Full Solutions
MATH215 December 2011

April 4, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

SOLUTION. Following the hints we multiply the equation by $\exp\left(\int -4\frac{1}{t}dt\right) = \exp(-4\ln(t)) = 1/t^4$ yielding:

$$\frac{1}{t^4}y' - \frac{4}{t^5}y = -\frac{2}{t^6}$$

where we can recognize the left-hand side as the derivative of a single term:

$$\left(\frac{1}{t^4}y\right)' = -\frac{2}{t^6}$$

Integrating both sides yields

$$\begin{aligned}\frac{1}{t^4}y &= \frac{2}{5t^5} + C\\ y &= \frac{2}{5t} + Ct^4\end{aligned}$$

for an arbitrary constant C.
Remark (not part of solution): Technically $\int \frac{-4}{t}dt = -4\ln(t) + C$ but by multiplying the entire equation by the integrating factor, we inevitably divide by it and this constant of integration cancels out.

Question 1 (b)

SOLUTION. As the exam hint suggests, we make the substitution $w = v^{-1/2}$ so that

$$w' = -\frac{1}{2}v^{-3/2}v'.$$

Under this substitution, $t^2w' + 2tw - w^3 = 0$ becomes

$$\begin{aligned}t^2(-\frac{1}{2}v^{-3/2}v') + 2tv^{-1/2} - v^{-3/2} &= 0\\ v' - \frac{4}{t}v + \frac{2}{t^2} &= 0\end{aligned}$$

where we multiplied the equation by $-2\frac{v^{3/2}}{t^2}$.
This is now the same ODE as in part (a), so the solution is that $v = \frac{2}{5t} + Ct^4$. Hence we have

$$w = \left(\frac{2}{5t} + Ct^4\right)^{-1/2}.$$

Question 2 (a)

SOLUTION. We begin with assuming there is a governing equation $M(x, y) = 0$ and differentiating this with respect to x.
Then, $\frac{d}{dx}M(x, y) = M_x(x, y) + M_y(x, y)\frac{dy}{dx} = 0$.
Hence, $M_x = ax^3e^{x+y} + bx^4e^{x+y} + 2x$ and $M_y = x^4e^{x+y} + 2y$.
By Clairaut's theorem, there must be equality of mixed partial derivatives, and $M_{xy} = M_{yx}$. In our problem,

$$
\begin{aligned}
M_{xy} &= ax^3e^{x+y} + bx^4e^{x+y} \\
M_{yx} &= 4x^3e^{x+y} + x^4e^{x+y} \\
&\implies a = 4, b = 1
\end{aligned}
$$

by comparing the coefficients of x^3e^{x+y} and x^4e^{x+y}.

Question 2 (b)

SOLUTION. By differentiating $M(x, y) = 0$ with respect to x, we arrive at $M_x + M_y\frac{dy}{dx} = 0$ and we identify

$$
\begin{aligned}
M_x &= 3x^2y + 8xy^2 \\
M_y &= x^3 + 8x^2y + 12y^2.
\end{aligned}
$$

By computing

$$M_{xy} = M_{yx} = 3x^2 + 16xy$$

we see that it is exact and hence partial integration can be used to solve it. Note that if

$$M_x = 3x^2y + 8xy^2$$

then by integrating partially with respect to x, we have

$$M = x^3y + 4x^2y^2 + f(y)$$

for some function f that does not depend on x. If we differentiate

$$M = x^3y + 4x^2y^2 + f(y)$$

with respect to y keeping x constant then

$$M_y = x^3 + 8x^2y + f'(y) = x^3 + 8x^2y + 12y^2$$

from our initial observations. We therefore find

$$f'(y) = 12y^2$$

so that $f(y) = 4y^3 + C$ for an arbitrary constant C.
Therefore,

$$M(x, y) = x^3y + 4x^2y^2 + 4y^3 + C = 0$$

is the general solution to the ODE.

Question 3 (a)

SOLUTION. Setting $\frac{dy}{dt} = (y^2-1)(y-2)^2 = 0$ we have $y = \pm 1, 2$ as our equilibrium points.

