Full Solutions
MATH105 April 2013

April 4, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

SOLUTION. Since we have the normal form of the plane, we find the vector, normal to the plane, in the equation.

$$-2x + y = 3z + 1 \qquad \text{which is} \qquad (x, y, z) \cdot \underbrace{\begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}}_{\text{normal vector} n} = 1$$

All equations of planes, parallel to the plane Q, have the same normal vector n.
This means the equation looks like
$(x, y, z) \cdot \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix} = a \in \mathbb{R}$
With the point $(-1, 1, 2)$ lying in the plane, we find a by
$a = (-1, 1, 2) \cdot \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix} = -3$
This gives us the equation of the plane

$$(x, y, z) \cdot \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix} = -3 \qquad \text{which is} \qquad -2x + y - 3z = -3$$

Question 1 (b)

SOLUTION. In the equation of a level curve we want to know all $x, y \in \mathbb{R}$, such that $f(x, y)$ is constant. Since the point $(1, -2)$ shall lie in the level curve, we search for the set satisfying
$f(x, y) = f(1, -2)$.
This means
$3xy^2 + 2y - 1 = 7$
We can simplify this equation to
$y(3xy + 2) = 8$

Question 1 (c)

SOLUTION. We start by noting that the denominator is in the form

$$a^2 - x^2$$

with $a = 5$. This is a good indicator that we want to do a trig substitution with

$$x = a \sin\theta$$

because

$$a^2 \sin^2\theta + a^2 \cos^2\theta = a^2$$

and hence

$$a^2 - a^2 \sin^2\theta = a^2\cos^2\theta$$

We will use this identity to handle the square root in the denominator. So, let

$$\begin{aligned} x &= 5\sin\theta \\ dx &= 5\cos\theta d\theta. \end{aligned}$$

We must also modify the limits of integration since we have a new variable. When

$$x = 0 = 5\sin\theta$$

solving for θ gives $\theta = 0$. Solving

$$x = 5/2 = 5\sin\theta$$

for θ gives $\theta = \frac{\pi}{6}$. Substituting these values into our integral we obtain

$$\int_0^{5/2} \frac{dx}{\sqrt{25-x^2}} = \int_0^{\pi/6} \frac{5\cos\theta d\theta}{\sqrt{25-(5\sin\theta)^2}} = \int_0^{\pi/6} \frac{5\cos\theta d\theta}{\sqrt{(5)^2-(5)^2(\sin\theta)^2}}.$$

We can use the identity above we get

$$\int_0^{\pi/6} \frac{5\cos\theta d\theta}{\sqrt{(5)^2-(5)^2(\sin\theta)^2}} = \int_0^{\pi/6} \frac{5\cos\theta d\theta}{\sqrt{(5)^2(\cos\theta)^2}} = \int_0^{\pi/6} 1 d\theta = \frac{\pi}{6}$$

Since this is a definite integral, the result is a number and we do not need to return to the previous variables.

Question 1 (d)

SOLUTION 1. There are multiple correct answers to this problem. The following is but one of three possible interpretations:
This is a left Riemann sum from 1.5 to 5.5 with $n = 4$ rectangles.
If we look at our left endpoint formula in the hint, we realize that

$$\begin{aligned} a &= 1.5 \\ n &= 4 \\ \Delta x &= \frac{b-a}{n} = 1. \end{aligned}$$

This last relation tells us that b=n+a=5.5.

SOLUTION 2. There are multiple correct answers to this problem. The following is but one of three possible interpretations:
This is a midpoint Riemann sum from 1 to 5 with $n = 4$ rectangles.
We can recognize this by writing

$$\sum_{k=0}^{3} f(1.5+k) = \sum_{k=0}^{3} f(1+(k+1/2))$$

which is in the form of our midpoint formula in the hint if we take

$$\begin{aligned} a &= 1 \\ n &= 4 \\ \Delta x &= \frac{b-a}{n} = 1. \end{aligned}$$

This last relation tells us that b=n+a=5.

SOLUTION 3. There are multiple correct answers to this problem. The following is but one of three possible interpretations:
This is a right endpoint Riemann sum from 0.5 to 4.5 with $n = 4$ rectangles.
We can recognize this by writing

$$\sum_{k=0}^{3} f(1.5+k) = \sum_{k=0}^{3} f(0.5+(k+1))$$

which is in the form of our right endpoint formula in the hint if we take

$$\begin{aligned} a &= 0.5 \\ n &= 4 \\ \Delta x &= \frac{b-a}{n} = 1. \end{aligned}$$

This last relation tells us that b=n+a=4.5.

