Full Solutions
MATH100 December 2012

April 4, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

SOLUTION. Since $\frac{d}{dx}\sin(x) = \cos(x)$ and $\frac{d}{dx}e^x = e^x$, we have that the function given by $f(x) = 2\sin(x) - e^x + C$ where C is any constant are functions that have the given derivative.
Since we are also given that $f(0) = 0$, we plug this information in to see that $0 = f(0) = 2\sin(0) - e^0 + C = 0 - 1 + C$. Isolating for C gives $C = 1$. Thus the required function is given by

$$f(x) = 2\sin(x) - e^x + 1$$

Question 1 (b)

SOLUTION. Let $f(x) = x^4$, so $f'(x) = 4x^3$. The linear approximation of $f(x)$ at 2 is given by

$$\begin{aligned} L(x) &= f(2) + f'(2)(x-2) \\ &= 2^4 + 4 \cdot 2^3(x-2) \\ &= 16 + 32(x-2) \end{aligned}$$

Thus, $(2.001)^4 \approx L(2.001) = 16 + 32 \cdot (2.001 - 2) = 16 + \frac{32}{1000} = \frac{16032}{1000}$

Question 1 (c)

SOLUTION. Following the hints, we have

$$\ln y = \ln\left((\sin x)^{\sin x}\right) = \sin x \ln(\sin x)$$

Differentiating both sides yields

$$\begin{aligned} \frac{y'}{y} &= (\cos x)\ln(\sin x) + (\sin x)\tfrac{d}{dx}\ln(\sin x) \quad \text{(product rule)} \\ \frac{y'}{y} &= (\cos x)\ln(\sin x) + (\sin x)(\cos x) \cdot \frac{1}{\sin x} \quad \text{(chain rule)} \\ \frac{y'}{y} &= (\cos x)\ln(\sin x) + \cos x \\ y' &= y\cos(x)(\ln(\sin x) + 1) \end{aligned}$$

Finally, plugging in y yields the final answer

$$y' = (\sin x)^{\sin x}\cos(x)(\ln(\sin x) + 1)$$

Question 2 (a)

SOLUTION 1. Applying the chain rule twice gives

$$\begin{aligned} f'(x) &= e^{(\sin x)^2}\tfrac{d}{dx}(\sin x)^2 \\ &= e^{(\sin x)^2}(2\sin x)(\cos x) \end{aligned}$$

SOLUTION 2. Taking logarithms and differentiating gives

$$\begin{aligned}
\ln(f(x)) &= (\sin x)^2 \\
\frac{f'(x)}{f(x)} &= 2\sin x \cos x \\
f'(x) &= 2f(x)\sin x \cos x \\
f'(x) &= 2e^{(\sin x)^2}\sin x \cos x
\end{aligned}$$

Question 2 (b)

SOLUTION. Implicitly differentiating gives

$$\begin{aligned}
y' + \cos y + x\tfrac{d}{dx}(\cos y) &= -\sin x \\
y' + \cos y + x(-\sin y)(y') &= -\sin x
\end{aligned}$$

Plugging in $(x, y) = (0, 1)$ gives

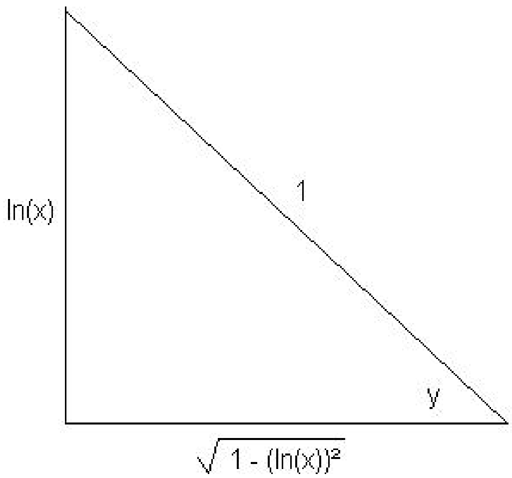
$$\begin{aligned}
y' + \cos(1) + 0(-\sin(1))(y') &= -\sin(0) \\
y' + \cos(1) &= 0 \\
y' &= -\cos(1)
\end{aligned}$$

completing the question.

