Full Solutions
MATH110 April 2012

December 6, 2014

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Educational Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Exam Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Educational Resources.

Question 1

Easiness: 85/100

SOLUTION. To find the equation of the tangent line we need a point and a slope. We already have the point, so it remains to find the slope of the tangent line. We will do so by using implicit differentiation.
Implicitly differentiating $(x^2+y^2)^2 = 18(x^2-y^2)$ gives:

$$2(x^2+y^2)(2x+2y\cdot y') = 18(2x-2y\cdot y')$$

We could solve for y' and then plug in the point $(2,\sqrt{2})$ to find the slope. However, it might be easier to plug in the point first and then solve for y', which is what I will do here:

$$\begin{aligned}
2((2)^2+(\sqrt{2})^2)(2(2)+2\sqrt{2}\cdot y') &= 18(2(2)-2\sqrt{2}\cdot y')\\
8+4\sqrt{2}\cdot y' &= 12-6\sqrt{2}\cdot y'\\
4\sqrt{2}\cdot y'+6\sqrt{2}\cdot y' &= 12-8\\
(10\sqrt{2})y' &= 4\\
y' &= \frac{2}{5\sqrt{2}}
\end{aligned}$$

Using the point slope formula, with slope $\frac{2}{5\sqrt{2}}$ and point $(2,\sqrt{2})$, we get:

$$y-\sqrt{2} = \frac{2}{5\sqrt{2}}(x-2)$$

$$y = \frac{2}{5\sqrt{2}}x - \frac{4}{5\sqrt{2}} + \sqrt{2}$$

as our tangent line.

Question 2 (a)

Easiness: 84/100

SOLUTION. In order to estimate $\ln(1.1)$ we need to find a linear approximation of $\ln(x)$ at some point a. Because I know that $\ln(1) = 0$ and *1* is close to *1.1*, $x = 1$ would be a good choice for a.
The linear approximation of $f(x)$ at a is the same as finding the equation of the tangent line to $f(x)$ at a. Our function is $f(x) = \ln x$ and we chose $a = 1$. We find the slope of the tangent line/linear approximation by taking the derivative $f'(x) = 1/x$ and then calculating $f'(1) = 1$. The tangent line/linear approximation runs though the point $(1, f(1))$ or *(1,0)*.
Using our slope of *1* and the point *(1,0)*, we find the equation of the tangent line/linear approximation to be *L(x) = x - 1*. To complete the approximation of $\ln(1.1)$, we plug *1.1* into *L(x)* to get:

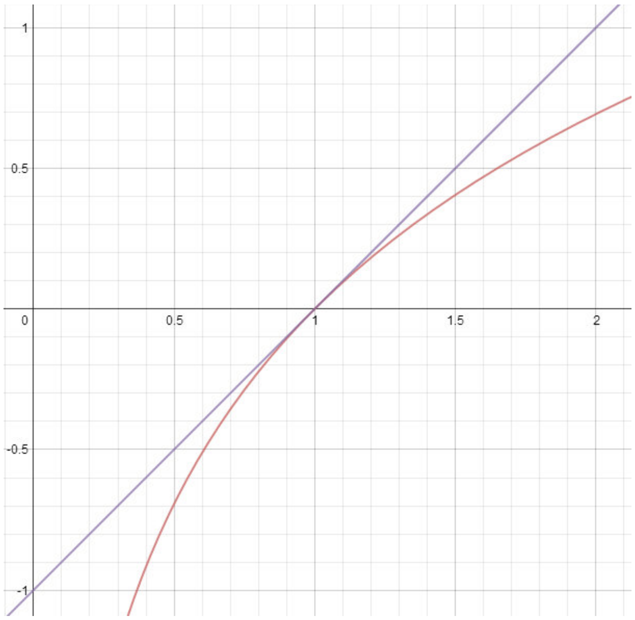
$$L(1.1) = 1.1-1 = 0.1$$

So $\ln(1.1) \approx 0.1$.