Question 1 (e)

SOLUTION. Because of the absolute value in the integral, we need to split it at *0*.

$$\begin{aligned} \int_{-1}^{2} |2x| dx &= \int_{-1}^{0} -2x dx + \int_{0}^{2} 2x dx \\ &= -[x^2]_{-1}^{0} + [x^2]_{0}^{2} \\ &= -(0^2 - (-1)^2) + (2^2 - 0^2) \\ &= 1 + 4 = 5 \end{aligned}$$

Question 1 (f)

SOLUTION. Since we don't want to calculate $F(x)$, we define the antiderivative $A(t)$ of $\frac{1}{t^3+6}$. This also means that $A'(t) = \frac{1}{t^3+6}$.
We use $A(t)$ to express $F(x)$:

$$F(x) = \int_{0}^{\cos(x)} \frac{1}{t^3+6} dt = [A(t)]_{0}^{\cos(x)} = A(\cos(x)) - A(0)$$

Now we can differentiate $F(x)$ by using the chain rule
$F'(x) = A'(\cos(x)) \cdot (-\sin(x))$
Note that $A(0)$ is a constant for x and cancels by differentiating.
We use that we know $A'(t)$, the derivative of $A(t)$, and find
$F'(x) = \frac{1}{(\cos x)^3 + 6}(-\sin(x)) = -\frac{\sin(x)}{(\cos x)^3 + 6}$
Then, we apply π,
$F'(\pi) = -\frac{\sin(\pi)}{(\cos \pi)^3 + 6} = 0$

Question 1 (g)

SOLUTION. We proceed directly using the hints. First, the area is given by

$$\int_0^1 \frac{1}{(2x-4)^2} dx$$

Using the substitution u = *2x-4* we find du = *2dx*. When x = *0* we find u = *-4*, when x = *1* we find u = *-2*. Hence

$$\begin{aligned}
\int_0^1 \frac{1}{(2x-4)^2} dx &= \int_{-4}^{-2} \frac{1}{u^2} \frac{du}{2} \\
&= \frac{1}{2} \int_{-4}^{-2} u^{-2} du \\
&= \frac{1}{2} \left(-u^{-1}\right)\Big|_{-4}^{-2} \\
&= -\frac{1}{2}((-2)^{-1} - (-4)^{-1}) \\
&= -\frac{1}{2}\left(-\frac{1}{2} + \frac{1}{4}\right) \\
&= \frac{1}{8}
\end{aligned}$$

Question 1 (h)

SOLUTION. We will evaluate this integral using integration by parts. The integration by parts formula says that the following holds:

$$\int u\, dv = uv - \int v\, du.$$

If we let

$$u = \ln(x), \quad dv = \frac{1}{x^7}\, dx$$

then we have

$$du = \frac{1}{x}\, dx, \quad v = -\frac{1}{6}x^{-6}$$

and we can use the integration by parts formula to write

$$\begin{aligned}\int \frac{\ln(x)}{x^7}\,dx &= \ln(x)\cdot\left(-\frac{1}{6}x^{-6}\right) - \int\left(-\frac{1}{6}x^{-6}\right)\frac{1}{x}\,dx\\ &= -\frac{\ln(x)}{6x^6} + \int \frac{1}{6x^7}\,dx\\ &= -\frac{\ln(x)}{6x^6} - \frac{1}{36x^6} + C\end{aligned}$$

where C is a constant of integration.
Therefore, $\int \frac{\ln(x)}{x^7}\,dx = -\frac{\ln(x)}{6x^6} - \frac{1}{36x^6} + C$

Question 1 (i)

SOLUTION. This question is challenging, but can be solved in an elegant way using a Taylor series. To begin with, rewrite the sum as:

$$\sum_{k=0}^{\infty}\frac{1}{e^k k!} = \sum_{k=0}^{\infty}\frac{e^{-k}}{k!} = \sum_{k=0}^{\infty}\frac{(e^{-1})^k}{k!}$$

Next, set $x = e^{-1}$ to obtain

$$\sum_{k=0}^{\infty}\frac{(e^{-1})^k}{k!} = \sum_{k=0}^{\infty}\frac{x^k}{k!}$$

The last sum, however, is the Taylor series of the exponential function,

$$\sum_{k=0}^{\infty}\frac{x^k}{k!} = e^x$$

Plugging $x = e^{-1}$ back in we find that our final answer is

$$\sum_{k=0}^{\infty}\frac{1}{e^k k!} = e^{\left(e^{-1}\right)}$$

Question 1 (j)

