Full Solutions
MATH105 April 2012

April 4, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

SOLUTION. We can apply the ratio test here. Let $a_k = (-1)^k 2^{k+1} x^k$, so that the series converges absolutely if
$\lim_{k\to\infty} |\frac{a_{k+1}}{a_k}| < 1$.
In this case, we require
$\lim_{k\to\infty} |\frac{(-1)^{k+1}2^{k+2}x^{k+1}}{(-1)^k 2^{k+1}x^k}| = |(-1)(2)(x)| = 2|x| < 1$ and hence absolute convergence is obtained for $|x| < 1/2$.
The radius of convergence is 1/2.

Question 1 (b)

SOLUTION. The trick here is to recognize the series given as a geometric series.
The question provides the formula $1 + r + r^2 + ... = \sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ for $|r| < 1$.
Essentially, would like to evaluate the series by finding a suitable r so that this formula can be applied.
Let's first re-express $\sum_{k=0}^{\infty}(-1)^k 2^{k+1}x^k = \sum_{k=0}^{\infty} 2(-1)^k 2^k x^k$ as $2\sum_{k=0}^{\infty}(-1)^k 2^k x^k$ so that all terms in the summation have an exponent of k and the indexing starts at 0.
Now we note that $2\sum_{k=0}^{\infty}(-1)^k 2^k x^k = 2\sum_{k=0}^{\infty}[(-1)(2)(x)]^k = 2\sum_{k=0}^{\infty}(-2x)^k$.
The formula provided now works with $r = -2x$. The series sums to $2 \times \frac{1}{1-(-2x)} = \frac{2}{1+2x}$.

Question 1 (c)

SOLUTION. For $|r|$<1 we use that

$$\frac{1}{1-r} = 1 + r + r^2 + r^3 + \dots$$

and integrate both sides to obtain

$$\begin{aligned}
-\ln|1-r| &= \int \frac{1}{1-r}dr \\
&= \int (1 + r + r^2 + r^3 + \dots)dr \\
&= r + r^2/2 + r^3/3 + r^4/4 + ... + C
\end{aligned}$$

Since $|r| < 1$ we have $|$1-$r| =$ 1-r, therefore

$$-\ln(1-r) = r + r^2/2 + r^3/3 + r^4/4 + \cdots + C$$

To solve for the integration constant C we plug in $r = 0$:

$$\begin{aligned}
-\ln(1-r) &= -\ln 1 = 0 \\
&= 0 + 0^2/2 + 0^3/3 + 0^4/4 + ... + C = C
\end{aligned}$$

so $C = 0$.
So we obtain the intermediate result that

$$\ln(1-r) = -r - r^2/2 - r^3/3 - r^4/4 + \cdots = -\sum_{k=1}^{\infty} r^k/k.$$

To arrive at the summation notation, we noted that every term has the same sign (so there are no terms like $(-1)^k$ floating around), every integer power of r is included starting from r^1, and each r^k is divided by k. As a last step, the Taylor series for $\ln(1+2x)$ can be found by replacing r by $-2x$ in the series above. We find:

$$\ln(1+2x) = -\sum_{k=1}^{\infty} \frac{(-2x)^k}{k}.$$

Question 2 (a)

SOLUTION. When we look at this series, all its terms are positive, and they are clearly decreasing

$$e^{-\sqrt{k}}$$

is getting smaller and smaller, as is $1/\sqrt{k}$.
The integral test can be applied here. We know that

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} e^{-\sqrt{k}} \text{ converges if and only if } I = \int_1^{\infty} \frac{1}{\sqrt{x}} e^{-\sqrt{x}} dx \text{ converges.}$$

The integral I is improper (the infinite range of integration), but it can be evaluated with substitution. Set $u = \sqrt{x}$ then $du = \frac{1}{2\sqrt{x}} dx$, and therefore $\frac{1}{\sqrt{x}} e^{-\sqrt{x}} dx = 2e^{-u} du$.

$$I = \lim_{R\to\infty} \int_1^R \frac{1}{\sqrt{x}} e^{-\sqrt{x}} dx = \lim_{R\to\infty} \int_1^R 2e^{-u} du = \lim_{R\to\infty} -2e^{-u}|_1^R = 2e^{-1}$$

From the above computation, I converges, and so does the original series.

