Full Solutions
MATH100 December 2013

April 4, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

SOLUTION.

$$\begin{aligned}\lim_{x\to -5}\frac{x^2+2x-15}{2x+10} &= \lim_{x\to -5}\frac{(x-3)(x+5)}{2(x+5)}\\ &= \lim_{x\to -5}\frac{x-3}{2}\\ &= \frac{(-5)-3}{2}\\ &= -4\end{aligned}$$

Question 1 (b)

SOLUTION. Firstly, we are given that $2x-3 \leq f(x) \leq x^2-4x+6$ for all $x \in [-1,\infty)$. Since $3 \in [-1,\infty)$, the initial condition of the squeeze theorem is satisfied.
Now, since $\lim_{x\to 3} 2x-3 = 3$ and$\lim_{x\to 3} x^2-4x+6 = 3$, we conclude by the squeeze theorem that $\lim_{x\to 3} f(x) = 3$.

Question 1 (c)

SOLUTION. Multiplying the numerator and denominator by $\sqrt{1+2x}+1$ yields:

$$\begin{aligned}\lim_{x\to 0}\frac{x}{\sqrt{1+2x}-1} &= \lim_{x\to 0}\left(\frac{x}{\sqrt{1+2x}-1}\cdot\frac{\sqrt{1+2x}+1}{\sqrt{1+2x}+1}\right)\\ &= \lim_{x\to 0}\frac{x(\sqrt{1+2x}+1)}{(\sqrt{1+2x})^2-1^2}\\ &= \lim_{x\to 0}\frac{x(\sqrt{1+2x}+1)}{(1+2x)-1}\\ &= \lim_{x\to 0}\frac{\sqrt{1+2x}+1}{2}\\ &= \frac{\sqrt{1+2\cdot 0}+1}{2}\\ &= 1\end{aligned}$$

Question 1 (d)

SOLUTION. Since the bacterial culture grows at a rate proportional to its size, we can say that $\frac{dy}{dt} = ky$, where $y(t)$ is the number of bacteria at time t (in hours).
The solution to this differential equation is:
$y(t) = y(0)e^{kt}$

We are given that $y(0) = 50$ and $y(1) = 250$, so we can solve for k as follows:

$$
\begin{aligned}
y(1) &= y(0)e^{k\cdot 1} \\
250 &= 50e^{k} \\
5 &= e^{k} \\
k &= \ln 5 \\
\\
\therefore y(t) &= 50e^{\ln(5)t} \\
&= 50\cdot 5^{t}
\end{aligned}
$$

Question 2 (a)

SOLUTION. Differentiating term-by-term and using the chain rule, we obtain:

$$
\begin{aligned}
y'(x) &= (e^{\sin x} + x)' \\
&= e^{\sin x}\cdot(\sin x)' + 1 \\
&= e^{\sin x}\cos x + 1 \\
\\
y'(0) &= e^{\sin(0)}\cos(0) + 1 \\
&= e^{0}\cdot 1 + 1 \\
&= 2
\end{aligned}
$$

The equation of the tangent line is therefore:

$$
\begin{aligned}
y &= 2(x-0)+1 \\
&= 2x+1
\end{aligned}
$$

Question 2 (b)

SOLUTION. First we take the natural logarithm of both sides:

$y = (\ln x)^x \Leftrightarrow \ln y = \ln((\ln x)^x) \Leftrightarrow \ln y = x\cdot\ln(\ln x)$ (by the laws of logarithms).
We then differentiate both sides with respect to x , using the product and chain rules as needed:

$$
\begin{aligned}
\frac{d}{dx}(\ln y) &= \frac{d}{dx}(x\cdot\ln(\ln x)) \\
\frac{1}{y}\cdot\frac{dy}{dx} &= 1\cdot\ln(\ln x) + x\cdot\frac{1}{\ln x}\cdot\frac{d}{dx}(\ln x) \\
\frac{dy}{dx} &= y\cdot\left(\ln(\ln x) + x\cdot\frac{1}{\ln x}\cdot\frac{1}{x}\right) \\
\frac{dy}{dx} &= (\ln x)^x\left(\ln(\ln x) + \frac{1}{\ln x}\right)
\end{aligned}
$$

Question 2 (c)