SOLUTION. Given n, an even integer, and an upper bound $K \geq |f^{(4)}(x)|$ for all x in the interval $[a, b]$, the error that comes from using Simpson's Rule is as follows:

$$|I - S_n| \leq \frac{K(b-a)^5}{180 \cdot n^4}$$

where $I = \int_1^5 \frac{1}{x}\,dx$. We know that $a = 1$, $b = 5$ and $n = 4$. To find an upper bound on the fourth derivative, we first compute that

$$
\begin{aligned}
f(x) &= \frac{1}{x} \\
f'(x) &= -\frac{1}{x^2} \\
f''(x) &= 2\frac{1}{x^3} \\
f'''(x) &= -6\frac{1}{x^4} \\
f''''(x) &= 24\frac{1}{x^5}
\end{aligned}
$$

As the fourth derivative $f''''(x) = \dfrac{24}{x^5}$ is a decreasing function in x, its maximum occurs at the smallest possible value, when $x = 1$. Hence a value of K is given by
$K = 24$
Hence, we have a bound on the error is given by
$\left|\int_1^5 \frac{1}{x}\,dx - S_n\right| \le \frac{24(5-1)^5}{180 \cdot 4^4} = \frac{24 \cdot 4^5}{180 \cdot 4^4} = \frac{8}{15}$

Question 1 (k)

SOLUTION. We proceed by using the geometric series summation given by

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

valid for $|x| < 1$. To mimic this, we proceed via

$$
\begin{aligned}
\frac{1}{2x-1} &= -\frac{1}{1-2x} \\
&= -\sum_{n=0}^{\infty} (2x)^n
\end{aligned}
$$

valid when $|2x| < 1$, that is, when $|x| < 1/2$.

Question 1 (l)

SOLUTION. We proceed in the hint and try to solve for k where
$1 = \displaystyle\int_{-\infty}^{1} ke^{-x}e^{(-e^{-x})}$
First, we change the improper integral to
$1 = \displaystyle\lim_{b\to-\infty}\int_{b}^{1} ke^{-x}e^{(-e^{-x})}$
Let $u = -e^{-x}$ so that $du = e^{-x}dx$ and the endpoints change to
$u(1) = -e^{-1} \qquad u(b) = -e^{-b}$
This gives

$$\begin{aligned} 1 &= \lim_{b\to-\infty} \int_b^1 ke^{-x}e^{(-e^{-x})} \\ &= \lim_{b\to-\infty} \int_{-e^{-b}}^{-e^{-1}} ke^u\,du \\ &= \lim_{b\to-\infty} ke^u\Big|_{-e^{-b}}^{-e^{-1}} \\ &= \lim_{b\to-\infty} k(e^{-e^{-1}} - e^{-e^{-b}}) \end{aligned}$$

Since $\lim_{b\to-\infty} -b = +\infty$ we have that $\lim_{b\to-\infty} e^{-b} = +\infty$. Using this yields that $\lim_{b\to-\infty} -e^{-b} = -\infty$ and hence $\lim_{b\to-\infty} e^{-e^{-b}} = 0$. Thus the second term in the long equation above vanishes and we finally conclude that
$1 = ke^{-e^{-1}}$
Solving gives $e^{e^{-1}} = k$

Question 1 (m)

SOLUTION. Remember that the cumulative distribution function, $F(x)$, is defined in terms of the probability density function, $f(x)$, as follows:

$$F(t) = \int_{-\infty}^{t} f(x)\,dx.$$

Hence, to determine the cumulative distribution function for the given probability density function is a matter of plugging the particular density function into the definition. (Remember that since the given density function is only defined for $x \geq 1$, we can assume that the density function evaluates to 0 for all $x < 1$):

$$F(t) = \int_{-\infty}^{t} f(x)\,dx = \int_1^t 3x^{-4}\,dx = -x^{-3}|_1^t = 1 - t^{-3}.$$

Therefore, the cumulative distribution is $F(t) = 1 - \frac{1}{t^3}$

Question 1 (n)

SOLUTION. The expected value of a random variable X with probability density function $f(x)$, which we denote by $\mathbb{E}[X]$, is given by :

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} xf(x)\,dx.$$

In this case, the expected value is given by

$$\begin{aligned} \mathbb{E}[X] &= \int_1^4 x\left(\frac{2}{9\sqrt[3]{x}}\right)\,dx = \frac{2}{9}\int_1^4 x^{2/3}\,dx \\ &= \frac{2}{9}\left(\frac{3}{5}x^{5/3}\right)\Big|_1^4 = \frac{2}{15}\left(4^{5/3} - 1\right). \end{aligned}$$

Therefore,

$$\mathbb{E}[X] = \frac{2}{15}\left(4^{5/3} - 1\right).$$

Question 2 (a)