Question 3 (b)

SOLUTION. From part (a) we have that the equilibrium points are $\pm 1, 2$ and therefore these points divide our derivative space into four regions where the derivative can be positive or negative, $(-\infty,-1)\cup(-1,1)\cup(1,2)\cup(2,\infty)$. We have that

$$\frac{dy}{dt} = (y^2-1)(y-2)^2$$

and we first note that $(y-2)^2$ is always positive (unless $y=2$) and so the sign of the derivative will be determined by the first term only. For this term we get

$$\begin{aligned} y^2-1 &< 0, \quad (-1,1) \\ y^2-1 &> 0, \quad (-\infty,-1)\cup(1,\infty). \end{aligned}$$

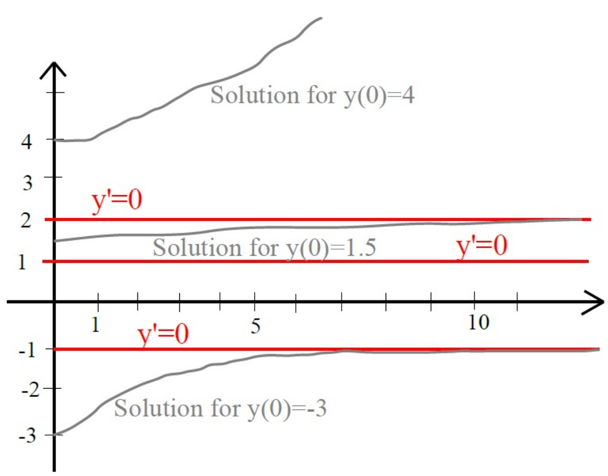
Therefore

$$\begin{aligned} \frac{dy}{dt} &> 0, \quad (-\infty,-1)\cup(1,2)\cup(2,\infty) \\ \frac{dy}{dt} &< 0, \quad (-1,1). \end{aligned}$$

Note: We separate the domain of increase for the derivative at the point y=2 because strictly speaking the derivative is not increasing at this point.

Question 3 (c)

SOLUTION. From (a) we know that the fixed points are given by $y_0 = -1, 1, 2$.
In part (b) we calculated that y is increases in $(-\infty,-1)\cup(1,\infty)$ and decreases in (-1,1).
This means, that starting at $y(0) = 1.5$, the solution increases towards the critical point $y_0 = 2$. Hence $\lim_{t\to\infty} y(t) = 2$.
Starting at $y(0) = -3$, the solution increases towards the critical point $y_0 = -1$. Hence $\lim_{t\to\infty} y(t) = -1$.
Starting at $y(0) = 4$, the solution increases. Since there is no larger critical point, the solution increases indefinitely. Hence $\lim_{t\to\infty} y(t) = \infty$.



Question 4 (a)

SOLUTION. Given the equation

$$2u'' + 2\gamma u' + 18u = 0,$$

we can obtain the characteristic equation by substituting a solution

$$u(t) = \exp(rt)$$

and looking at the resulting polynomial. Doing this we get

$$2r^2 + 2\gamma r + 18 = 0$$

which has roots

$$\begin{aligned} r &= \frac{-2\gamma \pm \sqrt{4\gamma^2 - 4(2)(18)}}{4} \\ &= -\frac{\gamma}{2} \pm \frac{\sqrt{\gamma^2 - 36}}{2} \\ &= \frac{1}{2}\left(-\gamma \pm \sqrt{\gamma^2 - 6^2}\right) \end{aligned}$$

.

Now we study the behaviour of the roots of the equation. Note that $\gamma > 0$ is necessary for physical damping.

- When $\gamma^2 - 36 < 0$, there are complex roots to the system, and there are oscillations. Thus the system is *underdamped*.
- When $\gamma^2 - 36 > 0$, the square root always yields a real value. Since $\sqrt{\gamma^2 - 36} < \gamma$ (assuming $\gamma > 0$), it follows that both roots are negative (so the amplitude decays) and real (so there are no oscillations). Thus, the system is *overdamped*.

Thus, there is overdamping for $\gamma > 6$ and underdamping for $\gamma < 6$ (and therefore critical damping if $\gamma = 6$).

Question 4 (b)

SOLUTION. Notice that the ODE

$$u'' + 6u' + 25u = 15\cos(5t)$$

has a nonzero right-hand side. We will begin by solving the homogeneous equation

$$u'' + 6u' + 25u = 0.$$

By making the ansatz $u = \exp(rt)$ we arrive at

$$r^2 + 6r + 25 = 0$$

with roots $r = -3 \pm 4i$. Complex roots of the form $a + ib$ lead to solutions of the form

$$\begin{aligned} &e^{at} \sin(bt) \\ &e^{at} \cos(bt). \end{aligned}$$

Therefore the homogeneous solution is

$$u_h = C_1 e^{-3t} \sin(4t) + C_2 e^{-3t} \cos(4t).$$

The right-hand side is sinusoidal with argument $5t$. We need to think of which functions (and their derivatives) that could lead to this possible right-hand side. The only such possible functions are

$$\begin{gathered} \cos(5t) \\ \sin(5t) \end{gathered}$$

and so, we will find a particular solution of the form

$$u_p = A \sin(5t) + B \cos(5t).$$

Taking this particular solution and plugging it in to the ODE, we have:

$$\begin{aligned} (-25A \sin(5t) - 25B \cos(5t)) + 6(5A \cos(5t) - 5B \sin(5t)) + 25(A \sin(5t) + B \cos(5t)) &= 15 \cos(5t) \\ -30B \sin(5t) + 30A \cos(5t) &= 15 \cos(5t) \end{aligned}$$

