Full Solutions
MATH105 April 2011

April 4, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

SOLUTION. The computation of the given integral involves several steps.
Step 1. In the first step we apply the substitution rule with $u = \ln(x)$. This implies $du/dx = 1/x$ and in particular $dx = xdu$. Hence,

$$\int \cos(\ln(x))\,dx = \int \cos(u) \cdot x\,du$$

We have to rewrite x in terms of u. This can be done by exponentiating the substitution equation

$$e^u = x.$$

We arrive at

$$\int \cos(u) \cdot x\,du = \int \cos(u)e^u\,du$$

Step 2. We apply integration by parts with $f(u) = \cos(u)$ and $g'(u) = e^u$:

$$\int \cos(u)e^u\,du = \cos(u)e^u + \int \sin(u)e^u\,du$$

Integration by parts onto the integral in the right-hand side with $f(u) = \sin(u)$ and $g'(u) = e^u$ yields

$$\int \sin(u)e^u\,du = \sin(u)e^u - \int \cos(u)e^u\,du$$

Step 3. Step 1 and Step 2 together results in the equation

$$\begin{aligned}\int \cos(u)e^u\,du &= \cos(u)e^u + \int \sin(u)e^u\,du \\ &= \cos(u)e^u + \sin(u)e^u - \int \cos(u)e^u\,du\end{aligned}$$

Step 4. We bring the integral on the right-hand side of the equation over to the left-hand side:

$$2\int \cos(u)e^u\,du = \cos(u)e^u + \sin(u)e^u = e^u(\sin(u) + \cos(u))$$

Finally, we divide the equation by 2:

$$\int \cos(u)e^u\,du = \frac{e^u}{2}(\sin(u) + \cos(u)) + C$$

where C is an arbitrary constant
Step 5. The last step is to rewrite our results in terms of x. Remember, that our very first step was the substitution $u = \ln(x)$. Therefore

$$\begin{aligned}\int \cos(\ln(x))\,dx &= \int \cos(u)e^u\,du \\ &= \frac{e^u}{2}(\sin(u) + \cos(u)) + C \\ &= \frac{x}{2}(\sin(\ln(x)) + \cos(\ln(x))) + C\end{aligned}$$

where we used $e^u = x$.

Question 1 (b)

SOLUTION. We will first compute the indefinite integral and evaluate our results at the boundaries of the integral to get the final result.
The first step is to split a $\cos(x)$-factor from $\cos^3(x)$ in the integrand:

$$\int \cos^3(x)\sin^2(x)\,dx = \int \cos(x)\cos^2(x)\sin^2(x)\,dx.$$

Now we use the trigonometric Pythagoras to write $\cos^2(x) = 1 - \sin^2(x)$ and arrive at

$$\int \cos(x)\cos^2(x)\sin^2(x)\,dx = \int \cos(x)(1-\sin^2(x))\sin^2(x)\,dx.$$

The next step is to use the substitution rule with $u = \sin(x)$. This implies $du/dx = \cos(x)$ or equivalently $dx = du/\cos(x)$. Substituting this into the integral yields

$$\begin{aligned}\int \cos(x)(1-\sin^2(x))\sin^2(x)\,dx &= \int \cos(x)(1-u^2)u^2\,\frac{du}{\cos(x)}\\ &= \int (1-u^2)u^2\,du\\ &= \int u^2 - u^4\,du\\ &= \frac{u^3}{3} - \frac{u^5}{5} + C.\end{aligned}$$

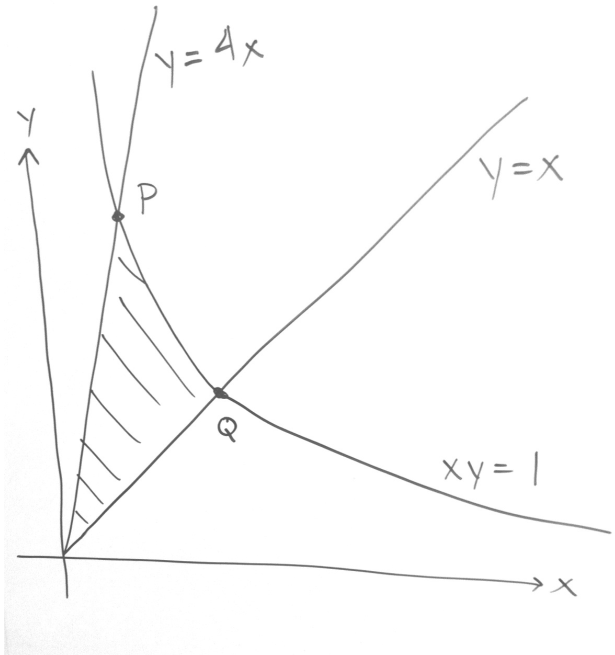
Rewriting this integral in terms of x by using the substitution equation $u = \sin(x)$ results in