SOLUTION. For this question, we apply the limit comparision test. Let

$$a_k = \frac{\sqrt[3]{k}}{k^2 - k}.$$

We wish to define another series $\sum b_k$ such that

$$c = \lim_{k\to\infty} \frac{a_k}{b_k}$$

is positive and finite (i.e. $< \infty$). By comparing the dominant powers in the numerator *(1/3)* and denominator *(2)*, we are inspired *(1/3-2 = -5/3)* to define b_k as

$$b_k = k^{-5/3} = \frac{1}{k^{5/3}}.$$

The terms a_k, b_k are positive for all $k \geq 2$. By the limit comparison test, if c is positive and finite, then both series converge or both diverge. We can see that $\sum_k b_k$ will converge by the p-series test, since the power on k in the denominator, 5/3, is greater than 1. So if c is finite, then $\sum_k a_k$ converges.

$$c = \lim_{k\to\infty} \frac{\sqrt[3]{k} \times k^{5/3}}{k^2 - k} = \lim_{k\to\infty} \frac{k^2}{k^2 - k} = \lim_{k\to\infty} \frac{1}{1 - 1/k} = 1 < \infty.$$

So c is positive and finite. Therefore, by the limit comparison test, the sum $\sum_{k=2}^{\infty} \frac{\sqrt[3]{k}}{k^2-k}$ converges.

Question 2 (b)

SOLUTION. We apply the ratio test for this problem. Let

$$c = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|, \quad a_k = \frac{k^{10}10^k(k!)^2}{(2k)!}.$$

By the ratio test, if $c < 1$, then the series converges. If $c > 1$, the series diverges. If $c = 1$, then no conclusion can be drawn. For $k > 0$, all a_k are positive so we can remove the absolute value signs in c when evaluating it.

$$\begin{aligned} c &= \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} \\ &= \lim_{k\to\infty} \frac{(k+1)^{10}10^{k+1}((k+1)!)^2}{(2(k+1))!} \frac{(2k)!}{k^{10}10^k(k!)^2} \\ &= \lim_{k\to\infty} \frac{(k+1)^{10}}{k^{10}} \frac{10^{k+1}}{10^k} \frac{((k+1)!)^2}{(k!)^2} \frac{(2k)!}{(2k+2)!} \end{aligned}$$

Using the fact that $(k+1)! = (k+1)\cdot k!$ and $(2k+2)! = (2k+2)\cdot(2k+1)\cdot(2k)!$ we can simplify the fractions with factorial terms, giving us the following:

$$\begin{aligned} c &= \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^{10} \cdot 10 \cdot (k+1)^2 \cdot \frac{1}{(2k+2)(2k+1)} \\ &= \lim_{k\to\infty} \left(1+\frac{1}{k}\right)^{10} \cdot \frac{10(k+1)^2}{(2k+2)(2k+1)} \end{aligned}$$

Dividing both the numerator and denominator by k^2 gives:

$$\begin{aligned} &= \lim_{k\to\infty} \left(1+\frac{1}{k}\right)^{10} \frac{10(1+1/k)^2}{(2+2/k)(2+1/k)} \\ &= \frac{10}{4} > 1 \end{aligned}$$

Since $c > 1$, this series diverges by the ratio test.

Question 2 (c)

SOLUTION 1. For this series, we apply the integral test. Notice that the terms of the sum are positive, decreasing and continuous for all $k \geq 3$, the lower bound of the sum. Let

$$M = \int_3^\infty \frac{dx}{x\ln(x)\ln(\ln(x))} = \lim_{b\to\infty} \int_3^b \frac{dx}{x\ln(x)\ln(\ln(x))}.$$

By the integral test, if M converges (or diverges), then so too does the series. To evaluate the integral M, we use a change of variable. We let $u = \ln(\ln(x))$. Under this change of variable, we have via the chain rule

$$u = \ln(\ln(x)) \quad \to \quad du = \frac{1}{x\ln(x)}\, dx.$$

We rewrite M using our change of variable:

$$\begin{aligned} M &= \lim_{b\to\infty} \int_{\ln(\ln(3))}^{\ln\ln b} \frac{du}{u} \\ &= \lim_{b\to\infty} \ln(u)\Big|_{\ln(\ln(3))}^{\ln\ln b} \\ &= \lim_{b\to\infty} \ln\ln\ln b - \ln\ln\ln(3) \end{aligned}$$

and this last limit diverges. So M diverges. Therefore, the given series diverges by the integral test.

