Full Solutions
MATH152 April 2010

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How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question A 01

SOLUTION. The length of a vector, $\|\mathbf{x}\|$ where $\mathbf{x}$ has components $[x_1, x_2, x_3]$ is,

$$||\mathbf{x}|| = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

Since $\mathbf{x}$=[1,-1,-1] then

$$||\mathbf{x}|| = \sqrt{1+1+1} = \sqrt{3}.$$

Question A 02

SOLUTION. If we think of denoting the unit vectors in the x, y, z directions as $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ respectively then we can turn our vectors $\mathbf{x}$ and $\mathbf{y}$ into the following matrix,
$\begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & -1 \\ 1 & 1 & 2 \end{bmatrix}$.
Firstly notice that we align the components of the vectors such that the x components are in the column with the heading $\hat{\mathbf{i}}$, the y components are in the column with the heading $\hat{\mathbf{j}}$, and the z components are in the column with the heading $\hat{\mathbf{k}}$. Secondly, this may look odd as a matrix since one of the rows is a vector. This matrix is artificial but if we treat it as if it were a standard matrix and computed its determinant, we would get

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & -1 \\ 1 & 1 & 2 \end{vmatrix} = -1\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}} = [-1, -3, 2]$$

which is precisely the definition of the cross product. Therefore $\mathbf{x} \times \mathbf{y} = [-1, -3, 2]$.

Question A 03

SOLUTION. From the hint we see that two vectors $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent if they are multiples of one another. This would mean that we can write,

$$\mathbf{x} = k\mathbf{y}$$

for some constant k. This is equivalent to

$$\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

or

$$\begin{aligned} 1 &= k \\ -1 &= k \\ -1 &= 2k \end{aligned}$$

The first equation tells us k=1 but the other two equations then say that 1=-1 and 2=-1 which are both false and therefore there does not exist a unique k such that $\mathbf{x}$=$k\mathbf{y}$ and thus $\mathbf{x}$ and $\mathbf{y}$ are not multiples of one another so they are **not** linearly dependent.

Question A 04

SOLUTION. Recall the conditions for a unique solution, no solution and an infinite number of solutions. For a unique solution, the row reduced matrix must be the identity matrix so that each unknown has precisely one determined value (hence unique). For no solutions to occur we need either a contradiction (i.e. trying to determine that an unknown takes on two values) or we need an impossible situation (i.e. trying to determine that 0x=7). For an infinite number of solutions we need a free variable in the form 0z=0.
We have 4 equations and 3 unknowns so our matrix is 4×3. We are trying to determine three variables with 4 conditions. Recall that to determine n unknowns we need exactly n equations (i.e. we only need 3 conditions here). Therefore, the only way we will get a unique solution is if one of the equations is redundant information (so that in actuality we only have 3 conditions). This would appear in the matrix as a row of zeros (since one row will be a multiple of another). For example, consider a system row reduced as follows

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

If we call our unknowns, x,y, and z then we see that x=7, y=2, z=4 and then the last row tells us 0x+0y+0z=0 which holds true and so we have a **unique** solution.
If the 4th condition was not simply redundant information but rather a independent condition then we could row reduce to get something of the form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 12 \end{array}\right].$$

Here we still are determining that x=7, y=2, and z=4 but we also require that 0x+0y+0z=12 which doesn't hold true for these values (in fact it will never hold true) and therefore all 4 conditions can't be satisfied so this results in **no solution**.
We have considered when there is no redundancy in the constraints and when one condition is redundant. What if multiple conditions are redundant? Let's say we have a model with the following constraints

$$\begin{aligned} 1x + 2y + 3z &= 5 \\ 1x + 3y + 6z &= 2 \\ 2x + 4y + 8z &= 10 \\ 3x + 6y + 12z &= 15. \end{aligned}$$

Notice that equation 3 is just 2 times equation 1 and equation 4 is 3 times equation 1. Therefore, really 3 of the four equations are just saying that $1x+2y+3z=5$. The row reduced matrix would be

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 11 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