In comparing the coefficients of $\sin(5t)$ and $\cos(5t)$, we find that

$$\begin{aligned} B &= 0 \\ A &= 1/2. \end{aligned}$$

The general solution to the ODE will be $u = u_h + u_p$, giving

$$u = \frac{1}{2} \sin(5t) + C_1 e^{-3t} \sin(4t) + C_2 e^{-3t} \cos(4t).$$

The initial conditions tell us

$$\begin{aligned} u(0) &= 1 \\ u'(0) &= -\frac{11}{2} \end{aligned}$$

From this, we have,

$$u(0) = C_2 = 1$$

and

$$u'(0) = \left(\frac{5}{2} \cos(5t) - 3C_1 e^{-3t} \sin(4t) + 4C_1 e^{-3t} \cos(4t) - 3C_2 e^{-3t} \cos(4t) - 4C_2 e^{-3t} \sin(4t) \right) \Bigg|_{t=0} = \frac{5}{2} + 4C_1 - 3 = \frac{-11}{2}.$$

Therefore we get that

$$C_1 = -5/4.$$

The solution is thus

$$u(t) = \frac{1}{2}\sin(5t) - \frac{5}{4}e^{-3t}\sin(4t) + e^{-3t}\cos(4t).$$

Question 4 (c)

SOLUTION. We are given the solution to the homogeneous equation. To find the solution with the nonzero right-hand side, we will try variation of parameters.
Given t^2 and t^3 satisfy the homogeneous ODE, we will make an ansatz

$$y = f(t)t^2 + g(t)t^3$$

where f(t) and g(t) are unknown functions. If we differentiate this we get

$$y' = f'(t)t^2 + 2f(t)t + g'(t)t^3 + 3g(t)t^2$$

and to avoid having more than a single derivative upon f and g in computing y'', we will set

$$f'(t)t^2 + g'(t)t^3 = 0$$

so that

$$y' = 2f(t)t + 3g(t)t^2.$$

Notice that by making the assumption, the original first derivative remains unchanged. It is important that we make this step, otherwise when we try to solve the ODE we will have one equation for two unknowns. This assumption adds extra information that allows us to solve the problem. Differentiating again, we get

$$y'' = 2f'(t)t + 2f(t) + 3g'(t)t^2 + 6g(t)t.$$

Now we take our derivatives and substitute them into the original equation:

$$t^2(2f'(t)t + 2f(t) + 3g'(t)t^2 + 6g(t)t) - 4t(2f(t)t + 3g(t)t^2) + 6(f(t)t^2 + g(t)t^3) = 3t.$$

After expanding and simplifying, we get

$$2t^3f'(t) + 3t^4g'(t) = 3t.$$

Now we have two first-order ODEs for f and g:

$$\begin{aligned} t^2f' + t^3g' &= 0 \\ 2t^3f' + 3t^4g' &= 3t. \end{aligned}$$

The first equation implies

$$f' = -tg'$$

which can be substituted into the second equation:

$$\begin{aligned} -2t^4g' + 3t^4g' &= 3t \\ t^4g' &= 3t \\ g' &= \frac{3}{t^3} \\ g &= \frac{-3}{2t^2} + C_1. \end{aligned}$$

Also, from above,

$$\begin{aligned} f' &= -tg' = \frac{-3}{t^2} \\ f &= \frac{3}{t} + C_2. \end{aligned}$$

Having solve for f(t) and g(t) we get that the general solution to our problem is

$$y = f(t)t^2 + g(t)t^3 = C_1t^3 + C_2t^2 + \frac{3}{2}t.$$

Question 5 (a)

SOLUTION. The Laplace transform, F(s), for a function f(t) is given by

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty f(t)\exp(-st)dt.$$

Since our function is piecewise defined we can split up the integral and write

$$\begin{aligned} F(s) &= \int_0^1 f(t)dt + \int_1^2 f(t)\exp(-st)dt + \int_2^\infty f(t)\exp(-st)dt \\ F(s) &= \int_0^1 0dt + 2\pi\int_1^2 \exp(-st)dt + \pi\int_2^\infty \exp(-st)dt. \end{aligned}$$

The first integral is just zero. For the second integral we get,

$$2\pi\int_1^2 \exp(-st)dt = -\frac{2\pi}{s}\exp(-st)\Big|_1^2 = -\frac{2\pi}{s}\exp(-2s) + \frac{2\pi}{s}\exp(-s).$$

For the third integral we get

$$\pi\int_2^\infty \exp(-st)dt = -\frac{\pi}{s}\exp(-st)\Big|_2^\infty = \frac{\pi}{s}\exp(-2s).$$