SOLUTION 2. We proceed similarly to solution 1 except we alter the approach to the integral. For this series, we apply the integral test. Notice that the terms of the sum are positive, decreasing and continuous for all $k \geq 3$, the lower bound of the sum. Let

$$M = \int_3^\infty \frac{dx}{x\ln(x)\ln(\ln(x))} = \lim_{b\to\infty} \int_3^b \frac{dx}{x\ln(x)\ln(\ln(x))}.$$

By the integral test, if M converges (or diverges), then so too does the series. To evaluate the integral$M$, we use a change of variable. We let $u = \ln(x)$. Under this change of variable, we have

$$u = \ln(x) \quad \to \quad du = \frac{1}{x}\,dx.$$

We rewrite M using our change of variable:

$$M = \lim_{b\to\infty} \int_{\ln(3)}^{\ln b} \frac{du}{u \ln u}$$

Now we do another substitution. We let $w = \ln(u)$. Under this change of variable, we have via the chain rule

$$w = \ln(u) \quad \to \quad dw = \frac{1}{u}\,du.$$

We rewrite M using our change of variable:

$$\begin{aligned} M &= \lim_{b\to\infty} \int_{\ln(\ln(3))}^{\ln\ln b} \frac{dw}{w} \\ &= \lim_{b\to\infty} \ln(w)\Big|_{\ln(\ln(3))}^{\ln\ln b} \\ &= \lim_{b\to\infty} \ln\ln\ln b - \ln\ln\ln(3) \end{aligned}$$

and this last limit diverges. So M diverges. Therefore, the given series diverges by the integral test.

Question 3 (a)

SOLUTION. First, we recognize that the differential equation given in this initial value problem is a separable differential equation. Hence, we move all terms involving x, y to opposite sides of the equal sign and take the indefinite integral:

$$\int y\,dy = \int \frac{dx}{x^2 + x}.$$

The integral on the left hand side is easy, but some work is required to solve the integral on the right hand side. First, we rewrite it as

$$\int y\,dy = \int \frac{1}{x(x+1)}\,dx$$

We proceed by using partial fractions to split up the integral. Notice that the fraction can be split up as

$$\frac{1}{x(x+1)} = \left(\frac{A}{x} + \frac{B}{x+1}\right) = \frac{A(x+1) + B(x)}{x(x+1)}.$$

This gives

$$1 = A(x+1) + B(x)$$

Setting x to 0 gives $A = 1$ and setting x to -1 gives $B = -1$. Thus, we can write out integral as

$$\int y\,dy = \int \frac{1}{x(x+1)}\,dx$$
$$\int y\,dy = \int \left(\frac{1}{x} - \frac{1}{x+1}\right)\,dx.$$

Now we integrate both sides of the equation and add the arbitary constant, C, to the right hand side:

$$\frac{1}{2}y^2 = \ln|x| - \ln|x+1| + C.$$

Isolating for y we get:

$$y(x) = \pm\sqrt{2\ln|x| - 2\ln|x+1| + K}.$$

where $K = 2C$. Since the initial condition, $y(1) = 2 > 0$, we take the positive solution for y:

$$y(x) = \sqrt{2\ln|x| - 2\ln|x+1| + K}.$$

To solve for K, we apply the initial condition to the solution:

$$\begin{aligned} y(1) = \sqrt{2\ln|1| - 2\ln|2| + K} &= 2 \\ \sqrt{K - 2\ln(2)} &= 2 \\ K - 2\ln(2) &= 4 \\ K &= 2\ln(2) + 4. \end{aligned}$$

Therefore, the solution to the initial value problem is:

$$y(x) = \sqrt{2\ln|x| - 2\ln|x+1| + 2\ln(2) + 4}.$$

Question 3 (b)

SOLUTION. We begin evaluating this integral by making a change of variable to deal with the square root:

$$u = \sqrt{x} \quad \rightarrow \quad du = \frac{1}{2\sqrt{x}}\,dx = \frac{1}{2u}\,dx.$$

And hence,

$$\int_0^4 f''(\sqrt{x})\,dx = 2\int_0^2 uf''(u)\,du.$$

Now using integration by parts we evaluate the integral. Let $v = u$ and $dw = f''(u)\,du$ so that we have $dv = du$ and $w = f'(u)$ (which holds by the Fundamental Theorem of Calculus as $f'(u)$ is an antiderivative of $f''(u)$. Thus

$$2\int_0^2 uf''(u)\,du = 2\left(uf'(u)\Big|_0^2 - \int_0^2 f'(u)\,du\right)$$

Now, we once again use the Fundamental Theorem of Calculus and note that $f(u)$ is an antiderivative of $f'(u)$ and this gives

$$\begin{aligned} 2\left(uf'(u)\Big|_0^2 - \int_0^2 f'(u)\,du\right) &= 2\left(uf'(u) - f(u)\right)\Big|_0^2 \\ &= 2\left(2f'(2) - f(2) + f(0)\right) \end{aligned}$$