Question 2 (b)

Easiness: 100/100

SOLUTION. If you draw a picture of $\ln(x)$ and the linear approximation $y = x-1$ (see below), you can see that the linear approximation is above the graph of $\ln(x)$. This means that the value of the linear approximation will be greater than the value of the actual function.



Another way of showing this is to prove that $\ln(x)$ is concave down. A concave down function has all of its tangent lines above the graph of the function and so the same conclusion follows as above. We know $\ln(x)$ is concave down because its second derivative is $-1/x^2$ which is always negative.

Question 3 (a)

Easiness: 96/100

SOLUTION. Let $f(x)$ be continuous on an interval $[a, b]$ and differentiable on (a, b).
Mean Value Theorem: Then there exists a c in *(a,b)* such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Rolle's Theorem: In addition to the conditions of continuity and differentiability, if $f(a) = f(b)$, then there exists a c in *(a,b)* such that $f'(c) = 0$.

Question 3 (b)

Easiness: 45/100

SOLUTION. If $x_1 = x_2$, the assumption is obviously true, so suppose $x_1 \neq x_2$. Since $f'(x) = 0$ everywhere, we can deduce that f is differentiable everywhere, and hence also continuous everywhere, in particular on the interval $[x_1, x_2]$. We can then apply the Mean Value Theorem, which tells us that there exists some c in the interval (x_1, x_2) such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

