Full Solutions
MATH221 April 2013

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How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

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Question 1

SOLUTION. We can write this system in matrix form:

$$\begin{bmatrix} 3 & 4 & 7 \\ -9 & 6 & 9 \\ 45 & -12 & h \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ k \end{bmatrix}$$

To know the number of solutions, we want to look at the row reduced form of the augmented matrix. First, perform the operations $L_3 \rightarrow L_3+5L_2$ and $L_2 \rightarrow L_2-3L_1$

$$\left[\begin{array}{cccc} 3 & 4 & 7 & 3 \\ -9 & 6 & 9 & 3 \\ 45 & -12 & h & k \end{array}\right] \sim \left[\begin{array}{cccc} 3 & 4 & 7 & 3 \\ 0 & 18 & 30 & 12 \\ 0 & 18 & 45+h & 15+k \end{array}\right]$$

Then, perform $L_3 \rightarrow L_3 - L_2$

$$\sim \left[\begin{array}{cccc} 3 & 4 & 7 & 3 \\ 0 & 18 & 30 & 12 \\ 0 & 0 & 15+h & 3+k \end{array}\right]$$

Noting that the first two rows of the matrix give a matrix of rank at least 2, the number of solutions depend only on the last row. We read the last row of this matrix and we obtain the equation

$$(15+h)z = 3+k$$

We can now read off the solutions from this equation base on whether or not $15+h$ and $3+k$ are equal to zero.
a) No solution if $15+h=0, 3+k \neq 0$
b) One solution if $15+h \neq 0$
c) Infinite solution if $15+h=0, 3+k=0$
Notice that this convers all possible values for $15+h$ and $3+k$.

Question 2 (a)

SOLUTION. Since the matrix A and B are related by the left multiplication of that invertible matrix M, they share many characteristic, including rank and they row space. The reason being that the left multiplication by M can be understood as simply being a set of (invertible) row operations and the rank and row space are properties that are preserved under (invertible) row operations.
Now, as B is in row echelon form, it is easier to read this information from the matrix B, so we will use to answer the question. As B has 3 pivots, we see that the rank is 3. We can also take each row that has a pivot to form a basis of the row space, thus our basis is given by:
$\begin{bmatrix} 1 & 2 & 1 & 3 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -3 & 5 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & -7 \end{bmatrix}$

Question 2 (b)

SOLUTION. By the Rank-Nullity theorem, the dimension of the nullspace of A is equal to the number of column minus the rank of A (which we found in part (a)), thus it is *5 - 3 = 2*.
The nullspace is unaffected by left multiplication by an invertible matrix, that is $Bv = MAv = 0$ if and only if $Av = 0$. Thus, we can use the matrix B to compute the nullspace.
To find a basis of the nullspace, we will further simplify B by $L_1 \rightarrow L_1-2L_2$, followed by $L_2 \rightarrow L_2-5L_3$ and $L_1 \rightarrow L_1+7L_3$ to get the matrix

$$\begin{bmatrix} 1 & 0 & 7 & 0 & -39 \\ 0 & 1 & -3 & 0 & 31 \\ 0 & 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Again, further row operations won't change the nullspace.
Applying this matrix to a vector $\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}$ and solving for zero we get the relations

$$\begin{aligned} x_1 + 7x_3 - 39x_5 &= 0 \\ x_2 - 3x_3 + 31x_5 &= 0 \\ x_4 - 7x_5 = 0 \end{aligned}$$

We know that the dimension of the nullspace is 2, thus there must be 2 free variables, namely x_5, x_3. So, we can read off the basis as $\begin{bmatrix} 39 & -31 & 0 & 7 & 1 \end{bmatrix}, \begin{bmatrix} -7 & 3 & 1 & 0 & 0 \end{bmatrix}$.