Plugging the values of the terms given in the statement of the question gives:

$$\int_0^4 f''(\sqrt{x})\,dx = 2\left(2(4) - 3 + 1\right) = 12.$$

Question 4 (a)

Solution. The critical points of a multivariable function $f(x, y)$ are the values of (x, y) where the following conditions are both simultaneously true:

$$\begin{cases} \frac{\partial f}{\partial x}(x, y) = 0 \\ \frac{\partial f}{\partial y}(x, y) = 0 \end{cases}$$

Since $f(x, y) = xye^{-2x-y}$, the above conditions become:

$$\begin{cases} ye^{-2x-y} - 2xye^{-2x-y} = 0 \\ xe^{-2x-y} - xye^{-2x-y} = 0 \end{cases}$$

To solve these equations, we begin by factoring the left-hand sides, giving us

$$\begin{cases} y(1 - 2x)e^{-2x-y} = 0 \\ x(1 - y)e^{-2x-y} = 0 \end{cases}$$

Since the exponential function is never zero, the equations above are equivalent to

$$\begin{cases} y(1 - 2x) = 0 & (1) \\ x(1 - y) = 0. & (2) \end{cases}$$

We now proceed to determine the values of (x, y) such both Eq. (1) and (2) hold. The solutions to Eq. (1) are $y = 0$ and $x = 1/2$. We take each of these solutions and plug them into Eq. (2).
If $y = 0$, then Eq. (2) becomes:

$$x(1 - 0) = 0 \quad \to \quad x = 0.$$

Hence one of the critical points is $(x, y) = (0, 0)$.
If $x = 1/2$, then Eq. (2) becomes:

$$\frac{1}{2}(1 - y) = 0 \quad \to \quad y = 1.$$

So, another critical point is $(x, y) = (1/2, 1)$.

Therefore, the critical points of f are $(x, y) = (0, 0), \left(\dfrac{1}{2}, 1\right)$.

Question 4 (b)

Solution. From part a), the critical points are $(x, y) = (0, 0)$, $\left(\frac{1}{2}, 1\right)$ and the first partial derivatives are

$$\frac{\partial f}{\partial x} = y(1-2x)e^{-2x-y} = 0 \quad \frac{\partial f}{\partial y} = x(1-y)e^{-2x-y} = 0$$

The second partial derivatives are given by

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= y(-2)e^{-2x-y} + y(1-2x)(-2)e^{-2x-y} \\
&= -2ye^{-2x-y}(1+1-2x) \\
&= -4ye^{-2x-y}(1-x) \\
&= 4ye^{-2x-y}(x-1)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial x \partial y} &= y(1-2x)e^{-2x-y} \\
&= (1-2x)e^{-2x-y} + y(1-2x)(-1)e^{-2x-y} \\
&= (1-2x)e^{-2x-y}(1+y) \\
&= (1-y-2x+2xy)e^{-2x-y}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial y^2} &= x(-1)e^{-2x-y} + x(1-y)(-1)e^{-2x-y} \\
&= xe^{-2x-y}(-1-1+y) \\
&= xe^{-2x-y}(y+2)
\end{aligned}$$

To classify the critical points, we need to compute the Hessian matrix, H, of the function $f(x, y)$:

$$H(x, y) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 4y(x-1)e^{-2x-y} & (1-y-2x+2xy)e^{-2x-y} \\ (1-y-2x+2xy)e^{-2x-y} & x(y-2)e^{-2x-y} \end{bmatrix}.$$

Evaluating H at the critical point $(x, y) = (0, 0)$ gives

$$H(0, 0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The determinant of H at the point $(0, 0)$ is equal to $(0 \cdot 0) - (1 \cdot 1) = -1$, which is less than zero. Hence the point $(0, 0)$ is a saddle point.

Evaluating H at the critical point $(x, y) = (1/2, 1)$ gives

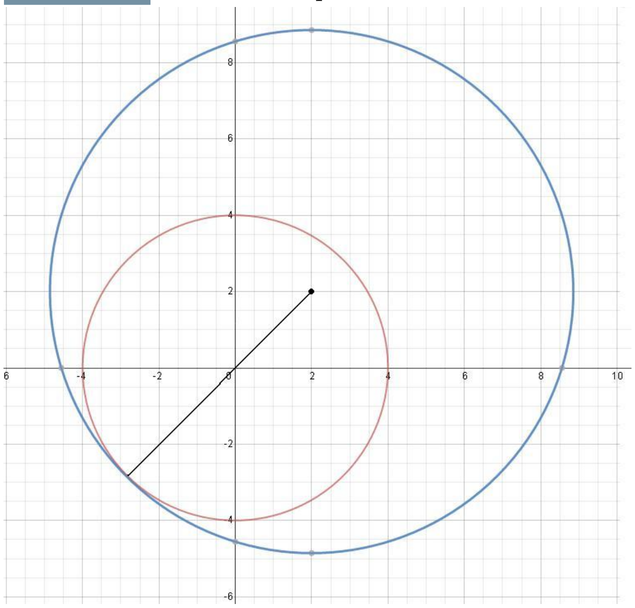
$$H\left(\frac{1}{2}, 1\right) = \begin{bmatrix} -2e^{-2} & 0 \\ 0 & -\frac{1}{2}e^{-2} \end{bmatrix}.$$

