Full Solutions
MATH103 April 2012

April 4, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

SOLUTION. The difference between the highest exponents is 1.1-0.1 $= 1$. By the series p test (with $p = 1$) this series **diverges**.
More precisely we can prove
$\sum_{n=1}^{\infty} \frac{n^{0.1}}{n^{0.99}+n^{1.1}+1} > \sum_{n=1}^{\infty} \frac{n^{0.1}}{n^{1.1}+n^{1.1}+n^{1.1}} = \sum_{n=1}^{\infty} \frac{n^{0.1}}{3n^{1.1}} = \frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n} = \infty$
The last sum diverges because of the series p test with p=*1*.

Question 1 (b)

SOLUTION. The difference between the highest exponents is 3 - 2 $= 1$. By the intergral p test (with $p = 1$) this integral diverges.
More precisely we estimate the integral with

$$\int_1^\infty \frac{x^2}{x^3+2x+5}dx > \int_1^\infty \frac{x^2}{x^3+2x^3+5x^3}dx = \frac{1}{8}\int_1^\infty \frac{1}{x}dx$$

Here we see exactly how to use the integral p test. We also can continue with the anti-derivative

$$\frac{1}{8}\int_1^\infty \frac{1}{x}dx = \frac{1}{8}\lim_{\alpha\to\infty}[\ln(x)]_1^\alpha = \frac{1}{8}\lim_{\alpha\to\infty}\ln(\alpha) = \infty$$

Question 1 (c)

SOLUTION. Since $0 \leq 1+\sin x \leq 2$ for all x, we use the integral comparison test to see that

$$\int_1^\infty \frac{1+\sin x}{x^2}dx \leq \int_1^\infty \frac{2}{x^2}\,dx$$

The latter integral converges, by the integral p test (with $p = 2$). Hence, again by the integral comparison test, the former integral also **converges**.

Question 1 (d)

SOLUTION. We use the substitution

$$u = \cos(x)$$

then we have

$$du = -\sin(x)\,dx$$

and

$$\sin^2(x) = 1 - u^2$$

Therefore

$$\sin^3(x)\,dx = (1-u^2)(-1)du = (u^2-1)du$$

Consequently

$$\int \sin^3(x)dx = \int (u^2-1)du = \frac{u^3}{3} - u = \frac{\cos^3(x)}{3} - \cos(x)$$

And so we can compute the integral

$$\begin{aligned}\int_0^\pi \sin^3(x)\,dx &= \frac{\cos^3(x)}{3} - \cos(x)\Big|_0^\pi \\ &= (-1/3+1) - (1/3-1) \\ &= 4/3\end{aligned}$$

and the correct answer is E.

Question 1 (e)

SOLUTION. Using integration by parts, we get

$$\int x^2e^{-x}dx = -x^2e^{-x} + \int 2xe^{-x}dx$$

And using integration by parts again on this integral we obtain

$$\int 2xe^{-x}dx = -2xe^{-x} + 2\int e^{-x}dx = -2xe^{-x} - 2e^{-x} + C$$

we get

$$\int x^2e^{-x}dx = (-x^2-2x-2)e^{-x} + C$$

Now that we have the antiderivative, we can compute the improper integral

$$\int_0^\infty x^2e^{-x}dx = \lim_{b\to\infty}\left((-x^2-2x-2)e^{-x}\Big|_0^b\right) = 2$$

And so the correct answer is E.

Question 1 (f)

SOLUTION 1. Recall that

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots,$$

therefore

$$x\ln(1-x) = -x^2 - \frac{x^3}{2} - \frac{x^4}{3} - \cdots$$

The coefficient of x^3 term is -1/2 and so the correct answer is D.

SOLUTION 2. If you don't remember the Taylor series about $x = 0$ of $\ln(1-x)$ you can simply re-compute it. Recall that the Taylor series of a function f about the point $x = a$ is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \ldots$$

For the function

$$f(x) = \ln(1-x)$$

we have

$$f'(a) = -\frac{1}{1-a} \quad f''(a) = -\frac{1}{(1-a)^2} \quad f^{(3)}(a) = -\frac{2}{(1-a)^3}$$

and so at $a = 0$ we obtain

$$f'(0) = -1 \quad f''(0) = -1 \quad f^{(3)}(x) = -2$$

And so the Taylor series of $\ln(1-x)$ around $x = 0$ is

$$-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + \ldots$$

And hence the Taylor series of $x\ln(1-x)$ around $x = 0$ is

$$-x^2 - \frac{1}{2}x^3 - \frac{1}{3}x^4 + \ldots$$

And we can observe that the x^3 coefficient is -1/2 which is the answer D.