Question 2 (c)

Solution. The column space is the same as the range of the matrix, which we know has has dimension 3, thus we are looking for a basis made up of 3 elements.
While row operations (or left multiplication by an invertible matrix) changes the column space, it still preserves the relations among the columns. The reason for this being is that left matrix multiplication acts on the columns of a matrix as if they were vectors, thus, for example, if c_1 is the first column of a matrix B, then Mc_1 is the first column of the matrix MB, furthermore if we had $c_1 = c_2 + c_3$, a relationship among the columns of A, then $Mc_1 = Mc_2 + Mc_3$ will be a relationship in the columns of MA. Since M is invertible you can reverse this step, so that linearly independent columns in B correspond to linearly independent columns in A.
Due to this fact, we can read the relations that the columns of the matrix A by looking at those off of the matrix B, in particular we can simply read which column of A are linear independent by looking at the columns of B that contain a pivot (i.e. a set of columns that is linearly independent in B).

The column space of A has a basis equal to $\begin{bmatrix} 1 \\ 3 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ -3 \end{bmatrix}$.

Question 2 (d)

Solution. If we write the system of equations as an extended matrix, it will be exactly the matrix A. Thus, thinking of A as an extended matrix, we know it represents 4 equations in 4 variables. Row operations lead to an equivalent system of equations, with the same solutions. Hence we can instead choose the extended matrix B, even better, we can further use the reduced matrix we used in part (b), i.e. the matrix:
$$\begin{bmatrix} 1 & 0 & 7 & 0 & -39 \\ 0 & 1 & -3 & 0 & 31 \\ 0 & 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
to determine the answer. From the above matrix, we can read the equations:

$$\begin{aligned} x + 7z &= -39 \\ y - 3z &= 31 \\ w &= -7 \end{aligned}$$

From this, we can quickly see that $w = 0$ leads to no solution (i.e. it would lead to saying *0 = -7*).
If z=*0*, then we get the solution

$$x = -39, \quad y = 31, \quad w = -7$$

Question 3

SOLUTION. As that matrices map the square to the square, it will map the corners to the corners, thus we will see all possible ways to map the four corners to the four new corners.

- First, linear transforms must bring the vector $(0, 0)$ to the vector $(0, 0)$. So one of the corners is taken care of already.
- Next, note that there is a relation between the corners of the original square, that is $(1, 0)+(0, 1) = (1, 1)$. As linear transformation preserve sums, *T(a+b) = T(a)+T(b)*, we need to have the same relation in the target corners, which we do $(1, 1) + (-1, 1) = (0, 2)$. Hence any linear transform between the two squares must send *(1,1)* to *(0,2)*.
- Thus, we are left with sending $(0, 1), (1, 0)$ to $(1, 1), (-1, 1)$, and there are two ways to do so. As $(0, 1), (1, 0)$ forms a basis of the matrix, knowing where theses two vectors go also pins down the two matrices uniquely. We simply need to put the result in the columns of the transformation matrix:

$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

Question 4

SOLUTION. Let's say we order a amount of type A, similarly b of type B and c of type C. Thus, the total cost can be read as

$$300a + 200b + 400c$$

which is the objective function we must *minimize*.
The variables a, b, c are not arbitrary, but must be chosen so that we get 70oz of gold and 500oz of silver. A choice of a,b,c leads to two equations relating to the amount of ore purchased:
Amount of gold: $a + b + 2c = 70$oz
Amount of silver: $20a + 10b + 10c = 500$oz
Put this into an extended matrix we get:
$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 70 \\ 20 & 10 & 10 & 500 \end{array}\right]$
Which we can simplify to the matrix:
$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -20 \\ 0 & 1 & 3 & 90 \end{array}\right]$.
Thus, we can consider c to be a free variable. Solving this system we get

$$a = -20 + c$$
$$b = 90 - 3c$$

Having written each quantity with respect to c we can put it in the cost equation

$$\begin{aligned} C = 300a + 200b + 400c &= 300(-20 + c) + 200(90 - 3c) + 400c \\ &= -6000 + 300c + 18000 - 600c + 400c \\ &= 12000 + 100c \end{aligned}$$

which we now have to minimize. C is an increasing function (it is linear with positive slope) and so choosing the smallest value of c possible will yield the minimum. It looks like this value may be c=*0* however looking at the other formulas tells us that if c were zero then a would be -20 and b would be 90. We can't order negative quantities of anything so we actually have hidden constraints, namely that

$$a \geq 0, \qquad b \geq 0, \qquad c \geq 0.$$

Using our relations for a and b, this tells us that

$$\begin{aligned} c &\geq 20 \\ c &\leq 30 \end{aligned}$$

Since we know the cost C is an increasing function, we want to take the smallest c allowable which is $\boldsymbol{c = 20}$. Using the equations above we find that this yields $\boldsymbol{a = 0}$ and $\boldsymbol{b = 30}$.