However, we know $f'(x) = 0$ everywhere, so $f'(c) = 0$ and we have

$$0 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

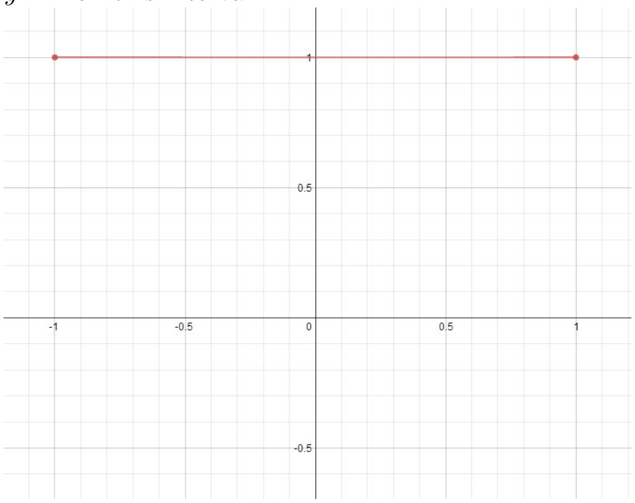
and by cross multiplication,

$$0 = f(x_2) - f(x_1).$$

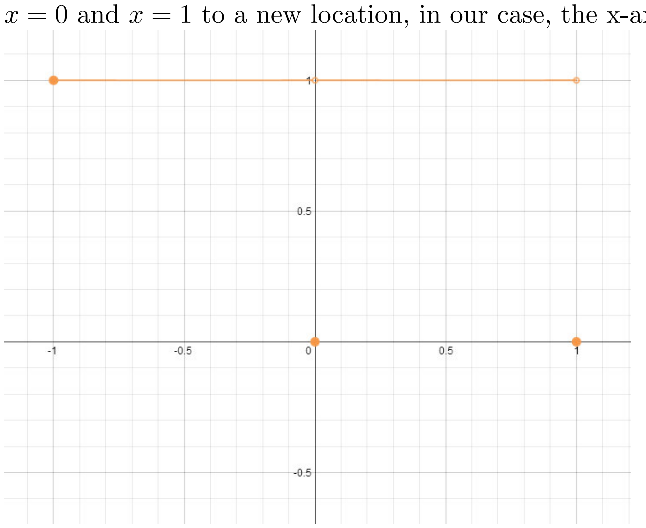
Question 4

Easiness: 31/100

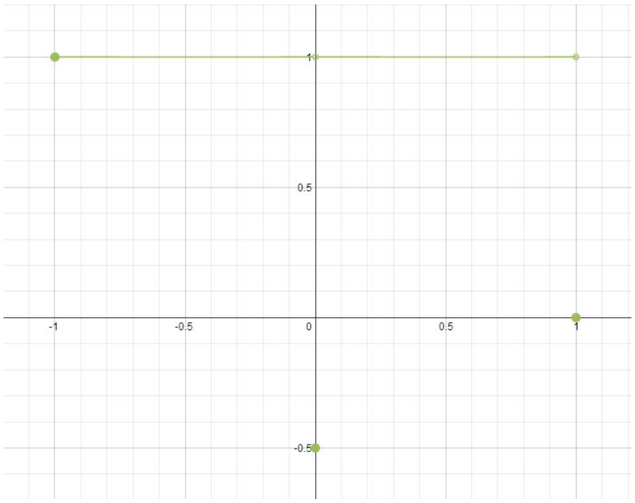
SOLUTION. As was suggested in the hint, one way to approach this problem is to draw a function that satisfies the first criterion and then modify it to fit each following criterion. It is worth noting that there are many, many ways to draw a picture satisfying all 4 criteria - the solution that follows is just one example. Our first criterion is that the function be defined on the interval [-1,1]. We will draw just a straight line at $y = 1$ on this interval:



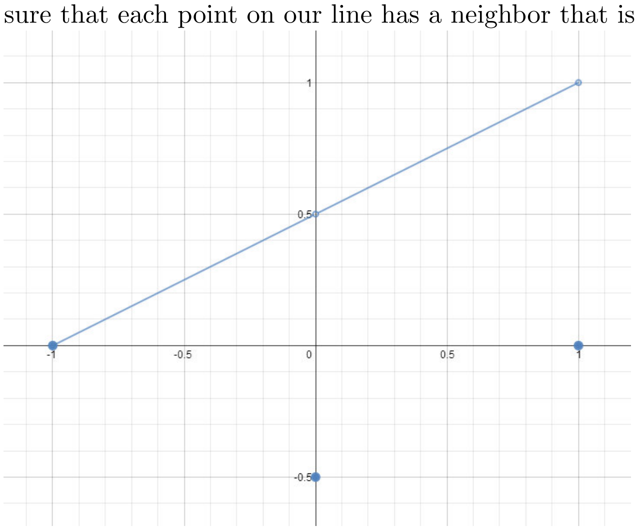
Now we add the criterion that the function be discontinuous at $x = 0$ and $x = 1$. One simple way to do this (without making the function undefined at those points, remember part 1!) is to move the two points at $x = 0$ and $x = 1$ to a new location, in our case, the x-axis.



Now we want f to have a global minimum at $x = 0$. This is easy enough - we'll just move our point that's now at the origin down until it is the lowest point on the entire graph.



Finally, we can't have a local or global maximum. Right now, our straight line is both a local and global maximum because each point on the line is greater than or equal to its neighboring points. In order to make sure that each point on our line has a neighbor that is greater than itself, we will tilt the line, like so:



There are two important things to note here: first, this change has not altered any of the previous criteria (still defined on [-1,1], still discontinuous at appropriate points, etc.) Second, look at the right endpoint of the graph. Because this point is undefined, f cannot have a global or local maximum at this point, and any point on the line cannot be a maximum either.

Thus we have drawn a function that satisfies all four criteria. We now simply translate our picture into a piecewise function. The "line" will be two pieces, and each point at $x = 0$ and $x = 1$ will be another piece each.

$$f(x) = \begin{cases} 1/2x + 1/2 & -1 \leq x < 0 \\ -1/2 & x = 0 \\ 1/2x + 1/2 & 0 < x < 1 \\ 0 & x = 1 \end{cases}$$

Question 5 (a)

Easiness: 92/100

SOLUTION. The numerator and denominator are polynomials, which are defined everywhere. However, since $f(x)$ is a fraction, we need to avoid points where the denominator is zero, i.e. where $x^2 - 1 = 0$. This occurs when $x = 1$ and $x = -1$. So the domain of $f(x)$ is