Question 1 (g)

SOLUTION. The Taylor expansion of $\cos(x)$ is

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots$$

Keeping the terms up to x^2, we get

$$\frac{1-\cos(x)}{x} \approx \frac{x^2/2}{x} = \frac{x}{2}$$

Therefore

$$\int_0^1 \frac{1-\cos(x)}{x}\,dx \approx \int_0^1 \frac{x}{2}\,dx = \left.\frac{x^2}{4}\right|_0^1 = 1/4$$

So 1/4 is a good approximation. Note that taking more terms of the Taylor expansions of $\cos(x)$ would simply yield an even better approximation.

Question 2 (a)

SOLUTION. On the interval $[1,\infty)$ we have that $2x$-1 ≥ 1 and so

$$\frac{e^{-x}}{2x-1} \leq e^{-x}$$

for all $x \geq 1$. Therefore

$$\int_1^\infty \frac{e^{-x}}{2x-1}\,dx \leq \int_1^\infty e^{-x}\,dx$$

by the integral comparison test. Since the latter integral

$$\int_1^\infty e^{-x}\,dx$$

converges (see lecture notes or direct calculation), the former integral also converges, by the integral comparison test.

Question 2 (b)

SOLUTION. We solve for y to obtain

$$y = \pm a\sqrt{1-\frac{x^2}{b^2}}.$$

It suffices to take the values $y \geq 0$ and then multiply the resulting arc-length by 2. Then the intersection points with $y = 0$ are $x =$ -b and $x = b$. Taking the derivative of $y = a\sqrt{1-\frac{x^2}{b^2}}$ yields

$$y'(x) = \frac{1}{2}a\left(1-\frac{x^2}{b^2}\right)^{-1/2}\left(-\frac{2x}{b^2}\right) = -\frac{ax}{b^2\sqrt{1-\frac{x^2}{b^2}}}$$

Plugging into the formula for the arc length we obtain our final answer (don't forget the 2)

$$L = 2\int_{-b}^{b} \sqrt{1+(y'(x))^2}\,dx = \int_{-b}^{b} 2\sqrt{1+\frac{a^2x^2}{b^4(1-\frac{x^2}{b^2})}}\,dx.$$

Question 2 (c)

SOLUTION. The centre of mass is given by

$$\bar{x} = \frac{I}{M}$$

where

$$I = \int_0^\pi x \cos\left(\frac{x}{2}\right)\, dx$$
$$M = \int_0^\pi \cos\left(\frac{x}{2}\right)\, dx$$

We start with the simpler integral M and use the substitution $w = x/2$. Then $dw = dx/2$, or equivalently, $dx = 2dw$. Further, when $x = \pi$ we have $w = \pi/2$ and when $x = 0$ we have $w = 0$. The integral M therefore becomes

$$\begin{aligned} M &= \int_0^\pi \cos\left(\frac{x}{2}\right)\, dx \\ &= \int_0^{\pi/2} \cos(w) 2\, dw \\ &= 2\sin(w)|_0^{\pi/2} \\ &= 2\sin(\pi/2) - 2\sin(0) = 2. \end{aligned}$$

Next we calculate I. This is done using integration by parts: Choose $u = x$ and $dv = \cos(x/2)dx$. Then $du = dx$ and from the integral we just solve, we know that $v = 2\sin(w) = 2\sin(x/2)$. Hence

$$\begin{aligned} I &= \int_0^\pi x \cos\left(\frac{x}{2}\right)\, dx = 2x\sin(x/2)|_0^\pi - \int_0^\pi 2\sin(x/2)\, dx \\ &= 2\pi - 0 - (-\,4\cos(x/2)|_0^\pi) \\ &= 2\pi + 0 - 4 = 2\pi - 4. \end{aligned}$$

For the second integral we again used our favourite substitution $w = x/2$. The center of mass is thus

$$\bar{x} = \frac{I}{M} = \frac{2\pi - 4}{2} = \pi - 2.$$

Question 2 (d)