In this example we have a free variable because we only have 2 pivots. Therefore let z=t. Then we conclude that

$$\begin{aligned}
x &= 11 \\
y &= -3 - 2z = -3 - 2t \\
z &= t
\end{aligned}$$

or

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}.$$

Here we see that there are actually an **infinite number of solutions** depending on the parameter t. Therefore we have constructed examples where we can have **unique solutions**, **no solutions**, or an **infinite number of solutions** and thus we see the answer is **(e)**

Question A 05

SOLUTION. Recall that $n \times n$ matrices have n eigenvalues. Here since n=2 we have 2 eigenvalues which are given as 2 and 3. Also recall that if A is not invertible then there exists at least one non-zero vector $\mathbf{x}$ such that A$\mathbf{x}$=0. Therefore, we see that if A is not invertible then there is at least one zero eigenvalue with eigenvector $\mathbf{x}$. Conversely, if 0 is an eigenvalue then there exists a non-zero vector $\mathbf{x}$ such that A$\mathbf{x}$=0 which means A is not invertible. This concludes that A being non-invertible implies a zero eigenvalue and also that a zero eigenvalue implies that A is non-invertible. We just wrote that the only eigenvalues are 2 and 3 so therefore, 0 is not an eigenvalues which means A must be invertible.

Question A 06

SOLUTION. A requirement for a matrix to be invertible is that its determinant is non-zero. Conversely, non-invertible matrices have zero determinants. Therefore, if we compute the determinant of the matrix and force it to be zero, that will give a requirement on a. We can compute the determinant of the matrix by cofactor expansion (we will choose the first row),

$$\begin{vmatrix} 1 & 0 & a \\ -1 & 1 & 1 \\ 2 & 1 & 0 \end{vmatrix} = 1 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + a \begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} = -3a - 1.$$

We want this determinant to vanish and so we require -3a-1=0 or a=-1/3. Therefore, if a=-1/3 then the matrix will have a zero determinant and thus *not* be invertible.

Question A 07

SOLUTION. We see from the hint that the formula for an inverse of 2×2 matrices of the form

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

In this example we have

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

and so

$$\det(A) = 1 - 2 = -1.$$

Therefore we have that

$$A^{-1} = -1 \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}.$$

Question A 08

SOLUTION. We can write the dot product between two vectors $\mathbf{u}$ and $\mathbf{v}$ as

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos(\theta)$$

where θ is the angle between vectors $\mathbf{u}$ and $\mathbf{v}$. Therefore we have that

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}.$$

Here our two vectors are

$$\begin{aligned} \mathbf{u} &= [1, 2, 3] \\ \mathbf{v} &= [-1, 5, -2] \end{aligned}$$

We have that

$$\begin{aligned} |\mathbf{u}| &= \sqrt{1 + 4 + 9} = \sqrt{14} \\ |\mathbf{v}| &= \sqrt{1 + 25 + 4} = \sqrt{30} \end{aligned}$$

and that

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = -1 + 10 - 6 = 3.$$

Therefore,

$$\cos(\theta) = \frac{3}{\sqrt{14}\sqrt{30}} = \frac{3}{\sqrt{420}}.$$

Question A 09

SOLUTION. From the current loop rule we know that each loop gets its own current which is an unknown for the system. We also need an unknown for the voltage drop that occurs across each current source. Since we have three loops we have **3** loop currents, i_1, i_2, i_3 and we have one current source so we have **1** unknown voltage drop v_1 for current source E. Therefore in total we have **4** unknowns for our linear system,

$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ v_1 \end{bmatrix}.$$

Question A 10

SOLUTION. To be consistent with the labs, we will denote everything in terms of voltage drops so positive numbers are voltage drops and negative numbers are voltage gains.
In the second loop we move clockwise and start with the 2Ω resistor. The voltage across this resistor (and any resistor) is given by Ohm's Law V=IR so $V_{2\Omega} = 2i_2$ where we have a positive sign because we're moving with the current and so there is a voltage drop. Next we come to the 4Ω resistor which has both loop current 2 and loop current 3 flowing through it. Therefore there is a voltage drop contribution from loop current 2 by virtue that we're moving in the direction of that current but there is also a voltage gain from loop current 3 going in the opposite direction. Therefore we get that $V_{4\Omega} = 4(i_2 - i_3)$. Next we reach the battery. The voltage gain is from the minus (short) terminal to the positive (long) terminal. Since we're going from the long to short terminal we are experiencing a voltage drop of 7V. Next is the 3Ω resistor and since we're moving with the current $V_{3\Omega} = 3i_2$. Finally we reach the current source; since we are traversing it with the arrow we have a voltage gain and the value is one of our unknowns, v_1. The Kirchhoff rule states that the sum of voltage drops and gains in a loop must be zero and so we have that