$$\int \cos^3(x)\sin^2(x)\,dx = \frac{\sin^3(x)}{3} - \frac{\sin^5(x)}{5} + C.$$

Therefore, we can compute our original definite integral as follows:

$$\begin{aligned}\int_0^{\pi/2} \cos^3(x)\sin^2(x)\,dx &= \left(\frac{\sin^3(x)}{3} - \frac{\sin^5(x)}{5}\right|_0^{\pi/2}\\ &= \left(\frac{\sin^3(\pi/2)}{3} - \frac{\sin^5(\pi/2)}{5}\right) - \left(\frac{\sin^3(0)}{3} - \frac{\sin^5(0)}{5}\right)\\ &= \left(\frac{1}{3} - \frac{1}{5}\right) - \left(\frac{0}{3} - \frac{0}{5}\right)\\ &= \frac{2}{15}.\end{aligned}$$

Question 2



SOLUTION.
To compute the area, we need to know the points of intersection of the curves. For $y=4x$ and $xy=1$, i.e. point P, this is when $x(4x) = 1$, i.e. $x^2 = 1/4$. Equivalently $x=1/2$, and $y=2$.
For $y=x$ and $xy=1$, i.e. point Q, this is $(x,y) = (1,1)$.
Thus, the total area is

$$\begin{aligned}
A &= \int_0^{1/2} (4x - x)\, dx + \int_{1/2}^{1} \left(\frac{1}{x} - x\right)\, dx \\
&= 3\int_0^{1/2} x\, dx + \int_{1/2}^{1} \frac{1}{x}\, dx - \int_{1/2}^{1} x\, dx \\
&= \frac{3}{2}x^2\Big|_0^{1/2} + \ln x|_{1/2}^{1} - \frac{x^2}{2}\Big|_{1/2}^{1} \\
&= \frac{3}{8} + \underbrace{\ln 1}_{=0} - \underbrace{\ln(1/2)}_{=-\ln 2} - \left(\frac{1}{2} - \frac{1}{8}\right) \\
&= \frac{3}{8} + \ln 2 - \frac{1}{2} + \frac{1}{8} = \ln 2.
\end{aligned}$$

Question 3

SOLUTION. Notice from the definition of $f(x)$ that if $k < 0$ then $f(x)$ is always 0, which cannot be a probability density function.
The same is true if $k = 0$. Therefore we only need to consider positive values of k as candidates.
We require that

$$\int_0^{k/2} f(x)dx = 1$$

Thus,

$$\int_0^{k/2} \frac{2k}{(k+x)(k-x)} dx = 1.$$

To deal with the integral, we use partial fractions. We write

$$\frac{2k}{(k+x)(k-x)} = \frac{A}{k+x} + \frac{B}{k-x}.$$

Now we multiply the equation through by $(k+x)(k-x)$ to find $2k = A(k-x) + B(k+x)$.
To find A and B we can choose "convenient" values for x.
Setting $x = k$ yields $2k = 0 + B(k+k)$ so $B = 1$.
Setting $x = -k$ yields $2k = A(k-(-k)) + 0$ so $A = 1$.
Therefore,

$$\begin{aligned}
\int_0^{k/2} f(x)dx &= \int_0^{k/2} \left(\frac{1}{k+x} + \frac{1}{k-x} \right) dx \\
&= (\ln|k+x| - \ln|k-x|)|_0^{k/2} \\
&= (\ln(3k/2) - \ln(k/2)) - (\ln k - \ln k) \\
&= \ln(3) + \ln(k) - \ln(2) - \ln(k) + \ln(2) \\
&= \ln(3)
\end{aligned}$$

Note this is independent of k and it will never be 1. Thus no such k exist!