Question 5

SOLUTION. We first carry out operations to isolate X

$$\begin{aligned} A + (BX)^T &= C \\ (BX)^T &= C - A \\ ((BX)^T)^T &= (C - A)^T \\ BX &= (C - A)^T \\ B^{-1}BX &= B^{-1}(C - A)^T \\ X &= B^{-1}(C - A)^T \end{aligned}$$

We now simply compute $(C - A)^T$ and B^{-1} and multiply them together.
$(C - A)^T$:

$$\begin{aligned} \left(\begin{bmatrix} -1 & 9 & 0 \\ 3 & 7 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)^T &= \left(\begin{bmatrix} -3 & 8 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)^T \\ &= \begin{bmatrix} -3 & 0 & 0 \\ 8 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

We can invert B by using an extended matrix and doing the following:

$$\begin{aligned} \begin{bmatrix} 0 & -5 & 1 & 1 & 0 & 0 \\ -3 & 7 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} &\sim \begin{bmatrix} 0 & -5 & 1 & 1 & 0 & 0 \\ 0 & 4 & -1 & 0 & 1 & 3 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \\ &\sim \begin{bmatrix} 0 & -5 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & -3 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \\ &\sim \begin{bmatrix} 0 & 0 & 1 & -4 & -5 & -15 \\ 0 & 1 & 0 & -1 & -1 & -3 \\ 1 & 0 & 0 & -1 & -1 & -2 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & 0 & -1 & -1 & -2 \\ 0 & 1 & 0 & -1 & -1 & -3 \\ 0 & 0 & 1 & -4 & -5 & -15 \end{bmatrix} \end{aligned}$$

So

$$B^{-1} = \begin{bmatrix} -1 & -1 & -2 \\ -1 & -1 & -3 \\ -4 & -5 & -15 \end{bmatrix}$$

You are then left with multiplying the two matrices out, which gives the final result of:

$$\begin{bmatrix} -5 & -2 & -2 \\ -5 & -2 & -3 \\ -28 & -10 & -15 \end{bmatrix}$$

Question 6

SOLUTION. The first thing we will do is simplify the matrix by row and column operations. We start by replacing row 4 with row 2 plus row 4 (this does not change the determinant) giving:

$$\begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 2 & 6 & 3 & -3 & 7 \\ -3 & -4 & -3 & 2 & -8 \\ 0 & 0 & 1 & -1 & 0 \\ 1 & -2 & 7 & 3 & 3 \end{bmatrix}$$

This step helps to simplify the row reductions. Next, using row 1 as a pivot, we replace row 2 with row 2 minus 2 times row 1, row 3 with row 3 plus three times row 1 and row 5 with row 1 minus row 1 giving

$$\begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 0 & 2 & -1 & -1 & 1 \\ 0 & 2 & 3 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & -4 & 5 & 4 & 0 \end{bmatrix}$$

Note that none of these changes affected the determinant either. Next, we replace row 3 with row 3 minus row 2 and we replace row 5 with row 5 plus two times row 2 to get

$$\begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 0 & 2 & -1 & -1 & 1 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 2 & 2 \end{bmatrix}$$

Again none of these changes affect the determinant. Next, using row 4 as a pivot, we replace row 3 with row 3 minus four times row 4 and replace row 5 with row 5 minus 3 times row 4 to get

$$\begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 0 & 2 & -1 & -1 & 1 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 5 & 2 \end{bmatrix}$$

Again this doesn't change the determinant. Next, use row 3 as a pivot and replace row 5 with row 5 minus five quarters of row 3 to get