Question 2 (b)

SOLUTION. When we look at this series, we note that the leading power of the numerator is k^4 and the leading power of the denominator is k^5. Thus, for k very large, we would expect the terms in the series to look like $k^4/k^5 = 1/k$.
We will use the limit comparison test with

$$a_k = \frac{k^4 - 2k^3 + 2}{k^5 + k^2 + k} \text{ and } b_k = \frac{1}{k}$$

We compute

$$\begin{aligned}
\lim_{k\to\infty} \frac{a_k}{b_k} &= \lim_{k\to\infty} \left(\frac{k^4 - 2k^3 + 2}{k^5 + k^2 + k}\right) / \left(\frac{1}{k}\right) \\
&= \lim_{k\to\infty} \frac{k^5 - 2k^4 + 2k}{k^5 + k^2 + k} \\
&= \lim_{k\to\infty} \frac{k^5/k^5 - 2k^4/k^5 + 2k/k^5}{k^5/k^5 + k^2/k^5 + k/k^5} \\
&= \lim_{k\to\infty} \frac{1 - 2/k + 2/k^4}{1 + 1/k^3 + 1/k^4} \\
&= \frac{1 - 0 + 0}{1 + 0 + 0} = 1
\end{aligned}$$

As this limit is nonzero and finite, $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=1}^{\infty} b_k$ converges.
The series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges (it's worth remembering - or being able to show with the integral test - that $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if $p > 1$ and diverges if $p \leq 1$).
The conclusion is that the series diverges!

Question 2 (c)

SOLUTION. Series of this form suggest the ratio test: given powers such as 2^k and factorials, we know that in taking the ratio of successive terms, many simplifications are possible.
Let us denote $a_k = \frac{2^k (k!)^2}{(2k)!}$ so that $a_{k+1} = \frac{2^{k+1}((k+1)!)^2}{(2(k+1))!}$.
The ratio test requires us to compute the limit

$$L = \lim_{k\to\infty} |\frac{a_{k+1}}{a_k}|$$

Theory tells us that

- if $0 \leq L < 1$ then the series converges;
- if $L = 1$, we cannot conclude anything from the test;
- if $L > 1$ then the series diverges.

We need to be careful with the factorials (!). A useful thing to recall is that $(k+1)! = (k+1)k!$.
We will compute:

$$\begin{aligned}
L = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty} \left|\frac{\frac{2^{k+1}((k+1)!)^2}{(2(k+1))!}}{\frac{2^k (k!)^2}{(2k)!}}\right| \\
&= \lim_{k\to\infty} \frac{2^{k+1}((k+1)!)^2(2k)!}{2^k(k!)^2(2(k+1))!} \\
&= \lim_{k\to\infty} \frac{2^{k+1-k} \times [(k+1)k!]^2 \times (2k)!}{(k!)^2 \times (2k+2)!} \\
&= \lim_{k\to\infty} \frac{2(k+1)^2(2k)!}{(2k+2)!} \\
&= \lim_{k\to\infty} \frac{2(k+1)^2(2k)!}{(2k+2)(2k+1)(2k)!} \\
&= 2\lim_{k\to\infty} \frac{(k+1)^2}{(2k+2)(2k+1)} \\
&= 2\lim_{k\to\infty} \frac{k^2+2k+1}{4k^2+6k+2} \\
&= 2\lim_{k\to\infty} \frac{k^2/k^2+2k/k^2+1/k^2}{4k^2/k^2+6k/k^2+2/k^2} \\
&= 2(\frac{1}{4}) = \frac{1}{2} < 1
\end{aligned}$$

As $0 \leq L < 1$, we determine the series converges.