SOLUTION. We first apply the product rule to differentiate $h(x)$:

$$\begin{aligned}h'(x) &= (f(x^2))' \cdot g(x) + f(x^2) \cdot g'(x)\\ &= f'(x^2) \cdot \frac{d}{dx}(x^2) \cdot g(x) + f(x^2) \cdot g'(x)\\ &= f'(x^2) \cdot 2x \cdot g(x) + f(x^2) \cdot g'(x)\end{aligned}$$

Now we evaluate $h'(-1)$:

$$\begin{aligned}h'(-1) &= f'((-1)^2) \cdot 2(-1) \cdot g(-1) + f((-1)^2) \cdot g'(-1)\\ &= -2 \cdot f'(1) \cdot g(-1) + f(1) \cdot g'(-1)\\ &= -2 \cdot (-2) \cdot 2 + (-1) \cdot 3\\ &= 5\end{aligned}$$

Question 3 (a)

SOLUTION. Differentiating both sides implicitly with respect to x,

$$\begin{aligned}\frac{d}{dx}(x^3 + xy) &= \frac{d}{dx}(5x - y^3)\\ \frac{d}{dx}(x^3) + \frac{d}{dx}(xy) &= \frac{d}{dx}(5x) - \frac{d}{dx}(y^3)\\ 3x^2 + 1 \cdot y + x \cdot \frac{d}{dx}(y) &= 5 - 3y^2 \cdot \frac{d}{dx}(y)\\ 3x^2 + y + x \cdot \frac{dy}{dx} &= 5 - 3y^2 \cdot \frac{dy}{dx}\end{aligned}$$

Solving for $\frac{dy}{dx}$ we find

$$\begin{aligned}x \cdot \frac{dy}{dx} + 3y^2 \cdot \frac{dy}{dx} &= 5 - 3x^2 - y\\ \frac{dy}{dx} \cdot (x + 3y^2) &= 5 - 3x^2 - y\\ \frac{dy}{dx} &= \frac{5 - 3x^2 - y}{x + 3y^2}\end{aligned}$$

Question 3 (b)

SOLUTION. When the particle is moving in the *negative direction*, its velocity is *negative*. Therefore we must look for the t for which $\frac{ds}{dt} = f'(t) < 0$:

$$\begin{aligned}s &= t^3 - \frac{21}{2}t^2 + 30t\\ \frac{ds}{dt} &= 3t^2 - 21t + 30\end{aligned}$$

Setting the derivative (i.e., the velocity function) equal to zero, we obtain $0 = 3t^2 - 21t + 30 = 3(t-2)(t-5)$. Hence the velocity is zero when $t = 2, 5$.
Plugging in values tells us that $\frac{ds}{dt} = f'(t) < 0$ on the interval $t \in (2, 5)$, where t is in seconds as stated above.

Question 3 (c)

SOLUTION. We are concerned with the continuity of f only at $x = 1$, though we need not be concerned about whether $f(1)$ is defined or not, since its polynomial'pieces' are defined on $(-\infty, \infty)$. It remains to find a c such that $\lim_{x\to 1} f(x) = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$.

- By the definition of $f(x)$ we have $f(1) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} cx^2 + 3 = c + 3$.
- Furthermore, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2x^3 - c = 2 - c$.

Hence we need to find a c such that $2 - c = c + 3$. It follows that the value c that makes $f(x)$ continuous on $(-\infty, \infty)$ is $c = \frac{-1}{2}$.

Question 4 (a)

SOLUTION. Newton's method is defined by $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, where x_n is the n-th approximation to a root of f.
We are given that $x_1 = -1$. Applying the equation above,

$$\begin{aligned} x_{1+1} &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ x_2 &= -1 - \frac{f(-1)}{f'(-1)} \\ &= -1 - \frac{(-1)^7 + 4}{7(-1)^6} \\ &= -1 - \frac{-1+4}{7\cdot 1} \\ &= -\frac{10}{7} \end{aligned}$$

Question 4 (b)

SOLUTION. Polynomial division shows us that $\frac{x^3+x^2}{x^2+1} = (x+1) + \frac{-x-1}{x^2+1}$. Therefore

$$\begin{aligned} \frac{x^3+x^2}{x^2+1} - (x+1) &= \frac{-x-1}{x^2+1} \\ \lim_{x\to\pm\infty} [f(x) - (x+1)] &= \lim_{x\to\pm\infty} \frac{-x-1}{x^2+1} \\ \lim_{x\to\pm\infty} [f(x) - (x+1)] &= 0 \end{aligned}$$

Hence $y = x + 1$ is a slant asymptote of $f(x)$.