$(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

Question 5 (b)

Easiness: 99/100

SOLUTION. **x-intercept**

$$f(x)$$

has an x-intercept when $f(x) = 0$. So

$$0 = \frac{2x^2}{x^2 - 1}$$

Cross multiplying gives $2x^2 = 0$, which simplifies to $x = 0$. So the x-intercept is $(0, 0)$
y-intercept

$$f(x)$$

has an y-intercept when $x = 0$. Plugging in $x = 0$ gives $f(0) = 0$.
So the y-intercept is the same as the x-intercept, $(0, 0)$

Question 5 (c)

Easiness: 100/100

SOLUTION. We take limits of the function at positive and negative infinity.

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{2x^2}{x^2 - 1} \\
&= \lim_{x\to\infty} \frac{2x^2}{x^2 - 1} \frac{1/x^2}{1/x^2} \\
&= \lim_{x\to\infty} \frac{2}{1 - 1/x^2} \\
&= 2
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} \frac{2x^2}{x^2 - 1} \\
&= \lim_{x\to-\infty} \frac{2x^2}{x^2 - 1} \frac{1/x^2}{1/x^2} \\
&= \lim_{x\to-\infty} \frac{2}{1 - 1/x^2} \\
&= 2
\end{aligned}$$

So the horizontal asymptotes at positive and negative infinity are both $y = 2$.

Question 5 (d)

Easiness: 60/100

SOLUTION. To find the vertical asymptotes, we will take the limit of the function at both $x = -1$ and $x = 1$.

$$\lim_{x\to-1} f(x) = \lim_{x\to-1} \frac{2x^2}{x^2 - 1}$$

This limit is undefined since the denominator goes to zero but the numerator is nonzero. We will try taking the left and right hand limit at x = *-1*

$$\lim_{x\to -1^-} \frac{2x^2}{x^2-1} = \lim_{x\to -1^-} \frac{2x^2}{(x+1)(x-1)} = \infty$$

Note that as x approaches -1 from the left, the numerator of the fraction will be positive. Both terms in the denominator will be negative, yielding an overall positive number. As the denominator is decreasing, this is the same as saying that the entire fraction is increasing. Thus the limit is positive infinity. Using similar reasoning we can solve the right hand limit to get

$$\lim_{x\to -1^+} \frac{2x^2}{x^2-1} = -\infty$$

Taking the limit at x = *1* will give us an undefined limit in the same way as before. Checking right and left hand limits gives

$$\lim_{x\to 1^-} \frac{2x^2}{x^2-1} = -\infty$$

$$\lim_{x\to 1^+} \frac{2x^2}{x^2-1} = \infty$$

Thus we have vertical asymptotes at x = *-1* and x = *1*, where the value of the function near the asymptote depends which side of the asymptote it is approaching.

Question 5 (e)

SOLUTION. We first find the derivative of *f(x)*. Using the quotient rule,

$$f'(x) = \frac{4x(x^2-1)-(2x)(2x^2)}{(x^2-1)^2} = \frac{4x^3-4x-4x^3}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}.$$

Since the denominator is always positive on the domain (squaring x^2-1 always yields a positive number), we only need check whether the numerator is positive or negative. When $x<0$, $-4x$ is positive and when $x>0$, $-4x$ is negative. Thus f is increasing on the interval $(\infty,-1)\cup(-1,0)$ and decreasing on the interval $(0,1)\cup(1,\infty)$.