Question 2 (c)

SOLUTION 1. Since
$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$
and
$\frac{d}{dx}\ln(x) = \frac{1}{x}$
The chain rule gives us
$\frac{d}{dx}\sin^{-1}(\ln(x)) = \frac{1}{\sqrt{1-(\ln(x))^2}} \cdot \frac{1}{x}$

SOLUTION 2. For an alternative solution, taking the sin of both sides yields
$\sin(y) = \ln(x)$
Taking implicit derivatives of both sides yields
$\cos(y)y' = \dfrac{1}{x}$
Solving for the derivative yields
$y' = \dfrac{1}{\cos(y)x}$
Next, we want to express the right hand side as a function of x. Since $\sin(y) = \ln(x)$ and $\sin(y)$ is opposite over hypotenuse, we can draw the following triangle, interpreting y as an angle:



where the last side (the base of the triangle) is computed using the Pythagorean theorem. Thus we see that
$\cos(y) = \sqrt{1-(\ln(x))^2}$
and plugging into the derivative yields
$y' = \dfrac{1}{x\sqrt{1-(\ln(x))^2}}$
completing the problem.

Question 2 (d)

SOLUTION. Taking the derivative gives

$$f'(x) = g'(2\sin(x))(2\cos(x))$$

Plugging in $x = \pi/4$ gives

$$\begin{aligned} f'(\pi/4) &= g'(2\sin(\pi/4))(2\cos(\pi/4)) \\ &= g'(2\cdot\tfrac{1}{\sqrt{2}})\cdot 2\cdot\tfrac{1}{\sqrt{2}} \\ &= g'(\sqrt{2})\cdot\sqrt{2} \\ &= \sqrt{2}\cdot\sqrt{2} \\ &= 2 \end{aligned}$$

Question 3 (a)

SOLUTION 1. We compute the equation of the line between the points $(-2,3)$ and $(0,5)$. First compute the slope via
$m = \frac{5-3}{0-(-2)} = \frac{2}{2} = 1.$
The y-intercept is given to be 5 since the point $(0,5)$ is on the line. Thus the equation of the line is $y = x+5$. As this line is tangent to the function at 1, we see that $m = 1 = f'(1)$ and $f(1) = y = (1)+5 = 6$ completing

the question.

Solution 2. According to the point slope formula given by
$y - y_0 = m(x - x_0)$
we can use the fact that our tangent line in question contains the point $(1, f(1))$ and has slope $f'(1)$ and thus, we have the equation of the line is given by
$y - f(1) = f'(1)(x - 1)$
Now, we know that the points $(-2, 3)$ and $(0, 5)$ lie on our cruve and this gives the two equations

$$\begin{aligned} 5 - f(1) &= f'(1)(0 - 1) \\ 3 - f(1) &= f'(1)(-2 - 1) \end{aligned}$$

Simplifying gives

$$\begin{aligned} 5 - f(1) &= -f'(1) \\ 3 - f(1) &= -3f'(1) \end{aligned}$$

Subtracting yields

$$2 = 2f'(1)$$

and so $f'(1) = 1$. Plugging back into either of the original equations yields $f(1) = 6$

Question 3 (b)

Solution. Following the advice of the hint since
$\lim_{x \to 4}(4x - 9) = 4(4) - 9 = 16 - 9 = 7$
and
$\lim_{x \to 4}(x^2 - 4x + 7) = (4)^2 - 4(4) + 7 = 16 - 16 + 7 = 7$
As these two limits are the same and our function $f(x)$ is bounded below and above by the two functions above, we have that $\lim_{x \to 4} f(x) = 7$ by the squeeze theorem.

Question 3 (c)

Solution. By the mean value theorem, we have a constant c between 1 and 4 such that
$\frac{f(4)-f(1)}{4-1} \leq f'(c) \leq -2$
Cross multiplying yields
$f(4) - f(1) \leq -2(4 - 1) = -6$
Since $f(1) = 3$, isolating for $f(4)$ gives
$f(4) \leq -6 + f(1) = -6 + 3 = -3$
Thus the largest value $f(4)$ can be is -3.