Question 4

SOLUTION. Let us write the supply and demand function as functions in the price p. The text says that at a price p the supplier produces $p-2$ units. Thus, the supply function must be

$$p = S(q) = q + 2$$

At a price p the consumer demands $65/p - 10$ units. In terms of the demand function this translates into

$$p = D(q) = \frac{65}{q+10}$$

By definition, the *market equilibrium* is the intersection point of the graph of $S(q)$ and $D(q)$. Let us denote this point by (q_e, p_e). To compute that point, we set both functions equal and compute a solution for q:

$$\begin{aligned}
S(q) &= D(q) \\
q + 2 &= \frac{65}{q+10}
\end{aligned}$$

And rearranging we obtain the equation

$$\begin{aligned}
(q+2)(q+10) &= 65 \\
q^2 + 12q - 45 &= 0
\end{aligned}$$

This is a quadratic equation and can be solved with the help of the *abc*-formula: the solutions to a quadratic equation of the form $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note, that in our problem the variable x is replaced by the variable q and we have $a = 1$, $b = 12$ and $x = -45$. Hence,

$$q = \frac{-12 \pm \sqrt{144 + 180}}{2} = -6 \pm 9$$

The solution $-6-9$ is not feasible, because quantities are always nonnegative. Hence, we arrive at the unique solution $q_e = -6 + 9 = 3$. We can calculate the associated price by using $D(q)$ or $S(q)$. Since $S(q)$ looks simpler than $D(q)$ we use $S(q)$ and get $p = S(q_e) = S(3) = 3 + 2 = 5$. Note, that this price is in between the range given in the problem. Let us summarize:
The market equilibrium is attained at $(q_e, p_e) = (3, 5)$
The second part of the question asks us to compute the *consumer surplus*. By definition, the consumer surplus is given by the formula

$$\text{consumer surplus} = \int_0^{q_e} (D(q) - p_e)\, dq$$

The following steps compute this integral:

$$\begin{aligned}
\int_0^{q_e} (D(q) - p_e)\, dq &= \int_0^3 \left(\frac{65}{q+10} - 5 \right) dq \\
&= 65 \int_0^3 \frac{1}{q+10}\, dq - \int_0^3 5\, dq \\
&= 65 \ln(q+10)\Big|_0^3 - 5q\Big|_0^3 \\
&= 65(\ln(13) - \ln(10)) - 15 \\
&= 65 \ln(13/10) - 15 \\
&\approx 2.05
\end{aligned}$$

The consumer surplus is approximately \$2.05

Question 5 (a)

SOLUTION. Let P be the initial deposit in the account and $A(t)$ be the amount of money in the account at time t. We denote the interest rate by i. The fact that we compound interest rate continuously translates into the differential equation

$$\frac{dA}{dt} = i \cdot A$$

In addition, at every time we add the amount $S(t)$ to the account. Hence, the differential equation changes to

$$\frac{dA}{dt} = i \cdot A + S$$

Furthermore, the interest rate grows proportionally with time. This means that $i = rt$ for some constant r. Thus, we arrive at

$$\frac{dA}{dt} = rt \cdot A + S$$

We also know that $A(0) = P$ because P is the initial deposit.

Question 5 (b)

SOLUTION. We now know that $S(t) = 0$ for all t and $r = 0.01$ as well $P = 100$. Thus, our differential equation specializes to

$$\frac{dA}{dt} = 0.01t \cdot A$$

with the initial condition $A(0) = 100$. This is a *separable* differential equation that can be written as

$$\frac{1}{A} \cdot \frac{dA}{dt} = 0.01t$$

The first step to solve this differential equation is to integrate both sides of the equation with respect to t:

$$\int \frac{1}{A} \cdot \frac{dA}{dt}\, dt = \int 0.01t\, dt$$

Informally, in the integrand of the integral on the left-hand side the dt terms cancel out each other. We arrive at

$$\begin{aligned} \int \frac{1}{A}\, dA &= 0.01 \cdot \frac{t^2}{2} + C_1 \\ &= 0.005t^2 + C_1 \end{aligned}$$