Question 5 (f)

SOLUTION. From the previous question, we know that

$$f'(x) = \frac{-4x}{(x^2-1)^2}.$$

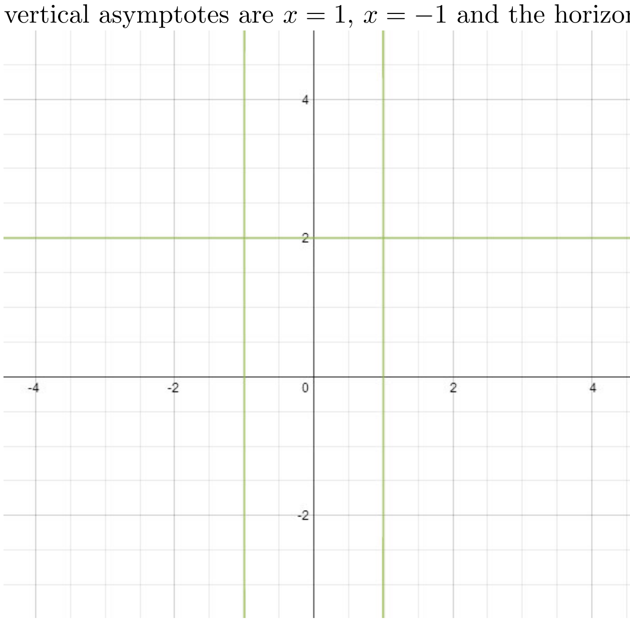
Using the quotient rule, we calculate the second derivative

$$
\begin{aligned}
f''(x) &= \frac{-4(x^2-1)^2 - (-4x)(2(x^2-1)(2x))}{(x^2-1)^4} \\
&= \frac{-4(x^2-1)^2 + 16x^2(x^2-1)}{(x^2-1)^4} \\
&= \frac{-4(x^2-1) + 16x^2}{(x^2-1)^3} \\
&= \frac{12x^2+4}{(x^2-1)^3}
\end{aligned}
$$

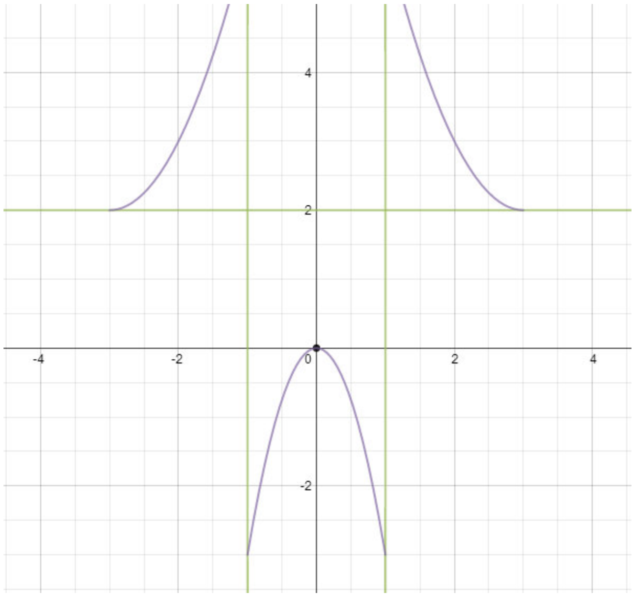
For this fraction, the numerator is always positive so we need only check the sign of the denominator. The denominator is zero when $x = \pm 1$. For $x < -1$, the denominator is positive, for $-1 < x < 1$ the denominator is negative, and then for $x > 1$ the denominator is again positive. Thus we have that the function is concave up on $(-\infty, -1) \cup (1, \infty)$ and concave down on $(-1, 1)$.

Question 5 (g)

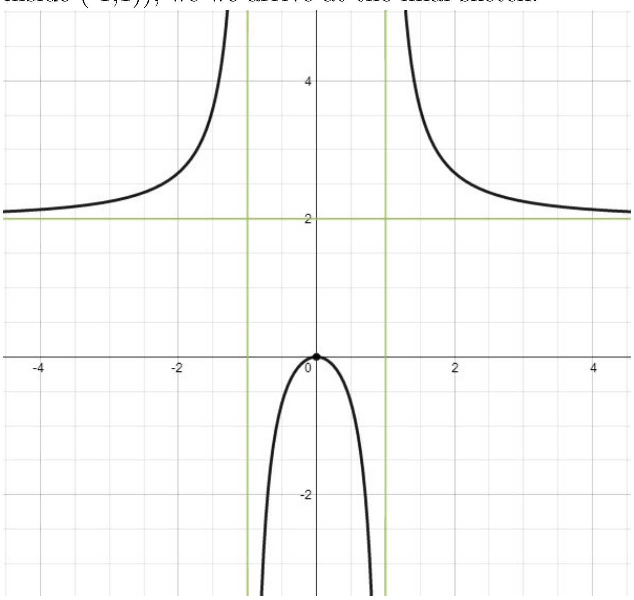
SOLUTION. As suggested in the hint, we start by drawing the vertical and horizontal asymptotes. The vertical asymptotes are $x = 1$, $x = -1$ and the horizontal asymptote is $y = 2$.