Question 3 (a)

SOLUTION. The integral will become less complicated if we make a substitution $u = \ln(x)$.

First substitution If $u = \ln(x)$ then $du = (1/x)\,dx$. We would like to express dx in terms of u and du. Isolating dx gives $dx = x\,du$. Solving for x in $u = \ln(x)$ gives $x = e^u$ and hence $dx = e^u\,du$. Therefore,

$$I = \int \sin(\ln(x))\,dx = \int e^u \sin(u)\,du$$

First integration by parts To find this integral, we will need to integrate by parts twice.
Let $f = e^u$ and $dg = \sin(u)\,du$. Then $df = e^u\,du$ and $g = -\cos(u)$. Thus,

$$\begin{aligned} I &= \int f\,dg \\ &= fg - \int g\,df \\ &= e^u(-\cos(u)) - \int -\cos(u)e^u\,du \\ &= -e^u\cos(u) + \int e^u\cos(u)\,du \end{aligned}$$

Second integration by parts We repeat the process on the second integral. This time, we let $v = e^u$ and $dw = \cos(u)\,du$. Thus, $dv = e^u du$ and $w = \sin(u)$. Now we find:

$$\begin{aligned} I &= -e^u\cos(u) + \int v\,dw \\ &= -e^u\cos(u) + vw - \int w\,dv \\ &= -e^u\cos(u) + e^u\sin(u) - \int \sin(u)e^u\,du \end{aligned}$$

Inspect the result We recognize the integral above as I and we can now solve for I:

$$I = -e^u\cos(u) + e^u\sin(u) - I$$

so

$$2I = -e^u\cos(u) + e^u\sin(u)$$

Then

$$I = \frac{1}{2}e^u(\sin(u) - \cos(u)) + C$$

We added the arbitrary constant C because we are computing an indefinite integral.

Bring x back Finally, the integral we desire can be found by replacing u by $\ln(x)$ so that $e^u = x$:

$$I = \frac{1}{2}x(\sin(\ln(x)) - \cos(\ln(x))) + C$$

Question 3 (b)

SOLUTION. One way to evaluate $\int_0^1 \frac{1}{x^2-5x+6}dx$ is to use partial fractions.
We note that $x^2 - 5x + 6 = (x-2)(x-3)$, and so we decompose the fraction into:

$$\frac{1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$$

which becomes

$$1 = A(x-3) + B(x-2)$$

after multiplying through by the common denominator of $(x-2)(x-3)$. If we expand the equation out and find the coefficients of the respective powers of x then:

$$0x + 1 = (A+B)x + (-3A - 2B)$$

- From matching the x coefficients, $A + B = 0$ so $A = -B$.
- From matching the constant coefficients, $-3A - 2B = 1$ which becomes $-3(-B) - 2B = 1$ in using the relation above.

This tells us that $B = 1$ and therefore $A = -1$.
The integral is easy to evaluate now:

$$\begin{aligned}\int_0^1 \frac{1}{x^2-5x+6}dx &= \int_0^1 (\frac{1}{x-3} - \frac{1}{x-2})dx \\ &= [\ln|x-3| - \ln|x-2|]|_0^1 \\ &= (\ln 2 - \ln 1) - (\ln 3 - \ln 2) \\ &= 2\ln 2 - \ln 3\end{aligned}$$

We used the fact that $\ln 1 = 0$ above.