$$0 = 2i_2 + 4(i_2 - i_3) + 7 + 3i_2 - v_1$$

as the linear equation for loop 2.

Question A 11

SOLUTION. To figure out the current on the current source E, we must choose which direction we will cross the source. It shouldn't matter which direction we take and so we will show both possibilities.
Traverse the current source from bottom to top:
Here we are moving with current 2 ($+i_2$) and against current 1 ($-i_1$). E has a current of 9A in the upward direction (indicated by the arrow). Since we are moving upwards then the current is +9A. Therefore,

$$i_2 - i_1 = 9A.$$

Traverse the current source from bottom to top:
Here we are moving with current 1 ($+i_1$) and against current 2 ($-i_2$). E has a current of 9A in the upward direction (indicated by the arrow). Since we are moving downward then the current is -9A. Therefore,

$$i_1 - i_2 = -9A.$$

Notice that these are the same equation as they should be.
Full Solution for Those Interested:
If people are looking to practice there work with circuits, we can continue work from here and from A10 to solve for the unknown voltages and loop currents. Based on the same reasoning in A10 we can get that the Kirchhoff linear equation for loop 1 is

$$-6 + i_1 + v_1 = 0$$

where recall that positive numbers mean voltage drops (across the resistors and across the current source since it points upwards while our clockwise current points downwards there) and that negative numbers mean voltage gains (across the battery since we move from the negative to positive terminal). We already have the second loop from A10 as

$$2i_2 + 4(i_2 - i_3) + 7 + 3i_2 - v_1 = 0$$

and so the third and final loop is

$$4(i_3 - i_2) + 5i_3 + 8 = 0$$

where the 4Ω resistor has a voltage drop since we're moving in the direction of current loop 3 but a voltage gain from current loop 2.
So far we only have 3 equations for 4 unknowns; the last one comes from the current source. From above we already have that

$$i_2 - i_1 = 9.$$

We can write this in a matrix problem as

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 9 & -4 & -1 \\ 0 & -4 & 9 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ v_1 \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \\ -8 \\ 9 \end{bmatrix}.$$

We can write this is an augmented matrix as

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 6 \\ 0 & 9 & -4 & -1 & -7 \\ 0 & -4 & 9 & 0 & -8 \\ -1 & 1 & 0 & 0 & 9 \end{array}\right]$$

which we can row reduce to get

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -8.4595 \\ 0 & 1 & 0 & 0 & 0.5405 \\ 0 & 0 & 1 & 0 & -0.6486 \\ 0 & 0 & 0 & 1 & 14.4595 \end{array}\right].$$

Therefore we see that our solution is

$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ v_1 \end{bmatrix} = \begin{bmatrix} -8.4595 \\ 0.5405 \\ -0.6486 \\ 14.4595 \end{bmatrix}.$$

Do not worry about the presence of a negative sign, a negative loop current just means that the current (i_1 and i_3 in this case) actually flows counter-clockwise, not clockwise like we assumed.

Question A 12

SOLUTION. These commands have the following effects, in sequence:
"A = zeros(2,3);" creates a matrix of zeroes having 2 rows and 3 columns. "A(1,1) = 2;" changes the value in the first row, first column entry to 2. "A(2,3) = 3;" changes the value in the second row, third column entry to 3.
The result is the following matrix, "A".

