Full Solutions
MATH101 April 2010

April 4, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

SOLUTION. $\int_0^1 \sqrt{x^3} + x^{\frac{2}{3}} dx = \int_0^1 x^{\frac{3}{2}} + x^{\frac{2}{3}} dx = \left(\frac{2x^{\frac{5}{2}}}{5} + \frac{3x^{\frac{5}{3}}}{5} \right) \Big|_0^1 = \frac{2}{5} + \frac{3}{5} = 1$

Question 1 (b)

SOLUTION. $\int_0^{\pi} |\cos x| dx = \int_0^{\pi/2} |\cos x| dx + \int_{\pi/2}^{\pi} |\cos x| dx$
Remember that cos is positive on $[0, \pi/2]$ and negative on $[\pi/2, \pi]$ so

$$\begin{aligned}
\int_0^{\pi/2} |\cos x| dx + \int_{\pi/2}^{\pi} |\cos x| dx &= \int_0^{\pi/2} \cos x dx + \int_{\pi/2}^{\pi} -\cos x dx \\
&= \sin(x)|_0^{\pi/2} - \sin(x)|_{\pi/2}^{\pi} \\
&= \sin(\pi/2) - \sin(0) - (\sin(\pi) - \sin(\pi/2)) \\
&= 1 - 0 - (0 - 1) = 2
\end{aligned}$$

Question 1 (c)

SOLUTION. We want

$$\begin{aligned}
0 &= \frac{1}{b} \int_0^b (x-1) dx \\
0 &= \frac{1}{b} (\frac{x^2}{2} - x)|_0^b \\
0 &= \frac{b^2}{2} - b \\
b &= \frac{b^2}{2} \\
2b &= b^2
\end{aligned}$$

Since $b > 0$, we can divide both sides by b to find $b = 2$.

Question 1 (d)

SOLUTION. From the Maclaurin polynomial we see that
$\sin(t^2) = t^2 - \frac{t^6}{3!} + \dots$
so

$$\begin{aligned}
\int_0^x \sin(t^2) dt &= \int_0^x (t^2 - \frac{t^6}{3!} + \dots) dt \\
&= \frac{t^3}{3} - \frac{t^7}{42} + \dots \Big|_0^x \\
&= \frac{x^3}{3} - \frac{x^7}{42} + \dots
\end{aligned}$$

Which means $b = -\frac{1}{42}$

Question 1 (e)

SOLUTION. First, we compute the area of the curve in the region specified.
$A = \int_1^2 \frac{1}{x^2}\, dx = -x^{-1} \mid_1^2 = -\frac{1}{2} + \frac{1}{1} = \frac{1}{2}$
Next, using the formula for the x-coordinate of the centroid, we have

$$\begin{aligned}
\bar{x} &= \frac{1}{\frac{1}{2}} \int_1^2 x \cdot \frac{1}{x^2}\, dx \\
&= 2 \int_1^2 \frac{1}{x^1}\, dx \\
&= 2\ln(x) \mid_1^2 \\
&= 2(\ln(2) - \ln(1)) \\
&= 2\ln(2) \\
&= \ln(4)
\end{aligned}$$

completing the problem.

Question 1 (f)

SOLUTION. Letting $u = 2x$ we get $du = 2dx$ so

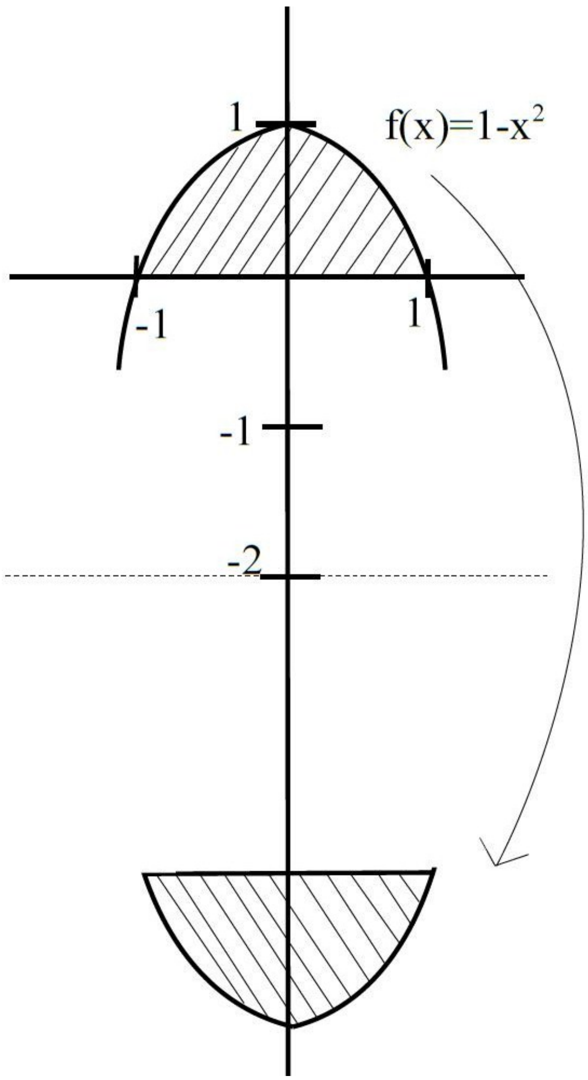
$$\begin{aligned}
\int_b^{-1} e^{2x} dx &= \frac{1}{2} \int_{2b}^{-2} e^u du \\
&= \frac{1}{2} e^u \Big|_{2b}^{-2} \\
&= \frac{1}{2e^2} - \frac{e^{2b}}{2}
\end{aligned}$$

When $b \to -\infty$ we have $\frac{e^{2b}}{2} \to 0$ so

$$\int_{-\infty}^{-1} e^{2x} dx = \frac{1}{2e^2} - 0 = \frac{1}{2e^2}$$

Question 2 (a)

SOLUTION 1. We see the function and a sketch of the volume in the following figure.