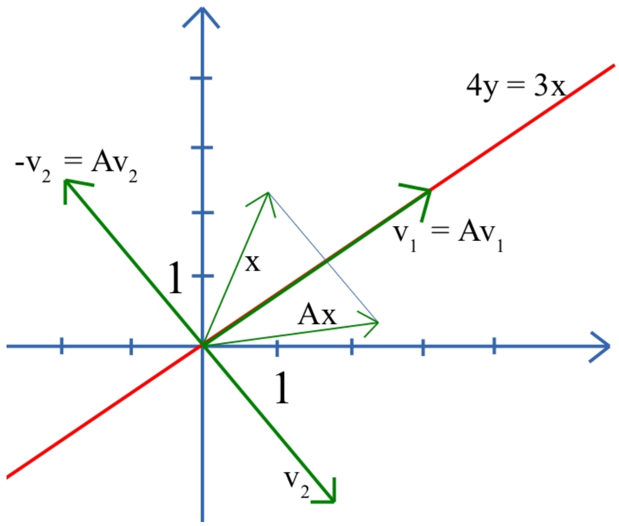
$$\begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 0 & 2 & -1 & -1 & 1 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

None of these row operations have thus far changed the determinant. Now, we flip rows 3 and 4 which will introduce an extra negative sign in our determinant. So the quantity we want is $-\det(M)$ where M is defined by:

$$M = \begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 0 & 2 & -1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

This matrix is diagonal and so the determinant of this matrix is just the product of the diagonal entries giving 16. Hence our answer is $-\det(M) = -16$ as required.

Question 7 (a)



Solution.

- For any non-trivial reflection, the eigenvalues are always *1* and *-1*.
- For the eigenvalue 1, we can pick any vector on the line itself. Such a vector will not be moved by the linear transformation. So, for us, we pick $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ as an eigenvector to the eigenvalue 1. Note that *(x,y) = (4,3)* is a point on the line *3x=4y* since *3(4)=4(3)*.
- The eigenvalue of *-1* comes from picking a vector in the line that is perpendicular (i.e. orthogonal) to the line $4y = 3x$. Any such vector will simply change its sign, thus will correspond to the eigenvalue *-1*. We choose $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$ as an eigenvector to the eigenvalue -1.

Question 7 (b)

Solution. *' What we know from part a)'*

Using the eigenvectors $\begin{bmatrix} 4 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \end{bmatrix}$, and the eigenvalues $1, -1$, we know that $A\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and $A\begin{bmatrix} -3 \\ 4 \end{bmatrix} = -\begin{bmatrix} -3 \\ 4 \end{bmatrix}$.

' Finding the formula for the matrix A'

Then, we can write the information from above in matrix notation $A\begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

We define the matrix B, where the columns of B are the eigenvectors and write the equation as $AB = B\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

This leads us to $A = B\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}B^{-1}$.

' Calculating A'

First the inverse of B is given by the usual method for 2x2 matrices (where $25 = \det(B)$):

$B^{-1} = \frac{1}{25}\begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}$

We can now compute A as claimed:

$A = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\left(\frac{1}{25}\begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}\right) = \frac{1}{25}\begin{bmatrix} 7 & 24 \\ 24 & -7 \end{bmatrix}$.

Question 7 (c)

SOLUTION 1. From the description of A, applying the matrix 1995 to R^2 is simply applying the same reflection 1995 times, but as reflections have order 2, i.e. applying it twice does nothing, and 1995 is an odd number, we see that $A^{1995} = A$.

SOLUTION 2. Alternatively, we can also show this by straightforward computation, using the breakdown in part b):

$$\begin{aligned} A^{1995} &= \left(B\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}B^{-1}\right)^{1995} \\ &= B\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{1995}B^{-1} \\ &= B\begin{bmatrix} 1^{1995} & 0 \\ 0 & (-1)^{1995} \end{bmatrix}B^{-1} \\ &= B\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}B^{-1} \\ &= A \end{aligned}$$

Question 8

SOLUTION. To find the diagonal matrix D, we need to compute the determinant of the matrix A-tI.

$$\begin{aligned} \det(A - tI) &= \det\left(\begin{bmatrix} 2-t & 0 & 3 \\ 0 & 3-t & 0 \\ 4 & 0 & 1-t \end{bmatrix}\right) \\ &= (3-t)\det\left(\begin{bmatrix} 2-t & 3 \\ 4 & 1-t \end{bmatrix}\right) \\ &= (3-t)\left((2-t)(1-t) - 12\right) \\ &= (3-t)(t-5)(t+2) \end{aligned}$$

Which means that eigenvalues are 3, 5 and -2. To find the eigenvectors (which will be the columns of the matrix P), we pick vectors of the kernels of $A - tI$ where $t = 3$, 5, -2.

