Full Solutions
MATH105 April 2010

April 4, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

SOLUTION. Given the function

$$f(x, y) = 8x^{1/5}y^{4/5}$$

We can compute each of $f(2x, 2y)$ and $2f(x, y)$.

$$\begin{aligned} f(2x, 2y) &= 8(2x)^{1/5}(2y)^{4/5} \\ &= 8(2)^{1/5+4/5}x^{1/5}y^{4/5} \\ &= 16x^{1/5}y^{4/5} \end{aligned}$$

and

$$\begin{aligned} 2f(x, y) &= 2(8x^{1/5}y^{4/5}) \\ &= 16x^{1/5}y^{4/5} \end{aligned}$$

And so we obtain that

$$f(2x, 2y) - 2f(x, y) = 0$$

Question 1 (b)

SOLUTION. We are asked here to compute the second derivative of the given function with respect to y. So we start by taking the first derivative with respect to y and obtain

$$\frac{\partial f}{\partial y}(x, y) = 2xe^{2y} + 2y$$

(Remember that while doing this, we treat the variable x as if it was a constant). And now we go on for the second derivative with respect to y and obtain

$$\frac{\partial^2 f}{\partial y^2}(x, y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}(x, y)\right) = 4xe^{2y} + 2$$

Question 1 (c)

SOLUTION. For $f(x, y)$ to have a local maximum or minimum at a point (a, b), that point needs to be a critical point, that is

$$f_x(a, b) = f_y(a, b) = 0$$

For the function

$$f(x, y) = x^3y - 8y - 5x$$

we have the partial derivatives

$$f_x = 3x^2y - 5 \quad \text{and} \quad f_y = x^3 - 8$$

To look for critical points, we need to find points which yield a zero in both partial derivatives. Looking in particular at the partial derivative with respect to y we observe that if

$$f_y = 0$$

then

$$x^3 - 8 = 0$$

and so

$$x = 2.$$

Taking this value of x and substituting it into the equation for the partial derivative with respect to x yields

$$3 \cdot 2^2 y - 5 = 0$$

so

$$12y = 5$$

or

$$y = 5/12$$

We find only one critical point,

$$(a, b) = \left(2, \frac{5}{12}\right).$$

Since in order to be a relative maximum or minimum, the point must be a critical point, we have found that the only point that **may** be a maximum or minimum is (2,5/12).
Continuing the Problem:
If we assume the second derivative test is applicable (sometimes it is inconclusive), we need

$$f_{xx}(a, b)f_{yy}(a, b) - f_{xy}^2(a, b) > 0$$

to have a relative maximum or minimum.
If we compute the second derivatives, we find

$$\begin{aligned} f_{xx}(x, y) &= \frac{\partial}{\partial x}(3x^2y - 5) = 6xy \\ f_{yy}(x, y) &= \frac{\partial}{\partial y}(x^3 - 8) = 0 \\ f_{xy}(x, y) &= \frac{\partial}{\partial y}(3x^2y - 5) = 3x^2 \end{aligned}$$

and so

$$\begin{aligned}
f_{xx}(a,b) &= 6 \cdot 2 \cdot \frac{5}{12} = 5 \\
f_{yy}(a,b) &= 0 \\
f_{xy}(a,b) &= 3 \cdot 2^2 = 12
\end{aligned}$$

Thus,

$$f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b) = 0 - 12^2 = -144$$

Since this is a negative number, this critical point is a saddle point and so the conclusion is that the critical point is not a relative maximum or minimum and since there is only one critical point, there are no relative maxima/minima for $f(x,y)$.