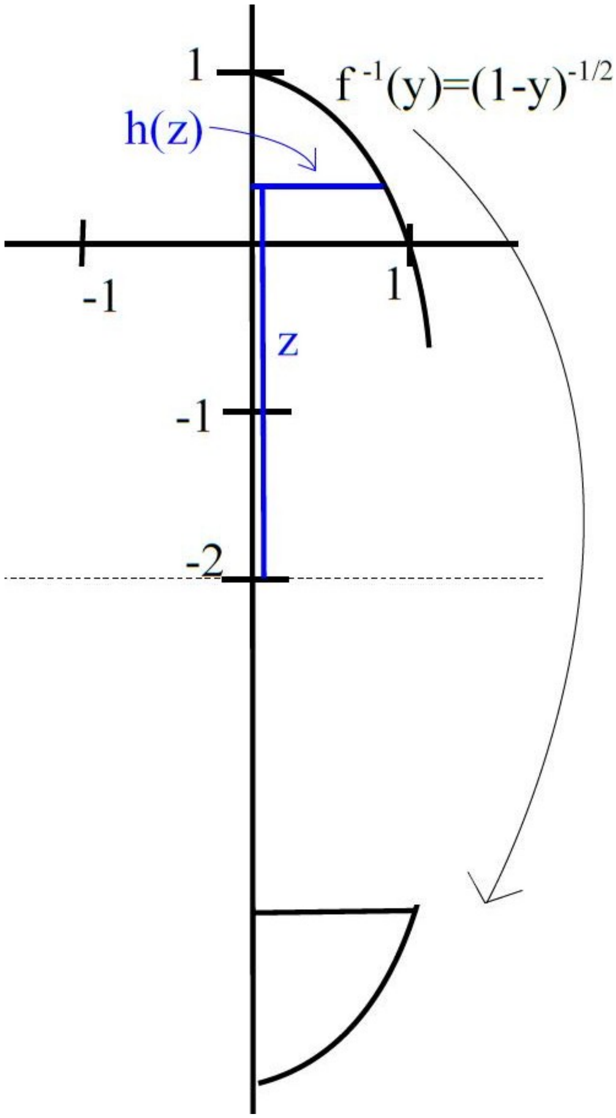
We use the shell method

$$V = 2\pi \int h(z) z \, dz$$

to find the volume. Therefore we need to determine what is the height $h(z)$ and what is the integral variable z.
On the next figure we see, that the height is the inverse of the function

$$y = f(x) = 1 - x^2.$$

And for this height the integral variable z is the distance from y to the rotating axis, which is $y = -2$.



So, we find $z = y + 2$ and for the height we calculate

$$f(x) = y = 1 - x^2$$

$$x^2 = 1 - y$$

$$f^{-1}(y) = x = \sqrt{1-y}.$$

For convenience, we calculate half of the volume and drop the left half. Then we can take

$$h(z) = \sqrt{1-y}$$

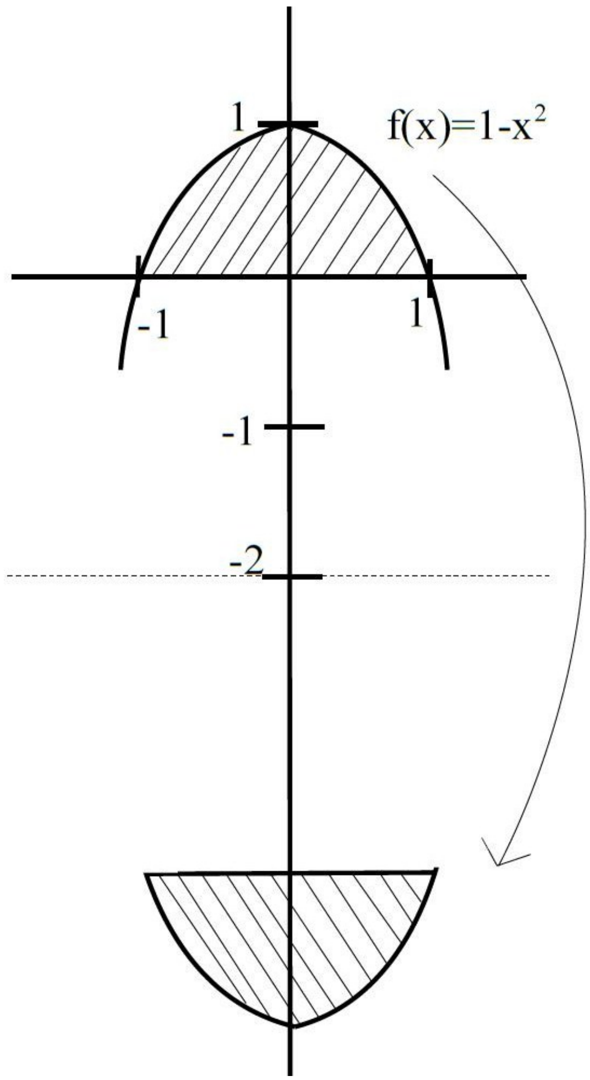
for $z = y + 2$.
The boundaries of the integral must be the left and right edge of the interval for $y = [0, 1]$. Now we need to calculate the integral

$$
\begin{aligned}
\frac{1}{2}V &= 2\pi \int h(z) z\, dz \\
&= 2\pi \int_0^1 (y+2)\sqrt{1-y}\, dy \\
&= 2\pi \overbrace{\int_0^1 \underbrace{y}_{=u} \underbrace{\sqrt{1-y}}_{=dv}\, dy}^{\text{use integration by parts}} + 2\pi \int_0^1 2\sqrt{1-y}\, dy \\
&= 2\pi \left[y(1-y)^{\frac{3}{2}}(-\frac{2}{3}) \right]_0^1 + 2\pi \frac{2}{3} \int_0^1 (1-y)^{\frac{3}{2}}\, dy + 2\pi 2 \left[(1-y)^{\frac{3}{2}} \right]_0^1 \\
&= 0 + 2\pi \frac{2}{3} \left[(1-y)^{\frac{5}{2}}(-\frac{2}{5}) \right]_0^1 - 0 + 2\pi \frac{4}{3} \\
&= 2\pi \frac{4}{15} + 2\pi \frac{4}{3} \\
&= \frac{16\pi}{5}
\end{aligned}
$$

So, we get for the volume

$$
V = \frac{32\pi}{5}.
$$

Solution 2. We see the function and a sketch of the volume in the following figure.