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Question A 13

SOLUTION. The first line of code creates the 3 by 3 zeros matrix:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then we enter the for-loop. In the first iteration, the entry *A(1,1)* is set to *-5* and *A(1,2)* is set to *1*, resulting in

$$A = \begin{bmatrix} -5 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In the second iteration the entry *A(2,2)* is set to *-5* and *A(2,3)* is set to *1*, hence

$$A = \begin{bmatrix} -5 & 1 & 0 \\ 0 & -5 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Finally, the entry *A(3,3)* is set to *-2*. So the final answer is

$$A = \begin{bmatrix} -5 & 1 & 0 \\ 0 & -5 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

Question A 14

SOLUTION. The first line of code creates a *2*x*3* matrix with the row vector [1 2 3] as its first row and the row vector [4 5 6] as its second row. Hence

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

The second line tells MATLAB to return a vector of all entries in A that have a second argument *1*, hence *A(1,1)* and *A(2,1)*. This is a column vector. Therefore the final answer is

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Question A 15

Solution 1. We calculate the characteristic polynomial.

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \det\left(\begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}\right) \\ &= (3-\lambda)^2 - 1 = \lambda^2 - 6\lambda + 8 \end{aligned}$$

Setting $p(\lambda) = 0$ we obtain

$$\lambda = \frac{6 \pm \sqrt{6^2 - 4(1)(8)}}{2} = \frac{6 \pm 2}{2}$$

Therefore the eigenvalues are $\lambda = 2$ and $\lambda = 4$.

Solution 2. Using Hint 2 we can set up the following two equations:

$$\begin{aligned} \operatorname{trace}(A) &= 6 = \lambda_1 + \lambda_2 \\ \det(A) &= 8 = \lambda_1 \lambda_2 \end{aligned}$$

From here we quickly see the solutions $\lambda = 2$ and $\lambda = 4$.

Question A 16

Solution. Following the hint gives us

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} as + bt \\ cs + dt \end{bmatrix} = s \begin{bmatrix} a \\ c \end{bmatrix} + t \begin{bmatrix} b \\ d \end{bmatrix}$$

Comparing to the original equation gives

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$$

Question A 17

Solution 1. All points on the line y=mx are given by [x0,mx0]=x0[1,m]. If we have a general point in space [x,y] then the projection matrix P should map the point [x,y] to x0[1,m]. Since this projection must hold for all points, it certainly must hold for the basis vectors. Therefore we try the vector [1,0]. Since we have,

$$P = \begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix}$$

then

$$P \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4/5 \\ -2/5 \end{bmatrix}$$

which of course is just the first column of the transformation matrix (recall this is exactly how we form transformation matrices). We can write,

$$\begin{bmatrix} 4/5 \\ -2/5 \end{bmatrix} = 4/5 \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}$$

which in the form x0[1,m] tells us that x0=4/5 and m=-1/2. Therefore the slope of the line is -1/2. We can then write that the line is y=-1/2x. To check our answer we could use the second basis vector [0,1] which would produce the second column of the matrix and once again we'd conclude the slope is -1/2.

Solution 2. To begin with, since P projects on a line, the vector Pv is on the line for any vector v.
Let *(x_0,y_0)* be a vector on the line that P projects onto. Projecting a vector that is already on the line onto the line, does not change the vector. In mathematical notation, this means that $P \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ and hence *(x_0,y_0)* is an eigenvector with eigenvalue 1.
To find this eigenvector, find the nullspace of

$$P - I = \begin{bmatrix} -1/5 & -2/5 \\ -2/5 & -4/5 \end{bmatrix}$$

Row reducing the matrix above yields

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

and hence *(x_0,y_0) = (-2,1)* is a vector on the line that P projects onto. Plugging this into the equation of the line, $y = mx$ yields *1 = m(-2)* or $m = -\frac{1}{2}$

Question A 18

Solution. The equation of the line is

$$\mathbf{x} = (1, 1, 0) + t(1, 0, -2)$$

where $\mathbf{x} = (x, y, z)$. We can write the line as three separate equations

$$\begin{aligned} x &= 1 + t \\ y &= 1 \\ z &= -2t. \end{aligned}$$

If we use this for the equation of the plane, we get

$$\begin{aligned} x+y+z &= 2 \\ 1+t+1-2t &= 2 \\ -t &= 0 \end{aligned}$$

and therefore we conclude that t=0. We plug this value of t back into the equation of the line to get that the point on the line that intersects the plane is (1,1,0).