Question 4 (c)

SOLUTION. By definition of the Taylor polynomials,

$$\begin{aligned}
T_2(x) &= \sum_{i=0}^{2} \frac{f^{(i)}(a)}{i!}(x-a)^i \\
&= \frac{f^{(0)}(a)}{0!}(x-a)^0 + \frac{f^{(1)}(a)}{1!}(x-a)^1 + \frac{f^{(2)}(a)}{2!}(x-a)^2 \\
&= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2
\end{aligned}$$

Computing the values of the derivatives of f at 1, we find that $f(1) = 4$, $f'(1) = 12$, $f''(1) = 24$. Therefore

$$\begin{aligned}
T_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 \\
&= 4 + 12(x-1) + 12(x-1)^2 \\
&= 12x^2 - 12x + 4
\end{aligned}$$

Question 5

SOLUTION. The solution of the differential equation $\frac{dT}{dt} = k(T - T_s)$ is

$T - T_s = (T_0 - T_s)e^{kt}$, where T is the temperature of an object, T_s is the temperature of its surroundings, and k is some constant.
We are given that $T_s = 40$, $T_0 = 70$, $T(0.5) = 60$, where t is in hours. Thus we begin by solving for k:

$$\begin{aligned}
T(t) - T_s &= (T_0 - T_s)e^{kt} \\
60 - 40 &= (70 - 40)e^{k\cdot 0.5} \\
e^{0.5k} &= \frac{2}{3} \\
\frac{1}{2}k &= \ln\frac{2}{3} \\
k &= \ln\frac{4}{9}
\end{aligned}$$

$$\begin{aligned}
\therefore T(t) &= 30e^{\ln\left(\frac{4}{9}\right)\cdot t} + 40 \\
&= 30\left(\frac{4}{9}\right)^t + 40
\end{aligned}$$

Finally, solving for t when $T(t) = 50$ yields:

$$
\begin{aligned}
50 &= 30\left(\frac{4}{9}\right)^t + 40 \\
\frac{1}{3} &= \left(\frac{4}{9}\right)^t \\
\ln\frac{1}{3} &= t\ln\frac{4}{9} \\
t &= \frac{\ln\frac{1}{3}}{\ln\frac{4}{9}} = \frac{\ln 3}{\ln\frac{9}{4}} \text{ hours}
\end{aligned}
$$

Question 6

SOLUTION 1. By the limit definition of the derivative of f at x,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{\frac{x+h}{2(x+h)+1}-\frac{x}{2x+1}}{h} \\
&= \lim_{h\to 0}\frac{(x+h)(2x+1)-x(2(x+h)+1)}{h(2(x+h)+1)(2x+1)} \\
&= \lim_{h\to 0}\frac{(2x^2+x+2xh+h)-(2x^2+2xh+x)}{h(2(x+h)+1)(2x+1)} \\
&= \lim_{h\to 0}\frac{h}{h(2(x+h)+1)(2x+1)} \\
&= \lim_{h\to 0}\frac{1}{(2(x+h)+1)(2x+1)} \\
&= \frac{1}{(2(x+0)+1)(2x+1)} \\
&= \frac{1}{(2x+1)^2}
\end{aligned}
$$

Use of the quotient rule confirms that this is indeed $f'(x)$:

$$
\begin{aligned}
f'(x) &= \frac{1\cdot(2x+1)-x\cdot 2}{(2x+1)^2} \\
&= \frac{1}{(2x+1)^2}
\end{aligned}
$$

Solution 2. By the alternative limit definition of the derivative of f at a,

$$\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \\
&= \lim_{x\to a} \frac{\frac{x}{2x+1} - \frac{a}{2a+1}}{x-a} \\
&= \lim_{x\to a} \frac{\frac{x(2a+1)-a(2x+1)}{(2x+1)(2a+1)}}{x-a} \\
&= \lim_{x\to a} \frac{\frac{(2ax+x)-(2ax+a)}{(2x+1)(2a+1)}}{x-a} \\
&= \lim_{x\to a} \frac{\frac{x-a}{(2x+1)(2a+1)}}{x-a} \\
&= \lim_{x\to a} \frac{1}{(2x+1)(2a+1)} \\
&= \frac{1}{(2a+1)^2} \\
f'(x) &= \frac{1}{(2x+1)^2}
\end{aligned}$$