We use the washers method

$$V = \pi \int (R_O(x)^2 - R_I(x)^2)\, dx$$

to find the volume, where $\mathrm{R_O}$ is the outer radius of the washer and $\mathrm{R_I}$ is inner radius. From the diagram above, we see that $R_O(x) = 1 - x^2 + 2 = 3 - x^2$, and $R_I(x) = 2$.
We are integrating along x, so the boundaries of the integral must be $x = [-1, 1]$. Now we need to calculate the integral

$$
\begin{aligned}
V &= \pi \int (R_O(x)^2 - R_I(x)^2)\, dx \\
&= \pi \int_{-1}^{1} ((3-x^2)^2 - 4)\, dx \\
&= \pi \int_{-1}^{1} (9 - 6x^2 + x^4 - 4)\, dx \\
&= \pi \int_{-1}^{1} (5 - 6x^2 + x^4)\, dx \\
&= \pi \left[5x - 2x^3 + \frac{1}{5}x^5\right]_{-1}^{1} \\
&= \pi \left[\left(5 - 2 + \frac{1}{5}\right) - \left(5(-1) - 2(-1)^3 + \frac{1}{5}(-1)^5\right)\right] \\
&= \pi \left(10 - 4 + \frac{2}{5}\right) \\
&= \frac{32\pi}{5}
\end{aligned}
$$

Question 2 (b)

SOLUTION. We can consider one loop of the polar curve to be defined by the values of θ in between successive points where $r = 0$. Consider that $\cos(z) = 0$, where $z = -\pi/2$ and $\pi/2$, so we can take θ to be vary in between $\theta = -\pi/4$ and $\pi/4$ to define a loop.
Now consider that the area of a wedge with angle $\Delta\theta$ is equal to $r^2\Delta\theta/2$ where r is the'radius' of the wedge (i.e: from the point to the rounded boundary). To compute the total area of the loop we simply integrate in between the appropriate angles.

$$
\begin{aligned}
A &= \int_{-\pi/4}^{\pi/4} \frac{1}{2} r(\theta)^2\, d\theta \\
&= 8 \int_{-\pi/4}^{\pi/4} \cos^2(2\theta)\, d\theta
\end{aligned}
$$

Using a trigonometric identity, we can simplify this integral so we are not dealing with the square of a trig function.

$$
\begin{aligned}
A &= 8 \int_{-\pi/4}^{\pi/4} \cos^2(2\theta)\, d\theta \\
&= 4 \int_{-\pi/4}^{\pi/4} 1 + \cos(4\theta)\, d\theta \\
&= 2\pi
\end{aligned}
$$

Therefore, the area of one loop of the polar curve $r = 4\cos(2\theta)$ is 2π.

Question 2 (c)

Solution. Using the formula in the hint, we see that we require the derivative of the curve in question. Notice that

$$\frac{dy}{dx} = 2x - \frac{1}{8x}$$

and so we compute the arc length to be

$$\begin{aligned}
\int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx &= \int_1^2 \sqrt{1 + \left(2x - \frac{1}{8x}\right)^2}\, dx \\
&= \int_1^2 \sqrt{1 + 4x^2 - \frac{1}{2} + \frac{1}{64x^2}}\, dx \\
&= \int_1^2 \sqrt{4x^2 + \frac{1}{2} + \frac{1}{64x^2}}\, dx \\
&= \int_1^2 \sqrt{(2x + \frac{1}{8x})^2}\, dx \\
&= \int_1^2 (2x + \frac{1}{8x})\, dx \\
&= (x^2 + \frac{1}{8}\ln|x|)\Big|_1^2 \\
&= (2)^2 + \frac{1}{8}\ln(2) - (1)^2 - \frac{1}{8}\ln(1) \\
&= 3 + \frac{1}{8}\ln(2).
\end{aligned}$$

Question 3 (a)

Solution 1. We look for an expression in partial fractions of the form:
$\frac{x^2-9}{x(x^2+9)} = \frac{A}{x} + \frac{Bx+C}{x^2+9}$
Which translates to:
$x^2 - 9 = Ax^2 + 9A + Bx^2 + Cx$
From this, we get the following system of equations:

$$\begin{aligned}
A + B &= 1 \\
9A &= -9 \\
C &= 0
\end{aligned}$$

And so $A = -1$ and $B = 2$. Then
$\frac{x^2-9}{x(x^2+9)} = \frac{-1}{x} + \frac{2x}{x^2+9}$
Now we compute the integral

$$\begin{aligned}
\int \frac{x^2 - 9}{x(x^2 + 9)} dx &= \int \frac{-dx}{x} + \int \frac{2x dx}{x^2 + 9} \\
&= -\ln|x| + \ln|x^2 + 9| + C
\end{aligned}$$

Where for the last equality we use the substitution $u = x^2 + 9$.