Question 4 (a)

Solution. To make this function continuous everywhere, we need
$\lim_{x \to 1} f(x) = f(1) = (1)^2 + a = 1 + a$
In particular, we require that the limit on the left exists. To do this, check left and right hand limits and find a value of a that makes them equal. First
$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^2 + a = 1^2 + a = 1 + a$

and for the other side
$\lim_{x\to 1^-} f(x) = 2x - 3 = 2(1) - 3 = -1$
Thus we need 1 + a = -1 and so $\boldsymbol{a}$ **= -2.**

Question 4 (b)

SOLUTION 1. Plugging in $x = 0$ in the limit gives an indeterminant form $0/0$ and so we may attempt to use L'Hopital's rule to get
$\lim_{x\to 0} \frac{e^{3x^2}-1}{\sin(x^2)} = \lim_{x\to 0} \frac{6xe^{3x^2}}{2x\cos(x^2)} = \lim_{x\to 0} \frac{3e^{3x^2}}{\cos(x^2)} = \frac{3e^0}{1} = 3$

SOLUTION 2. Using Taylor Polynomials we get

$$\begin{aligned}
\lim_{x\to 0} \frac{e^{3x^2}-1}{\sin(x^2)} &= \lim_{x\to 0} \frac{(1+(3x^2)^1/1+(3x^2)^2/2+\ldots)-1}{x^2-x^6/3+\ldots} \\
&= \lim_{x\to 0} \frac{3x^2+(3x^2)^2/2+\ldots}{x^2-x^6/3+\ldots} \\
&= \lim_{x\to 0} \frac{3+9x^2/2+\ldots}{1-x^4/3+\ldots} \\
&= 3
\end{aligned}$$

Question 4 (c)

SOLUTION. Proceeding as suggested in the hint, we have

$$\begin{aligned}
\lim_{x\to\infty} \left(1+\frac{5}{x}\right)^x &= \lim_{x\to\infty} e^{\ln\left(1+\frac{5}{x}\right)^x} \\
&= e^{\lim_{x\to\infty} x\ln\left(1+\frac{5}{x}\right)} \\
&= e^{\lim_{x\to\infty} \frac{\ln\left(1+\frac{5}{x}\right)}{\frac{1}{x}}}
\end{aligned}$$

Where we first use that $\ln(a^b) = b\ln(a)$, and then multiplied the top and bottom by $1/x$.
Let us now look at the exponent in more detail. For $x\to\infty$ the numerator and the denominator both go to 0 (since *ln(1) = 0*). We hence use L'Hospital's rule for the exponent:

$$\lim_{x\to\infty} \frac{\ln\left(1+\frac{5}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{5}{x}}\cdot\frac{-5}{x^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{5}{1+\frac{5}{x}}$$

Where we cancel the $-\frac{1}{x^2}$ in the second step. The remaining limit is straight forward and evaluates to 5. Hence our final answer is

$$\lim_{x\to\infty} \left(1+\frac{5}{x}\right)^x = e^5$$

Question 5 (a)

SOLUTION. Proceeding as the hint, let $y(t) = Ce^{kt}$ be the percentage of material remaining after t years. At time 0, we have 100% left, so we write:

$$\begin{aligned} 100 &= y(0) \\ &= Ce^{k0} \\ &= C \end{aligned}$$

Next, we know that after 10 years, there is 12.5% left, thus using $C = 100$:

$$\begin{aligned} 12.5 &= y(10) \\ &= 100e^{10k} \end{aligned}$$

Dividing by 100 and taking natural logarithms of both sides will give:
$\ln(12.5/100) = \ln(e^{10k})$
Because e^x and $\ln(x)$ are inverses, the right side simplifies to:
$\ln(12.5/100) = 10k$
And we can solve for k:
$k = \ln(12.5/100)/10 = \ln(1/8)/10$.
Now, we are looking for the percentage remaining after 15 years (5 years after the initial 10 years), or $y(15)$, so plugging in the data gives $y(15) = 100e^{15(\ln(1/8)/10)} = 100e^{\ln((1/8)^{15/10})} = 100(1/8)^{3/2}$
completing the question.