The determinant of H at the point $(1/2, 1)$ is equal to $(-2e^{-2}) \cdot \left(-\frac{1}{2}e^{-2}\right) - 0 \cdot 0 = e^{-4}$ which is greater than zero and so $(1/2, 1)$ is not a saddle point of f and must be either a local max or min. Since $\frac{\partial^2 f}{\partial x^2}\left(\frac{1}{2}, 1\right) = -2e^{-2}$ is less than zero, the point $(1/2, 1)$ is a local maximum of f.

Therefore, (x,y) = (0,0) is a saddle point and (x,y) = (1/2,1) is a local maximum.

Question 5 (a)

SOLUTION. Examine the picture below



The red circle describes the contaminated region. Our blue circle has to be just large enough to completely contain the red circle. Hence, we need to find the distance between any point on the red circle and the centre of the blue circle, *(2,2)*. The radius of the blue circle is then the smallest number that is bigger than all these distances.
The distance between any point *(x,y)* and *(2,2)* is given by $\sqrt{(x-2)^2+(y-2)^2}$. Hence, the radius r we are looking for is the maximum value of distances from points on the red circle to the centre of the blue circle:
$r = \sqrt{(x-2)^2+(y-2)^2}$
subject to this point (x, y) lying on the red circle
$x^2 + y^2 = 16$
Maximizing this distance gives the smallest possible radius such that the blue circle still encloses the entire contamination area.

Question 5 (b)

SOLUTION. Let $g(x, y) = x^2 + y^2 - 4$ be the constraint function. We thus need to solve the system of equations given by

$$g(x,y) = 0$$
$$\nabla f(x,y) = \lambda \nabla g(x,y)$$

Plugging in our values gives

$$x^2 + y^2 - 4 = 0$$
$$(-6xy, 6 - 3y^2 - 3x^2) = \lambda(2x, 2y)$$

Expanding the second equation gives the following three equations

$$\begin{aligned} x^2 &= 4 - y^2 \\ -6xy &= 2\lambda x \\ 6 - 3y^2 - 3x^2 &= 2\lambda y \end{aligned}$$

Substituting the first equation in the third equation gives $6 - 3y^2 - 3(4 - y^2) = 2\lambda y$ which simplifies to

$$-6 - 3y^2 + 3y^2 = 2\lambda y$$

Reducing gives $-3 = \lambda y$. This tells us that $\lambda \neq 0$ and thus we can divide by it to get that $\frac{-3}{\lambda} = y$. Looking at the second equation from the group of three above gives us that

$$0 = 2\lambda x + 6xy = x(2\lambda + 6y)$$

Case 1: $\boldsymbol{x = 0}$

Plugging $x = 0$ into $g(x, y)$ gives $y = \pm 2$.

Case 2: $\boldsymbol{x \neq 0}$

In this case we divide the equation above by x and obtain
$2\lambda = -6y = -6\left(\frac{-3}{\lambda}\right) = \frac{18}{\lambda}$
Solving gives $\lambda = \pm 3$. Thus $y = \frac{-3}{\lambda} = \mp 1$. Plugging these y values into $g(x, y)$ gives $x = \pm\sqrt{3}$.

Compare function values at all candidate points

We found the following six solutions:

$$(0, -2), (0, 2), (-\sqrt{3}, -1), (\sqrt{3}, -1), (-\sqrt{3}, 1), (\sqrt{3}, 1)$$

Plugging in each of the values gives

$$\begin{aligned} f(0, 2) &= 6(2) - (2)^3 - 3(0)^2(2) = 4 \\ f(0, -2) &= 6(-2) - (-2)^3 - 3(0)^2(-2) = -4 \\ f(-\sqrt{3}, -1) &= 6(-1) - (-1)^3 - 3(-\sqrt{3})^2(-1) = 4 \\ f(\sqrt{3}, -1) &= 6(-1) - (-1)^3 - 3(\sqrt{3})^2(-1) = 4 \\ f(-\sqrt{3}, 1) &= 6(1) - (1)^3 - 3(-\sqrt{3})^2(1) = -4 \\ f(\sqrt{3}, 1) &= 6(1) - (1)^3 - 3(\sqrt{3})^2(1) = -4 \end{aligned}$$

and thus the minimum value is -4 and the maximum value is 4.