Question A 19

SOLUTION. Consider a vector, v, that lies on the line $y=x$.
Applying QR to this vector will first rotate the vector so that it lies parallel to the y-axis, then reflect the vector over $y=x$ so that it lies parallel to the x-axis.
Applying RQ to this vector will first reflect the vector over the line $y=x$, leaving it unchanged, then will rotate the vector so that it lies parallel to the y-axis.
By this example, QR and RQ have different effects on v and thus do not commute. i.e:

$$QR \neq RQ.$$

Question A 20

SOLUTION. Notice that I can write [0,1] in terms of the other two vectors [2,3] and [1,1] via,

$$\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix} - 2\begin{bmatrix}1\\1\end{bmatrix}.$$

Now transformation are linear. This means that T(ax)=aT(x) and T(x+y)=T(x)+T(y). Therefore,

$$\begin{aligned} T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) &= T\left(\begin{bmatrix}2\\3\end{bmatrix} - 2\begin{bmatrix}1\\1\end{bmatrix}\right) \\ &= T\left(\begin{bmatrix}2\\3\end{bmatrix}\right) - 2T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) \\ &= \begin{bmatrix}1\\2\\8\end{bmatrix} - 2\begin{bmatrix}5\\2\\1\end{bmatrix} = \begin{bmatrix}-9\\-2\\6\end{bmatrix}. \end{aligned}$$

Therefore

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-9\\-2\\6\end{bmatrix}.$$

Question A 21

SOLUTION. Just like in A20, we can write,

$$\begin{bmatrix}1\\0\end{bmatrix} = 3\begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}2\\3\end{bmatrix}.$$

Now transformation are linear. This means that T(ax)=aT(x) and T(x+y)=T(x)+T(y). Therefore,

$$\begin{aligned} T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) &= T\left(3\begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}2\\3\end{bmatrix}\right) \\ &= 3T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) - T\left(\begin{bmatrix}2\\3\end{bmatrix}\right) \\ &= 3\begin{bmatrix}5\\2\\1\end{bmatrix} - \begin{bmatrix}1\\2\\8\end{bmatrix} = \begin{bmatrix}14\\4\\-5\end{bmatrix}. \end{aligned}$$

Therefore

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}14\\4\\-5\end{bmatrix}$$

and from A20 we have,

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-9\\-2\\6\end{bmatrix}$$

so therefore

$$T = \begin{bmatrix}14 & -9\\4 & -2\\-5 & 6\end{bmatrix}$$

where we recall that the columns of the transformation matrix are just the transformation of the basis vectors.

Question A 22

SOLUTION. This is straight forward:

$$2u - z = 2(3+2i) - (1-i) = 6 + 4i - 1 + i = (6-1) + (4+1)i = 5 + 5i$$

Question A 23

SOLUTION. Using the hint we find

$$\overline{u} = \overline{3+2i} = 3 - 2i.$$

Question A 24

SOLUTION. Using the hint and the facts $\overline{z} = 1 + i$ and i^2 = *-1* we compute

$$\begin{aligned}\frac{u}{z} &= \frac{3+2i}{1-i} \\ &= \frac{(3+2i)(1+i)}{(1-i)(1+i)} \\ &= \frac{3+3i+2i+2i^2}{1-i^2} \\ &= \frac{1+5i}{2} \\ &= \frac{1}{2} + \frac{5}{2}i\end{aligned}$$

Question A 25

SOLUTION. Our complex number is z=1+i. Therefore,

$$|z| = \sqrt{1+1} = \sqrt{2}$$

and

$$\operatorname{Arg}(z) = \arctan\left(\frac{1}{1}\right) = \arctan(1) = \frac{\pi}{4}$$

since the vector 1+i is in the first quadrant. Therefore,

$$1 + i = \sqrt{2}\exp\left(\frac{\pi}{4}i\right)$$

and

$$(1+i)^{10} = 2^{10/2}\exp\left(\frac{10\pi}{4}i\right) = 32\exp\left(\frac{5\pi}{2}i\right).$$