Question 5 (b)

SOLUTION 1. From part a, we know that
$y(t) = 100e^{t\ln(1/8)/10} = 100e^{\ln((1/8)^{t/10})} = 100(1/8)^{t/10}$
Writing as a power of 1/2, we have
$y(t) = 100(1/8)^{t/10} = 100(1/2^3)^{t/10} = 100(1/2)^{3t/10}$
In order for the rightmost expression to give us *50*, the exponent needs to equal *1*. Thus, the value of t that gives $y(t) = 50$ is $t = 10/3$

SOLUTION 2. From part a, we have
$y(t) = 100e^{t\ln(1/8)/10} = 100e^{\ln((1/8)^{t/10})} = 100(1/8)^{t/10}$
Because *y(t)* represents the percentage of Rhodium-101 remaining at time t, to find half-life (which is when there is 50% remaining), we set $y(t) = 50$ and solve for t.

$$\begin{aligned} 50 &= 100(1/8)^{t/10} \\ 1/2 &= (1/8)^{t/10} \\ 1/2 &= (1/2^3)^{t/10} \\ 1/2 &= (1/2)^{3t/10} \\ 1 &= \frac{3t}{10} \\ \frac{10}{3} &= t \end{aligned}$$

completing the question.

Question 6 (a)

SOLUTION. Differentiating $f(x) = 2x^{3/5} + 3x^{-2/5}$ gives
$f'(x) = \dfrac{6x^{-2/5}}{5} + \dfrac{-6x^{-7/5}}{5} = \dfrac{6x^{-7/5}}{5}(x-1)$.
Thus the critical numbers are $x = 1$ and $x = 0$ (notice here our derivative and our function are both undefined). So we check the intervals $(-\infty, 0), (0, 1), (1, \infty)$ individually.
Case 1

$$(-\infty, 0)$$

We test that the derivative at a point in the interval, say $x = -1$, which gives a value of $12/5 > 0$. Thus the function is increasing on this interval.
Case 2

$$(0, 1)$$

We test that the derivative at a point in the interval, say $x = 1/2$, which gives a value of $6 \cdot 2^{7/5}/5(-1/2) < 0$. Thus the function is decreasing on this interval.
Case 3

$$(1, \infty)$$

We test that the derivative at a point in the interval, say $x = 2$, which gives a value of $6 \cdot 2^{-7/5}/5 > 0$. Thus the function is increasing on this interval.
Summary
Overall, the function in increasing on $(-\infty, 0) \cup (1, \infty)$ and decreasing on $(0, 1)$.

Question 6 (b)

SOLUTION. As stated in the hint, local extrema can only occur at critical points that are also points in the domain. Based on our work in part a, we have two critical points, but only $x = 1$ is in the domain. Thus, we only need to check if $x = 1$ is a minimum or maximum. Here since we have a decreasing function left of *1* and an increasing function right of *1*, we have that $x = 1$ is a local minimum.

Question 6 (c)

SOLUTION. We are given that
$f''(x) = -(12/25)x^{-7/5} + (42/25)x^{-12/5} = (1/25)x^{-12/5}(-12x + 42)$
and this second derivative is undefined at $x = 0$. The second derivative is 0 when $-12x + 42 = 0$, that is, when $x = 42/12 = 7/2$. So we check the intervals $(-\infty, 0), (0, 7/2), (7/2, \infty)$ individually.
Note that $x^{-12/5} = \frac{1}{x^{12/5}} = \frac{1}{\sqrt[5]{x^{12}}} > 0$ for all values of $x \neq 0$.
Case 1

$$(-\infty, 0)$$

We test that the second derivative at a point in the interval, say $x = -1$, which gives a value of $54/25 > 0$. Thus the function is concave up on this interval.
Case 2

$$(0, 7/2)$$

We test that the derivative at a point in the interval, say $x = 1$, which gives a value of $30/25 > 0$. Thus the function is concave up on this interval.

Case 3

$$(7/2, \infty)$$

We test that the derivative at a point in the interval, say $x = 10$, which gives a value of $(1/25) * 10^{-12/5} * (-78) < 0$. Thus the function is concave down on this interval.
This completes the question.

Question 6 (d)

SOLUTION. By part c, we see immediately that concavity changes from concave up to concave down at the point $x = 7/2$. This is the only inflection point.