Use of the quotient rule confirms that this is indeed $f'(x)$:

$$\begin{aligned}
f'(x) &= \frac{1\cdot(2x+1) - x\cdot 2}{(2x+1)^2} \\
&= \frac{1}{(2x+1)^2}
\end{aligned}$$

Question 7

Solution. If the distance between the spectator and the launchpad is x and the distance between the launchpad and the rocket is y, the distance z between the rocket and the spectator is $z = \sqrt{x^2+y^2}$. We are given that $x = 4$ km and $y = 3$ km. Thus $z = 5$ km.
Note that since the distance between the spectator and the launchpad is not changing, $\frac{dx}{dt} = 0\,\mathrm{km/s}$. Differentiating both sides implicitly with respect to time t:

$$\begin{aligned}
z^2 &= x^2 + y^2 \\
2z\cdot\frac{dz}{dt} &= 2x\cdot\frac{dx}{dt} + 2y\cdot\frac{dy}{dt} \\
\frac{dz}{dt} &= \frac{1}{z}\left(x\cdot\frac{dx}{dt} + y\cdot\frac{dy}{dt}\right) \\
&= \frac{1}{5\,\mathrm{km}}\left(4\,\mathrm{km}\cdot 0\,\mathrm{km/s} + 3\,\mathrm{km}\cdot 0.7\,\mathrm{km/s}\right) \\
&= 0.42\,\mathrm{km/s}
\end{aligned}$$

Question 8 (a)

Solution. By the definition of a **critical number**, we must find the x for which $f'(x) = 0$ or for which $f'(x)$ does not exist:

$$\begin{aligned} f(x) &= 63x^{\frac{2}{7}} - 14x^{\frac{9}{7}} \\ f'(x) &= \frac{2}{7} \cdot 63x^{\frac{2}{7}-1} - \frac{9}{7} \cdot 14x^{\frac{9}{7}-1} \\ &= 18x^{-\frac{5}{7}} - 18x^{\frac{2}{7}} \\ &= 18x^{-\frac{5}{7}}(1-x) \end{aligned}$$

Setting $f'(x) = 0$ and applying the zero product property:

$$\begin{aligned} 18x^{-\frac{5}{7}}(1-x) &= 0 \\ 18x^{-\frac{5}{7}} = 0, \quad 1-x &= 0 \\ \frac{18}{x^{\frac{5}{7}}} = 0, \quad 1-x &= 0 \\ x &= 1 \end{aligned}$$

Note that $18x^{-\frac{5}{7}}$ is never equal to zero on the interval $(-\infty, \infty)$. However, $18x^{-\frac{5}{7}}$ is undefined at $x = 0$, so $f'(x)$ does not exist at $x = 0$.
Therefore the critical numbers of f are $x = 0, 1$.

Question 8 (b)

SOLUTION. Where $f'(x) > 0$, $f(x)$ is *increasing*; where $f'(x) < 0$, $f(x)$ is *decreasing*.
From part (a), we note that $f'(x)$ may change sign at its critical points $x = 0$ and $x = 1$. We can construct a sign table to organize our calculations:

	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
$18x^{-\frac{5}{7}}$	$-$	$+$	$+$
$1-x$	$+$	$+$	$-$
$f'(x) = (18x^{-\frac{5}{7}})(1-x)$	$-$	$+$	$-$

From this table we observe that $f(x)$ is **increasing** on $x \in (0, 1)$ and**decreasing** on $x \in (-\infty, 0) \cup (1, \infty)$.