Question 1 (d)

Solution. We can find f up to an arbitrary constant by integrating $f'(x)$.

$$\begin{aligned}
f(x) &= \int (\frac{1}{\sqrt{x}} + x^2) dx \\
&= \int (x^{-1/2} + x^2) dx \\
&= 2x^{1/2} + \frac{x^3}{3} + C
\end{aligned}$$

Imposing that $f(1) = 2$, we must have $2 + 1/3 + C = 2$ or that $C = -1/3$.
The final answer is:

$$f(x) = 2\sqrt{x} + \frac{x^3}{3} - 1/3.$$

Question 1 (e)

Solution. This is simple matter of evaluating an integral. From the definition of the average value of a function on an interval, we can write an expression for the average value of $1/x$ on the interval $[1, e]$:

$$\begin{aligned}
f_{avg} &= \frac{1}{e-1} \int_1^e \frac{1}{x} \, dx \\
&= \frac{1}{e-1} \ln(x) \Big|_1^e \\
f_{avg} &= \frac{1}{e-1}
\end{aligned}$$

So the average value of $1/x$ on the interval $[1, e]$ is $(e-1)^{-1}$.

Question 1 (f)

SOLUTION. With $n = 2$ and an interval of $[0, 4]$ we have

$$\Delta x = \frac{4-0}{2} = 2$$

and

$$x_j = 0 + j\Delta x = 2j \quad \text{for } j = 0, 1, 2$$

i.e. our subinterval endpoints are 0, 2 and 4.
To approximate with midpoints, we need to take sample points

$$x_1^* = 1 \quad \text{and} \quad x_2^* = 3$$

(these are the midpoints of x_0 and x_1, and x_1 and x_2 respectively).
The area under the curve is approximated by the midpoint Riemann sum, which yields

$$\begin{aligned}\int_0^4 f(x)\,dx &\approx \Delta x(f(x_1^*) + f(x_2^*)) \\ &= 2\left(\frac{1}{1+\sqrt{1}} + \frac{1}{1+\sqrt{3}}\right) \\ &= 1 + \frac{2}{1+\sqrt{3}}\end{aligned}$$

Question 1 (g)

SOLUTION. Using the disc method of integration, we know that the volume of the solid can be expressed as follows:

$$V = \int_0^m \pi(\sqrt{x})^2\,dx = \int_0^m \pi x\,dx = \pi\frac{m^2}{2}.$$

We want the value of m such that the volume is equal to 2π, so we solve the equation:

$$\pi\frac{m^2}{2} = 2\pi \quad \rightarrow \quad m = \pm 2.$$

Taking the positive solution gives us our answer

$$m = 2.$$

Question 1 (h)

SOLUTION. Apply integration by substitution. Let $u = \ln(x)$ (and thus, $du = dx/x$). Then it is a simple matter of applying the power law of integration and making sure to substitute back to express the integral in terms of x:

$$\begin{aligned}\int \frac{\sqrt{\ln x}}{x}\,dx &= \int \sqrt{u}\,du \\ &= \frac{2}{3}u^{3/2} + C \\ &= \frac{2}{3}(\ln x)^{3/2} + C\end{aligned}$$

where C is an arbitary constant.

Question 1 (i)

SOLUTION. Use integration by parts with

$$u = x \quad \text{and} \quad dv = e^{2x}$$

we obtain:

$$\begin{aligned}\int xe^{2x}\,dx &= \frac{xe^{2x}}{2} - \int \frac{1}{2}e^{2x}\,dx \\ &= \frac{xe^{2x}}{2} - \frac{1}{4}e^{2x} + C \\ &= \frac{(2x-1)e^{2x}}{4} + C\end{aligned}$$

where C is a constant.

Question 1 (j)

SOLUTION. As the hint suggests, we should calculate all the information we need to begin. First, we find the width of the intervals:

$$\Delta x = \frac{b-a}{n} = \frac{4-(-2)}{3} = 2.$$

We also need the endpoints of the intervals:

$$x_k = a + k\Delta x,$$

for k = 0 to k = 3. Thus the points are -2, 0, 2, and 4. Then the formula for the Trapezoidal rule is

$$T_n = \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{n-1}) + f(x_n)),$$

so we can fill in our information:

$$T_3 = \frac{2}{2}\left(\frac{1}{1+\left(\frac{-2}{2}\right)^4} + 2\frac{1}{1+\left(\frac{0}{2}\right)^4} + 2\frac{1}{1+\left(\frac{2}{2}\right)^4} + \frac{1}{1+\left(\frac{4}{2}\right)^4}\right),$$