Solution 2. We want to evaluate the integral

$$\int \frac{x^2-9}{x(x^2+9)}\,dx.$$

As a first step, we split up the integral:

$$\int \frac{x^2-9}{x(x^2+9)}\,dx = \int \frac{x^2}{x(x^2+9)}\,dx - \int \frac{9}{x(x^2+9)}\,dx$$

We can evaluate the first integral by canceling a factor of x, and then substitute *u= x^2+9, du = 2dx*:

$$\begin{aligned}\int \frac{x^2}{x(x^2+9)}\,dx &= \int \frac{x}{x^2+9}\,dx = \int \frac{1}{2u}\,du \\ &= \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+9) + C\end{aligned}$$

The second integral can be evaluated using a partial fraction decomposition

$$\frac{-9}{x(x^2+9)} = \frac{A}{x} + \frac{Bx}{x^2+9} + \frac{C}{x^2+9},$$

which we can solve for $-9 = A(x^2+9) + Bx^2 + Cx$. When x = *-1*, we get *-9 = 10A+B-C*, when *x=1*, we get *-9 = 10A+B+C*. Subtracting the two equations gives C = *0*. Finally, when x = *2*, we get *-9 = 13A + 4B*. So *10A + B = 13 A+ 4B*, which yields A = *-B*. Thus *-9 = 10A-A = 9A*, so that A = *-1* and B = *1*. Therefore

$$\int \frac{-9}{x(x^2+9)}\,dx = \int \frac{-1}{x}\,dx + \int \frac{x}{x^2+9}\,dx = -\ln|x| + C + \int \frac{x}{x^2+9}\,dx.$$

We evaluated the last integral earlier in this problem and found $\int \frac{x}{x^2+9}\,dx = \frac{1}{2}\ln(x^2+9)$. So

$$\int \frac{-9}{x(x^2+9)}\,dx = \frac{1}{2}\ln(x^2+9) - \ln|x| + C,$$

and therefore the final answer is

$$\int \frac{x^2-9}{x(x^2+9)}\,dx = \ln(x^2+9) - \ln|x| + C.$$

Solution 3. We want to evaluate the integral

$$\int \frac{x^2-9}{x(x^2+9)}\,dx.$$

This can be rewritten as

$$\begin{aligned}\int \frac{2x^2-(x^2+9)}{x(x^2+9)}\,dx &= \int \frac{2x^2}{x(x^2+9)}\,dx - \int \frac{x^2+9}{x(x^2+9)}\,dx \\ &= \int \frac{2x}{x^2+9}\,dx - \int \frac{1}{x}\,dx.\end{aligned}$$

We evaluate the first integral using the substitution *u = xˆ2+9, du = 2xdx*, the second integral is just *ln|x| + C*. Hence

$$\begin{aligned}\int \frac{2x^2 - (x^2+9)}{x(x^2+9)}\,dx &= \int \frac{1}{u}\,du - \ln|x| - C \\ &= \ln|u| - \ln|x| - C \\ &= \ln|x^2+9| - \ln|x| - C\end{aligned}$$

Note: Since C is a constant, we can write -C or +D, for D=-C. Either solution is fine.

Question 3 (b)

SOLUTION. Using the change of variable

$$x = 2\tan(\theta)$$

and taking a derivative with respect to x on both sides we obtain

$$1 = 2\sec^2(\theta)\frac{d\theta}{dx}$$

and since $1 + \tan^2(\theta) = \sec^2(\theta)$ we have

$$\begin{aligned}(4+x^2)^{3/2} &= (4 + 4\tan^2(\theta))^{3/2} \\ &= 4^{3/2}(1+\tan^2(\theta))^{3/2} \\ &= 8(\sec^2(\theta))^{3/2} \\ &= 8\sec^3(\theta)\end{aligned}$$

and so our integral becomes

$$\begin{aligned}\int \frac{1}{(4+x^2)^{3/2}}\,dx &= \int \frac{1}{8\sec^3(\theta)} 2\sec^2(\theta)\,d\theta \\ &= \int \frac{1}{4\sec(\theta)}\,d\theta \\ &= \frac{1}{4}\int \cos(\theta)\,d\theta \\ &= \frac{1}{4}\sin(\theta) + c\end{aligned}$$

Now using that

$$\frac{x}{2} = \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

we get that

$$\cos(\theta) = \frac{2\sin(\theta)}{x}$$

and since $\sin^2(\theta)+\cos^2(\theta)=1$ we have that

$$\sin^2(\theta)+\frac{4\sin^2(\theta)}{x^2}=1 \quad \text{so} \quad \sin^2(\theta)(1+\frac{4}{x^2})=1$$

and so finally we can solve for $\sin(\theta)$ and get

$$\sin(\theta)=\frac{x}{\sqrt{4+x^2}}$$

which we can now plug back into our integral and conclude that

$$\int \frac{1}{(4+x^2)^{3/2}}\,dx=\frac{x}{4\sqrt{4+x^2}}+c$$

Question 3 (c)

SOLUTION. For the integral

$$\int x\arctan(x)\,dx,$$

we start by using integration by parts. Let

$$\begin{aligned} u &= \arctan(x) & v &= \frac{x^2}{2} \\ du &= \frac{dx}{1+x^2} & dv &= xdx \end{aligned}$$

This gives

$$\int x\arctan(x)\,dx=\arctan(x)\frac{x^2}{2}-\frac{1}{2}\int\frac{x^2}{1+x^2}\,dx$$

To evaluate the last integral, we use the fact that $x^2=1+x^2-1$ to see that

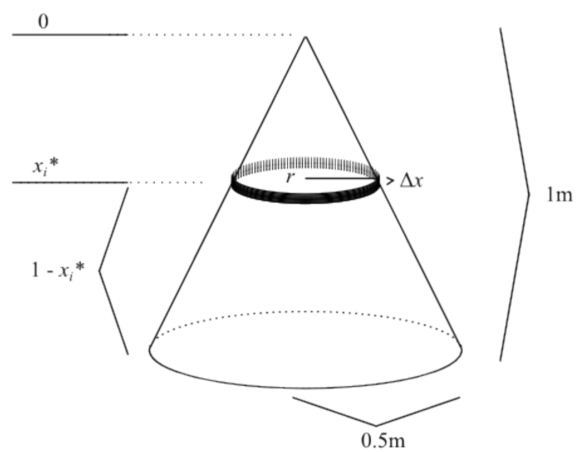
$$\begin{aligned} \int x\arctan(x)\,dx &= \arctan(x)\frac{x^2}{2}-\frac{1}{2}\int\frac{x^2}{1+x^2}\,dx \\ &= \arctan(x)\frac{x^2}{2}-\frac{1}{2}\int\frac{1+x^2-1}{1+x^2}\,dx \\ &= \arctan(x)\frac{x^2}{2}-\frac{1}{2}\int dx+\frac{1}{2}\int\frac{1}{1+x^2}\,dx \\ &= \arctan(x)\frac{x^2}{2}-\frac{1}{2}x+\frac{1}{2}\arctan(x)+C \end{aligned}$$

completing the question.