Adding the two integrals together we get

$$F(s) = -\frac{2\pi}{s}\exp(-2s) + \frac{2\pi}{s}\exp(-s) + \frac{\pi}{s}\exp(-2s) = \frac{\pi}{s}\left(2\exp(-s) - \exp(-2s)\right).$$

Question 5 (b)

SOLUTION. Given that in part (a) of this problem we were asked to compute the Laplace transform of f(t), it stands to reason that using Laplace transforms will be a good way to solve this problem. We begin by taking the Laplace transform of the entire differential equation. We can do this because the Laplace Transform is a **linear operator**, i.e.

$$\mathcal{L}(af(t) + bg(t)) = a\mathcal{L}(f(t)) + b\mathcal{L}(g(t)).$$

Taking the Laplace transform, we get

$$\mathcal{L}(y'') - 5\mathcal{L}(y') + 6\mathcal{L}(y) = \mathcal{L}(f(t)).$$

Define the following

$$\begin{aligned} F(s) &= \mathcal{L}(f(t)) \\ Y(s) &= \mathcal{L}(y(t)). \end{aligned}$$

From part (a), we have that

$$F(s) = \frac{\pi}{s}\left(2\exp(-s) - \exp(-2s)\right).$$

If we look at the table at the back of the exam we can retrieve the Laplace transforms for derivatives (of course we could also compute these if we needed to). We see that the general rule for the Laplace transform of the nth derivative of a function, g(t) is

$$\mathcal{L}(g^{(n)}(t)) = s^n G(s) - s^{n-1}g(0) - \cdots - g^{(n-1)}(0)$$

and therefore we can write

$$\begin{aligned} \mathcal{L}(y') &= sY(s) - y(0) = sY - 1 \\ \mathcal{L}(y'') &= s^2 Y(s) - sy(0) - y'(0) = s^2 Y - s \end{aligned}$$

where we have used that $y(0) = 1$ and $y'(0) = 0$ as given in the question. Substituting these expressions into the original differential equation we get,

$$s^2 Y(s) - s - 5(sY(s) - 1) + 6Y = F(s).$$

Simplifying this expression we get

$$(s^2 - 5s + 6)Y(s) = F(s) + s - 5$$

which we can solve for Y(s)to get

$$Y(s) = \frac{F(s) - 4s}{(s-3)(s-2)} = \frac{F(s)}{(s-3)(s-2)} + \frac{s-5}{(s-3)(s-2)}$$

where we have factored the quadratic. While it may look like we have solved the problem, we need to present the solution in terms of y(t), the original function. To do this we need to compute the inverse Laplace transform of Y(s). While we could attempt to do this manually using the formula for inverse Laplace transforms, we should attempt to put it in a form where we can use the chart at the end of the exam. Once again, the inverse Laplace transform is a linear operator so we have

$$y(t) = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{F(s)}{(s-3)(s-2)}\right) + \mathcal{L}^{-1}\left(\frac{s-5}{(s-3)(s-2)}\right)$$

and we can compute these two quantities separately. We will start with the second inverse Laplace transform. Looking at our chart at the end of the exam we see an inverse Laplace transform for functions of the form

$$\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) = \exp(at).$$

However, our problem has a product of linear terms in the denominator and this suggests that we split our problem up using partial fraction decomposition. Doing this we get

$$\frac{s-5}{(s-3)(s-2)} = \frac{3}{(s-2)} - \frac{2}{(s-3)}.$$

We now have this in a form where we can apply the inverse Laplace transform but we will do that at the end. We now focus on the first term,

$$\frac{F(s)}{(s-3)(s-2)} = \frac{\frac{\pi}{s}\left(2\exp(-s) - \exp(-2s)\right)}{(s-3)(s-2)}.$$

Now there are several ways we can proceed. One way is to consider

$$G(s) = \frac{1}{(s-3)(s-2)}$$

and then our term looks like F(s)G(s). We could then invert this using the convolution identity

$$\mathcal{L}^{-1}(F(s)G(s)) = \int_0^t f(t-\tau)g(t)dt.$$

While this is valid, it could lead to confusion since f(t) is piecewise defined and so we would have to be careful about the different cases that come out of the integration. Another way we can do the inversion is to include the 1/s terms in F(s) along with the other linear terms in the denominator of G(s). In this way we write

$$\frac{F(s)}{(s-3)(s-2)} = \frac{\pi\left(2\exp(-s) - \exp(-2s)\right)}{s(s-3)(s-2)} = 2\pi\exp(-s)G(s) - \pi\exp(-2s)G(s)$$

where we now define

$$G(s) = \frac{1}{s(s-3)(s-2)}.$$

The advantage of this form is that the table at the end of the exam lists

$$\mathcal{L}^{-1}\left(\exp(-cs)F(s)\right) = u_c(t)f(t-c)$$

where u_c is the Heaviside function defined as

$$u_c = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}.$$

We can use this identity if we once again write G(s) using partial fraction decomposition. Doing this we get

$$G(s) = \frac{1}{s(s-3)(s-2)} = \frac{1}{6s} + \frac{1}{3(s-3)} - \frac{1}{2(s-2)}.$$