$$T_3 = \frac{1}{1+1} + 2\frac{1}{1+0} + 2\frac{1}{1+1} + \frac{1}{1+16},$$

$$T_3 = \frac{1}{2} + 2 + 1 + \frac{1}{17},$$

$$T_3 = \frac{121}{34} \approx 3.5588.$$

Notice if we actually evaluate the integral,

$$\int_{-2}^{4} \frac{1}{1+\left(\frac{x}{2}\right)^4} \mathrm{d}x = 3.8742$$

so we see that the trapezoidal rule with only 4 points is already in fairly good agreement.

Question 1 (k)

SOLUTION. The instantaneous rate of change of capital has two components:

- We continuously add money at a rate of \$3,000 per year;
- We earn 6% interest compounded continuously.

If we denote by *A(t)* the balance of the account at time t, then we can write the differential equation

$$\frac{dA}{dt} = 3000 + 6 \qquad\qquad = 3000 + 0.06\,A(t)$$

We solve this equation using separation of variables:

$$\int \frac{dA}{3000 + 0.06A} = \int dt$$

Solving both sides separately yields

$$\frac{1}{0.06} \ln|3000 + 0.06A| + C_1 = t + C_2$$

Since A is positive, as the amount of money in the account, we can drop the absolute value sign. This yields

$$\ln(3000 + 0.06A) = 0.06t + C_3$$

or, equivalently

$$3000 + 0.06A = e^{0.06t + C_3} = e^{0.06t}e^{C_3} = C_4 e^{0.06t}.$$

C_4 is another arbitrary, but positive constant. Hence we obtain for *A(t)*:

$$A(t) = \frac{1}{0.06}\left(C_4 e^{0.06t} - 3000\right).$$

Now it's time to think about the initial condition. Since the account does not start with any balance initially, we know that *A(0) = 0*. This yields for C_4 that

$$0 = \frac{1}{0.06}\left(C_4 e^0 - 3000\right), \quad \text{which implies} \quad C_4 = 3000.$$

With this we finally find the expression for the balance in dollars at t years:

$$A(t) = \frac{3000}{0.06}\left(e^{0.06t} - 1\right).$$

This implies that at $t = 5$ years the balance is

$$A(5) = \frac{3000}{0.06}\left(e^{0.06 \cdot 5} - 1\right) = 17492.94$$

Reality check: Without interest we would expect to see *5 x 3000 = 15 000* dollars in the account. The amount with interest should be in the same order of magnitude and a little larger.

Question 1 (l)

SOLUTION. The constant k needs to be chosen such that *f(x)* integrates to 1:

$$\begin{aligned}\int_1^4 kx^{3/2}\,dx &= k\int_1^4 x^{3/2}\,dx = k\frac{2}{5}x^{5/2}\Big|_1^4 \\ &= k\frac{2}{5}(4^{5/2} - 1^{5/2}) = k\frac{62}{5},\end{aligned}$$

hence we choose *k=5/62*.

Question 1 (m)

SOLUTION. The cumulative distribution function, $F(x)$, can be obtained by integrating the probability density function from negative infinity to x. i.e:

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

(Note that we can assume that $f(x) = 0$ if $x > 2$ or $x < 1$). Applying the defintion of the cumulative distribution to this problem gives:

$$F(x) = \begin{cases} \int_{-\infty}^{x} 0\,dt & \text{if } x < 1 \\ \int_{-\infty}^{1} 0\,dt + \int_1^x 2(t-1)\,dt & \text{if } 1 \le x \le 2 \\ \int_{-\infty}^{1} 0\,dt + \int_1^2 2(t-1)\,dt + \int_2^x 0\,dt & \text{if } x > 2 \end{cases}$$

Simplifying gives:

$$F(x) = \begin{cases} 0 & \text{if } x < 1 \\ x^2 - 2x + 1 & \text{if } 1 \le x \le 2 \\ 1 & \text{if } x > 2 \end{cases}$$