Question 3 (d)

SOLUTION. Following the hint, we try the substitution $u = x^2$. Then $du = 2xdx$ and the endpoints will change to $u(0) = 0^2 = 0$ and $u(3) = 3^2 = 9$. Hence
$\int_0^3 x\sqrt{81 - x^4}\,dx. = \frac{1}{2}\int_0^9 \sqrt{81 - u^2}\,du$
This last value can be seen to be half the area of a quarter circle of radius 9. Hence, the value is
$\frac{1}{2} \cdot \frac{\pi(9)^2}{4} = \frac{81\pi}{8}$
as required.

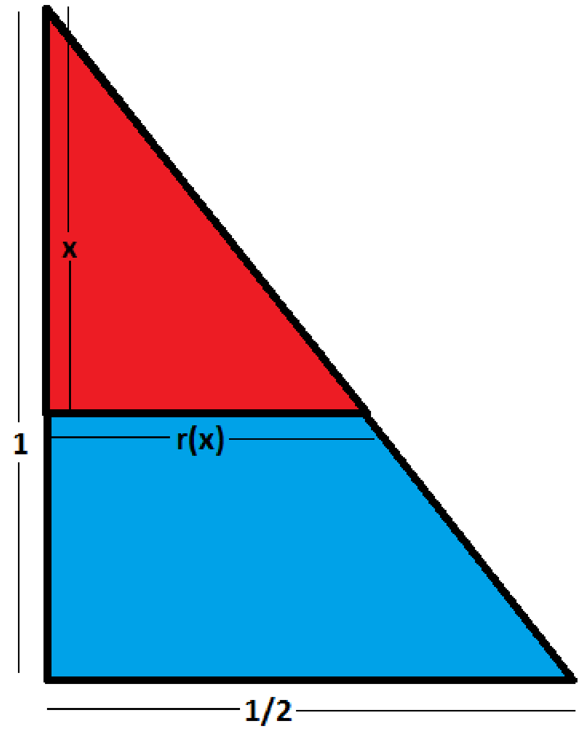
Question 4



SOLUTION. Consider the diagram for building the anthill .
We can think of building the hill as stacking several small cylinders of height Δx or dx (thinking forward to integration) from the ground (*x=1*) to the top (*x=0*). Consider the anthill at some intermediate distance from the top x (at a distance *1-x* from the ground). This point is represented in the diagram as x_i^*. The volume of a new cylinder to add onto the anthill will be

$$dV = \pi(r(x))^2\mathrm{d}x$$

where *dx* is the height of the cylinder and *r(x)* is the radius of the anthill at the current height. There is a similar triangle problem between the top of the completed anthill and the current disk we're looking



at and so we can determine the radius at our current height x via

$$\frac{1}{1/2} = \frac{x}{r(x)}$$

and so

$$r(x) = \frac{x}{2}.$$

Notice that indeed when we're at the top (x=*0*) then the radius is zero and when we're at the bottom (x=*1*) that the radius is 1/2. We therefore have that the new amount of sand volume we are adding is

$$dV = \frac{\pi}{4}x^2 \mathrm{d}x.$$

In the problem we are given a density $150 lb/ft^3$ which is actually a **force** density because lbs is a measure of force. This is telling us the force required to lift sand of a given volume which is precisely what we are trying to do with our cylinder of sand and so the force required to lift it is
`force = force density` $\times$ `volume`

$$dF = 150dV = \frac{150\pi}{4}x^2 \mathrm{d}x.$$

The work required to lift this cylinder to its position in the hill at x will be this force dF multiplied with the height we have to lift it from the ground *1-x*. Therefore the amount of work to lift this cylinder is

$$dW = (1-x)dF = \frac{150\pi}{4}x^2(1-x)\mathrm{d}x.$$

This is the work to build one small cylinder. The whole hill is built by adding cylinders for all values of x and so to get the total work we integrate

$$\begin{aligned}
\text{work} = \int_0^1 \mathrm{d}W &= \int_0^1 \frac{150\pi}{4} x^2 (1-x) dx \\
&= \left[\frac{150\pi x^3}{12} - \frac{150\pi x^4}{16}\right]_0^1 \\
&= \frac{150\pi}{12} - \frac{150\pi}{16} \\
&= \frac{25\pi}{8}
\end{aligned}$$

Let us do a quick sanity check on the units to see if we have missed out anything.
The volume is in ft^3, and the density is given in lb/ft^3, so the weight is in lb, finally work is force (weight) times distance (which is in ft), so we get our work to be in $ft \cdot lb$, which is right.
In conclusion, the work done by the ants to build the anthill is $\frac{25\pi}{8} ft \cdot lb$.