Question 8 (c)

SOLUTION. Where $f''(x) > 0$, $f(x)$ is *concave up*; where $f''(x) < 0$, $f(x)$ is *concave down*. From part (a), we know that

$$\begin{aligned} f'(x) &= 18\left(x^{-\frac{5}{7}} - x^{\frac{2}{7}}\right) \\ f''(x) &= 18\left(-\tfrac{5}{7} \cdot x^{-\frac{5}{7}-1} - \tfrac{2}{7} \cdot x^{\frac{2}{7}-1}\right) \\ &= 18\left(-\tfrac{5}{7}x^{-\frac{12}{7}} - \tfrac{2}{7}x^{-\frac{5}{7}}\right) \\ &= \tfrac{18}{7}x^{-\frac{12}{7}}(-5-2x) \end{aligned}$$

Setting $f''(x) = 0$ and applying the zero product property:

$$\frac{18}{7}x^{-\frac{12}{7}}(-5-2x) = 0$$

$$\frac{18}{7}x^{-\frac{12}{7}} = 0 \qquad -5-2x = 0$$

$$-5-2x = 0$$

$$x = -\frac{5}{2}$$

Note that $\frac{18}{7}x^{-\frac{12}{7}}$ is never equal to zero on the interval $(-\infty, \infty)$. However, $\frac{18}{7}x^{-\frac{12}{7}}$ is undefined at $x = 0$, so $f''(x)$ does not exist at $x = 0$.
Therefore, we note that $f''(x)$ may change sign at $x = 0$ and $x = -\frac{5}{2}$. We can construct a sign table to organize our calculations:

	$(-\infty, -\frac{5}{2})$	$(-\frac{5}{2}, 0)$	$(0, \infty)$
$\frac{18}{7}x^{-\frac{12}{7}}$	$+$	$+$	$+$
$-5-2x$	$+$	$-$	$-$
$f''(x) = (\frac{18}{7}x^{-\frac{12}{7}})(-5-2x)$	$+$	$-$	$-$

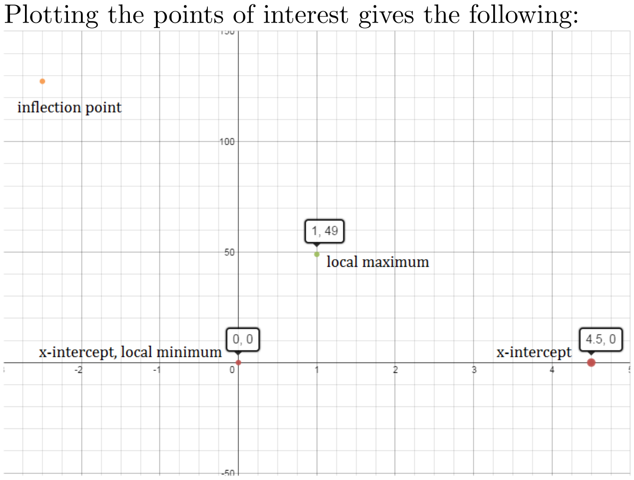
From this table we observe that $f(x)$ is **concave up** on $x \in (-\infty, -\frac{5}{2})$ and**concave down** on $x \in (-\frac{5}{2}, 0) \cup (0, \infty)$.

Question 8 (d)

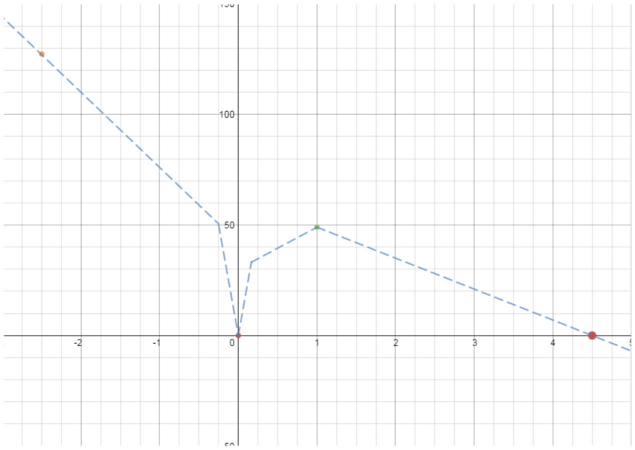
Solution. From parts (a), (b), and (c) we know the following about $f(x) = 63x^{2/7} - 14x^{9/7}$:

- The **critical numbers** of f (where f' is zero or does not exist) are $x = 0, 1$
- f is **increasing** on $x \in (0, 1)$ and **decreasing** on $x \in (-\infty, 0) \cup (1, \infty)$
- f has a **local minimum** at $x = 0$ and a **local maximum** at $x = 1$ (note the change in the increase/decrease of f)
- f is **concave up** on $x \in (-\infty, -\frac{5}{2})$ and **concave down** on $x \in (-\frac{5}{2}, 0) \cup (0, \infty)$
- f has an **inflection point** (changes concavity) at $x = -\frac{5}{2}$
- $f(x) = x^{\frac{2}{7}}(63 - 14x)$ and thus f has x-intercepts at $x = 0, \frac{9}{2}$
- $\lim_{x\to 0^-} f'(x) = -\infty$ and $\lim_{x\to 0^+} f'(x) = \infty$, thus the graph of f is very steep near $x = 0$

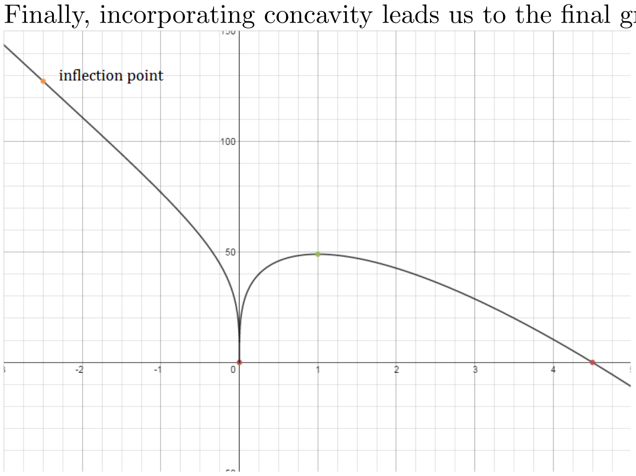
Plotting the points of interest gives the following:



Accounting for increase and decrease gives the following rough sketch:



Finally, incorporating concavity leads us to the final graph:



Question 9

SOLUTION. If the base of the rectangle measures $2b$, its area is $A(b) = 2b(8 - b^2) = 16b - 2b^3$. To maximize A, we find its critical points by finding A':

$$A(b) = 16b - 2b^3$$
$$A'(b) = 16 - 6b^2$$

$$0 = 16 - 6b^2$$
$$6b^2 = 16$$
$$b = \pm\sqrt{\frac{8}{3}}$$

A simple inspection shows that $b = \sqrt{\frac{8}{3}}$ is a maximum point of A since A' changes sign from positive to negative at that point. The value of *A(b)* at that b (i.e., the maximum area) is:

$$
\begin{aligned}
A\left(\sqrt{\frac{8}{3}}\right) &= 2\sqrt{\frac{8}{3}}\left(8-\left(\sqrt{\frac{8}{3}}\right)^2\right)\\
&= \frac{32}{3}\cdot\sqrt{\frac{8}{3}} = \frac{64\sqrt{2}}{3\sqrt{3}} = \frac{64\sqrt{6}}{9}
\end{aligned}
$$

Question 10 (a)

SOLUTION. The equation of the *first degree* **Maclaurin polynomial** for f is $T_1(x) = f(0) + f'(0)\cdot x$. For $f(x) = \ln(1-x^2)$ we have $f(0) = \ln(1-0^2) = 0$. To find $f'(0)$, we differentiate $f(x)$ using the chain rule:

$$
\begin{aligned}
f'(x) &= \frac{1}{1-x^2}\cdot\frac{d}{dx}(1-x^2)\\
&= \frac{-2x}{1-x^2}\\
f'(0) &= 0
\end{aligned}
$$

Thus the first degree Maclaurin polynomial for f is $T_1(x) = 0 + 0x = 0$.

Question 10 (b)

SOLUTION. To apply the Lagrange Remainder Formula, we first need to find an x such that $f(x) = \ln(1-x^2) = \ln\left(\frac{8}{9}\right)$, that is, $1-x^2 = \frac{8}{9} \Leftrightarrow x = \pm\frac{1}{3}$.

Since $\ln(1-x^2)$ is an even function, we choose the positive root $x = \frac{1}{3}$ without loss of generality.