Question 4 (a)

SOLUTION. Following the definition of a cdf (see the hint), we compute

- $\lim_{x\to-\infty} F(x) = a$ so $a = 0$

and

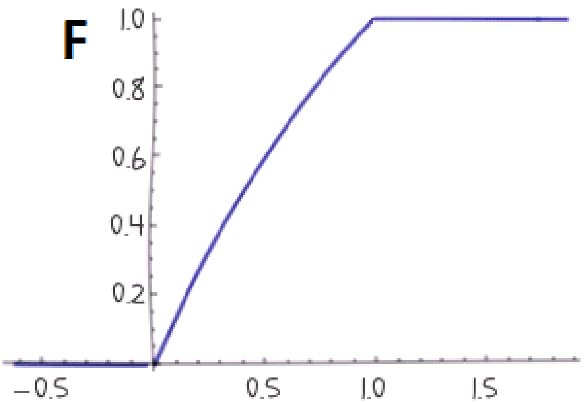
- $\lim_{x\to\infty} F(x) = b$ so $b = 1$.
- To impose continuity at $x = 1$, we need $\lim_{x\to1} F(x) = F(1) = b = 1$. Obviously $\lim_{x\to1^+} F(x) = 1$. So looking at the left limit will give us k:

$$\lim_{x\to1^-} F(x) = \lim_{x\to1^-} k\arctan x = k\arctan 1$$

Therefore $k = 1/(\arctan 1)$", or simply $k = \frac{4}{\pi}$.

- Since arctan is non-decreasing, so is F.

Hence our final answer is
$a = 0, \quad b = 1, \quad k = \frac{4}{\pi}$.



As a remark, it is important to know the exact trig values for special angles. Here

$$\tan \frac{\pi}{4} = 1$$

so $\arctan 1 = \frac{\pi}{4}$.

Question 4 (b)

SOLUTION. The cumulative distribution is obtained by integrating the probability density function, hence the probability density function (pdf) is the derivative of the cumulative distribution function.
From part (a),

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ \frac{4}{\pi} \arctan x & \text{if } 0 \le x \le 1, \\ 1 & \text{if } x \ge 1. \end{cases}$$

hence the pdf is the derivative:

$$f(x) = F'(x) = \begin{cases} 0 & \text{if } x < 0, \\ \frac{4}{\pi} \frac{1}{1+x^2} & \text{if } 0 < x < 1, \\ 0 & \text{if } x > 1. \end{cases}$$

If you are really picky, you might be concerned about the endpoints. Technically the derivative is not defined at the points 0 and 1 (due to "corners"); however, for continuous random variables, the probabilities will be unaffected by isolated points where the derivative of the cumulative distribution function is undefined. How we defined the pdf $f(x)$ above is completely fine.

Question 5 (a)

SOLUTION. By the Fundamental Theorem of Calculus, if

$$F(x) = \int_0^x \ln(2 + \sin t) dt$$

then

$$F'(x) = \ln(2 + \sin x).$$

Similarly,

$$G(y) = \int_y^0 \ln(2 + \sin t)dt = -\int_0^y \ln(2 + \sin t)dt$$

so the Fundamental Theorem of Calculus tells us that

$$G'(y) = -\ln(2 + \sin y).$$

We now can evaluate

$$F'(\pi/2) = \ln(2 + \sin(\pi/2)) = \ln(2 + 1) = \ln 3$$

and

$$G'(\pi/2) = -\ln(2 + \sin(\pi/2)) = -\ln(2 + 1) = -\ln 3.$$

Question 5 (b)

Solution. We write H as a sum of F and G:

$$H(x, y) = \int_y^0 \ln(2 + \sin t)dt + \int_0^x \ln(2 + \sin t)dt = G(y) + F(x).$$

Then

$$H_x = \frac{\partial}{\partial x}(F(x) + G(y)) = F'(x) + 0 = \ln(2 + \sin(x))$$

and similarly

$$H_y = \frac{\partial}{\partial y}(F(x) + G(y)) = 0 + G'(y) = -\ln(2 + \sin(y))$$

where we used the knowledge about the derivatives *F'(x)* and *G'(y)* from part (a). Finally, all that is left to do is to plug in the given values to find that

$$\nabla H\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = \left\langle F'\left(\frac{\pi}{2}\right), G'\left(\frac{\pi}{2}\right)\right\rangle = \langle \ln 3, -\ln 3\rangle$$

Question 6 (a)

SOLUTION. We wish to maximize the revenue $f(p,q) = pq$ subject to the constraint $g(p,q) = p^2+4q^2 = 800$.