Question 5 (a)

SOLUTION. Since the formula for the trapezoid rule can be given by

$$T_n = \frac{\Delta x}{2} \left(f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{n-1}) + f(x_n)\right)$$

and we know that we want to evaluate this rule using $n = 4$, we have

$$x_i = a + i\Delta x = 0 + i\frac{4-0}{4}$$

and thus, the trapezoidal approximation that we need is

$$\begin{aligned}
T_4 &= \frac{1}{2} \left(f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)\right) \\
&= \frac{1}{2}(1 + 2(2) + 2(0) + 2(-2) + (-4)) = \frac{-3}{2}
\end{aligned}$$

Question 5 (b)

SOLUTION. The formula for the Simpson's rule is given by

$$S_n = \frac{\Delta x}{3} \left(f(x_0) + 4f(x_1) + 2f(x_2) + ... + 4f(x_{n-1}) + f(x_n)\right)$$

When $n = 4$, we have that

$$x_i = a + i\Delta x = 0 + i\frac{4-0}{4} = i$$

and thus, the Simpson's rule approximation that we need is

$$
\begin{aligned}
S_n &= \frac{1}{3}\left(f(0)+4f(1)+2f(2)+4f(3)+f(4)\right)\\
&= \frac{1}{3}(1+4(2)+2(0)+4(-2)+(-4)) = -1
\end{aligned}
$$

Question 5 (c)

SOLUTION. Using the hint, we plug in $n = 4$ into the error bound and notice that the error lies between
$\frac{(-3)(4-0)^5}{180(4)^4} < \frac{K(b-a)^5}{180n^4} < \frac{(2)(4-0)^5}{180(4)^4}$
Evaluating these bounds, we get that
$\frac{-12}{180} < \frac{K(b-a)^5}{180n^4} < \frac{8}{180}$
and one last simplification yields
$\frac{-1}{15} < \frac{K(b-a)^5}{180n^4} < \frac{2}{45}$.
Now, let's use the the approximation and notice that by definition, we have that
$\left|\int_0^4 f(x)\,dx - S_4\right| < \frac{K(b-a)^5}{180n^4}$
Notice that in magnitude, the value
$\frac{1}{15} > \frac{2}{45}$
and so we have that the largest the integral can be is the S_4 approximation added onto this largest error bound. This gives
$\int_0^4 f(x)\,dx < S_4 + \frac{1}{15} = -1 + \frac{1}{15} = \frac{-14}{15}$
completing the question.

Question 6

SOLUTION 1. *There is a certain ambiguity to how the diagram presents this problem. While it talks of a standard swimming pool with an inclined plane along the bottom, the diagram looks as if the top-side is inclined. Therefore we will present a solution where the bottom is flat and the top side is inclined and another solution where the top is flat and the bottom side is inclined. The approach to both problems is the same but the numerical results will differ.*

Top Inclined, Bottom Flat:

Create a diagram (like that to the right) that just illustrates the side of pool we are interested in right|500px|thumbnail. We will use the standard coordinate system of x representing the horizontal direction and y representing the vertical direction.

Notice we can divide the figure into a rectangular portion and a triangular portion which in the diagram is separated by a dashed line. Since this triangular portion is actually a right triangle we can determine the missing size z to be

$$z = \sqrt{20^2 - 1^2} = \sqrt{399}.$$

Now we are interested in determining the hydrostatic force that is exerted on this pool wall. From the hint we know this depends on the pressure which we can write as,

$$P = \rho g(2 - y)$$

where we take $y = 0$ to be the bottom of the diagram. Notice that because of this definition of y, the depth, h, as defined in the hint is then actually h=2-y since the top of the surface is at y=2. We see that depending where we are along the pool side, we will exert a different pressure and therefore a different force. Since the pressure depends on y, let's consider a very small rectangle of height dy and length, L. The small amount of force exerted on this rectangle of area dA is,

$$\mathrm{d}F = P\mathrm{d}A = \rho g(2-y)Ldy.$$

Therefore the total amount of force exerted will be the sum of the little amounts of force on each rectangle as we go from height $y = 0$ to height $y = 2$. However, we have to be careful, because as we climb up the height of the pool side, the length of the small rectangles depend on our height y due to the pool getting narrower in the triangular region. We will then break up the problem into computing the total force on the rectangular portion, F_1 and the total force on the triangular portion, F_2. In the end the total force will be, $F = F_1 + F_2$.

Rectangular Portion:

In this region, for any given height, y, the length is a constant and in fact we already computed it,

$$L = \sqrt{399}.$$

Therefore, for this region we have that for every little rectangle,

$$\mathrm{d}F_1 = \rho g\sqrt{399}(2-y)\mathrm{d}y.$$

In order to add up the contributions from all the small rectangles we turn to integration,

$$\begin{aligned} F_1 = \int \mathrm{d}F_1 &= \int_0^1 \rho g\sqrt{399}(2-y)\mathrm{d}y \\ &= \rho g\sqrt{399}\left(2y - \frac{y^2}{2}\right)\Big|_0^1 \\ &= \frac{3\rho g\sqrt{399}}{2}. \end{aligned}$$

The limits of integration can be seen from the diagram since the rectangular region starts at the bottom and goes up 1m.

Triangular Portion:

This region is slightly more tricky to compute because the length changes with the height. The first thing we need to do is figure out a function for the slanted side of the triangle in terms of x and y. We see that it is a line so it will be of the form $y = mx + b$ with m the slope and b the y-intercept. We know that when the height is 1, we are at the left edge of the diagram which we will denote $x = 0$. When the height is 2, we are at the right edge of the diagram which we know is

$$x = \sqrt{399}.$$

Therefore we get that the slope is

$$m = \frac{2-1}{\sqrt{399}-0} = \frac{1}{\sqrt{399}}$$

and the y-intercept occurs when $x = 0$ so $b = 1$. Therefore, the slanted edge of the triangle is given by,

$$y = \frac{1}{\sqrt{399}}x + 1.$$

Now at any height, the length of our small rectangle will start at the slant edge,

$$x = \sqrt{399}(y-1)$$

and extend to the right edge of the diagram at $x = \sqrt{399}$. Therefore we have that the length at any given height y, is

$$L(y) = \sqrt{399} - \sqrt{399}(y-1) = \sqrt{399}(2-y).$$

The small amount of force on any rectangle in this region is,

$$\mathrm{d}F_2 = P\mathrm{d}A = \rho g y L \mathrm{d}y = \rho g\sqrt{399}(2-y)(2-y)\mathrm{d}y.$$

