Full Solutions
MATH307 December 2008

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Educational Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Exam Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Educational Resources.

Question 1

SOLUTION. The rank can be found from the rref of A or of A^T however rref(A^T) is easier to find for this matrix, and necessary to calculate N($\mathrm{A^T}$). So we start by calculating rref(A^T) .

$$A = \begin{pmatrix} 2 & 4 & -6 & 0 & -4 \\ -1 & -2 & 3 & 0 & 2 \\ 3 & 6 & -9 & 0 & -6 \end{pmatrix}, \quad A^T = \begin{pmatrix} 2 & -1 & 3 \\ 4 & -2 & 6 \\ -6 & 3 & -9 \\ 0 & 0 & 0 \\ -4 & 2 & -6 \end{pmatrix}$$

Now calculating rref(A^T) is quite easy because all of the rows are multiples of the first row and hence linearly dependent. See that $R_1 = \frac{R_2}{2} = \frac{-R_3}{3} = \frac{-R_5}{2}$. Hence

$$\text{rref}(A^T) = \begin{pmatrix} 1 & \frac{-1}{2} & \frac{3}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The rank of A is equal to the number of leading zeroes in *rref(A)* so the rank of A is 1.
Now to find the nullspace of A^T we start by writing the matrix as an equation equal to zero.

$$\begin{bmatrix} 1 & \frac{-1}{2} & \frac{3}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

So
$$\begin{bmatrix} x_1 - \frac{x_2}{2} + \frac{3x_3}{2} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
$x_1 - \frac{x_2}{2} + \frac{3x_3}{2} = 0 \therefore x_1 = \frac{x_2}{2} - \frac{3x_3}{2}$
Now since there are two independent variables, N(A) will have two vectors. We can find them by setting up a vector with the equation we found above.

$$\begin{bmatrix} \frac{x_2}{2} - \frac{3x_3}{2} \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix}$$

And therefore we find that

$$N(A^T) = \text{span}\left(\left\{\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix}\right\}\right)$$

Question 2

SOLUTION. The number of basis vectors equals the dimension of the vector space, so $\mathbb{P}_3$ has dimension 4. To begin with, we represent the basis monomials as vectors:

$$1 \leftrightarrow e_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \quad x \leftrightarrow e_2 = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \quad x^2 \leftrightarrow e_3 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \quad x^3 \leftrightarrow e_4 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

Then, a general polynomial $a_3x^3 + a_2x^2 + a_1x + a_0$ can represented as

$$a_3x^3 + a_2x^2 + a_1x + a_0 = a_3e_4 + a_2e_3 + a_1e_2 + a_0e_1 = \begin{bmatrix}a_0\\a_1\\a_2\\a_3\end{bmatrix}$$

in the given basis. Next we calculate how A acts on the basis:

$$\begin{aligned} Ae_1 &= 1'' = 0 \\ Ae_2 &= x'' = 0 \\ Ae_3 &= (x^2)'' = 2 = 2e_1 \\ Ae_4 &= (x^3)'' = 6x = 6e_2 \end{aligned}$$

This determines the columns of A, i.e.

$$A = \begin{bmatrix}0 & 0 & 2 & 0\\0 & 0 & 0 & 6\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}$$

To double check this we calculate

$$\begin{aligned} (a_3x^3 + a_2x^2 + a_1x + a_0)'' &\leftrightarrow A\begin{bmatrix}a_0\\a_1\\a_2\\a_3\end{bmatrix} \\ 6a_3x + 2a_2 &\leftrightarrow \begin{bmatrix}2a_2\\6a_3\\0\\0\end{bmatrix} \\ 6a_3e_2 + 2a_2e_1 &\leftrightarrow 6a_3\begin{bmatrix}0\\1\\0\\0\end{bmatrix} + 2a_2\begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad \checkmark \end{aligned}$$

Finally, since the second derivative of the first two monomials is zero, the two corresponding vectors form a basis of the null space of A:

$$N(A) = \text{span}\left(\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right\}\right)$$

Question 3

SOLUTION. Let A be the augmented matrix, that is,

$$A = \begin{bmatrix} 1 & -1 & 1 & 2 \\ -2 & 1 & \alpha & -3 \\ 1 & \alpha & -1 & 1 \end{bmatrix}$$

