&=C_2
\end{aligned}
$$

NOTE: Above plugging in 0 for C_1 immediately would make the above computation simpler. Hence, the final solution is
$y=e^{2x}\sin(x)+x^2+x.$
Phew!

Question 5

Solution. Our first step will be to extend the cup to look like a cone and label everything as shown in the diagram.



Now, we can compute the value of y quickly via similar triangles (the big triangle and the one with side length 3),

$$\frac{20+y}{6} = \frac{y}{3}$$

and this gives $y = 20$cm. However, notice in the question that there are several different units everywhere and so for consistency we should use a single unit (metres). As shown, let x_i^* be a sample point between 0 and 0.2 (0 to 20cm), measured from the top of the cup. Then, at this point, the radius, labeled as r_i is

$$\frac{r_i}{0.4 - x_i^*} = \frac{0.06}{0.4} = \frac{3}{20}$$

and isolating gives

$$r_i = \frac{3(0.4 - x_i^*)}{20} = \frac{1.2 - 3x_i^*}{20}$$

Now, using the volume of a cylinder formula, we have

$$V_i = \pi r_i^2 \Delta x = \pi \left(\frac{1.2 - 3x_i^*}{20} \right)^2 \Delta x$$

Using the mass formula with δ as density, we have

$$m_i = \delta V_i = 1000\pi \left(\frac{1.2 - 3x_i^*}{20} \right)^2 \Delta x$$

Using the force formula, we have

$$F_i = m_i g = (9.8)1000\pi\left(\frac{1.2-3x_i^*}{20}\right)^2\Delta x$$

and finally, using the work formula, we can compute that the work done at the sample point is

$$W_i = F_i d = (9.8)1000\pi\left(\frac{1.2-3x_i^*}{20}\right)^2\Delta x(0.1+x_i^*)$$

where we added 0.1m (10cm) to the distance d above since we are using a straw that is 10 cm above the cup. Hence, summing all these pieces gives

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n} W_i &= \lim_{n\to\infty}\sum_{i=1}^{n}(9.8)1000\pi\left(\frac{1.2-3x_i^*}{20}\right)^2\Delta x(0.1+x_i^*)\\ &= \int_0^{20}(9.8)1000\pi\left(\frac{1.2-3x}{20}\right)^2(0.1+x)\,dx\end{aligned}$$

completing the question (we do not need to evaluate this integral).

Question 6 (a)

Solution. With $k = 16$ and $K = 4$, we know from the problem that
$m(t) = \frac{4}{1+Ae^{-16t}}$
Solving for A, we use the fact that $m(0) = 2$ and see that
$2 = m(0) = \frac{4}{1+Ae^{-16(0)}} = \frac{4}{1+A}$
Cross multiplying, we see that
$2 + 2A = 4$
and solving gives $A = 1$. Thus, we have
$m(t) = \frac{4}{1+e^{-16t}}$.
We want to solve for t when $m(t) = 3$. Plugging this in yields
$3 = \frac{4}{1+e^{-16t}}$
Cross multiplying, we see that
$3 + 3e^{-16t} = 4$
Bringing the 3 over and dividing by 3 yields
$e^{-16t} = \dfrac{1}{3}$
Taking logarithms and then dividing by -16 gives
$t = \dfrac{-\ln(\frac{1}{3})}{16} = \dfrac{\ln(3)}{16}$
and this completes the problem.

Question 6 (b)

Solution. We want to find the steady states of this differential equations. To do so, set $\frac{dm}{dt} = 0$, i.e.

$$16m\left(1-\frac{m}{4}\right) - 12 = 0$$

Simplifying yields

$$16m - 4m^2 - 12 = 0$$

and once more

$$m^2 - 4m + 3 = 0$$

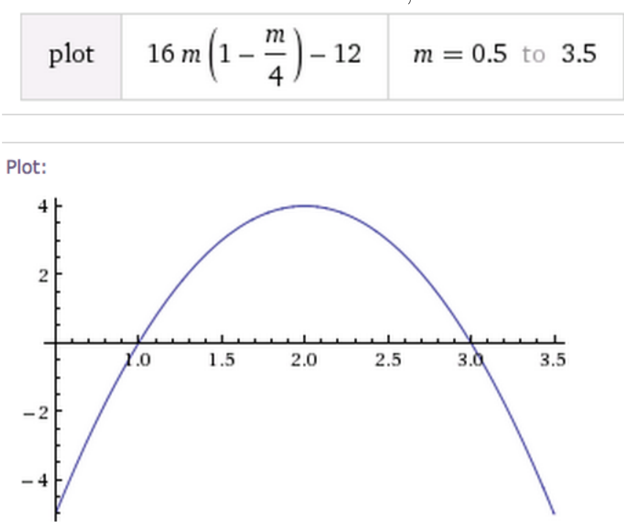
Factoring yields

$$(m-3)(m-1) = 0$$

and so the steady states are given by $m = 1$ and $m = 3$. It is quickly checked, e.g. by plugging in $m = 0$, $m = 2$, and $m = 4$, that

$$\frac{dm}{dt} = \begin{cases} \text{negative} & \text{if } m < 1 \text{ or } m > 3, \\ 0 & \text{if } m = 1 \text{ or } m = 3, \\ \text{positive} & \text{if } 1 < m < 3. \end{cases}$$

At time $t = 0$ we have $m(0) = 2$. Hence, despite the fishing, the population of fish will grow initially. But the growth will slow down as the population size approaches the steady state value $m = 3$. A steady state can never be reached in finite time, hence the answer is **No**, the fish population will never equal 3 million.



Good Luck for your exams!