Now

$$\exp\left(\frac{5\pi}{2}i\right) = \cos\left(\frac{5\pi}{2}\right) + i\sin\left(\frac{5\pi}{2}\right) = i$$

and therefore

$$(1+i)^{10} = 32i.$$

Question A 26

SOLUTION. We are told we have a transition matrix that represents the transition probabilities of a 12-state transition matrix. Therefore the matrix will be 12 x 12 which we will assume has been initialized for us as P in Matlab. We are also told that we are initially in state 1. Since the initial state vector, $\mathbf{x}$, lists the probability of being in any of the 12 states, we should have

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

To initialize this in Matlab we would type

```
x = [1;0;0;0;0;0;0;0;0;0;0;0];
```

Now, we are interested in knowing the probability of being in state 5 after 10 time steps (or transitions).

Solution 1: Matrix Powers

Recall that if $\mathbf{x}$ is the initial state vector with P, the transition matrix then $\mathbf{t_1} = P\mathbf{x}$ returns the vector corresponding to the probability of being in each of the 12 states after one time step. If we apply the transition matrix again then $\mathbf{t_2} = P\mathbf{t_1}$ we get how the transition vector from the first time step transitions to the second time step. Notice I can write,

$$\mathbf{t_2} = P\mathbf{t_1} = P(P\mathbf{x}) = P^2\mathbf{x}.$$

Therefore, I have that the transition vector, $\mathbf{b}$ after n time steps is

$$\mathbf{b} = P^n\mathbf{x}.$$

To figure out the transition vector after 10 time steps using Matlab we would type

```
b = P^(10)*x;
```

Now this returns the whole vector. Since we are interested in the probability of being in the 5th state after 10 time steps, we want to know the 5th entry of $\mathbf{b}$. Therefore we type

```
b(5)
```

Notice the lack of the semicolon tells us that the entry of b(5) will be output to the screen for us to see.

Solution 2: For Loops

Recall that if $\mathbf{x}$ is the initial state vector with P, the transition matrix then $\mathbf{t_1} = P\mathbf{x}$ returns the vector corresponding to the probability of being in each of the 12 states after one time step. If I want to know the transition to the second time step, I just need to perform a single transition on the new transition vector $\mathbf{t_1}$. Therefore, for each new time step I want a transition, I just have to multiply the previous transition vector with the transition matrix P. This type of iterated operation is perfectly suited for a for loop in Matlab. Notice, I want 10 time steps so I would keep applying the transition matrix to the generated transition vectors until I've done that 10 times.

```
for k = 1:10
x = P*x;
end
```

Notice here that I reinitialize **x** to be the new transition vector so that I can minimize the amount of code I have to write. Once this is done we have the transition vector after 10 time steps. However, we're interested in the probability of being in the 5th state. This is the 5th entry of the vector. Therefore we type,

```
x(5)
```

Notice the lack of semicolon tells us that the entry of x(5) will be output to the screen for us to see.

Question A 27

SOLUTION. If we have a linear system

$$A\mathbf{x} = \mathbf{b}$$

then Matlab can solve this via the \ operator. Therefore if we type

```
x=A\b;
```

then the vector x produced by Matlab is the solution to the linear system.

Note. Since A is invertible, it is possible to get the solution by typing

```
x=inv(A)*b;
```

but you are generally discouraged from using the inv() operation in Matlab to solve linear systems. (That's because calculating the inverse of a matrix A is an overkill here, there is faster ways of finding x that don't require the matrix inverse.)