SOLUTION. This requires partial fractions. In order to use partial fractions, the denominator must be factored. One way to find the factors is to find the zeroes of the denominator, and then use them to create the factors. To do so we set $3x^2$-$4x$+$1 = 0$. Using the quadratic formula, we get the following two zeroes:

$$\frac{4 \pm \sqrt{16-12}}{6} = \frac{4 \pm 2}{6} = \frac{1}{3} \text{ or } 1.$$

These zeroes can then be put into factors as follows: $(x - 1/3)(x - 1)$
But because the first coefficient of the original denominator is *3*, we need to multiply these factors by 3, giving: $3(x - 1/3)(x - 1) = (3x - 1)(x - 1)$

If you're comfortable with factoring, instead of finding the zeroes and then generating the factors, you can also just factor the denominator directly.
Now we perform the decomposition into partial fractions:

$$\frac{2}{3x^2-4x+1} = \frac{2}{(3x-1)(x-1)} = \frac{A}{3x-1} + \frac{B}{x-1}.$$

When we try to recombine the final pair of fractions, we get:

$$\frac{A}{3x-1}\cdot\frac{(x-1)}{(x-1)} + \frac{B}{x-1}\cdot\frac{(3x-1)}{(3x-1)} = \frac{A(x-1)+B(3x-1)}{(3x-1)(x-1)}$$

Thus $2 = A(x\text{-}1)+B(3x\text{-}1)$. An easy way to find the values for A and B is to plug in $x = 1$ and $x = 1/3$. Plugging in x=1 yields B=1 and plugging in x=1/3 yields A = -3. We have now decomposed the fraction as follows:

$$\frac{2}{3x^2-4x+1} = \frac{-3}{3x-1} + \frac{1}{x-1}$$

Having now decomposed the expression using partial fractions, we can return to our original goal, which was to integrate the expression.

$$\begin{aligned}\int_2^4 \frac{2}{3x^2-4x+1}\,dx &= \int_2^4 \frac{2}{(3x-1)(x-1)}\,dx\\ &= \int_2^4 \left(\frac{-3}{3x-1} + \frac{1}{x-1}\right)dx\\ &= \left(\frac{-3}{3}\ln|3x-1| + \ln|x-1|\right)\Big|_2^4\\ &= -\ln(11)+\ln(3) - (-\ln(5)+\ln(1))\\ &= -\ln(11)+\ln(3)+\ln(5) = \ln\left(\frac{15}{11}\right).\end{aligned}$$

Question 2 (e)

SOLUTION. This game has 9 outcomes, but due to symmetry only 6 of them are distinct. The probabilities are distributed as follows (multiples of 1/64)

	pig 1 side	pig 1 feet	pig 1 snout
pig 2 side	4×4	4×3	4×1
pig 2 feet	3×4	3×3	3×1
pig 2 snout	1×4	1×3	1×1

Let X be the random variable that returns the score. Then

$$p(X=0) = \frac{16}{64} \qquad p(X=1) = 2\frac{12}{64} \qquad p(X=5) = 2\frac{4}{64}$$
$$p(X=3) = \frac{9}{64} \qquad p(X=7) = 2\frac{3}{64} \qquad p(X=10) = \frac{1}{64}$$

and thus the average μ is found to be

$$\begin{aligned}\mu &= 0p(X=0) + 1p(X=1) + 3p(X=3) + 5p(X=5) + 7p(X=7) + 10p(X=10)\\ &= 2\frac{12}{64} + 3\frac{9}{64} + 10\frac{4}{64} + 14\frac{3}{64} + 10\frac{1}{64} \quad = \frac{143}{64} \approx 2.23\end{aligned}$$

Question 2 (f)

SOLUTION. In order to see if $y(x)$ satisfies the differential equation, we need to find y', substitute y and y' into the left side of the differential equation, and see if the resulting expression is equal to *1*.
Thus, first, we calculate y'.

$$y'(x) = \frac{x\left(\frac{1}{x}+0\right) - (\ln(x)+2)}{x^2} = \frac{-1-\ln(x)}{x^2}$$

Then we plug y and y' into the left hand side of the differential equation.

$$\begin{aligned}x^2y' + xy &= x^2\left(\frac{-1-\ln(x)}{x^2}\right) + x\left(\frac{\ln(x)+2}{x}\right)\\ &= -1-\ln(x)+\ln(x)+2 = 1\checkmark\end{aligned}$$

Because this equals the right hand side of the original differential equation, the solution $y(x)$ is correct.