In order to get the total force on the triangular region we integrate all these little $\mathrm{d}F_2$ over every height in the triangular region. This region starts at $y = 1$ and ends at $y = 2$. Therefore these are our lower and upper limits of integration respectively. We get for our total force in this region,

$$\begin{aligned} F_2 = \int \mathrm{d}F_2 &= \int_1^2 \rho g\sqrt{399}(2-y)^2\mathrm{d}y \\ &= \rho g\sqrt{399}\int_1^2 (y^2-4y+4)\mathrm{d}y \\ &= \rho g\sqrt{399}\left(\frac{y^3}{3} - 2y^2 + 4y\right)_1^2 \\ &= \frac{1}{3}\rho g\sqrt{399}. \end{aligned}$$

We can now compute the total force as,

$$F = F_1 + F_2 = \rho g\sqrt{399}\left(\frac{3}{2} + \frac{1}{3}\right) = \frac{11}{6}\rho g\sqrt{399}$$

which using the numbers provided in the question, $\rho = 1000$ and $g = 9.8$, we get,

$$F \approx 358883.9N$$

which is a very large force but not unsurprising considering the power that water has.
Caution!!
You may be tempted to want to set up an integral with x instead of y, i.e. a diagram that looks like that to the right right|500px|thumbnail.
The problem is that, as we mentioned above, the pressure is a function of height. Based on this formulation and this diagram, the small rectangles take every height value from the base of the rectangle to the slanted edge and so the pressure varies along the whole length of the rectangle. You may think that a solution to this problem is to take your rectangle of height, L and width $\mathrm{d}x$ and splice it up into even **smaller** rectangles of say height $\mathrm{d}y$ and width $\mathrm{d}x$. You then would have an integration problem on your small rectangle from the diagram and then another integration problem as you vary x and look at other small rectangles. This solution technique is very valid and is known as double integration, something that will be covered in more advanced courses.

Solution 2. *There is a certain ambiguity to how the diagram presents this problem. While it talks of a standard swimming pool with an inclined plane along the bottom, the diagram looks as if the top-side is inclined. Therefore we will present a solution where the bottom is flat and the top side is inclined and another* *solution where the top is flat and the bottom side is inclined. The approach to both problems is the same but the numerical results will differ.*

Top Flat, Bottom Inclined:

Create a diagram (like that to the right) that just illustrates the side of pool we are interested in right|500px|thumbnail. We will use the standard coordinate system of x representing the horizontal direction and y representing the vertical direction.

Notice we can divide the figure into a rectangular portion and a triangular portion which in the diagram is separated by a dashed line. Now we are interested in determining the hydrostatic force that is exerted on this pool wall. From the hint we know this depends on the pressure which we can write as,

$$P = \rho g(2 - y)$$

where we take $y = 0$ to be the bottom of the diagram. Because of this choice for y, we have that the depth, h, as defined in the hint is actually h=2-y which is why that shows up in the hydrostatic pressure. We see that depending where we are along the pool side, we will exert a different pressure and therefore a different force. Since the pressure depends on the height, y, let's consider a very small rectangle of height dy and length, L. The small amount of force exerted on this rectangle of area dA is,

$$\mathrm{d}F = P\mathrm{d}A = \rho g(2 - y)Ldy.$$

Therefore the total amount of force exerted will be the sum of the little amounts of force on each rectangle as we go from height $y = 0$ to height $y = 2$. However, we have to be careful, because as we climb up the height of the pool side, the length of the small rectangles depend on our height y due to the pool getting narrower in the triangular region. We will then break up the problem into computing the total force on the rectangular portion, F_1 and the total force on the triangular portion, F_2. In the end the total force will be, $F = F_1 + F_2$.

Rectangular Portion:

In this region, for any given height, y, the length is a constant and in fact it is the length of the pool,

$$L = 20.$$

Therefore, for this region we have that for every little rectangle,

$$\mathrm{d}F_1 = \rho g 20(2 - y)\mathrm{d}y.$$

In order to add up the contributions from all the small rectangles we turn to integration,

$$\begin{aligned} F_1 = \int \mathrm{d}F_1 &= \int_1^2 \rho g 20(2 - y)\mathrm{d}y \\ &= \rho g 20 \left(2y - \frac{y^2}{2}\right)\Big|_1^2 \\ &= \frac{\rho g 20}{2} = 10\rho g. \end{aligned}$$

The limits of integration can be seen from the diagram since the rectangular region starts at $y = 1$ and goes to the top of the pool at $y = 2$.

Triangular Portion:

This region is slightly more tricky to compute because the length changes with the height. The first thing we need to do is figure out a function for the slanted side of the triangle in terms of x and y. We see that it is a line so it will be of the form $y = mx + b$ with m the slope and b the y-intercept. We know that when the height is 1, we are at the left edge of the diagram which we will denote $x = 0$. When the height is 0, we are at the right edge of the diagram which we know is $x = 20$.

Therefore we get that the slope is

$$m = \frac{0-1}{20-0} = -\frac{1}{20}$$

and the y-intercept occurs when $x = 0$ so $b = 1$. Therefore, the slanted edge of the triangle is given by,

$$y = -\frac{1}{20}x + 1.$$

Now at any height, the length of our small rectangle will start at the slant edge,

$$x = -20(y-1)$$

and extend to the right edge of the diagram at $x = 20$. Therefore we have that the length at any given height y, is

$$L(y) = 20 - (-20(y-1)) = 20y.$$

The small amount of force on any rectangle in this region is,

$$\mathrm{d}F_2 = P\mathrm{d}A = \rho g(2-y)L\mathrm{d}y = \rho g 20(2-y)y\mathrm{d}y.$$

In order to get the total force on the triangular region we integrate all these little $\mathrm{d}F_2$ over every height in the triangular region. This region starts at $y = 0$ and ends at $y = 1$. Therefore these are our lower and upper limits of integration respectively. We get for our total force in this region,

$$\begin{aligned} F_2 = \int \mathrm{d}F_2 &= \int_0^1 \rho g 20(2y - y^2)\mathrm{d}y \\ &= \rho g 20 \left(y^2 - \frac{y^3}{3} \right) \Big|_0^1 \\ &= \frac{40}{3}\rho g. \end{aligned}$$