Question A 28

SOLUTION. a. $(AB)C = A(BC)$

This is **true** and is known as the distributive property. Notice we are not changing the order in which the multiplication occurs, just which multiplications we are performing first.

b. $(A + B)(A - B) = A^2 - B^2$

This is **false**. Let's carefully expand out the left-hand side,

$$A^2 - AB + BA - B^2$$

however, since matrices do not in general commute then AB is not the same as BA and those terms do not cancel like they do with real numbers.

c. $A^T B^T = (AB)^T$

This is **false**. The property of transposes is that

$$(AB)^T = B^T A^T$$

however even without remembering the correct form of the property we can conclude that the presented form is wrong. Let A be size n x m and B be size m x n such that $m \neq n$. Notice that AB is a sensible operation here and produces a matrix n x n. If we then transpose the resulting matrix it is also n x n. However, notice that A^T is m x n and B^T is n x m. Therefore, $A^T B^T$ is m x m which since $m \neq n$ is not the same size as n x n and therefore the property can not hold for all matrices.

d. $(A^T)^{-1} = (A^{-1})^T$

This is **true** and is a consequence of the transpose property,

$$(AB)^T = B^T A^T.$$

e. $(rA)^{-1} = \frac{1}{r} A^{-1}$

This is **true** (as long as the scalar is non-zero which is given) and is a property of inverses.

Therefore, a,d, and e are true.

Question A 29

SOLUTION. a. $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$
This is **true**. Recall that the cross product between two vectors **a** and **b** produces a vector which is orthogonal (perpendicular) to both the original vectors. The dot product between two orthogonal vectors is zero. Therefore since $\mathbf{a} \times \mathbf{b}$ is orthogonal to **a** then the dot product should indeed vanish.
b. $\mathbf{a}, \mathbf{b}.\mathbf{c}$ are all linearly independent
This is **false**. We can easily come up with a counter-example of three vectors which are not linearly independent. Consider the vector [1,2,3] and [1,0,0] which are indeed in R3. The vector [3,2,3] is also in R3 but can be written as [1,2,3] + 2 [1,0,0] and therefore is linearly dependent on the other two vectors.
c. $\mathbf{a} \cdot \mathbf{a} = ||\mathbf{a}||^2$
This is **true**. Let $\mathbf{a} = [a_1, a_2, a_3]$ then $\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2 = ||\mathbf{a}||^2$.
d. $\mathbf{a} \cdot \mathbf{b} = 0$
This is **false**. We can easily construct two vectors in R3 which do not have a zero dot product. For example consider [1,1,1] and [2,2,2], their dot product is 6.
e. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is not defined
This is **false**. The cross product is always defined for vectors in R3. The cross product also transforms two vectors in R3 to another vector belonging in R3. Therefore the triple cross product here is just the cross product between two vectors in R3 which is defined.
Therefore, only (a) and (c) are true.

Question A 30

SOLUTION. Let Δ be the determinant. Using the fact that adding and subtracting rows from one another within a matrix does not affect the value of the determinant, we subtract the first row from each of the remaining rows:

$$\Delta = \left| \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 4 & 3 & 2 \\ 1 & 1 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 & 5 \end{bmatrix} \right| = \left| \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 3 & 2 & 1 \\ 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} \right|.$$

Now we notice that the resulting matrix is upper-triangular, and thus the determinant is just the product of the entries on the diagonal: $\Delta = 1 \times 1 \times 3 \times 2 \times 4 = 24$.

Question B 01 (a)

SOLUTION. Notice that we identify the speed that Hartosh can paint with as x_1, the speed that Mark can paint with as x_2, and the speed that Keiko can paint with as x_3 (all measured in rooms per hour). Our job is to translate each piece of information provided into a linear equation.
Hartosh paints twice as fast as Mark
This tells us that whatever Mark's speed is (x_2), Hartosh's speed (x_1) is twice that and therefore we get

$$x_1 = 2x_2$$

or, to write it like a linear equation with all three variables,

$$x_1 - 2x_2 + 0x_3 = 0.$$

Hartosh and Keiko paint a home with 6 rooms in 8 hours
The total number of rooms that any person can paint in 8 hours is their speed multiplied by 8. Therefore the number of rooms that Hartosh, Mark, and Keiko paint in 8 hours is $8x_1$, $0x_2$ and $8x_3$ respectively where we notice that Mark painted 0 rooms because he wasn't involved in this particular project. The total number of rooms that get painted in that time is 6. Therefore, the sum of all the rooms that each person paints, must total to 6. With this in mind we have,

$$8x_1 + 0x_2 + 8x_3 = 6.$$

All three together paint a home with 14 rooms in 16 hours
This is just like the last equation where Hartosh and Keiko painted the 6 rooms, but now all three people are contributing. Therefore, by similar logic to above we have

$$16x_1 + 16x_2 + 16x_3 = 14.$$

Now, we can summarize all this as the linear system

$$\begin{aligned} x_1 - 2x_2 + 0x_3 &= 0 \\ 8x_1 + 0x_2 + 8x_3 &= 6 \\ 16x_1 + 16x_2 + 16x_3 &= 14. \end{aligned}$$