Question 3 (a)

SOLUTION. The equation of a circle with radius 1 is $x^2+y^2=1$ or, once we solve for y, $y=\sqrt{1-x^2}$ (which gives the upper half of the circle). Because we are rotating around the x-axis, we will be integrating between two points on the x-axis. The left endpoint is clearly *0*. The right endpoint is a little trickier: it will be the point where $y=x$ and $y=\sqrt{1-x^2}$ intersect. This point of intersection is found by setting these two curves equal and solving:

$$\begin{aligned}x &= \sqrt{1-x^2}\\ x^2 &= 1-x^2\\ x^2 &= \frac{1}{2}\\ x &= \pm\frac{1}{\sqrt{2}}\end{aligned}$$

.
Because we are in the first quadrant, the intersection point must be positive. Thus we will be integrating between 0 and $\frac{1}{\sqrt{2}}$.
If we use the Disc Method, the outside radius of the disc will be the arc of the circle, given by $y=\sqrt{1-x^2}$ and the inside radius will be given by the line $y=x$. Plugging these, and our endpoints, into the integral, we get:

$$
\begin{aligned}
V &= \int_0^{\frac{1}{\sqrt{2}}} \pi \left[\left(\sqrt{1-x^2} \right)^2 - (x)^2 \right] dx \\
&= \int_0^{\frac{1}{\sqrt{2}}} \pi \left[1 - 2x^2 \right] dx \\
&= \pi \left[x - \frac{2x^3}{3} \right]_0^{\frac{1}{\sqrt{2}}} \\
&= \pi \left[\frac{1}{\sqrt{2}} - \frac{1}{3\sqrt{2}} \right] \\
&= \frac{\sqrt{2}}{3} \pi
\end{aligned}
$$

Question 3 (b)

SOLUTION 1. Again using the Disc Method, this time we are rotating around the y-axis. Because A runs along the y-axis, this means that the inner radius of the disc will be 0. The outer radius of the disc will be given by first the line $x = y$, and then by the circle, which, when solved for x is given by $x = \sqrt{1-y^2}$. The first radius, $x = y$ will apply from $y = 0$ until the intersection point of the line and the circle. The second radius, $x = \sqrt{1-y^2}$ will apply from the intersection point to $y = 1$.
To find the intersection point of the line and circle, we use the same process as the previous question, setting the two equations equal to each other and solving for y. In this case, the intersection point will be at $y = \frac{1}{\sqrt{2}}$. We now have all the pieces we need to set up the integral and solve, as follows.

$$
\begin{aligned}
V &= \int_0^{\frac{1}{\sqrt{2}}} \pi y^2 dy + \int_{\frac{1}{\sqrt{2}}}^1 \pi \left(\sqrt{1-y^2} \right)^2 dy \\
&= \int_0^{\frac{1}{\sqrt{2}}} \pi y^2 dy + \pi \int_{\frac{1}{\sqrt{2}}}^1 1 - y^2 \, dy \\
&= \pi \left[\frac{y^3}{3} \Big|_0^{\frac{1}{\sqrt{2}}} + \pi \left[y - \frac{y^3}{3} \Big|_{\frac{1}{\sqrt{2}}}^1 \right.\right. \\
&= \pi \left[\frac{\left(\frac{1}{\sqrt{2}}\right)^3}{3} \right] + \pi \left[1 - \frac{1^3}{3} - \frac{1}{\sqrt{2}} + \frac{\left(\frac{1}{\sqrt{2}}\right)^3}{3} \right] \\
&= \frac{\pi}{6\sqrt{2}} + \pi \left[\frac{3}{3} - \frac{1}{3} - \frac{1}{\sqrt{2}} + \frac{1}{6\sqrt{2}} \right] \\
&= \frac{\pi}{6\sqrt{2}} + \pi \left[\frac{6\sqrt{2} - 2\sqrt{2} - 6 + 1}{6\sqrt{2}} \right] \\
&= \pi \left[\frac{1}{6\sqrt{2}} + \frac{4\sqrt{2} - 5}{6\sqrt{2}} \right] \\
&= 2\pi \left[\frac{\sqrt{2} - 1}{3\sqrt{2}} \right] \\
&= \pi \left[\frac{2 - \sqrt{2}}{3} \right]
\end{aligned}
$$