Question 6 (b)

SOLUTION. We are maximizing $f(p,q) = pq$ subject to $g(p,q) = p^2 + 4q^2 = 800$. Solving the constraint for zero we obtain $h(p,q) = p^2 + 4q^2 - 800 = 0$, so that the Lagrange system is

$$\nabla f(p,q) = \lambda \nabla h(p,q)$$

Computing the gradients,

$$f_p = (pq)_p = q \qquad f_q = (pq)_q = p$$

Also,

$$h_p = (p^2 + 4q^2)_p = 2p \qquad h_q = (p^2 + 4q^2)_q = 8q$$

Therefore,

$$< q, p >= \lambda < 2p, 8q >$$

which gives us two equations, namely:

$$\begin{aligned} q &= 2\lambda p \quad \text{(equation 1)} \\ p &= 8\lambda q \quad \text{(equation 2)} \end{aligned}$$

In substituting $q = 2\lambda p$ into $p = 8\lambda q$, we get

$$p = 8\lambda(2\lambda p) = 16\lambda^2 p$$

We can factor this to find $p(1 - 16\lambda^2) = 0$ so that either

1. $p = 0$
2. $1 - 16\lambda^2 = 0$, i.e. $\lambda = \pm 1/4$.

1. If $p = 0$ then $q = 2\lambda(0) = 0$, and $p = q = 0$ cannot satisfy the constraint. **We reject this case.**
2. If $\lambda = \pm 1/4$ then $q = 2(\pm 1/4)p = \pm p/2$. Substituting this into the constraint gives:

$$0 = p^2 + 4(\pm p/2)^2 - 800 = 2p^2 - 800$$

which mean $p^2 = 400$, or $p = \pm 20$.
We take the positive root (since price should be positive). At a price of \$20 (with a demand of 10/day), the revenue is maximized.
Aside: it wouldn't make sense for either p or q to be 0 (there would be no revenue), so we can safely divide the equations (1) and (2) without the risk of dividing by 0. Or to look at it another way: if $q = 0$ then from the equations that forces p=0 (and vice versa), and p=q=0 could not satisfy the constraint. If we divide the equations then $q/p = (2\lambda p)/(8\lambda q) = p/(4q)$. Cross-multiplying gives the relation $p^2 = 4q^2$ which could be used in the constraint equation to find p and q.

Question 7 (a)

SOLUTION. To find the critical points of $f(x, y) = xye^y + \frac{1}{2}x^2 - 2$, we need to set both partial derivatives to 0:

$$f_x = ye^y + x = 0$$

and, **using the product rule**,

$$f_y = xe^y + xye^y = x(e^y + ye^y) = 0$$

To solve the simultaneous system, we should first factor the equations as best we can (this makes it easier to find conditions that are necessary for the derivatives to vanish).
Nothing can be done for the first equation so we leave it:

$$(1) \qquad f_x = ye^y + x = 0$$

For the second equation, we write:

$$(2) \qquad f_y = xe^y(1 + y) = 0$$

It's easier to work with the equation above first. From above, we consider the possibilities that:

- $x = 0$,
- $e^y = 0$ (this is impossible), or
- $1+y = 0$, which happens when $y = -1$.

Let's now use the possibilities above in the equation (1): $f\text{x} = ye^y + x = 0$.

- If $x = 0$, then $ye^y + 0 = 0$ implies $ye^y = 0$, so $y = 0$ (recall $e^y \neq 0$). **Thus,** $(0, 0)$ **is a critical point.**
- If $y = -1$ then $-e^{-1} + x = 0$ implies $x = e^{-1}$. **Thus,** $(e^{-1}, -1)$ **is a critical point.**

Question 7 (b)