Question 1 (n)

SOLUTION 1. From the definition of the variance we can directly compute its value.

$$\begin{aligned}
\mathrm{Var}(X) &= \int_a^b (x-\overline{X})^2 f(x)\,dx = \int_1^4 \left(x-\frac{7}{3}\right)^2 \frac{1}{2\sqrt{x}}\,dx \\
&= \int_1^4 \left(\frac{x^2}{2\sqrt{x}} - \frac{7}{3}\frac{x}{\sqrt{x}} + \frac{49}{9}\frac{1}{2\sqrt{x}}\right) dx \\
&= \int_1^4 \left(\frac{1}{2}x^{3/2} - \frac{7}{3}x^{1/2} + \frac{49}{18}x^{-1/2}\right) dx \\
&= \left(\frac{1}{2}\frac{2}{5}x^{5/2} - \frac{7}{3}\frac{2}{3}x^{3/2} + \frac{49}{18}2x^{1/2}\right)\Big|_1^4 \\
&= \frac{1}{5}32 - \frac{14}{9}8 + \frac{49}{9}2 - \left(\frac{1}{5} - \frac{14}{9} + \frac{49}{9}\right) \\
&= \frac{34}{45}.
\end{aligned}$$

SOLUTION 2. Alternatively, we can also use the identity

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2.$$

We're already given $\mathbb{E}(X) = 7/3$ so we just need $\mathbb{E}(X^2) = \int_1^4 x^2 f(x)dx$.
We compute this integral, finding $\mathbb{E}(X^2) = \int_1^4 x^2 \frac{1}{2\sqrt{x}}dx = \int_1^4 \frac{1}{2}x^{3/2}dx = \frac{1}{5}x^{5/2}|_1^4 = \frac{1}{5}(32-1) = 31/5$.
Thus $\mathrm{Var}(X) = \frac{31}{5} - (\frac{7}{3})^2 = 34/45$.

Question 2

SOLUTION. First we find the sale level A, where $D(A) = S(A)$:

$$\begin{aligned}
D(A) &= S(A) \\
\frac{42}{A+2} &= A+3 \\
42 &= A^2 + 5A + 6 \\
A^2 + 5A - 36 &= 0 \\
(A+9)(A-4) &= 0 \\
A &= -9 \text{ or } A = 4
\end{aligned}$$

The sale level must be positive, so $A = 4$. At this sale level, the price is $p_A = 7$.
The consumer surplus is:

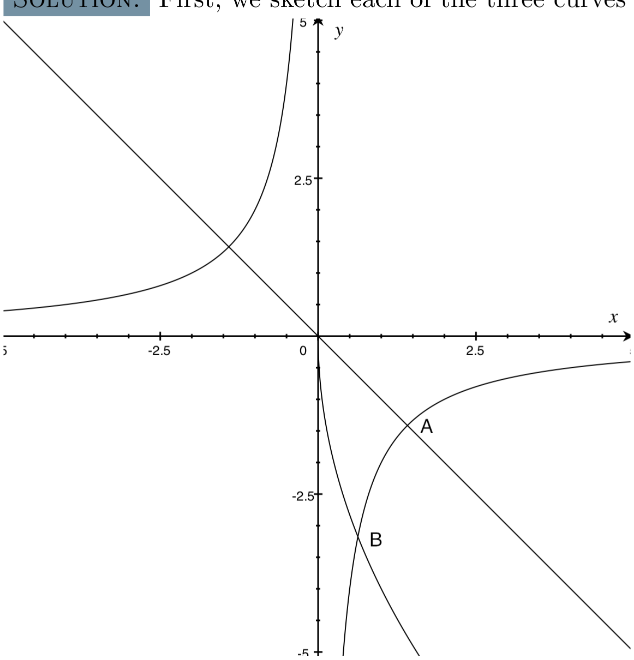
$$
\begin{aligned}
CS &= \int_0^A D(x)dx - Ap_A \\
&= \int_0^4 \frac{42}{x+2}dx - (4)(7) \\
&= 42\ln(x+2)\Big|_0^4 - 28 \\
&= 42(\ln(6) - \ln(2)) - 28 \\
&= 42\ln(3) - 28 \\
&\approx 18.14
\end{aligned}
$$