We know that the function increases to the left of $x = -1$, increases again between -1 and 0, and then decreases from 0 to 1, and then again after $x = 1$. Furthermore, the graph must go through the x-axis at the point (0,0) but not anywhere else. This suggests the following picture:



Confirming that this picture matches the concavity found in part f (concave up outside of (-1,1), concave up inside (-1,1)), we we arrive at the final sketch:



Question 6 (a)

SOLUTION. The differential equation above says that the derivative of the function $P(t)$ is a constant multiple of the original function. A function that has this property is the exponential function, e^t. An exponential function will satisfy the differential equation when its derivative is equal to 10^8 times the same exponential function. This occurs when we include a factor of 10^8 in the exponent of the e^t so we have the function $P(t) = e^{10^8 t}$. We can check by differentiation that this satisfies the differential equation $\frac{dP}{dt} = 10^8 P$.

Question 6 (b)

SOLUTION. We will call our given number of neutrons at the start (when t = 0), P_0. Note that when we calculate the number of neutrons at the start, we get $P(0) = 1$. So in order for our function $P(t)$ to give us the starting number of neutrons, P_0, we need to multiply $e^{10^8 t}$ by P_0 to get our final formula, $P(t) = P_0 e^{10^8 t}$. To calculate how long it takes for the neutrons to double, we set our function $P(t)$ equal to double the starting amount, or $2P_0$. This gives us

$$2P_0 = P_0 e^{10^8 t}$$

After canceling the P_0 from both sides, taking a natural logarithm of both sides, and simplifying using log rules we get

$$\ln(2) = \ln(e^{10^8 t})$$

$$\ln(2) = 10^8 t$$

$$\frac{\ln(2)}{10^8} = t$$

Question 7

SOLUTION. The question is asking us to find the "strongest" beam, thus we are optimizing strength, which is given by the formula $S = kwh^2$. There are two variables here: width (w) and height (h). In order to simply the formula to one variable, we want to relate w and h using another formula. Looking at the diagram of the log, it would make sense to use the Pythagorean theorem, as w and h are the two side of a right triangle. Thus we have

$$w^2 + h^2 = 50^2$$

Now we can solve this equation for w or h and substitute it back into our formula $S = kwh^2$. Which variable should I choose? Either one will work, but because I have an h^2 in my strength formula and an h^2 in the Pythagorean theorem, I will solve for h^2. Solving for h^2 in the Pythagorean theorem gives

$$h^2 = 50^2 - w^2.$$

Substituting this back into my strength formula gives

$$S = kw(50^2 - w^2)$$

Which is now a function in one variable (w) since k is a constant. Note that because the width of the beam is constrained by the diameter of the log, the domain for this function is $[0, 50]$.
I know that my function S will be maximized at either the endpoints of the domain or at a critical point. The endpoints are $w = 0$ and $w = 50$. To find critical points, I first find the derivative:

$$S' = 50^2 k - 3kw^2$$

And then set"S"' equal to zero and solve.

$$\begin{aligned}
0 &= 50^2k - 3kw^2 \\
0 &= k(50^2 - 3w^2) \\
0 &= 50^2 - 3w^2 \\
3w^2 &= 50^2 \\
w &= \pm\frac{50}{\sqrt{3}}
\end{aligned}$$