To begin the row-reduction, we add two multiples of the first line to the second line and also subtract the first line from the third to obtain

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ -2 & 1 & \alpha & -3 \\ 1 & \alpha & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & -1 & \alpha+2 & 1 \\ 0 & \alpha+1 & -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & -(\alpha+2) & -1 \\ 0 & \alpha+1 & -2 & -1 \end{bmatrix}$$

Then, subtract a factor of *(α+2)* times the second row from the last row and simplify

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & -(\alpha+2) & -1 \\ 0 & \alpha+1 & -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & -(\alpha+2) & -1 \\ 0 & 0 & (-(\alpha+2))(-(\alpha+1))-2 & -1+\alpha+1 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & -(\alpha+2) & -1 \\ 0 & 0 & \alpha(\alpha+3) & \alpha \end{bmatrix}$$

We can now investigate what happens for different values of α. Notice that the first line doesn't depend on α and so there is nothing to conclude for the first line. For the second line, α could make one of the terms vanish but there will still be a solution to that equation if this happens. Therefore, the real power of α comes from the third line. We can distinguish three cases for α:

- When α = *0* the matrix will be \begin{bmatrix}

1 & -1 & 1 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, and there will be **infinitely many solutions**.

- When α = *-3* the matrix will be \begin{bmatrix}

1 & -1 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & -3 \end{bmatrix}, and there will **be no solution**.

- When α is neither 0 nor -3, there will be a **unique solution**.

Question 4

SOLUTION. To begin with, we note that the columns of A are already orthogonal,

$$\begin{bmatrix} 0 & 1 & 0 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = 0$$

So we can simply put the normalized columns of A into the matrix Q:

$$Q = \begin{bmatrix} 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{bmatrix}$$

As a last step, we calculate $R = Q^T A$ to find

$$R = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \end{bmatrix}$$

Question 5

SOLUTION. Using the definition of a determinant along the first column, we have

$$\begin{aligned} \det(A) &= 0 \cdot \begin{vmatrix} 8 & 3 & 1 \\ -1 & -4 & 3 \\ 2 & 3 & 1 \end{vmatrix} + (-1)(-2) \cdot \begin{vmatrix} 2 & 3 & -1 \\ -1 & -4 & 3 \\ 2 & 3 & 1 \end{vmatrix} + 0 \cdot \begin{vmatrix} 2 & 3 & -1 \\ 8 & 3 & 1 \\ 2 & 3 & 1 \end{vmatrix} + 0 \cdot \begin{vmatrix} 2 & 3 & -1 \\ 8 & 3 & 1 \\ -1 & -4 & 3 \end{vmatrix} \\ &= (2)[(2)(-4)(1) + (-1)(3)(-1) + (2)(3)(3) - (-1)(-4)(2) - (3)(-1)(1) - (2)(3)(3)] \\ &= (2)(-10) \\ &= -20 \end{aligned}$$

As

$$\det(A^3) = \det(A)^3$$

we see that

$$\det(A^3) = (-20)^3 = -8000$$

Question 6 (a)

SOLUTION. First, we can show that the matrix Q is symmetric by showing that $Q{=}Q^T$:

$$Q^T = (I - 2uu^T)^T = I^T - (2uu^T)^T = I - 2(u^T)^T u^T = I - 2uu^T = Q$$

Next, we can show the matrix Q is orthogonal by showing that $Q^T Q{=}I$. Since we know that Q is symmetric, $Q^T Q{=}QQ$.

$$\begin{aligned} Q^T Q = (I - 2uu^T)(I - 2uu^T) &= I - 4uu^T + 4(uu^T)(uu^T) \\ &= I - 4uu^T + 4u(u^T u)u^T \end{aligned}$$

Notice that $u^T u$ is just a number. In fact, $u^T u = \|u\|^2$ and since u is unit vector $\|u\|^2 = 1$. Thus, indeed Q is orthonogal:

$$Q^T Q = I - 4uu^T + 4u(1)u^T = I - 4uu^T + 4uu^T = I.$$

Question 6 (b)

SOLUTION. Recall the definition of the eigenvalue, we know that $-i$ is one of Q's eigenvalues when $det(Q + iI) = 0$, and $Q + iI$ is invertible if and only if $det(Q + iI) \neq 0$. From part a, we know that Q is a symmetric matrix. A symmetric matrix will only have real eigenvalues and since $-i$ is not real, it is not an eigenvalue of Q and $det(Q + iI)$ is never equal to zero.
Therefore, the matrix $Q + iI$ is invertible.