The integral on the left-hand side is just $\ln|A| + C_2$ for a constant C_2. Since from the context A is always nonnegative, we can ignore the absolute value brackets. In addition, we can combine the two constants C_1 and C_2 to a new constant that we denote by C_3. We arrive at

$$\ln(A) = 0.005t^2 + C_3$$

Applying the exponential function on both side of the equation results in

$$A = e^{0.005t^2 + C_3} = C_4 e^{0.005t^2}$$

We use the initial condition $A(0) = 100$ to determine the constant C_4:

$$100 = A(0) = C_4 e^{0.005 \cdot 0^2} = C_4 e^0 = C_4$$

Hence, the solution to the initial value problem is given by the function

$$A(t) = 100e^{0.005 \cdot t^2}$$

Question 5 (c)

SOLUTION. We are looking for the time t such that $A(t) = 100e^2$. A computation shows

$$\begin{aligned} A(t) &= 100e^2 \\ 100e^{0.005\cdot t^2} &= 100e^2 \\ e^{0.005\cdot t^2} &= e^2 \\ 0.005\cdot t^2 &= 2 \\ t^2 &= 2/0.005 = 400 \\ t &= \pm\sqrt{400} = \pm 20 \end{aligned}$$

Since the question asks for the time t after the initial deposit, the negative solution isn't feasible. We arrive at $t = 20$ years.

Question 6

SOLUTION. We already have given the cost function

$$C(x, y) = \frac{1}{200}x^2 + 6x + 4y + 4000$$

Since we sell x T-Shirts of type A for \$15 each and y T-Shirts of type B for \$10 each, the revenue must be given by

$$R(x, y) = 15x + 10y$$

The profit is by definition given by the formula

$$\begin{aligned} P(x, y) &= R(x, y) - C(x, y) \\ &= 15x + 10y - (\frac{1}{200}x^2 + 6x + 4y + 4000) \\ &= -\frac{1}{200}x^2 + 9x + 6y - 4000 \end{aligned}$$

There are essentially two ways of solving the problem. Let us describe both in full detail.
Solution via Lagrange Multiplier. The fact that the company only produces 1000 T-Shirts is reflected in the *constraint equation* $x + y = 1000$. Thus, the constraint function is given by

$$g(x, y) = x + y - 1000$$

and the constraint translates into $g(x, y) = 0$. The *objective function* is the profit function $P(x, y)$. We have

$$\begin{aligned} P_x &= -\frac{1}{100}x + 9, \\ P_y &= 6 \end{aligned}$$

as well as

$$g_x = g_y = 1.$$

Thus, for the Lagrange equation in y,

$$\begin{aligned} P_y &= \lambda \cdot g_y \\ 6 &= \lambda \end{aligned}$$

and for the Lagrange equation in x,

$$\begin{aligned} P_x &= \lambda \cdot g_x \\ -\frac{1}{100}x + 9 &= \lambda = 6. \end{aligned}$$

We can solve these Lagrange equations to get $x = 300$. Since we also must have $g(x, y) = x+y-1000 = 0$ we get $y = 700$ as the unique solution of the Lagrange multiplier problem. The profit we make is $P(300, 700) = 2450$. That this is really a maximum and not a minimum can be shown be computing the profit of another distribution of T-Shirts. For instance $x = 0$ and $y = 1000$ yields $P(0, 1000) = 2000$. This argument works because we only have one global extremum under the constraint $g(x, y) = 0$.
Solution without Lagrange Multipliers. The question does not state that we have to use Lagrange Multipliers to solve this problem. We can use the following solution as well. First, we solve the constraint equation $x + y = 1000$ for either x or y. Since the profit function is quadratic in x and only linear in y we should solve the constraint equation for y to make life simpler. We get $y = 1000 - x$. We replace y by this expression in the profit function. This results in

$$\begin{aligned} P &= -\frac{1}{200}x^2 + 9x + 6y - 4000 \\ &= -\frac{1}{200}x^2 + 9x + 6(1000 - x) - 4000 \\ &= -\frac{1}{200}x^2 + 9x + 6000 - 6x - 4000 \\ &= -\frac{1}{200}x^2 + 3x + 2000 \end{aligned}$$

Now the profit is transformed into a function in just x and we can use methods of differential calculus learned in MATH 104 or 184 to find the global maximum. First we see that P describes a parabola. Since the sign in front of the quadratic term is negative, the focal point of the parabola must be the maximum which is the only critical point of P. Thus, it is enough to find the zeros of $P'(x)$. We have $P'(x) = -1/100 \cdot x + 3$ and this equals to zero only for $x = 300$. Since $y = 1000 - x$ we must have $y = 700$.
Important Remark. If the question does not clearly mention a specific method you have to apply to solve the problem, it is your choice which method you use. Some students find the Lagrange Multiplier solution simpler, some not. Decide based on your strengths!