By Taylor's Remainder theorem (as applied to Maclaurin polynomials),

$$
T_1(x) + \min\{R_1(x)\} \le f(x) \le T_1(x) + \max\{R_1(x)\}
$$

where $\min\{R_1(x)\}$ and $\max\{R_1(x)\}$ denote the minimum and maximum values of $R_1(x)$ on the interval $[0, x]$, respectively.

Since we know that $T_1(x) = 0$ from part (a), we simply consider:

$$
\min\{R_1(x)\} \le \ln\left(\frac{8}{9}\right) \le \max\{R_1(x)\}
$$

We now calculate the upper and lower bounds of the error. To do so, we substitute $x = \frac{1}{3}$ into the Lagrange Remainder Formula:

$$
R_1\left(\frac{1}{3}\right) = \frac{f''(c)}{2!}\left(\frac{1}{3}\right)^2 = \frac{f''(c)}{18}
$$

for some c in in the interval $[0, \frac{1}{3}]$. To find the error bounds, we maximize this function as a function of c. But observe that

$$f''(x) = \frac{-2(x^2+1)}{(1-x^2)^2}$$

is monotonically decreasing on the interval $[0, \frac{1}{3}]$ (the denominator is increasing while the numerator is decreasing (note the minus sign), so overall $f''(x)$ is decreasing). Hence the minimum and maximum values of the error are attained at $c = \frac{1}{3}$ and $c = 0$, respectively.
Plugging this in we obtain the required lower and upper bound for the error:

$$\frac{f''(\frac{1}{3})}{18} \leq \ln\left(\frac{8}{9}\right) \leq \frac{f''(0)}{18}$$

$$\frac{1}{18} \cdot \frac{-2((\frac{1}{3})^2+1)}{(1-(\frac{1}{3})^2)^2} \leq \ln\left(\frac{8}{9}\right) \leq \frac{1}{18} \cdot \frac{-2(0^2+1)}{(1-0^2)^2}$$

$$\frac{1}{18} \cdot \frac{-180}{64} \leq \ln\left(\frac{8}{9}\right) \leq \frac{1}{18} \cdot -2$$

$$-\frac{5}{32} \leq \ln\left(\frac{8}{9}\right) \leq -\frac{1}{9}$$

Question 11

SOLUTION. First we observe that 4^x and $3\cos x$ are both continuous on $\mathbb{R}$. It follows that $f(x) = 4^x - 3\cos x$ is also continuous on $\mathbb{R}$. Note that $f(x) = 0 \iff 4^x = 3\cos x$. We must show that $f(x)$ has at most one zero.
Optional: Show that a zero *exists*.
Plugging in the values $f(0) = 4^0 - 3\cos(0) = 1 - 3 = -2$ and $f\left(\frac{\pi}{2}\right) = 4^{\pi/2} - 3\cos\left(\frac{\pi}{2}\right) = 4^{\pi/2}$ we see that $f(0) < 0 < f\left(\frac{\pi}{2}\right)$. Thus by the Intermediate Value Theorem, $\exists c \in \left(0, \frac{\pi}{2}\right) : f(c) = 0$. In particular, this implies that this positive number c satisfies $4^c = 3\cos c$.
Necessary: Show that the zero is *unique*.
If $f(x)$ is monotonically increasing (or monotonically decreasing) for all positive values, then $f(x)$ can only have one zero in the interval $(0, \infty)$. Hence we calculate

$$f'(x) = 4^x \ln 4 + 3\sin x.$$

$f(x)$ is increasing where $f'(x) > 0$, i.e., where $4^x \ln 4 > -3\sin x$. We consider the positive real numbers as follows:
Case I: $0 < x < \pi$. Since $4^x \ln 4 > 0$ and $-3\sin x < 0$ on this interval, $4^x \ln 4 > -3\sin x$ on the interval $(0, \pi)$.
Case II: $x \geq \pi$. Since $-3 \leq -3\sin x \leq 3$ and $4^x \ln 4 \geq 4^\pi \ln 4 > 3\ln e = 3$ on this interval, $4^x \ln 4 > -3\sin x$ on the interval $[\pi, \infty)$.
It follows that $f'(x) > 0$ on the interval $(0, \infty)$ and that $f(x)$ is monotonically increasing on the interval $(0, \infty)$. We therefore conclude that if there exists a *positive* number c such that $4^c = 3\cos c$, this number must be *unique*.

Good Luck for your exams!