We can now return to the full problem and actually compute the various inverse Laplace transforms using these identities.

$$\begin{aligned} y(t) = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{F(s)}{(s-3)(s-2)}\right) &+ \mathcal{L}^{-1}\left(\frac{s-5}{(s-3)(s-2)}\right) \\ &= 2\pi\mathcal{L}^{-1}(\exp(-s)G(s)) - \pi\mathcal{L}^{-1}(\exp(-2s)G(s)) + 3\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) - 2\mathcal{L}^{-1}\left(\frac{1}{s-3}\right) \\ &= 2\pi u_1 g(t-1) - \pi u_2 g(t-2) + 3\exp(2t) - 2\exp(3t) \end{aligned}$$

where

$$g(t) = \mathcal{L}^{-1}G(s) = \mathcal{L}^{-1}\left(\frac{1}{6s} + \frac{1}{3(s-3)} - \frac{1}{2(s-2)}\right) = \frac{1}{6} + \frac{1}{3}\exp(3t) - \frac{1}{2}\exp(2t).$$

Notes:
Had we decided to use the convolution identity we would get the same results, we would just have to either explicitly write down the different cases or we would need to independently identify the different Heaviside functions that are appearing.
There are two handy things we can confirm to at least give us some confidence in our solution. Firstly, if we substitute in t=0 then we can easily see that both of the initial conditions are satisfied. Secondly, if t<1 then the source term f(t) contributes nothing and so we have

$$y(t) = 3\exp(2t) - 2\exp(3t)$$

which we can verify, by direct substitution or otherwise, solves the homogeneous problem.
Finally, if reading the Heaviside notation is awkward, we present the final solution in terms of cases:

$$y(t) = \begin{cases} 3\exp(2t) - 2\exp(3t), & t < 1 \\ \frac{\pi}{3} + \frac{2\pi}{3}\exp(3(t-1)) - \pi\exp(2(t-1)) + 3\exp(2t) - 2\exp(3t), & 1 \leq t < 2 \\ \frac{\pi}{6} + \frac{2\pi}{3}\exp(3(t-1)) - \pi\exp(2(t-1)) - \frac{\pi}{3}\exp(3(t-2)) + \frac{\pi}{2}\exp(2(t-2)) + 3\exp(2t) - 2\exp(3t), & t \geq 2 \end{cases}.$$

Question 5 (c)

SOLUTION. From the chart at the end of the exam we have the following identity

$$\mathcal{L}(f^{(n)}(t)) = s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$$

and we should always consider this identity first if we see products of polynomials and transforms. Comparing this to what we have,

$$s^2 F(s) - 2s - 3,$$

we see that since $f(0) = 2$ and $f'(0) = 3$ then

$$s^2 F(s) - 2s - 3 = \mathcal{L}(f''(t)).$$

Therefore we have

$$G(s) = \exp(-4s)\mathcal{L}(f''(t)).$$

We now need to compute $\mathcal{L}^{-1}(G(s))$. Returning again to our chart we have the identity

$$\mathcal{L}^{-1}(\exp(-cs)F(s)) = u_c(t)f(t-c)$$

where u_c is the Heaviside function defined as

$$u_c = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}.$$

Using this identity, we can immediately write

$$g(t) = \mathcal{L}^{-1}(G(s)) = \mathcal{L}^{-1}(\exp(-4s)\mathcal{L}(f''(t))) = u_4(t)f''(t-4).$$

Therefore we have that

$$g(5) = u_4(5)f''(1) = f''(1) = 2.$$

Question 6 (a)

SOLUTION. We have

$$A - \lambda I = \begin{pmatrix} 3-\lambda & 1 \\ -4 & -1-\lambda \end{pmatrix}$$

which has determinant

$$(3-\lambda)(-1-\lambda) - (-4)(1) = \lambda^2 - 2\lambda + 1.$$

Setting $\lambda^2 - 2\lambda + 1 = 0$ we find $\lambda = 1$ as the only eigenvalue.

Question 6 (b)

SOLUTION. __NOTOC__
To begin with we matrix M in Jordan normal form, such that $A = SMS^{-1}$. To do we need to find the (generalized) eigenvector of A.

Eigenvector v Knowing $\lambda = 1$, we will proceed to find our eigenvectors by solving $(A - \lambda I)v = 0$.
$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
which gives the homogeneous system
$\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$.
Placing this in reduced-row form we get

$$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

so that $v_2 = -2v_1$. We take

$$v = (1, -2)^T$$

as the eigenvector.
Unfortunately, we only find one eigenvector from this, and we'll need to consider generalized eigenvectors and the Jordan canonical form in order to solve this problem.