Question 7

SOLUTION. The derivative of f with respect to x is

$$f_x(x, y) = (1/x) + 2bx + 3ay.$$

Also, the derivative of f with respect to y is

$$f_y(x, y) = (2/y) + 3ax.$$

For a critical point at $(1, 3)$, we require that $f_x(1, 3) = f_y(1, 3) = 0$. Using the above two equations, this gives us the following system to solve for a and b:

1. $f_x(1, 3) = (1/1) + 2b(1) + 3a(3) = 1 + 2b + 9a = 0$
2. $f_y(1, 3) = (2/3) + 3a(1) = 2/3 + 3a = 0.$

Equation 2 tells us that $a = -2/9$.
Substituting this value for a into equation 1 tells us that $1+2b+9(-2/9) = 1+2b-2 = 2b-1 = 0$. Therefore, we have that $b = 1/2$.
This critical point is a saddle point. To show this, we solve for the second derivatives f_{xx}, f_{xy} and f_{yy}.
First,

$$f_{xx}(x, y) = -(1/x)^2 + 2b.$$

Therefore,

$$f_{xx}(1, 3) = -(1) + 2(1/2) = 0.$$

Second,

$$f_{yy}(x, y) = -2(1/y)^2.$$

Therefore,

$$f_{yy}(1, 3) = -2/9$$

Third,

$$f_{xy}(x, y) = 3a.$$

Therefore,

$$f_{xy}(1, 3) = -2/3.$$

We can classify the critical points by looking at the formula,

$$D = f_{xx}f_{yy} - (f_{xy})^2.$$

If $D < 0$ then the critical point is a saddle point. If $D > 0$ then we look to the sign of f_{xx}. If $f_{xx} > 0$ then we have a local minimum and if $f_{xx} < 0$ then we have a local maximum. For our problem,

$$D = 0\left(\frac{2}{9}\right) - \left(-\frac{2}{3}\right)^2 = -\frac{4}{9} < 0.$$

Therefore since $D < 0$ the critical point is a saddle point.
Advanced:
Using the second partial derivatives of a function $f(x)$ we can define a general Hessian matrix as

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

which for our specific example is

$$\begin{bmatrix} 0 & -2/3 \\ -2/3 & -2/9 \end{bmatrix}.$$

The eigenvalues of this matrix are given by the solving the formula $(-\lambda)(-2/9 - \lambda) - 4/9 = 0$ for λ. That is,

$$\lambda^2 + (2/9)\lambda - 4/9 = 0.$$

Solving gives us the answer

$$\lambda = (1/9)(\pm\sqrt{37} - 1).$$

Because one eigenvalue is positive and one eigenvalue is negative, we conclude that the Hessian matrix is indefinite, and thus the point $(1, 3)$ is a saddle point.

Question 8 (a)

Solution. The marginal productivity of labour is the rate of change of productivity with respect to the labour investment, so in this case, it corresponds to the partial derivative

$$\begin{aligned} \frac{\partial f}{\partial x}(x, y) &= \frac{\partial}{\partial x}(160x^{3/4}y^{1/4}) \\ &= 160\frac{3}{4}x^{-1/4}y^{1/4} \\ &= 120x^{-1/4}y^{1/4} \end{aligned}$$

And the marginal productivity of capital is the rate of change of productivity with respect to the capital investment, so in this case, it corresponds to the partial derivative

$$\begin{aligned} \frac{\partial f}{\partial y}(x, y) &= \frac{\partial}{\partial y}(160x^{3/4}y^{1/4}) \\ &= 160x^{3/4}\frac{1}{4}y^{-3/4} \\ &= 40x^{3/4}y^{-3/4} \end{aligned}$$