SOLUTION. To classify the critical points, we use the second derivative test.
From $f_x = ye^y + x$, we compute:

$$f_{xx} = 1$$

and

$$f_{xy} = e^y + ye^y = e^y(1 + y)$$

From $f_y = xe^y(y + 1)$, we find:

$$f_{yy} = x[e^y(y + 1) + e^y] = xe^y(y + 2)$$

Hence,

$$D = f_{xx}f_{yy} - f_{xy}^2 = xe^y(y + 2) - e^{2y}(1 + y)^2$$

- At (0,0), we find $D = 0 - 1 = -1 < 0$. As $D < 0$, we have a **saddle point**.
- At $(e^{-1}, -1)$, we find $D = e^{-2} - 0 > 0$, so there is either a local maximum or local minimum. From $f_{xx}(e^{-1}, -1) = 1 > 0$, it must be a **local minimum**.

Question 8 (a)

SOLUTION. To evaluate $S = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$, we need to make it look more like $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$ for some x.
Both sums start with $n = 0$, have $(-1)^n$, and have a $2n+1$ in the denominator. Therefore, we need that $x^{2n+1} = \frac{1}{3^n}$. Following the hint, we write $\frac{1}{3^n} = \frac{1}{(\sqrt{3})^{2n}}$ which we can make even more similar to x^{2n+1} by writing it as $(\frac{1}{\sqrt{3}})^{2n}$.
Let's write out the two sums again and see how close we are:
$\arctan x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$ and we wish to evaluate $S = \sum_{n=0}^{\infty}(-1)^n \frac{(1/\sqrt{3})^{2n}}{2n+1}$.
If $x = 1/\sqrt{3}$, the only difference between the two sums now is a factor of $(1/\sqrt{3})^1$, which doesn't depend on n. We can multiply S by $1/\sqrt{3}$ (and put that factor in the sum) and also divide S by that same factor (keeping it outside the sum):

$$\begin{aligned}
S &= \frac{1}{1/\sqrt{3}} \sum_{n=0}^{\infty}(-1)^n \frac{(1/\sqrt{3})^{2n+1}}{2n+1} \\
&\overset{\text{recognize sum as} \arctan(1/\sqrt{3})}{=} \sqrt{3}\arctan(1/\sqrt{3}) \\
&= \sqrt{3}\frac{\pi}{6}
\end{aligned}$$

Question 8 (b)

SOLUTION. We wish to find the limit of the sequence $a_k = \frac{k!\sin^3(k)}{(k+1)!}$.
Note: this is a sequence, not a series! The ratio test does not apply here, despite the fact we see factorials.
Let's first simplify this a little. Note that $(k+1)! = (k+1)k!$, so really

$$a_k = \frac{k!\sin^3(k)}{(k+1)k!} = \frac{\sin^3(k)}{k+1}$$

We now observe that the denominator goes to infinity for large values of k, while the enumerator remains bounded between -1 and +1. Mathematically, $-1 \leq \sin^3(k) \leq 1$. Hence, we expect the sequence to converge to zero.
To prove this, we use the squeeze theorem:
$\lim_{k\to\infty} \frac{-1}{k+1} \leq \lim_{k\to\infty} \frac{\sin^3(k)}{k+1} \leq \lim_{k\to\infty} \frac{1}{k+1}$.
The first and third limits are both 0, hence
$0 \leq \lim_{k\to\infty} \frac{\sin^3(k)}{k+1} \leq 0$
so the squeeze theorem tells us
$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{\sin^3(k)}{k+1} = 0$.

Question 8 (c)

SOLUTION. We begin by rewriting the equation by using exponent and logarithm rules:

$\frac{dy}{dx} = xe^{x^2 - \ln(y^2)} = xe^{x^2}e^{-\ln(y^2)} = xe^{x^2}e^{\ln(y^{-2})} = xe^{x^2}\frac{1}{y^2}$.
From $\frac{dy}{dx} = xe^{x^2}\frac{1}{y^2}$, we can separate variables:
$y^2 dy = xe^{x^2} dx$.
Now we integrate:

$$\int y^2 dy = \int xe^{x^2} dx \implies \frac{y^3}{3} = \frac{1}{2}e^{x^2} + C_1.$$