Eigenvalue $t = 3$.

$$\begin{bmatrix} 2-3 & 0 & 3 \\ 0 & 3-3 & 0 \\ 4 & 0 & 1-3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 3 \\ 0 & 0 & 0 \\ 4 & 0 & -2 \end{bmatrix}$$

Thus an eigenvector corresponding to the eigenvalue $t = 3$ is given by

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Eigenvalue t = 5.

$$\begin{bmatrix} 2-5 & 0 & 3 \\ 0 & 3-5 & 0 \\ 4 & 0 & 1-5 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 3 \\ 0 & -2 & 0 \\ 4 & 0 & -4 \end{bmatrix}$$

Thus an eigenvector corresponding to the eigenvalue $t = 5$ is given by

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Eigenvalue t = -2.

$$\begin{bmatrix} 2-(-2) & 0 & 3 \\ 0 & 3-(-2) & 0 \\ 4 & 0 & 1-(-2) \end{bmatrix} = \begin{bmatrix} 4 & 0 & 3 \\ 0 & 5 & 0 \\ 4 & 0 & 3 \end{bmatrix}$$

Thus an eigenvector corresponding to the eigenvalue $t =$ -2 is given by

$$\begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$$

can be chosen as an eigenvector.
So the matrices D and P are given by

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad P = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & -4 \end{bmatrix}$$

Now, as $A = PDP^{-1}$, then

$$\begin{aligned} A^{-k} &= \left(PDP^{-1}\right)^{-k} \\ &= PD^{-k}P^{-1} \\ &= P \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -2 \end{bmatrix}^{-k} P^{-1} \\ &= P \begin{bmatrix} 3^{-k} & 0 & 0 \\ 0 & 5^{-k} & 0 \\ 0 & 0 & (-2)^{-k} \end{bmatrix} P^{-1} \end{aligned}$$

So we can see that as k gets large, D^{-k} get closers to the zero matrix. Hence the right hand side, and with it the matrix $A_{\text{-}k}$, also approaches the zeros matrix.

Question 9

SOLUTION. **Determine eigenvalues of $B = 10A$**

We will first find the eigenvalues of the matrix $B = \left[\begin{array}{cc} 8 & 5 \\ 2 & 5 \end{array}\right]$ to avoid dealing with fractions. The eigenvalues of the matrix A, will simply be $\frac{1}{10}$ of this integer matrix B.
To find the eigenvalues we calculate as usual:

$$\begin{aligned} \det\left(\left[\begin{array}{cc} 8-t & 5 \\ 2 & 5-t \end{array}\right]\right) &= (8-t)(5-t)-(5)(2) \\ &= t^2-13t+30 \\ &= (t-3)(t-10) \\ &= 0 \end{aligned}$$

Determine eigenvectors of $B = 10A$

For the eigenvectors with **eigenvalue 3** we look at $\left[\begin{array}{cc} 8-3 & 5 \\ 2 & 5-3 \end{array}\right] = \left[\begin{array}{cc} 5 & 5 \\ 2 & 2 \end{array}\right]$ which gives the eigenvector $\left[\begin{array}{c} 1 \\ -1 \end{array}\right]$.

For the **eigenvalue 10**, we look at the $\left[\begin{array}{cc} -2 & 5 \\ 2 & -5 \end{array}\right]$ which gives the eigenvectors $\left[\begin{array}{c} 5 \\ 2 \end{array}\right]$.