The question asks us to compute those marginal productivities for 81 units of labour and 16 units of capital, that is to compute

$$\frac{\partial f}{\partial x}(81, 16) \quad \text{and} \quad \frac{\partial f}{\partial y}(81, 16)$$

which yields

$$\begin{aligned}\frac{\partial f}{\partial x}(81,16) &= 120 \cdot 81^{-1/4} 16^{1/4} \\ &= 120 \cdot \frac{1}{3} \cdot 2 \\ &= 80\end{aligned}$$

and

$$\begin{aligned}\frac{\partial f}{\partial y}(81,16) &= 40 \cdot 81^{3/4} 16^{-3/4} \\ &= 40 \cdot 27 \cdot \frac{1}{8} \\ &= 135\end{aligned}$$

So the marginal productivity of labour is 80 (dollars of productivity per dollar of labour investment) and the marginal productivity of capital is 135 (dollars of productivity per dollar of capital investment).

Question 8 (b)

SOLUTION. From the hint we see that a linear approximation based on a given point is

$$g(x,y) \approx f(x_0,y_0) + \left(\frac{\partial f}{\partial x}(x_0,y_0)\right) \cdot (x-x_0) + \left(\frac{\partial f}{\partial y}(x_0,y_0)\right) \cdot (y-y_0)$$

but how does this relate to the problem at hand? We want to know how **small** changes from our current labour/capital setup of (81,16) will affect the overall productivity. If we used our linear approximation formula above then if $f(x,y) > f(x_0,y_0)$ then the new labour/capital pairing of (x,y) is better than the old pairing $(x_0,y_0) = (81,16)$.
If we use the first policy and only increase labour by 1 unit ($x - x_0 = 1, y - y_0 = 0$), then since the marginal productivity of labour at the current level ($f_x(81,16)$) is 80, the productivity will be raised by approximately 80 dollars since

$$f(x,y) - f(81,16) = f_x(81,16)(1) + f_y(81,16)(0) = 80.$$

On the other hand, if we use the second policy, an increase of labour by 1/2 unit ($x - x_0 = 1/2$) will increase productivity by

$$f_x(81,16)(x-x_0) = 80(1/2) = 40$$

dollars; and an increase of capital by 1/3 unit ($y - y_0 = 1/3$) will increase productivity by

$$f_y(81,16)(y-y_0) = 135(1/3) = 45$$

dollars. The two combined yield an increase of productivity of approximately

$$f(x,y) - f(81,16) = f_x(81,16)(1/2) + f_y(81,16)(1/3) = 85$$

dollars.

So based on these approximations, the second policy should yield a higher productivity.

Question 8 (c)

SOLUTION. In part (a) we computed the partial derivatives, allowing us to write the gradient vector

$$\nabla f(x,y) = (120x^{-1/4}y^{1/4}, 40x^{3/4}y^{-3/4})$$

hence (as calculated previously)

$$\nabla f(81,16) = (80,135)$$

Recall that a perpendicular vector to (a,b) with the same length is is (b,-a) and (-b,a). Therefore, the perpendicular vectors (of the same length as the gradient itself) are the vectors

$$(-135,80) \quad \text{and} \quad (135,-80)$$

Question 9 (a)

SOLUTION. No, this limit does not exist. We can see this by considering two different paths to the origin when taking this limit. Let

$$f(x,y) = \frac{xy^2}{x^2+y^4}.$$

First, consider the given limit when the limit is taken by approaching on the x-axis. Notice in this case we have $f(x,0)=0$ for all x. Therefore,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,0) = \lim_{x\to 0} 0 = 0.$$

Now consider a different path. In order to find a path that makes the limit non-zero, we choose x and y such that the powers of the enumerator and denominator are equal. The highest power in the denominator is the 4 in y. So let us choose x such that the enumerator matches that power: Let us approach the origin along the curve $x = y^2$.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{y\to 0} f(y^2,y) = \lim_{y\to 0} \frac{y^4}{y^4+y^4} = \frac{1}{2} \neq 0.$$

So, depending on the path taken to the origin, we get a different value for the above limit. Therefore, the limit does not exist.