The producer surplus is:

$$
\begin{aligned}
PS &= Ap_A - \int_0^A S(x)dx \\
&= (4)(7) - \int_0^4 (x+3)dx \\
&= 28 - (x^2/2 + 3x)\Big|_0^4 \\
&= 28 - (16/2 + 12) \\
&= 8
\end{aligned}
$$

Question 3

SOLUTION. First, we sketch each of the three curves to see what is the region that we are talking about.



We will need to find the coordinates of points A and B which are the intersection of curves. For A we have

$$
\begin{aligned}
-x &= -\frac{2}{x} \\
x^2 &= 2 \\
x &= \pm\sqrt{2}
\end{aligned}
$$

We can see from the picture that we are interested in the point with a positive x coordinate, so

$$A = (\sqrt{2}, -\sqrt{2})$$

For B we have

$$
\begin{aligned}
-4\sqrt{x} &= -\frac{2}{x} \\
x^{3/2} &= \frac{1}{2} \\
x &= \left(\frac{1}{2}\right)^{2/3} = \frac{1}{\sqrt[3]{4}}
\end{aligned}
$$

And so since y = -2/x we have

$$B = \left(\frac{1}{\sqrt[3]{4}}, -2\sqrt[3]{4}\right)$$

This allows us to write the desired area as the sum of 2 integrals

$$
\begin{aligned}
\text{area} &= \int_0^{\frac{1}{\sqrt[3]{4}}} \big((-x) - (-4\sqrt{x})\big)\, dx + \int_{\frac{1}{\sqrt[3]{4}}}^{\sqrt{2}} \left((-x) - (-\frac{2}{x})\right)\, dx \\
&= \int_0^{\frac{1}{\sqrt[3]{4}}} \big(-x + 4\sqrt{x}\big)\, dx + \int_{\frac{1}{\sqrt[3]{4}}}^{\sqrt{2}} \left(-x + \frac{2}{x}\right)\, dx \\
&= \left[-\frac{x^2}{2} + \frac{4\cdot 2}{3}x^{3/2}\right]_0^{\frac{1}{\sqrt[3]{4}}} + \left[-\frac{x^2}{2} + 2\ln(x)\right]_{\frac{1}{\sqrt[3]{4}}}^{\sqrt{2}} \\
&= -\frac{(2^{-2/3})^2}{2} + \frac{8}{3}(2^{-2/3})^{3/2} + 0 - 0 - \frac{(\sqrt{2})^2}{2} + 2\ln(\sqrt{2}) + \frac{(2^{-2/3})^2}{2} - 2\ln(2^{-2/3}) \\
&= \frac{4}{3} - 1 + \ln(2) + \frac{4}{3}\ln(2) \\
&= \frac{1}{3} + \frac{7}{3}\ln(2)
\end{aligned}
$$

Question 4

SOLUTION. We are given that labour costs \\$108 per unit and capital costs \\$2 per unit, and we want to minimize cost. Thus the objective function is

$$C(x, y) = 108x + 2y.$$

The firm wants to produce 600 units, so using the production function we want $5x^{2/3}y^{1/3} = 600$. Thus the constraint function is

$$g(x, y) = x^{2/3}y^{1/3} - 120,$$

and the constraint is g(x, y) = 0.
The Lagrange multiplier method says that the gradients of the objective function and the constraint function should be proportional, or

$$\nabla C(x, y) = \lambda \nabla g(x, y).$$

The gradient of the objective function is

$$\nabla C(x, y) = \langle 108, 2 \rangle.$$

and the gradient of the constraint function is

$$\nabla g(x, y) = \langle \frac{2}{3}x^{-1/3}y^{1/3}, \frac{1}{3}x^{2/3}y^{-2/3} \rangle$$