The second integral was solved using the substitution $u = x^2$, so that $du = 2xdx$ and hence

$$\int xe^{x^2} dx = \int \frac{1}{2}e^u du = \frac{1}{2}e^u + C_1 = \frac{1}{2}e^{x^2} + C_1$$

We can solve for y:

$$y^3 = \frac{3}{2}e^{x^2} + 3C$$

so $y = \sqrt[3]{\frac{3}{2}e^{x^2} + 3C_1}$
Since C_1 is arbitrary, we can just as well write

$$y = \sqrt[3]{\frac{3}{2}e^{x^2} + C_2}$$

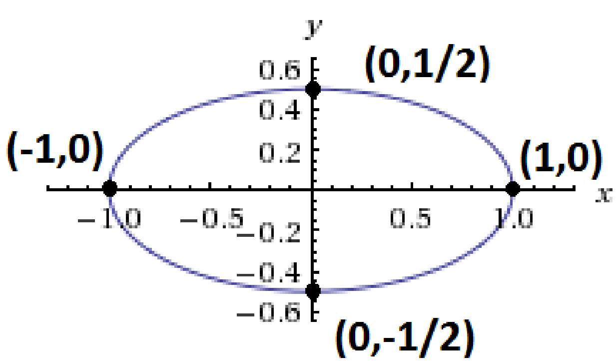
for C_2 an arbitrary constant.

Question 8 (d)

SOLUTION. If $z = e$ then
$e = e^{x^2+4y^2}$ and by taking the logarithm of both sides:

$$1 = x^2 + 4y^2$$

This is an ellipse.
The four easiest points are the intercepts.
If $x = 0$ then $4y^2 = 1$ and $y = \pm 1/2$ are the y-intercepts. If $y = 0$ then $x^2 = 1$ and $x = \pm 1$ are the x-intercepts. The sketch is below:



Question 8 (e)

SOLUTION. We consider all possible outcomes.
Draw -1 from bin 1 and 0 from bin 2

$$X = 2(-1) + 3(0) = -2.$$

Draw -1 from bin 1 and 2 from bin 2

$$X = 2(-1) + 3(2) = 4.$$

Draw +1 from bin 1 and 0 from bin 2

$$X = 2(1) + 3(0) = 2.$$

Draw +1 from bin 1 and 2 from bin 2

$$X = 2(1) + 3(2) = 8.$$

All possible values of X are: -2, 2, 4, 8.
Each outcome is equally likely since drawing from the bins is independent and within each bin the balls have equal probability (this isn't mentioned, but it seems like a reasonable assumption). Therefore, $\Pr(X = -2) = \Pr(X = 2) = \Pr(X = 4) = \Pr(X = 8) = 1/4$ (since the sum of all 4 probabilities must be 1).
The expected value of X is

$$\mathbb{E}(X) = -2/4 + 2/4 + 4/4 + 8/4 = 3.$$

Question 8 (f)

SOLUTION. The error bound for Simpson's rule requires us to find the fourth derivative of the integrand. For $f(x) = e^{-2x} + 3x^3$, we find:

$$f'(x) = -2e^{-2x} + 9x^2$$

$$f''(x) = 4e^{-2x} + 18x$$

$$f'''(x) = -8e^{-2x} + 18$$

and

$$f^{(4)}(x) = 16e^{-2x}$$

To find our K, we need to know the largest $|f^{(4)}|$ can be over the integration range $[0, 1]$.
$|f^{(4)}(x)| = 16e^{-2x}$ is a decreasing function, with its largest value on $[0, 1]$ being 16 at $x = 0$, so $K = 16$.
The error with $n = 6$ is bounded by $\frac{K(b-a)^5}{180n^4} = \frac{16(1-0)^5}{180\times6^4} = \frac{16}{180\times6^4}$.

Note: the question did not ask us to evaluate the $n = 6$ Simpson's rule approximation; we were only asked to bound its error.