Question 9 (b)

SOLUTION. To evaluate the slope of the curve at $x = 0$, we let

$$f(x) = \int_0^{2\sin(x)} e^{-t^2}\, dt$$

and evaluate $f'(0)$. By the fundamental theorem of calculus, we know that

$$f(x) = \int_0^{2\sin(x)} e^{-t^2}\,dt = F(2\sin(x)) - F(0)$$

where F is the antiderivative of the integrand (i.e

$$F'(t) = e^{-t^2}$$

). Taking the derivative with respect to x, we find

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[F(2\sin(x)) - F(0)\right] \\ &= F'(2\sin(x)) \cdot \frac{d}{dx}(2\sin(x)) \\ &= e^{-4\sin^2(x)} \cdot 2\cos(x) \\ f'(0) &= 2 \end{aligned}$$

So the slope of the curve $y = f(x)$ at $x = 0$ is 2.

Question 9 (c)

Solution 1. The Riemann sum formula using the right end points for a function $f(x)$ is
$f(x) = R_n = \sum_{k=1}^{n} f(x_k)\Delta x$
where
$x_k = a + k\Delta x \qquad \Delta x = \frac{b-a}{n}$
In our example, comparing the terms, we see that
$\Delta x = \frac{1}{n}$
and so
$b - a = 1$
Also, we have that
$f(x_k) = \ln\left(1 + \frac{k}{n}\right)$
So taking $f(x) = \ln(1 + x)$, we have
$x_k = a + k\Delta x = a + \frac{k}{n} = \frac{k}{n}$
so we have $a = 0$ and hence since $b - a = 1$ and so $b = b - 0 = 1$. Plugging all this information in, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln\left(1 + \frac{k}{n}\right) = \int_0^1 \ln(1 + x)\,dx$$

Solution 2. The Riemann sum formula using the right end points for a function $f(x)$ is
$f(x) = R_n = \sum_{k=1}^{n} f(x_k)\Delta x$
where
$x_k = a + k\Delta x \qquad \Delta x = \frac{b-a}{n}$
In our example, comparing the terms, we see that
$\Delta x = \frac{1}{n}$
and so
$b - a = 1$
Also, we have that
$f(x_k) = \ln\left(1 + \frac{k}{n}\right)$
So taking $f(x) = \ln(x)$, we have

$x_k = a + k\Delta x = a + \frac{k}{n} = 1 + \frac{k}{n}$
so we have $a = 1$ and hence since $b - a = 1$ and so $b - 1 = 1$ giving $b = 2$. Plugging all this information in, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln\left(1 + \frac{k}{n}\right) = \int_1^2 \ln(x)\, dx$$

Question 9 (d)

SOLUTION. We want to determine whether the integral

$$\int_0^2 \frac{2x}{1-x^2}\, dx$$

converges. We notice that the integrand is not defined at the point $x = 1$ so the integral is improper. Hence, we break up the integral at this discontinuity to get that
$\int_0^2 \frac{2x}{1-x^2}\, dx = \lim_{b\to 1^-} \int_0^b \frac{2x}{1-x^2}\, dx + \lim_{a\to 1^+} \int_a^2 \frac{2x}{1-x^2}\, dx$
To evaluate the first integral, we use a substitution namely $u = 1 - x^2$ and get that $du = -2xdx$. The endpoints change to $u(0) = 1 - (0)^2 = 1$ and $u(b) = 1 - b^2$

$$\begin{aligned}
\lim_{b\to 1^-} \int_0^b \frac{2x}{1-x^2}\, dx &= \lim_{b\to 1^-} \int_1^{1-b^2} \frac{-du}{u} \\
&= \lim_{b\to 1^-} \left(-\ln|u|\right)\Big|_1^{1-b^2} \\
&= \lim_{b\to 1^-} -\ln(1-b^2) + \ln(1)
\end{aligned}$$

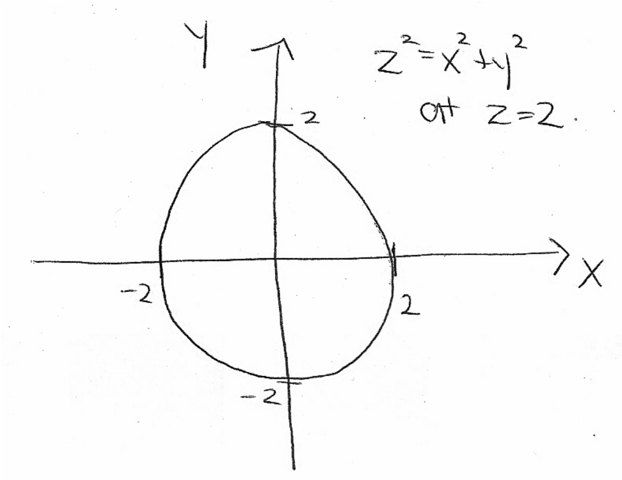
and this last limit evaluates to negative infinity. Hence, the integral diverges.