Solution 2. We can use the shell method to calculate the volume of this solid of revolution as well. If you don't know the shell method, don't worry about understanding this solution. Since we are using the shell method, we should integrate in the x-variable. The integral should be of the form

$$2\pi \int_a^b x(f(x) - g(x))dx$$

where a is the smallest x-value in the region (which will be zero), b is the largest x-value in the region (which was calculated in Problem 3a to be $\frac{1}{\sqrt{2}}$), f(x) is the largest y-value in the region at x (which will correspond to the top of each cylindrical shell), and g(x) is the smallest y-value in the region at x (which will correspond to the bottom of each cylindrical shell). So the volume is

$$\begin{aligned}
&2\pi \int_0^{\frac{1}{\sqrt{2}}} [x(\sqrt{1-x^2} - x)]dx \\
&= 2\pi \int_0^{\frac{1}{\sqrt{2}}} [x\sqrt{1-x^2}]dx - 2\pi \int_0^{\frac{1}{\sqrt{2}}} [x^2]dx
\end{aligned}$$

We now use a u-substitution u= 1 - xˆ2 in the first integral, and evaluate the second integral directly.

$$\begin{aligned}
&= \pi \int_{\frac{1}{2}}^1 [\sqrt{u}]du - 2\pi \frac{1}{3\sqrt{2}^3} \\
&= \frac{2\pi}{3}\left[u^{3/2}\right]_{\frac{1}{2}}^1 - \frac{2\pi}{6\sqrt{2}} \\
&= \frac{2\pi}{3} - \frac{2\pi}{3\sqrt{2}^3} - \frac{2\pi}{6\sqrt{2}} \\
&= \frac{2\pi}{3} - \frac{2\pi}{3\sqrt{2}} \\
&= \pi\left[\frac{2-\sqrt{2}}{3}\right].
\end{aligned}$$

Question 4 (a)

Solution. Following the instructions of the second hint, if we set dy/dt equal to zero and solve, we get:

$$0 = -\lambda t \left(\frac{y^2 - k^2}{y}\right)$$

This expression will be zero if $y^2 - k^2 = 0$, or alternatively, $y^2 = k^2$. This yields the two constant solutions of $y = +\ k$ and $y =$ -k.

Question 4 (b)

Solution. Separating variables and integrating as an indefinite integral,

$$\frac{y\,dy}{y^2-k^2} = -\lambda t\,dt$$
$$\int \frac{y\,dy}{y^2-k^2} = -\lambda \int t\,dt$$

We make a simple substitution in the y integral $u = y^2$-k^2, du = 2y dy to find

$$\int \frac{\frac{1}{2}du}{u} = -\lambda\frac{1}{2}t^2 + C_1$$
$$\frac{1}{2}\ln(|u|) = -\lambda\frac{1}{2}t^2 + C_1$$
$$\ln(|y^2-k^2|) = -\lambda t^2 + C_2,$$

with C_2=2C_1 is the constant of integration. Evaluating this at t=0, we find $C_2 = \ln(|y_0^2 - k^2|)$, hence

$$\ln(|y^2-k^2|) = \ln(|y_0^2-k^2|) - \lambda t^2$$
$$|y^2-k^2| = |y_0^2-k^2|e^{-\lambda t^2}$$
$$y(t)^2 = k^2 \pm |y_0^2-k^2|e^{-\lambda t^2}$$
$$y(t) = \pm\sqrt{k^2 \pm |y_0^2-k^2|e^{-\lambda t^2}}$$

To choose branches $\pm$ we need more specific initial conditions.

Question 4 (c)

SOLUTION. Substituting ${y_0}^2$=$(3k)^2$, we find

$$y(t) = \pm\sqrt{k^2 \pm |(3k)^2-k^2|e^{-\lambda t^2}}$$
$$= \pm\sqrt{k^2 \pm 8k^2e^{-\lambda t^2}}$$

When t=0, the argument to the square root is $k^2 \pm 8k^2$, which must be positive. This indicates that we need to take the positive branch inside the square root:

$$y(t) = \pm\sqrt{k^2 + 8k^2e^{-\lambda t^2}},$$

and because $y(0) > 0$, we need to take the positive branch outside the square root, giving the final answer

$$y(t) = \sqrt{k^2 + 8k^2e^{-\lambda t^2}}$$
$$= |k|\sqrt{1 + 8e^{-\lambda t^2}}$$
$$= k\sqrt{1 + 8e^{-\lambda t^2}}$$

where the last line holds as k is positive. As $t \to \infty$ we have that $y(t) \to +k$.