Question 7

SOLUTION. The infinity norm of a matrix is the infinity norm of the vector whose entries are the 1-norm of each row, so it will be

$$\|A\|_\infty = \max\{2 + 1 + 0, 1 + 2 + 1, 0 + 1 + 2\} = 4.$$

The 2-norm of a matrix is the largest of the absolute values of the eigenvalues. So we need to calculate the eigenvalues of A as the roots of the characteristic polynomial

$$\begin{aligned}
\det(A - \lambda I) &= \det \begin{vmatrix} -2-\lambda & 1 & 0 \\ 1 & -2-\lambda & 1 \\ 0 & 1 & -2-\lambda \end{vmatrix} \\
&= (-2-\lambda)\det \begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} - (1)\det \begin{vmatrix} 1 & 0 \\ 1 & -2-\lambda \end{vmatrix} \\
&= (-2-\lambda)((-2-\lambda)(-2-\lambda) - 1) - (-2-\lambda) \\
&= (-2-\lambda)(\lambda^2 + 4\lambda + 2)
\end{aligned}$$

Hence, the eigenvalues of A are $\lambda_1 = 2$ and $\lambda_{2,3} = \frac{-4\pm\sqrt{16-8}}{2} = -2 \pm \sqrt{2}$. The 2-norm of A is the largest of the absolute values of these and therefore $\|A\|_2 = 2 + \sqrt{2}$.

Question 8

SOLUTION. To begin with we rewrite the system in matrix form:

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

and call this matrix A. We need the eigenvalues of A and hence find the zeros of the characteristic polynomials

$$\begin{aligned}
\det(A - \lambda I) &= \det\left(\begin{bmatrix} 3-\lambda & 4 \\ 3 & 2-\lambda \end{bmatrix}\right) \\
&= (3-\lambda)(2-\lambda) - 12 \\
&= \lambda^2 - 5\lambda - 6 \\
&= (\lambda - 6)(\lambda + 1)
\end{aligned}$$

We see that $\lambda_1 = 6$ and $\lambda_2 = -1$.
Next, we calculate corresponding eigenvectors. For $\lambda_1 = 6$ we find the eigenvectors in the null space of $\begin{bmatrix} -3 & 4 \\ 3 & -4 \end{bmatrix}$ e.g. $v_1 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$. Similarly, for $\lambda_2 = -1$ we find the eigenvectors in the null space of $\begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix}$ e.g. $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Hence, the general solution of the linear differential equation is given by

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 4 \\ 3 \end{bmatrix} e^{6t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

To find the values of the constants c_1 and c_2 we plug in the values at the initial condition $t = 0$:

$$\begin{aligned} x(0) = 6 && = 4c_1 + c_2 \\ y(0) = 1 && = 3c_1 - c_2 \end{aligned}$$

which has the solution $c_1 = 1$, $c_2 = 2$. Therefore, the full solution of the linear differential equation with the given initial conditions is

$$u(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} e^{6t} + \begin{bmatrix} 2 \\ -2 \end{bmatrix} e^{-t}$$

Finally, since one of the eigenvalues is positive, the differential equation is unstable.

Question 9

Solution. Given $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $A^T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ we calculate $A^T A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, which has the eigenvalues 2 and 1. Hence, the singular values of A are $\sqrt{2}$ and 1.
Putting the singular values on the diagonal of Σ we find that $\Sigma = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Question 10

Solution. Calculating the Maclaurin expansion we find that

$$\left(e^{At}\right)^H = \left(\sum_{k=0}^{\infty} \frac{(At)^k}{k!} \right)^H = \sum_{k=0}^{\infty} \frac{(A^H t)^k}{k!} = e^{\left(A^H\right)t}$$

Thus,

$$\left(e^{At}\right)^H e^{At} = \left(e^{A^H t}\right) e^{At} = \left(e^{-At}\right) e^{At} = e^0 = I.$$

Good Luck for your exams!