Question 9 (e)

SOLUTION. As the hint suggests, we can check each equation with the point (0,0,0), which lies on the surface. The point (0,0,0) does not lie on the surface given by $x^2 + y^2 + z^2 = 1$, so we can rule that out.
Since the surface takes negative z values and $z = x^2 + y^2$ only takes non-negative values, we can rule this out as an option as well.
Finally, we notice that the surface is symmetric about the x, y plane (this means that for a given (x,y) we get a value z above the xy-plane and the same value (but negative), -z below the xy-plane. For this reason, we can rule out $z = x^2 - y^2$ since for a given (x,y) we only produce one z value.
Thus the surface is $z^2 = x^2 + y^2$.
We can check that this solution makes sense by solving the equation for z to get $z = \pm\sqrt{x^2 + y^2}$. The two solutions give identical surfaces above and below the x, y plane, the point (0,0,0) does indeed lie on the surface $z^2 = x^2 + y^2$, and as x and y increase, so does z. These all match the given graph.
The level curve at $z = 2$ is what we would see if we sliced into the surface horizontally at $z = 2$ and looked at it from above. We can find the equation by substituting into the equation for the surface

$$2^2 = x^2 + y^2.$$

This is simply a circle of radius 2, the graph as follows:



Question 9 (f)

SOLUTION. The variance for a discrete random variable is defined to be

$$\text{Var}(X) = \sum_{x_i} (x_i - \mu)^2 P(X = x_i)$$

where the values x_i, are the possible outcomes for the random variable X and μ is the mean:

$$\mu = \sum_{x_i} x_i P(X = x_i).$$

In this case, there are only two possible outcomes $X = 0$ or $X = 1$. If we define the probability $P(X = 1)$ with p then by the law that the sum of the probabilities for all events must sum to one, we know that $P(X = 0) = 1 - P(X = 1) = 1 - p$. Thus,

$$\mu = 0P(X = 0) + 1P(X = 1) = p.$$

Now we can use the definition of variance and get an equation for p.

$$\begin{aligned}
\text{Var}(X) = (0 - p)^2 \underbrace{P(X = 0)}_{=1-p} + (1 - p)^2 P(X = 1) &= 1/4 \\
-p^2(1 - p) + (1 - p)^2 p &= 1/4 \\
-p^2 + p - 1/4 &= 0 \\
p^2 - p + 1/4 &= 0 \\
(p - 1/2)^2 &= 0
\end{aligned}$$

Thus, the value of p such that Var(X) $= 1/4$ is $p = 1/2$.

Question 9 (g)

SOLUTION. The function f depends on the variables x and y which themselves are functions of the variable t. We have

$$z(t) = f(x(t), y(t)) \quad \text{and} \quad x(t) = e^{3t} \quad \text{and} \quad y(t) = \sin(2t)$$

Now, using the chain rule for multivariable functions we obtain

$$\frac{dz}{dt}(t) = \frac{\partial f}{\partial x}(x(t), y(t))\frac{dx}{dt}(t) + \frac{\partial f}{\partial y}(x(t), y(t))\frac{dy}{dt}(t)$$

which using the notation as in the question statement gives

$$z'(t) = f_X(x(t), y(t))\, x'(t) + f_Y\,(x(t), y(t))\; y'(t)$$

Now we just want to know what's going on at $t = 0$. We have

$$x(0) = e^0 = 1 \quad \text{and} \quad y(0) = \sin(0) = 0$$

and since

$$x'(t) = 3e^{3t} \quad \text{and} \quad y'(t) = 2\cos(2t)$$

we have

$$x'(0) = 3 \quad \text{and} \quad y'(0) = 2$$

So we can conclude that

$$\begin{aligned} z'(0) &= f_X(1,0)\cdot 3 + f_Y(1,0)\cdot 2 \\ &= 3\cdot 3 + (-1)\cdot 2 \\ &= 7 \end{aligned}$$

.
Notice that the provided information for $f(1,0) = 0$ was unnecessary.

Good Luck for your exams!