Then we have three equations that must be satisfied

$$108 = \lambda \frac{2}{3}x^{-1/3}y^{1/3},$$

$$2 = \lambda \frac{1}{3}x^{2/3}y^{-2/3},$$

$$x^{2/3}y^{1/3} - 120 = 0.$$

Solving the second equation for λ gives

$$\lambda = 6x^{-2/3}y^{2/3},$$

and substituting into the first equation to eliminate λ gives

$$108 = 6x^{-2/3}y^{2/3}\frac{2}{3}x^{-1/3}y^{1/3}.$$

Simplifying, we have

$$108 = 4x^{-1}y,$$

and then we solve for y to get

$$y = 27x.$$

Now substituting into the third equation, the constraint equation, gives

$$x^{2/3}(27x)^{1/3} - 120 = 0.$$

Simplifying, we have

$$3x = 120,$$

and thus

$$x = 40.$$

We then have that

$$y = 27x = 27(40) = 1080.$$

Therefore the cost is minimized when using 40 units of labour and 1080 units of capital.

Question 5 (a)

SOLUTION. At any given time, two things are happening: the owed money is growing at a rate of 5% and the amount of money is going down at a rate of A. Therefore, we write:

$$\frac{dy}{dt} = 0.05y - A.$$

As an initial condition, we impose that $y(0) = 0.8 \times 300,000 = 240,000$ i.e. the amount owed is eighty percent of the purchase price since twenty percent was initially paid.

Question 5 (b)

SOLUTION 1. From part (a), we have

$$\frac{dy}{dt} = 0.05y - A,$$

which we will solve by separation of variables. Knowing the solution will allow us to give a condition on A so that after 25 years, the balance owed is zero.
We notice that the equation is separable and rearrange to find

$$\frac{dy}{0.05y - A} = dt.$$

Now we integrate to get

$$\int \frac{dy}{0.05y - A} = 20 \ln |0.05y - A| = t + C.$$

Therefore,

$$
\begin{aligned}
\ln|0.05y - A| &= t/20 + C/20 \\
|0.05y - A| &= e^{t/20 + C/20} = e^{C/20}e^{t/20} \\
0.05y - A &= Fe^{t/20}
\end{aligned}
$$

where F is an arbitrary constant that comes out of the integration,

$$\pm e^{C/20}$$

in this case.
After one last rearrangement, we see that the amount of money owed after t years is

$$y(t) = 20A + 20Fe^{t/20}$$

or

$$y(t) = 20A + Ge^{20t}$$

for another arbitary constant G.
From the initial condition, $y(0) = 240000$ so $20A + G = 240000$ or $G = 240,000 - 20A$. Finally then our full equation for the amount of money owing, $y(t)$ is,

$$y(t) = (240000 - 20A)e^{t/20} + 20A.$$

Setting $y(25) = 0$ (which corresponds to having no money owing after 25 years) forces

$$(240000 - 20A)e^{25/20} + 20A = 0$$

so

$$A = \frac{240000e^{25/20}}{20(e^{25/20} - 1)} = \frac{12000e^{5/4}}{e^{5/4} - 1} = 16818.61.$$

Therefore, the annual payments are $A =$ \$16818.61.

Solution 2. From part (a) we have that

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 0.05y(t) - A$$

with the initial amount of money owed being $y(0) = 240000$. The goal is to solve this equation for the rate of y so that we can actually obtain the quantity we want, $y(t)$ which is the money owed after t years. Consider the substitution,

$$u(t) = 0.05y - A$$

which implies that

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 0.05\frac{\mathrm{d}y}{\mathrm{d}t}.$$