Find the generalized eigenvector w The generalized eigenvector satisfies $(A - \lambda I)w = v$ which gives us
$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.
Row reductions lead to
$\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
which has a solution set that can be parameterized by w_1:

$$\begin{pmatrix} w_1 \\ 1 - 2w_1 \end{pmatrix} = w_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

For simplicity, we can take $w_1 = 0$ so that our generalized eigenvector is

$$w = (0, 1)^T.$$

Setting up the Jordan normal form M Having found the generalized eigenvector, we get that the matrix A has Jordan Canonical form

$$A = SMS^{-1}$$

where

$$S = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

is the matrix with columns being the eigenvectors, and

$$M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is the matrix with the eigenvalues along the diagonal with 1's above them.
A quick computation yields

$$S^{-1} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Our system states that

$$x' = Ax$$

where

$$x = (x_1, x_2)^T.$$

We can therefore write,

$$\begin{aligned} x' &= SMS^{-1}x \\ S^{-1}x' &= MS^{-1}x \\ y' &= My \end{aligned}$$

where

$$y = S^{-1}x.$$

Solve the simplified system *y' = My* The system

$$y' = My = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

tells us

$$\begin{aligned} y_1' &= y_1 + y_2 \\ y_2' &= y_2 \end{aligned}$$

.
The second equation gives us

$$y_2 = ce^t.$$

Using this in the first equation means

$$\begin{aligned} y_1' &= y_1 + ce^t \\ y_1' - y_1 &= ce^t \\ e^{-t}y_1' - e^{-t}y_1 &= (e^{-t}y_1)' = c \\ e^{-t}y_1 &= ct + b \\ y_1 &= (ct + b)e^t \end{aligned}$$

which has been solved using integrating factors. Now we have

$$y = \begin{pmatrix} cte^t + be^t \\ ce^t \end{pmatrix}$$

Recover the solution $\boldsymbol{x}$ which immediately gives us our solution x,
$x = Sy = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} cte^t + be^t \\ ce^t \end{pmatrix} = \begin{pmatrix} cte^t + be^t \\ -2cte^t + (c-2b)e^t \end{pmatrix}$.

Question 6 (c)

SOLUTION. As there are constants and exponentials in this first-order linear ODE, we will use undetermined coefficients, postulating that the particular solution is
$x_p(t) = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} e^{2t} + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$.
so that
$x_p'(t) = \begin{pmatrix} 2B_1 \\ 2B_2 \end{pmatrix} e^{2t}$.
Substituting our solution into the system we get,

$$\begin{aligned} &Ax + \begin{pmatrix} -2e^{2t} \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix} \begin{pmatrix} B_1e^{2t} + C_1 \\ B_2e^{2t} + C_2 \end{pmatrix} + \begin{pmatrix} -2e^{2t} \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 3B_1 + B_2 - 2 \\ -4B_1 - B_2 \end{pmatrix} e^{2t} + \begin{pmatrix} 3C_1 + C_2 \\ -4C_1 - C_2 + 1 \end{pmatrix}. \end{aligned}$$

This must equal x_p' and so we have

$$\begin{aligned} \begin{pmatrix} 3B_1 + B_2 - 2 \\ -4B_1 - B_2 \end{pmatrix} e^{2t} + \begin{pmatrix} 3C_1 + C_2 \\ -4C_1 - C_2 + 1 \end{pmatrix} &= \begin{pmatrix} 2B_1 \\ 2B_2 \end{pmatrix} e^{2t} \\ \begin{pmatrix} B_1 + B_2 - 2 \\ -4B_1 - 3B_2 \end{pmatrix} e^{2t} + \begin{pmatrix} 3C_1 + C_2 \\ -4C_1 - C_2 + 1 \end{pmatrix} &= \mathbf{0} \end{aligned}$$

This gives us the system of equations

$$\begin{aligned} B_1 + B_2 - 2 &= 0 \\ -4B_1 - 3B_2 &= 0 \end{aligned}$$

(based on the e^{2t} terms)
and

$$\begin{aligned} 3C_1 + C_2 &= 0 \\ -4C_1 - C_2 &= -1 \end{aligned}$$

(based on comparing the constant vectors).
Looking at the first equation of the first system of equations, we have $B_1 = 2 - B_2$ so that in the second equation,

$$\begin{aligned} -4(2 - B_2) - 3B_2 &= 0 \\ B_2 &= 8. \end{aligned}$$

Using the first equation again, we get

$$B_1 = -6.$$

Looking at the first equation of the second system of equations, we have $C_2 = -3C_1$. From the second equation we get

$$\begin{aligned} -4C_1 - (-3C_1) &= -1 \\ C_1 &= 1. \end{aligned}$$

and hence

$$C_2 = -3$$

after substituting back into the first equation.
The general solution to the ODE system will be $u_p + u_h$ where u_h is our homogeneous solution from part (b). Thus,

$$x(t) = \begin{pmatrix} 1 \\ -3 \end{pmatrix} + \begin{pmatrix} -6 \\ 8 \end{pmatrix} e^{2t} + \begin{pmatrix} cte^t + be^t \\ -2cte^t + (c - 2b)e^t \end{pmatrix}.$$