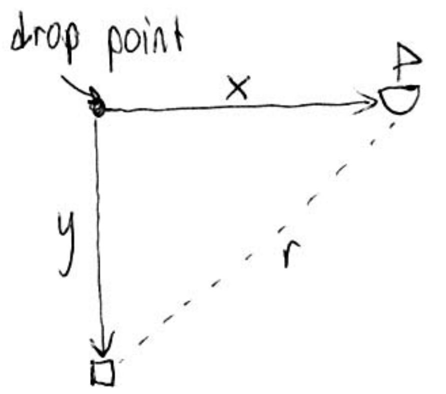
The negative square root is not in our domain, so we have one critical point, $w = 50/\sqrt{3}$.
I can check for the maximum in two ways. The first (and more rigorous) method is to plug the endpoints and critical point back into my function $S = kw(50^2 - w^2)$ and compare the values I get.

$$\begin{aligned}
w &= 0, & S &= k(0)(50^2 - 0) = 0 \\
w &= \frac{50}{\sqrt{3}}, & S &= k\frac{50}{\sqrt{3}}(50^2 - \frac{50^2}{3}) = \frac{2\cdot 50^3k}{(3\sqrt{3})} \\
w &= 50, & S &= k(50)(50^2 - 50^2) = 0
\end{aligned}$$

So S is maximized when $w = 50/\sqrt{3}$. Recalling that $h^2 = 50^2 - w^2$ we can solve for $h = 50\sqrt{\frac{2}{3}}$.
The second way to check for the maximum to realize that the endpoints will cause $S = kw(50^2 - w^2)$ to be zero (same process as checking the endpoints above), and to perform the first (or second) derivative test to check that $w = 50/\sqrt{3}$ is a maximum.

Question 8

Solution. Let x be the distance of the ship from the drop point, y be the distance of the probe from the drop point, and r be the distance between the ship and the probe. Drawing a labelled picture looks like this:



We want to find $\frac{dr}{dt}$, at time 10s after the probe is dropped. We know that $\frac{dx}{dt} = 5m/s$ and $\frac{dy}{dt} = 4m/s$.
By the Pythagorean Theorem,

$$x^2 + y^2 = r^2.$$

In order to find the relationship between the time rates of change of these variables, we differentiate both sides of this equation with respect to time, in order to obtain

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2r\frac{dr}{dt}$$

Dividing both sides by $2r$ in order to isolate $\frac{dr}{dt}$, we find

$$\frac{dr}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{r}.$$

We are interested in the value of $\frac{dr}{dt}$, at time $10s$ after the probe is dropped. At this point in time,

$$x = 5m/s \cdot 10s = 50m,$$

$$y = 4m/s \cdot 10s = 40m,$$

and

$$\begin{aligned} r &= \sqrt{x^2 + y^2} = \sqrt{(50m)^2 + (40m)^2} \\ &= 10\sqrt{5^2 + 4^2}\ m = 10\sqrt{41}\ m. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{dr}{dt} &= \frac{(50m)(5m/s) + (40m)(4m/s)}{10\sqrt{41}m} \\ &= \frac{25m/s + 16m/s}{\sqrt{41}} \\ &= \sqrt{41}\ m/s. \end{aligned}$$

That is, $10s$ after the probe is dropped, the distance between the probe and ship is increasing at a rate of $\sqrt{41}\ m/s$.

Question 9

SOLUTION. Because e^x is its own derivative, our anti-derivative for the function e^{3-x} will probably be very similar to the original function. In fact, $F(x) = -e^{3-x}$ will work; you can check by differentiating using the chain rule to get

$$F'(x) = -(-e^{3-x}) = e^{3-x} = f(x)$$

This is one anti-derivative. To find another one, we can simply add a constant to the anti-derivative shown above, say $F_2(x) = -e^{3-x} + 5$. When we differentiate, the constant will disappear, giving us the same derivative as before.

Good Luck for your exams!