Question 6 (e)

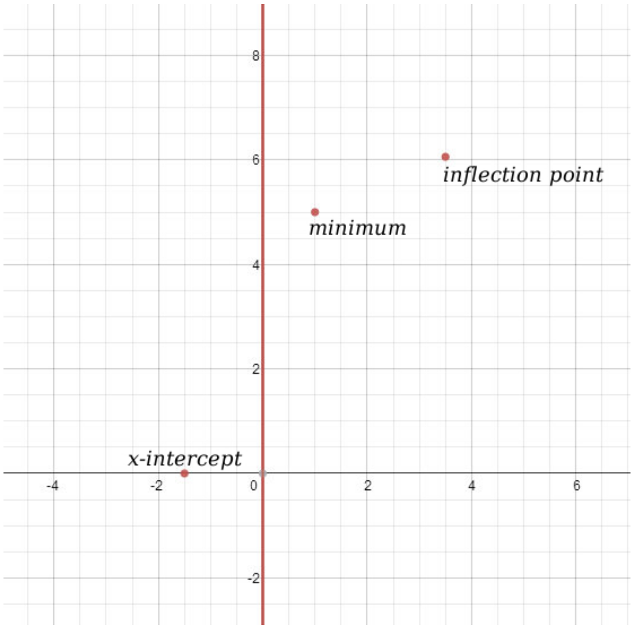
SOLUTION. As given in the original statement, the function has domain of all nonzero reals. The function itself is only not continuous at 0 so we check there for an asymptote.
The value

$$\lim_{x \to 0^+} (2x^{3/5} + 3x^{-2/5}) = \infty$$

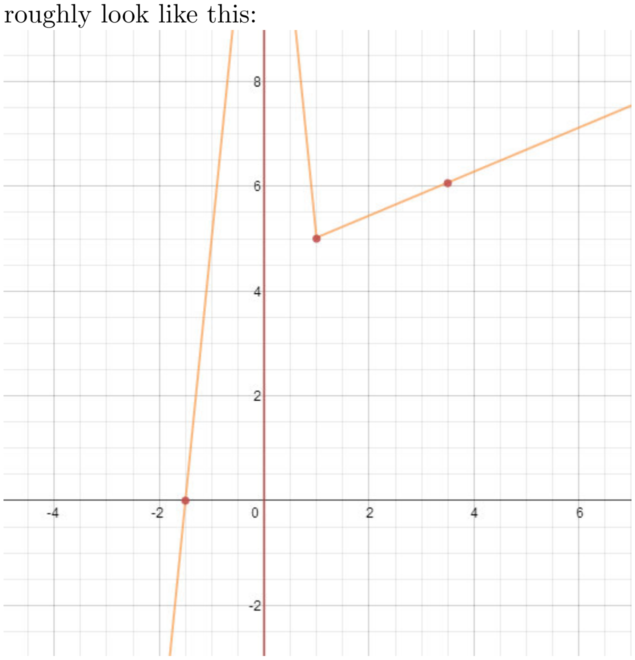
since this function when we plug in values of x slightly bigger than 0, the first term tends to 0 and the second term tends to positive infinity.
Notice that the left hand limit is also positive infinity since the second term can be written as $3x^{-2/5} = \dfrac{3}{\sqrt[5]{x^2}}$ which is always positive.

Question 6 (f)

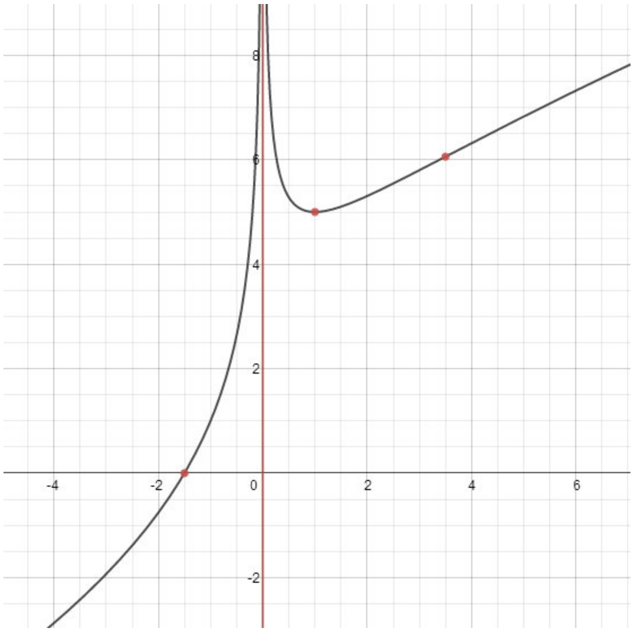
SOLUTION. Before assembling the graph, we will calculate the x-intercepts, found by setting the original function equal to zero and solving for x.
$2x^{3/5} + 3x^{-2/5} = 0$
Factoring out $x^{-2/5}$ gives:
$x^{-2/5}(2x + 3) = 0$
So there is an x-intercept at $x = -3/2$
To draw the graph, we will start by plotting the vertical asymptote at $x = 0$ and various points: the x-intercept at $(-3/2, 0)$, the minimum at $x = 1$ and the inflection point at $x = 7/2$. Because the question asks for the y-coordinate of the minimum (from part b), we plug $x = 1$ into $f(x)$ to get:
$f(1) = 2(1)^{3/5} + 3(1)^{-2/5} = 2 + 3 = 5$
So our minimum will be the point $(1, 5)$. All this gives a picture that looks like this:



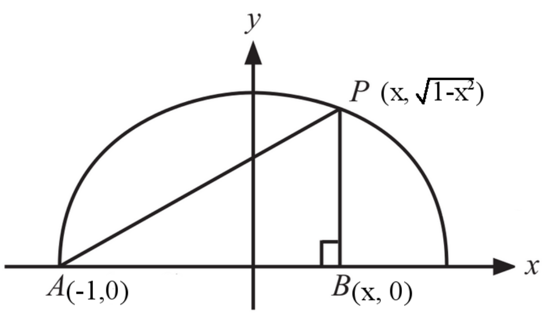
Using information from part a) about the intervals of increase and decrease, we know the graph should roughly look like this:



Finally, using part c), we know the function is concave up for $x < 7/2$ and concave down otherwise, which leads us to the final graph:



Question 7



SOLUTION.

We are trying to maximize the area of triangle ABP, which is given by the formula:
$A = \frac{1}{2}bh$
As shown in the picture above, the base of triangle ABP is $(1 + x)$ and the height is the y-value of point P, or $\sqrt{1 - x^2}$, so our final formula for area is:
$A(x) = \frac{1}{2}(1 + x)\sqrt{1 - x^2}$
where $-1 \leq x \leq 1$, as x cannot extend past the radius of the circle, which is *1*. The maximum of this function will occur either at the endpoints of the domain or critical points in the open interval $(-1, 1)$. Thus, the endpoints $x = -1$, $x = 1$ will be two of the points we test to find the maximum
To find remaining critical points in the interval $(-1, 1)$, we take the derivative and find where it is either equal to zero or undefined.
Using the product rule, we get:
$A'(x) = \frac{1}{2}\left(\sqrt{1 - x^2} + \frac{(1+x)(-2x)}{2\sqrt{1-x^2}}\right)$
Finding a common denominator, and combining terms on the top gives:

$$
\begin{aligned}
A'(x) &= \frac{1}{2}\left(\frac{2-2x^2-2x^2-2x}{2\sqrt{1-x^2}}\right)\\
&= \frac{1}{2}\left(\frac{2(1-x-2x^2)}{2\sqrt{1-x^2}}\right)\\
&= \frac{1}{2}\left(\frac{1-x-2x^2}{\sqrt{1-x^2}}\right)
\end{aligned}
$$

Setting equal to zero, we get:

$$
\begin{aligned}
0 &= \frac{1}{2}\left(\frac{1-x-2x^2}{\sqrt{1-x^2}}\right)\\
&= \frac{1-x-2x^2}{\sqrt{1-x^2}}\\
&= \frac{(1+x)(-2x+1)}{\sqrt{1-x^2}}
\end{aligned}
$$

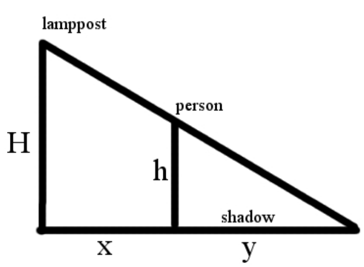
On the top, $x=-1$ and $x=1/2$ gives us zero, while on the bottom, $x=1$ and $x=-1$ makes the derivative undefined.
Thus, we test the endpoints, $x=-1, x=1$, and the critical point $x=1/2$ in the area formula $A(x)$ to determine which gives the maximum area.