Therefore we can rewrite (and simplify) our differential equation as

$$\frac{\mathrm{d}u}{\mathrm{d}t} = ru$$

which we recognize as the exponential model and so we know the solution is

$$u(t) = B\exp(0.05t)$$

with B an arbitrary constant. This solution can also be determined from separation of variables and integration. Putting back into our original variable, $y(t)$, we get

$$y(t) = \frac{B\exp(0.05t) + A}{0.05}.$$

which is our desired equation for the amount owed on the mortgage after time t. Using the initial condition

$$y(0) = \frac{B + A}{0.05} = 240000$$

we get that $B = 12000 - A$. Therefore we are able to write the amount of money owed solely in terms of the annual payments, A,

$$y(t) = \frac{(12000 - A)\exp(0.05t) + A}{0.05}.$$

Now when $t = 25$, we want that the mortgage is paid off, i.e., we want the amount of money we owe to be zero. Therefore we seek that $y(25) = 0$. Therefore,

$$\begin{aligned} 0 &= \frac{(12000 - A)\exp(0.05 \cdot 25)}{0.05} \\ A &= \frac{12000\exp(0.05 \cdot 25)}{\exp(0.05 \cdot 25) - 1} = 16818.61. \end{aligned}$$

Therefore, in order to pay off the mortgage in 25 years, we require that $A = 16818.61$. Recall that when we started paying the mortgage at year zero we owed 80% of the value or 240 000 dollars. If we want to find out how much we really paid, we just need to multiply our annual rate, A by 25 years to get \$420 465.25. Add this to the \$60 000 we already paid up front then our \$300 000 dollar house has actually cost us \$480 465.25!

Question 5 (c)

SOLUTION. In part (b) we saw that our payments totalled \$480,465.25. The house itself had a price of only \$300,000. Therefore, the difference of \$180,465.25 represents the interest paid during the entire mortgage.

Question 6

Solution. We can recognize this as the limit of a Riemann sum. Let's first put this into summation notation:

$$f(x) = \lim_{n\to\infty} \sum_{j=0}^{n-1} \frac{1}{2 + (1 + j(x-1)/n)^3} \cdot \frac{x-1}{n}$$

Note that when $j = 0$ this gives us the $1/3$ term in the sum. We put the factor $(x-1)/n$ first to resemble the general form of a (left) Riemann sum as given in Hint 3.
Ultimately, we aim to express $f(x)$ as a definite integral i.e. something of the form

$$\int_a^b g(s)ds = \lim_{n\to\infty} \sum_{j=0}^{n-1} g(s_j) \cdot \Delta s$$

We choose s here for ease of notation and not overuse the x symbol.
For x fixed, we can read off an interval spacing of

$$\Delta s = \frac{x-1}{n}$$

Let's rewrite the sum with Δs in place of $(x-1)/n$:

$$f(x) = \lim_{n\to\infty} \sum_{j=0}^{n-1} \frac{1}{2 + (1 + j\Delta s)^3} \cdot \Delta s.$$

This is suggestive of a left-endpoint Riemann sum, since we start with $j = 0$ and end at $j = n-1$. In that case, we can take our subinterval endpoints to be $s_j = 1 + j\Delta s$ (recall that when we do a Riemann sum, we have subintervals with endpoints that look like $s_j = a + j\Delta s$ and terms of the form $1 + j\Delta s$ are of this form). Let's rewrite this as

$$f(x) = \lim_{n\to\infty} \sum_{j=0}^{n-1} \frac{1}{2 + s_j^3} \cdot \Delta s$$

From here, we read off

$$g(s) = \frac{1}{2 + s^3}$$

Also, since $s_0 = 1 + 0\Delta s = 1$ is the lower bound and $s_n = 1 + n\Delta s = 1 + n(x-1)/n = x$ is the upper bound, our final result is that

$$f(x) = \int_1^x \frac{1}{2 + s^3} ds$$

Finally we can compute the equation of the tangent line to $f(x)$ at $x = 1$. All we need are $f(1)$ and $f'(1)$.

$$f(1) = \int_1^1 g(s)ds = 0$$

since we are integrating over a range of zero.
And by the Fundamental Theorem of Calculus,

$$f'(x) = \frac{1}{2+x^3}$$

so

$$f'(1) = \frac{1}{3}$$

Hence the equation of the tangent line is

$$y - 0 = \frac{1}{3}(x-1)$$

or simply

$$y = \frac{1}{3}(x-1)$$

Good Luck for your exams!