Question 8 (g)

SOLUTION. We wish to find $f(x)$ so that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2}\sqrt{1-\frac{k^2}{n^2}} = \int_0^1 f(x)dx.$$

This requires making the limit of the Riemann sum (the expression on the left) look more like:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \Delta x f(x_k).$$

Our interval is $[a, b] = [0, 1]$ and this makes $\Delta x = \frac{1-0}{n} = \frac{1}{n}$. We also know $x_k = 0 + k\Delta x = \frac{k}{n}$. Let's rewrite the sum in terms of Δx and x_k:

$$\begin{aligned}
\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2}\sqrt{1-\frac{k^2}{n^2}} &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\frac{k}{n}\sqrt{1-\frac{k^2}{n^2}} \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \Delta x(k\Delta x)\sqrt{1-(k\Delta x)^2} \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} (\Delta x)x_k\sqrt{1-x_k^2}.
\end{aligned}$$

In comparing what we want, $\lim_{n\to\infty} \sum_{k=1}^{n} \Delta x f(x_k)$, with $\lim_{n\to\infty} \sum_{k=1}^{n}(\Delta x)x_k\sqrt{1-x_k^2}$, we must $f(x) = x\sqrt{1-x^2}$.
Therefore $f(x) = x\sqrt{1-x^2}$.

Question 8 (h)

SOLUTION. To evaluate $I = \int \frac{\sqrt{25x^2-4}}{x}dx$, we will use trigonometric substitution.
The part with the square root is $25x^2 - 4$. We only have freedom over what we do to x, and we'd like to use a trig identity to help deal with the square root.
We recall $\sec^2\theta - 1 = \tan^2\theta$. If $25x^2 = 4\sec^2\theta$ then $\sqrt{25x^2-4} = \sqrt{4\tan^2\theta} = 2\tan\theta$. Let's make the change of variables $x = \frac{2}{5}\sec\theta$ (by solving $25x^2 = 4\sec^2\theta$ for x.
If $x = \frac{2}{5}\sec\theta$ then $dx = \frac{2}{5}\tan\theta\sec\theta d\theta$:

$$I = \int \frac{\overbrace{2\tan\theta}^{\sqrt{25x^2-4}}}{\underbrace{\frac{2}{5}\sec\theta}_{x}} \underbrace{\frac{2}{5}\tan\theta\sec\theta d\theta}_{dx} = 2\int \tan^2\theta d\theta.$$

Now, we can use a trig identity:
$I = 2\int\tan^2\theta d\theta = 2\int(\sec^2\theta - 1)d\theta = 2(\tan\theta - \theta) + C.$
Finally, we need to go back in terms of x.
If $x = \frac{2}{5}\sec\theta$ then $\sec\theta = \frac{5}{2}x$ and $1+\tan^2\theta = \sec^2\theta = \frac{25}{4}x^2$. This means we can replace $\tan\theta$ by $\sqrt{\frac{25}{4}x^2 - 1}$.
Also, $\theta = \text{arcsec}(\frac{5}{2}x)$.
Thus, our final answer is:

$I = 2(\sqrt{\frac{25}{4}x^2 - 1} - \text{arcsec}(\frac{5}{2}x)) + C.$

Question 8 (i)

SOLUTION. For a plane parallel to $3x - y + 4z = 13$, we can take its normal vector to be $< 3, -1, 4 >$ (two parallel planes have normal vectors that are scalar multiples of each other and we can just as well choose their normal vectors to be identical).
Thus, the plane we seek has equation $3(x - x_0) + (-1)(y - y_0) + 4(z - z_0) = 0$ where (x_0, y_0, z_0) is any point on the plane. We are told the plane passes through $(2, 1, -1)$ so we take $x_0 = 2$, $y_0 = 1$, and $z_0 = -1$ and the plane's equation is:
$3(x - 2) - (y - 1) + 4(z - (-1)) = 0$ or equivalently, $3x - y + 4z = 1$.

Good Luck for your exams!