We can now compute the total force as,

$$F = F_1 + F_2 = \rho g \left(10 + \frac{40}{3} \right) = \frac{70}{3}\rho g$$

which using the numbers provided in the question, $\rho = 1000$ and $g = 9.8$, we get,

$$F \approx 228666.7N$$

which is a very large force but not unsurprising considering the power that water has.
Caution!!
You may be tempted to want to set up an integral with x instead of y, i.e. a diagram that looks like that to the right right|500px|thumbnail.
The problem is that, as we mentioned above, the pressure is a function of height. Based on this formulation and this diagram, the small rectangles take every height value from the base of the rectangle to the slanted edge and so the pressure varies along the whole length of the rectangle. You may think that a solution to this problem is to take your rectangle of height, L and width $\mathrm{d}x$ and splice it up into even **smaller** rectangles

of say height dy and width dx. You then would have an integration problem on your small rectangle from the diagram and then another integration problem as you vary x and look at other small rectangles. This solution technique is very valid and is known as double integration, something that will be covered in more advanced courses.

Question 7

SOLUTION. Beginning with the equation

$$\frac{dL}{dt} = kL^2 \ln t$$

we separate the variables L and t to obtain

$$\frac{1}{L^2} dL = k \ln t \, dt$$

If we then integrate both sides of this equation (ignoring the constant of integration in the first case) we find that

$$\int \frac{1}{L^2} dL = -\frac{1}{L}$$

and using integration by parts we find that

$$\int k \ln t \, dt = k(t \ln t - t) + C$$

and so if we set the two of these equal to each other we find that

$$-\frac{1}{L} = k(t \ln t - t) + C$$

If we solve for L we find

$$L = -\frac{1}{k(t \ln t - t) + C}$$

Since $L(1) = 1$ it follows that $C = k - 1$ and so

$$L(t) = -\frac{1}{k(t \ln t - t) + k - 1}$$

Question 8

SOLUTION. The interval has size 1 so $\Delta x = \frac{1}{n}$. The interval is $[0, 1]$ so $x_i = \frac{i}{n}$. In this case $f(x) = 6 - 3x^2$ so the right Riemann sum is:

$$
\begin{aligned}
R_n &= \sum_{i=1}^{n}(6-3(\frac{i}{n})^2)(\frac{1}{n}) \\
&= \frac{6}{n}\sum_{i=1}^{n}1 - \frac{3}{n^3}\sum_{i=1}^{n}i^2 \\
&= \frac{6}{n}(n) - \frac{3}{n^3}\frac{n(n+1)(2n+1)}{6} \\
&= 6 - \frac{n(n+1)(2n+1)}{2n^3}
\end{aligned}
$$

So the integral is:

$$
\begin{aligned}
\int_0^1 (6-3x^2)dx &= \lim_{n\to\infty} R_n \\
&= \lim_{n\to\infty}(6 - \frac{n(n+1)(2n+1)}{2n^3}) \\
&= 6 - \lim_{n\to\infty}\frac{2n^3+3n^2+n}{2n^3} \\
&= 6 - \lim_{n\to\infty}\frac{2+3/n+1/n^2}{2} \\
&= 6 - \frac{2-0-0}{2} = 6-1 = 5
\end{aligned}
$$

Checking our answer, we see that
$\int_0^1 (6-3x^2)dx = (6x-x^3)\ |_0^1 = 6(1)-(1)^3-(6(0)-(0)^3) = 6-1 = 5$
which matches the above.

Question 9

Solution. First we solve the integral considering x as a constant:

$$
\begin{aligned}
g(x) &= \int_0^1 (xe^t - t)^2 dt \\
&= \int_0^1 (x^2e^{2t} - 2xe^t t + t^2)dt \\
&= x^2\int_0^1 e^{2t}dt - 2x\int_0^1 e^t t dt + \int_0^1 t^2 dt
\end{aligned}
$$

The first integral in the sum is
$\int_0^1 e^{2t}dt = \frac{1}{2}(e^2-e^0) = \frac{1}{2}(e^2-1)$
Now we solve $\int_0^1 e^t t dt$ by parts letting $u=t$ and $dv = e^t dt$ so that $du = dt$ and $v = e^t$. Hence

$$
\begin{aligned}
\int_0^1 e^t t dt &= te^t\Big|_0^1 - \int_0^1 e^t dt \\
&= (e-0)-(e-1) = 1
\end{aligned}
$$

The last integral in the sum is:
$\int_0^1 t^2 dt = \frac{1}{3}(1^3-0^3) = \frac{1}{3}$
So

$g(x) = \frac{x^2}{2}(e^2-1) - 2x + \frac{1}{3}$
The derivative is directly computed to be:
$g'(x) = x(e^2-1) - 2$
Which is zero at $x = \frac{2}{e^2-1}$.
To make sure it is a minimum we use the second derivative test. The second derivative is:
$g''(x) = e^2 - 1$
Which is always positive (Remember that $e > 2$). This means $x = \frac{2}{e^2-1}$ is the minimum for g. Plugging in this value gives

$$\begin{aligned}
g\left(\frac{2}{e^2-1}\right) &= \left(\frac{2}{e^2-1}\right)^2 \frac{e^2-1}{2} - 2\left(\frac{2}{e^2-1}\right) + \frac{1}{3} \\
&= \left(\frac{4}{(e^2-1)^2}\right)\frac{e^2-1}{2} - \frac{4}{e^2-1} + \frac{1}{3} \\
&= \frac{2}{e^2-1} - \frac{4}{e^2-1} + \frac{1}{3} \\
&= -\frac{2}{e^2-1} + \frac{1}{3} \\
&= -\frac{6+e^2+1}{3(e^2-1)} \\
&= \frac{e^2-7}{3(e^2-1)}
\end{aligned}$$

Good Luck for your exams!