Question 4 (d)

SOLUTION. Using similar reasoning as the previous problem,

$$
\begin{aligned}
y(t) &= \pm\sqrt{k^2 \pm |(k/2)^2 - k^2|e^{-\lambda t^2}} \\
&= \pm\sqrt{k^2 \pm |-3/4|k^2 e^{-\lambda t^2}} \\
&= \pm k\sqrt{1 \pm 3/4e^{-\lambda t^2}}
\end{aligned}
$$

When t=0, $y(0)$>0, so we need to take the positive branch outside the square root:

$$y(t) = +|k|\sqrt{1 \pm 3/4e^{-\lambda t^2}} = k\sqrt{1 \pm 3/4e^{-\lambda t^2}},$$

again the last equality holding since k is positive. If we evaluate this at t=0, the positive branch gives $y(0) = k\sqrt{1 + 3/4} = \sqrt{7}k/2$, which is wrong, since $y(0)$=y_0=$k/2$. Hence this is the wrong branch. Indeed, the negative branch gives $y(0) = k\sqrt{1 - 3/4} = k/2$. Our answer is therefore

$$y(t) = k\sqrt{1 - 3/4e^{-\lambda t^2}},$$

As $t \to \infty$ we have that $y(t) \to k$.

Question 4 (e)

SOLUTION. Using similar reasoning as the previous problems,

$$y(t) = \pm k\sqrt{1 \pm 3/4e^{-\lambda t^2}}$$

When t=0, $y(0)$<0, so we need to take the positive branch outside the square root:

$$y(t) = -k\sqrt{1 \pm 3/4e^{-\lambda t^2}},$$

If we evaluate this at t=0, the positive branch gives $y(0) = -k\sqrt{7/4}$, which is wrong, since $y(0)$=y$_0$=-$k/2$. Hence this is the wrong branch, as before. The negative branch gives $y(0) = -k\sqrt{1 - 3/4} = -k/2$. Our answer is therefore

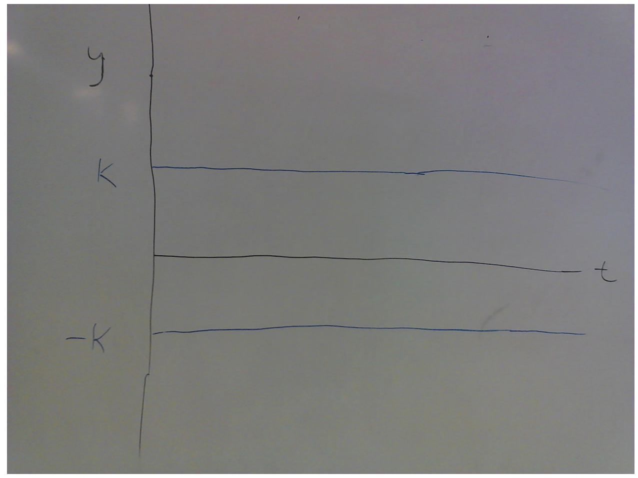
$$y(t) = -k\sqrt{1 - 3/4e^{-\lambda t^2}},$$

As $t \to \infty$ we have $y(t) \to -k$.

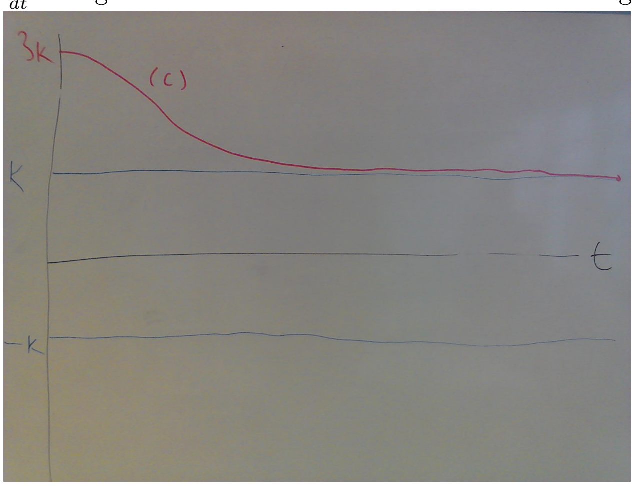
Question 4 (f)