$$
\begin{array}{llll}
A(-1) & =\frac{1}{2}(1+(-1))\sqrt{1-(-1)^2} & =\frac{1}{2}(0)(0) & =0\\
A(1) & =\frac{1}{2}(1+(1))\sqrt{1-(1)^2} & =\frac{1}{2}(2)(0) & =0\\
A(1/2) & =\frac{1}{2}(1+(1/2))\sqrt{1-(1/2)^2} & =\frac{1}{2}\left(\frac{3}{2}\right)\sqrt{\frac{3}{4}} & >0
\end{array}
$$

So the area is maximized when $x=1/2$. The associated point P is given by $\left(\frac{1}{2}, \sqrt{1-\left(\frac{1}{2}\right)^2}\right)$ or $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

Question 8

SOLUTION. To get started, we draw a diagram and label with variables.



Note that H and h (the height of the lamppost and the person) are constant but x and y (the distance of the person from the lamppost, and the length of the person's shadow) are changing.
We relate all the variables in the above picture by using similar triangles:
$\frac{y}{h}=\frac{x+y}{H}$
Cross-multiplying gives
$yH=xh+yh$
If we differentiate with respect to time, we get
$H\frac{dy}{dt}=h\frac{dx}{dt}+h\frac{dy}{dt}$
(Recall that H, h are constants, so they aren't differentiated.)

The question is asking us for the rate at which the person's shadow is changing, which is the same as the rate at which y is changing, or dy/dt. So we solve the above expression for dy/dt.
$H\frac{dy}{dt} - h\frac{dy}{dt} = h\frac{dx}{dt}$
$\frac{dy}{dt}(H-h) = h\frac{dx}{dt}$
$\frac{dy}{dt} = \frac{h\frac{dx}{dt}}{H-h}$
The rate v that is given in the question is the speed at which the person is walking, i.e. the rate at which x changes, so:
$\frac{dx}{dt} = v$
Plugging this into our previous formula, we get
$\frac{dy}{dt} = \frac{hv}{H-h}$
which is the final answer.

Question 9

SOLUTION 1. We proceed directly

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{1-2(x+h)^2}-\sqrt{1-2x^2}}{h} \\
&= \lim_{h\to 0} \left(\frac{\sqrt{1-2(x+h)^2}-\sqrt{1-2x^2}}{h}\right)\cdot\left(\frac{\sqrt{1-2(x+h)^2}+\sqrt{1-2x^2}}{\sqrt{1-2(x+h)^2}+\sqrt{1-2x^2}}\right) \\
&= \lim_{h\to 0} \frac{1-2(x+h)^2-(1-2x^2)}{h(\sqrt{1-2(x+h)^2}+\sqrt{1-2x^2})} \\
&= \lim_{h\to 0} \frac{1-2(x^2+2xh+h^2)-1+2x^2}{h(\sqrt{1-2(x+h)^2}+\sqrt{1-2x^2})} \\
&= \lim_{h\to 0} \frac{1-2x^2-4xh-2h^2-1+2x^2}{h(\sqrt{1-2(x+h)^2}+\sqrt{1-2x^2})} \\
&= \lim_{h\to 0} \frac{-4xh-2h^2}{h(\sqrt{1-2(x+h)^2}+\sqrt{1-2x^2})} \\
&= \lim_{h\to 0} \frac{-4x-2h}{\sqrt{1-2(x+h)^2}+\sqrt{1-2x^2}} \\
&= \frac{-4x-2(0)}{\sqrt{1-2(x+(0))^2}+\sqrt{1-2x^2}} \\
&= \frac{-4x}{\sqrt{1-2x^2}+\sqrt{1-2x^2}} \\
&= \frac{-4x}{2\sqrt{1-2x^2}} \\
&= \frac{-2x}{\sqrt{1-2x^2}}
\end{aligned}$$

Lastly, the domain of this function is anywhere that $1-2x^2 > 0$ as square roots cannot be evaluated for negative numbers and if the quantity were zero, then the derivative would have a zero denominator and so we want that the term inside the radical is strictly positive. This occurs when $-1/\sqrt{2} < x < 1/\sqrt{2}$