Eigenvalues and eigenvectors of A

For the matrix A this means that A has the

- eigenvalue $\frac{3}{10}$ with eigenvector v_1=\left[\begin{array}{cc}

1 \\ -1 \end{array} \right]

- eigenvalue 1 with eigenvector v_2=\left[\begin{array}{cc}

5 \\ 2 \end{array} \right]

Find $a, b \in \mathbb{R}$ such that $x_0 = av_1 + bv_2$

To find the closed formulae for the components of $\mathbf{x}_\mathrm{n}$ we decompose the vector x_0 into a sum of the eigenvectors of A. To do so, we use the extended matrix:

$$[v_1\ v_2 \mid x_0] = \left[\begin{array}{cc|c} 1 & 5 & 4 \\ -1 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 5 & 4 \\ 0 & 7 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 3/7 \\ 0 & 1 & 5/7 \end{array}\right]$$

Thus $a = 3/7$ and $b = 5/7$ and therefore

$$\left[\begin{array}{c} 4 \\ 1 \end{array}\right] = 3/7\left[\begin{array}{c} 1 \\ -1 \end{array}\right] + 5/7\left[\begin{array}{c} 5 \\ 2 \end{array}\right].$$

' *Computing x_n*'

We can now compute x_n as follows:

$$\begin{aligned} x_n = A^n x_0 &= A^n\left(\frac{3}{7}\left[\begin{array}{c} 1 \\ -1 \end{array}\right] + \frac{5}{7}\left[\begin{array}{c} 5 \\ 2 \end{array}\right]\right) \\ &= \frac{3}{7}A^n\left[\begin{array}{c} 1 \\ -1 \end{array}\right] + \frac{5}{7}A^n\left[\begin{array}{c} 5 \\ 2 \end{array}\right] \\ &= \frac{3}{7}\frac{3^n}{10^n}\left[\begin{array}{c} 1 \\ -1 \end{array}\right] + \frac{5}{7}1^n\left[\begin{array}{c} 5 \\ 2 \end{array}\right] \end{aligned}$$

Question 10

SOLUTION. We first note that there is a non-trivial linear relationship between the three vectors, namely $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 & 1 & 2 \end{bmatrix}$ Thus, we get that W has dimension 2 and thus $W^{\perp}$ is also 2 dimensional. Let V be the matrix of basis vectors to W,

$$V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 2 \end{bmatrix}.$$

If $x = [a, b, c, d]^T$ is a basis vector to $W^{\perp}$ then orthogonality tells us that

$$V^T x = 0$$
$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \mathbf{0}$$

Computing this we get the following relationships

$$a + c = 0$$
$$b + c + 2d = 0$$

Since the dimension (or rank) of V is two we will have two free parameters. In particular, this means that the basis is not unique (it never is). We choose a and d as free parameters to get

$$c = -a$$
$$b = -2d - c = -2d + a$$

and finally that

$$x = \begin{bmatrix} a \\ -2d + a \\ -a \\ d \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Therefore the two orthogonal vectors $\begin{bmatrix} 1 & 1 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 & 0 & 1 \end{bmatrix}$ are a basis for $W^{\perp}$. Notice that the dimension of $W^{\perp}$ is two, which happens to be the same dimension as that of W.

Question 11

SOLUTION. The least squares solution vector $\hat{\mathbf{x}}$ is a 2x1 vector. We can quickly find it using the normal equations.

$$A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$$
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}\hat{\mathbf{x}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix}\mathbf{b}$$
$$\begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}\hat{\mathbf{x}} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

To find $\hat{\mathbf{x}}$ from here we can either invert the matrix on the left hand side, or use gaussian elimination. We choose the latter.

$$\left[\begin{array}{cc|c} 4 & 2 & 5 \\ 2 & 6 & 6 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 3 \\ 4 & 2 & 5 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 3 \\ 0 & -10 & -7 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 9/10 \\ 0 & 1 & 7/10 \end{array}\right]$$

Hence

$$\hat{\mathbf{x}} = \begin{bmatrix} 9/10 \\ 7/10 \end{bmatrix}$$

Finally, how does this translate to the linear function whose graph best fits the four given points? In other words, find m and t such that

$$y = mx + t$$

best fits the four given points (x_1,y_1), (x_2,y_2), (x_3,y_3), (x_4,y_4). To answer this we need to understand how the matrix A and the vector $\boldsymbol{b}$ were chosen. We notice that $\boldsymbol{b}$ holds the y values. Further, the x values of the points, are in the second column of A. The first column of A is ones and this is because the same constant $\boldsymbol{t}$ appears for each (x,y) pair. In matrix notation

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \end{bmatrix}\begin{bmatrix} \hat{\mathbf{x}}_1 \\ \hat{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