If

$$x(0) = (1, -5)^T$$

then

$$\begin{pmatrix} 1 \\ -5 \end{pmatrix} = \begin{pmatrix} 1 - 6 + b \\ -3 + 8 + (c - 2b)) \end{pmatrix}.$$

The first component tells us that $b = 6$ while using this value of b in the second component gives $c = 2$. Therefore, the solution is
$x(t) = \left(\begin{array}{c} 1 \\ -3 \end{array} \right) + \left(\begin{array}{c} -6 \\ 8 \end{array} \right) e^{2t} + \left(\begin{array}{c} 2te^t + 6e^t \\ -4te^t - 10e^t \end{array} \right).$

Question 7 (a)

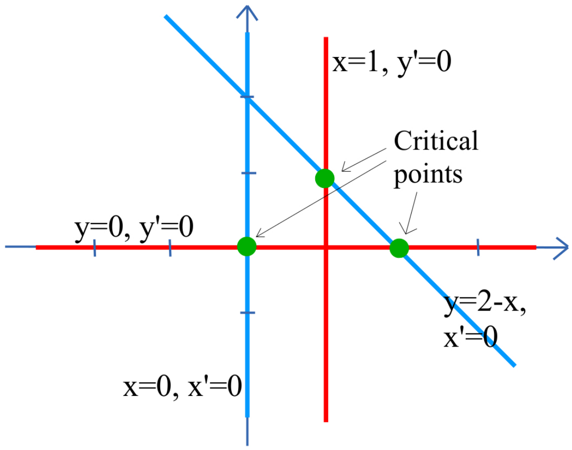
SOLUTION. For the nullclines, we consider $x' = x(2 - x - y) = 0$

$$x' = 0,$$

if $x = 0$ or $y = 2 - x$.
Now we consider $y' = y(1 - x) = 0$

$$y' = 0,$$

if $y = 0$ or $x = 1$.
For the critical points, $x' = y' = 0$ must hold. Hence, the critical points are $(x_0, y_0) = (0, 0), (2, 0), (1, 1)$.
Plotted in the phase space:



Question 7 (b)

SOLUTION. We will denote $f(x,y) = \frac{dx}{dt}(x,y) = x(2-x-y)$ and $g(x,y) = \frac{dy}{dt}(x,y) = y(1-x)$. Then the Jacobian matrix at at point (x,y) is

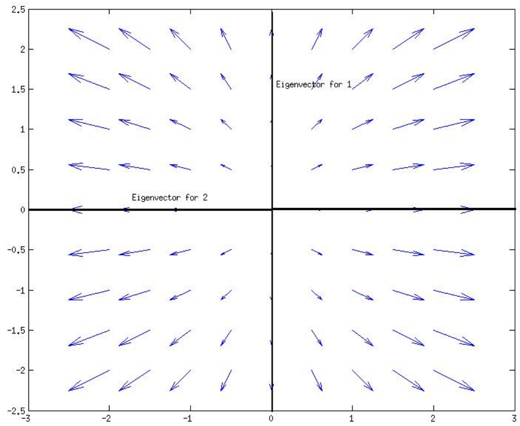
$$J(x,y) = \begin{pmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{pmatrix} = \begin{pmatrix} 2-2x-y & -x \\ -y & 1-x \end{pmatrix}.$$

Question 7 (c)

SOLUTION. Using the Jacobian from 7 (b) we find the linearization of the system in the critical points.
Critical point $y_0 = (0,0)$

$$J(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

- This matrix is diagonal and so the diagonal elements are the eigenvalues
- This matrix is *positive definite* since its two eigenvalues $1,2$ are positive. All positive eigenvalues indicate that the critical point $(0,0)$ is an **unstable node**.
- The eigenvector for the eigenvalue 1 is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
- The eigenvector for the eigenvalue 2 is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
- The linearized critical point in the phase space is on the following figure:



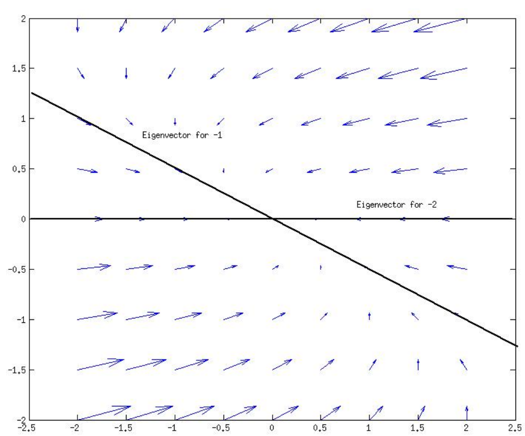
Critical point $y_0 = (2, 0)$

$$J(2,0) = \begin{pmatrix} -2 & -2 \\ 0 & -1 \end{pmatrix}.$$

- The eigenvalues are $-2, -1$, because a triangular matrix has its eigenvalues on the diagonal. The eigenvalues are both negative and hence, the matrix is *negative definite* and this critical point is a **stable node**.
- The eigenvector for the eigenvalue -2 is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ because of the triangular shape of the matrix.
- For the other eigenvector we need to work a little more and calculate the null space of $J(2,0) - (-1I)$:

$(J(2,0) + 1I)(v_1, v_2)^T = \begin{pmatrix} -1 & -2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -v_1 - 2v_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
Hence, $v_1 = -2v_2$

- We find the the eigenvector for -1 is $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.
- The linearized critical point in the phase space is on the following figure:



Critical point $y_0 = (1, 1)$

$$J(1,1) = \begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix}.$$

- For the eigenvalues we calculate

$$det \begin{pmatrix} -1-\lambda & -1 \\ -1 & -\lambda \end{pmatrix} = \lambda^2 + \lambda - 1 = 0$$
$$\lambda = \frac{-1\pm\sqrt{5}}{2}$$

- The eigenvalue for $\frac{-1-\sqrt{5}}{2}$ is the null space of

$$\begin{pmatrix} -1+\frac{1+\sqrt{5}}{2} & -1 \\ -1 & \frac{1+\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$v_1 = \frac{1+\sqrt{5}}{2} v_2$$

so that the eigenvector is $\begin{pmatrix} 1+\sqrt{5} \\ 2 \end{pmatrix}$.

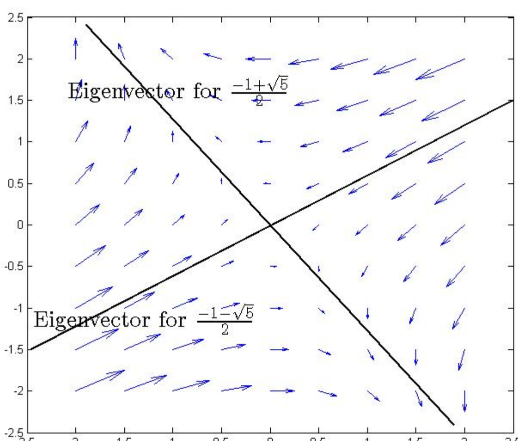
- The eigenvalue for $\frac{-1+\sqrt{5}}{2}$ is the null space of

$$\begin{pmatrix} -1+\frac{1-\sqrt{5}}{2} & -1 \\ -1 & \frac{1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$v_1 = \frac{1-\sqrt{5}}{2} v_2$$

and the eigenvector is $\begin{pmatrix} 1-\sqrt{5} \\ 2 \end{pmatrix}$.

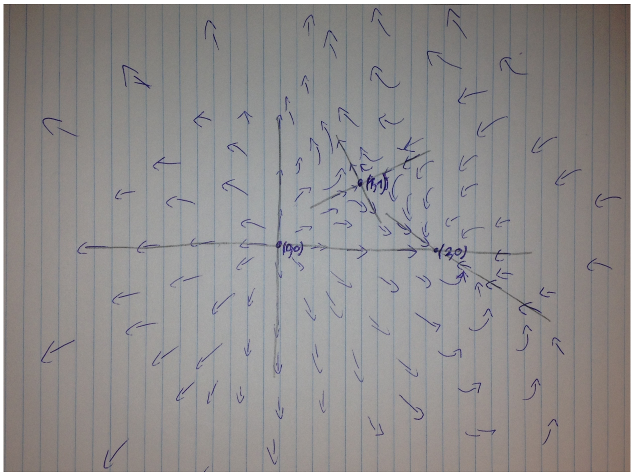
- Because one eigenvalue is positive, the other negative, the critical point is not asymptotically stable. Along the eigenvector for $\frac{-1+\sqrt{5}}{2}$, the solution moves away from the critical point, along the eigenvector for $\frac{-1-\sqrt{5}}{2}$, the solution moves towards the critical point. Since the eigenvalues do not have the same sign, the matrix is *indefinite* and the equilibrium is a **saddle point**.

- See the following figure:



Question 7 (d)

Solution.



Good Luck for your exams!