Question 6 (a)

SOLUTION. To determine the interval of convergence, we apply the ratio test. Let

$$a_k(x) = \frac{(x+1)^{2k}}{k^2 9^k}.$$

To determine the interval of convergence, we must find the values of x such that the value of the limit below is less than 1 (and then we need to check the endpoints). Evaluating gives

$$\begin{aligned}
\lim_{k\to\infty}\left|\frac{a_{k+1}(x)}{a_k(x)}\right| &= \lim_{k\to\infty}\left|\frac{(x+1)^{2k+2}}{(k+1)^2 9^{k+1}} \div \frac{(x+1)^{2k}}{k^2 9^k}\right| \\
&= \lim_{k\to\infty}\left|\frac{(x+1)^{2k+2}}{(k+1)^2 9^{k+1}}\frac{k^2 9^k}{(x+1)^{2k}}\right| \\
&= \lim_{k\to\infty}\left|\frac{1}{9}\left(\frac{k}{k+1}\right)^2 (x+1)^2\right| \\
&= \left|\frac{(x+1)^2}{9}\right| \\
&= \frac{(x+1)^2}{9}
\end{aligned}$$

where we drop the absolute value signs, since $[(x+1)^2]/9$ is always positive.
Now, when the above limit is less than one, we have:

$$\begin{aligned}
\frac{(x+1)^2}{9} &< 1 \\
(x+1)^2 &< 9 \\
-3 < x+1 &< 3 \\
-4 < x &< 2.
\end{aligned}$$

Similarly, we know what when $\frac{(x+1)^2}{9} > 1$ that is, when $x < -4$ and when $x > 2$, we have that the series diverges. So all that is left is to check the two endpoints. When $x = -4$, we have

$$\begin{aligned}
\sum_{k=1}^{\infty} a_k(-4) &= \sum_{k=1}^{\infty} \frac{(-4+1)^{2k}}{k^2 9^k} \\
&= \sum_{k=1}^{\infty} \frac{(-3)^{2k}}{k^2 9^k} \\
&= \sum_{k=1}^{\infty} \frac{9^k}{k^2 9^k} \\
&= \sum_{k=1}^{\infty} \frac{1}{k^2}
\end{aligned}$$

and this converges by the p-series test. Similarly, when $x = 2$, we have

$$\begin{aligned}
\sum_{k=1}^{\infty} a_k(2) &= \sum_{k=1}^{\infty} \frac{(2+1)^{2k}}{k^2 9^k} \\
&= \sum_{k=1}^{\infty} \frac{(3)^{2k}}{k^2 9^k} \\
&= \sum_{k=1}^{\infty} \frac{9^k}{k^2 9^k} \\
&= \sum_{k=1}^{\infty} \frac{1}{k^2}
\end{aligned}$$

and this also converges by the p-series test. Thus, the interval of convergence is $-4 \leq x \leq 2$

Question 6 (b)

SOLUTION. Let $b_n = a_n(x-1)^n$ Ratio test gives

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to\infty} \frac{a_{n+1}|x-1|}{a_n}$$

Next, the partial summation from the convergent sum given to us in the problem statement tells us that

$$\begin{aligned}
\sum_{k=1}^{n} \left(\frac{a_k}{a_{k+1}} - \frac{a_{k+1}}{a_{k+2}}\right) &= \left(\frac{a_1}{a_2} - \frac{a_2}{a_3}\right) + \left(\frac{a_2}{a_3} - \frac{a_3}{a_4}\right) + ... + \left(\frac{a_n}{a_{n+1}} - \frac{a_{n+1}}{a_{n+2}}\right) \\
&= \frac{a_1}{a_2} + \left(-\frac{a_2}{a_3} + \frac{a_2}{a_3}\right) + \left(-\frac{a_3}{a_4} + \frac{a_3}{a_4}\right) + ... + \left(-\frac{a_n}{a_{n+1}} + \frac{a_n}{a_{n+1}}\right) + \left(-\frac{a_{n+1}}{a_{n+2}}\right) \\
&= \frac{a_1}{a_2} - \frac{a_{n+1}}{a_{n+2}}
\end{aligned}$$

Since these partial sums converge to $\frac{a_1}{a_2}$, we know that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_{n+2}} = 0$$

and hence by relabeling

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 0$$

Thus the reciprocal limit from the ratio test above diverges and so the limit in the ratio test diverges unless $x = 1$. Thus the interval of convergence is just the point $I = \{1\}$.

Good Luck for your exams!