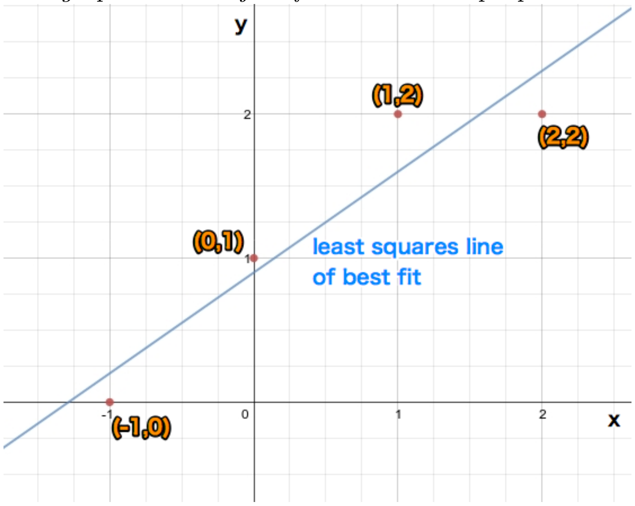
Reading $\mathbf{b} = A\hat{\mathbf{x}}$ line by line reveals

$$y_j = 1\hat{\mathbf{x}}_1 + x_j\hat{\mathbf{x}}_2 = \hat{\mathbf{x}}_2 x_j + \hat{\mathbf{x}}_1 = 7/10x_j + 9/10$$

for $j = 1,2,3,4$. Comparing to $y = mx + t$ we obtain

$$m = 7/10, \quad t = 9/10$$

The graph below is just for illustration purposes. It is not necessary to include a graph in your answer.



Therefore, the linear function is given by

$$y = \frac{7}{10}x + \frac{9}{10}$$

Question 12 (a)

SOLUTION. We recall that the formula to compute a projection onto a line spanned by the vector v is to take a vector x and mapping it to $\frac{x\cdot v}{v\cdot v}v$, that is you take the inner product of the vector x with the vector v divide it by the length of the vector v and apply this scalar to v. To get a matrix out of this, we can simply apply the projection to the standard basis of R^3 and see where each vector is mapped to.
First we note that, for us, the vector v is the normal vector of the plane, i.e. $v = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}$. This gives $v \cdot v = 1 + 1 + 1 = 3$. We are now ready to apply it to a standard basis vector, $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, which gets mapped by projection to the vecotr $\frac{1}{3}\begin{bmatrix} 1 & -1 & 1 \end{bmatrix}$. Similarly for the other basis vectors.
Thus our matrix for the projection is

$$\frac{1}{3}\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

Question 12 (b)

SOLUTION. If we apply the projection of (a) to the vector $\boldsymbol{u}$, we get a vector $\boldsymbol{w}$ that is orthogonal to the plane. To get $\boldsymbol{v}$, we simply subtract $\boldsymbol{w}$ from $\boldsymbol{u}$.
To verify that $\boldsymbol{v}$ is indeed on the plane, we compute that

$$P_1(v) = P_1(u - w) = P_1(u) - P_1(w) = w - w = 0,$$

which shows that the vector $\boldsymbol{v}$ has no competent that is orthogonal to the plane. Thus $\boldsymbol{v}$ must lie in the plane itself.
Let's perform the necessary calculations to get $\boldsymbol{w}$ and $\boldsymbol{v}$. We have

$$w = \frac{u \cdot \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}}{3}\begin{bmatrix} 1 & -1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & -1 & 1 \end{bmatrix}$$

And hence $v = u - w = \begin{bmatrix} \frac{2}{3} & \frac{4}{3} & \frac{2}{3} \end{bmatrix}$.
"**Note**: It is also easy to directly verify that $\boldsymbol{v}$ satisfies the equation $x - y + z = 0$ of the plane, since $2/3 - 4/3 + 2/3 = 0$.

Question 12 (c)

SOLUTION. We simply repeat the process we did in (b); for any vector u, we have that $u - P_1(u)$ is in the plane. Thus, the projection onto the plane is given by the linear transformation $P_2 = I - P_1$, where I is the identity matrix.
So, we compute it as follows:

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}.$$

Good Luck for your exams!