SOLUTION. First, we graph the constant solutions to the differential equation

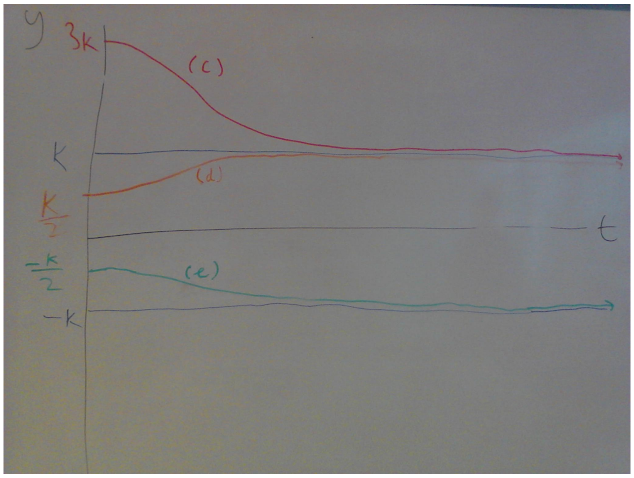
$$y = k$$



and $y = -k$. These are the solutions we got from part a.
Then, we graph the solution from part (c). Since 3k is larger than the constant solution k, it follows that the graph must stay above this line (since if y = k at any point t_0 , then y must be equal to k for all $t > t_0$). Furthermore, if this holds, then yˆ2 > kˆ2, so from the differential equation given in the problem statement, $\frac{dy}{dt}$ is negative and therefore the function is decreasing.



Now, we graph the solutions from part (d) and (e). Since $\frac{k}{2}$ and $-\frac{k}{2}$ are between -k and k, it follows that the graphs for each of these solutions must lie between the constant solutions y=-k and y=k. The graph from part d has $y(0) = \frac{k}{2}$. Note that for any $0 < y < \frac{k}{2}$, dy/dt is positive, so the solution to the differential equation given in (d) should be increasing. We know that the limit of the function as $t \to \infty$ is k. Similarly, if we plug a y-value such that -k < y < 0 into the differential equation, we see that the solution should be decreasing, so the graph of the solution described in part e should decrease to its limit -k.



Question 5 (a)

SOLUTION. In order to satisfy the second condition given in the hint, we must have the following equality hold:

$$1 = \int_0^{\sqrt{3}} \frac{k}{1+x^2}\,dx$$

However, this is easy enough to solve, especially if we remember that $\int \frac{1}{1+x^2}\,dx = \arctan x$.

$$1 = \int_0^{\sqrt{3}} \frac{k}{1+x^2}\,dx = k(\arctan\sqrt{3} - \arctan(0)) = \frac{k\pi}{3},$$

hence $k = \frac{3}{\pi}$.
(Or $k = \frac{1}{\arctan\sqrt{3}}$, we used $\arctan(\sqrt{3}) = \pi/3$.)

Question 5 (b)

SOLUTION. Using the value of k we found in the previous question, and the formula for the mean given in the hint, we know the mean is found using the following integral.

$$\int_0^{\sqrt{3}} \frac{3}{\pi}\frac{t}{1+t^2}\,dt$$

We can then solve the integral to find the mean.

$$\begin{aligned}\mu &= \int_0^{\sqrt{3}} \frac{3}{\pi}\frac{t}{1+t^2}\,dt \\ &= \frac{3}{2\pi}\ln(1+t^2)\Big|_0^{\sqrt{3}} \\ &= \frac{3}{2\pi}\ln(4) \\ &= \frac{3\ln(2)}{\pi}\end{aligned}$$

(or $\mu = \frac{\ln(2)}{\arctan\sqrt{3}}$).

Question 5 (c)

SOLUTION. Using the formula given in the hint, we know that the median m must satisfy the integral:

$$\frac{1}{2} = \int_0^m \frac{3}{\pi}\frac{1}{1+t^2}\,dt$$

Thus, we evaluate the integral above and then solve for m.

$$\begin{aligned}\frac{1}{2} &= \int_0^m \frac{3}{\pi}\frac{1}{1+t^2}\,dt \\ &= \frac{3}{\pi}\arctan(m)\end{aligned}$$

hence $m = \tan(\pi/6) = 1/\sqrt{3}$.
(Or $m = \tan(\frac{\arctan\sqrt{3}}{2})$.)

Good Luck for your exams!