SOLUTION 2. We could also have used the other formula

$$
\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \\
&= \lim_{x\to a} \frac{\sqrt{1-2x^2}-\sqrt{1-2a^2}}{x-a} \\
&= \lim_{x\to a} \left(\frac{\sqrt{1-2x^2}-\sqrt{1-2a^2}}{x-a}\right)\cdot\left(\frac{\sqrt{1-2x^2}+\sqrt{1-2a^2}}{\sqrt{1-2x^2}+\sqrt{1-2a^2}}\right) \\
&= \lim_{x\to a} \frac{1-2x^2-(1-2a^2)}{(x-a)(\sqrt{1-2x^2}+\sqrt{1-2a^2})} \\
&= \lim_{x\to a} \frac{1-2x^2-1+2a^2}{(x-a)(\sqrt{1-2x^2}+\sqrt{1-2a^2})} \\
&= \lim_{x\to a} \frac{-2x^2+2a^2}{(x-a)(\sqrt{1-2x^2}+\sqrt{1-2a^2})} \\
&= \lim_{x\to a} \frac{-2(x^2-a^2)}{(x-a)(\sqrt{1-2x^2}+\sqrt{1-2a^2})} \\
&= \lim_{x\to a} \frac{-2(x-a)(x+a)}{(x-a)(\sqrt{1-2x^2}+\sqrt{1-2a^2})} \\
&= \lim_{x\to a} \frac{-2(x+a)}{\sqrt{1-2x^2}+\sqrt{1-2a^2}} \\
&= \frac{-2(a+a)}{\sqrt{1-2a^2}+\sqrt{1-2a^2}} \\
&= \frac{-4a}{2\sqrt{1-2a^2}} \\
&= \frac{-2a}{\sqrt{1-2a^2}}
\end{aligned}
$$

The domain is computed as in solution 1.

Question 10

SOLUTION. To find the form of the function we compute the antiderivative,
$f(x) = \frac{x^4}{4} + C$.
To find the actual function we are interested in, we need to determine C. The extra information we are given is that we want the line $x+y=0$, which is equivalent to $y=-x$ to be tangent at some point on the function $f(x)$. The slope of this line is -1 and thus we need $f'(x) = x^3 = -1$ which happens when $x=-1$. Plugging this point into the line we have that $y=-(-1)=1$ and so since the line is tangent to the function there, $f(-1)=1$ as well. Therefore we have,

$$
\begin{aligned}
f(-1) = \frac{1}{4} + C &= 1 \\
C = 1 - \frac{1}{4} &= \frac{3}{4}.
\end{aligned}
$$

Thus, our desired function is $y = \dfrac{x^4}{4} + \dfrac{3}{4}$.

Question 11

SOLUTION. Following the hints, we calculate the Taylor polynomials up to degree four:

$$
\begin{aligned}
T_0(x) &= 1\\
T_1(x) &= 1\\
T_2(x) &= 1-\frac{x^2}{2}\\
T_3(x) &= 1-\frac{x^2}{2}\\
T_4(x) &= 1-\frac{x^2}{2}+\frac{x^4}{24}
\end{aligned}
$$

We see that $T_2(x) = T_3(x) = 1\text{-}x^2/2$ indeed matches the middle term in the question statement. Hence we can use the Taylor remainder formula either for *n=2* or *n=3*. We choose *n=3* because this gives us a smaller error term here; we divide the error by 4! instead of 3!.
Our interval is $[-1, 1]$, we have that for x values in $[-1, 1]$,

$$
\begin{aligned}
\cos(x) - \left(1-\frac{x^2}{2}\right) &= f(x) - T_3(x) = R_3(x)\\
&= \frac{f''''(c)}{4!}(x-0)^4 = \frac{\cos(c)}{24}x^4
\end{aligned}
$$

where c is an unknown value in $[-1, 1]$. As $-\frac{\pi}{2} \le -1 \le c \le 1 \le \frac{\pi}{2}$, we have that $0 \le \cos(c) \le 1$ (drawing a picture helps confirm these computations - the cosine function is always positive on this interval). Also, we have that $0 \le x^4 \le 1$ and so we conclude that
$0 \le \cos(x) - \left(1-\frac{x^2}{2}\right) = R_3(x) = \frac{\cos(c)}{24}x^4 \le \frac{1}{24}$
as required.

Good Luck for your exams!