To write this as a matrix problem, we note that the coefficients in front of each variable form the matrix A and everything to the right of the equals sign, form the vector $\mathbf{b}$. Therefore,

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 8 & 0 & 8 \\ 16 & 16 & 16 \end{bmatrix}$$

and

$$\mathbf{b} = \begin{bmatrix} 0 \\ 6 \\ 14 \end{bmatrix}.$$

Question B 01 (b)

SOLUTION. From B1(a) we have that

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 8 & 0 & 8 \\ 16 & 16 & 16 \end{bmatrix}$$

and

$$\mathbf{b} = \begin{bmatrix} 0 \\ 6 \\ 14 \end{bmatrix}.$$

The augmented matrix for $A\mathbf{x} = \mathbf{b}$ is $[A|\mathbf{b}]$. This says to take the matrix A and add the vector $\mathbf{b}$ as an extra column. Therefore the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 8 & 0 & 8 & 6 \\ 16 & 16 & 16 & 14 \end{array}\right]$$

Question B 01 (c)

SOLUTION. From B1(b) we have that the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 8 & 0 & 8 & 6 \\ 16 & 16 & 16 & 14 \end{array}\right]$$

which we now need to solve by Gaussian elimination. This means putting pivots along the diagonals of the matrix with value 1. Notice the pivot on the first row is the first column which already has a 1 and so we can leave that unchanged. We want all of the entries below this 1 (and in the same column) to vanish. To do this we subtract 8 multiples of row 1 from row 2 (8 because that's the number in the first column of the second row). For a similar reason, we subtract 16 multiples of row 1 from row 3. Therefore the new resulting augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 16 & 8 & 6 \\ 0 & 48 & 16 & 14 \end{array}\right].$$

We move to the second row, where the pivot is in the second column. We want this value to be 1 and so we multiply the second row by 1/16 to get

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 1 & 1/2 & 3/8 \\ 0 & 48 & 16 & 14 \end{array}\right].$$

We now must make the entries below this pivot (and in the same column) vanish. To do this we subtract 48 multiples of row 2 from row 3

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 1 & 1/2 & 3/8 \\ 0 & 0 & -8 & -4 \end{array}\right].$$

Now, if wanted we could remove the entries above the pivot as well. This is known as Gauss-Jordan elimination. By doing this we'd end up with zeros everywhere except for the augmented portion and possibly for the pivots. This is a very easy way to see the final solution since there are no more operations to perform. We will continue with just Gaussian elimination, which will result in an upper triangular matrix with zeros below each pivot. To find the solution from there will require a little more work, but we are compensating that by doing fewer row operations on the matrix.

Finally then, by continuing Gaussian elimination, we want to make the pivot in the third row (the number in the third column) have a value 1. We therefore multiply the third row by -1/8 to get

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 1 & 1/2 & 3/8 \\ 0 & 0 & 1 & 1/2 \end{array}\right].$$

We must now answer the problem which is to find how many rooms each person can paint in an hour (x_1, x_2, x_3). From the last row we have that

$$x_3 = \frac{1}{2}$$

and so Keiko paints 1/2 rooms per hour (or 1 room every 2 hours). From the second row we get (using Keiko's now known speed),

$$\begin{aligned} x_2 + \frac{1}{2}x_3 &= \frac{3}{8} \\ x_2 + \frac{1}{4} &= \frac{3}{8} \\ x_2 &= \frac{1}{8} \end{aligned}$$

and so Mark paints 1/8 rooms per hour (or 1 room every 8 hours). We can use both of these speeds now with the first row to get,

$$\begin{aligned} x_1 - 2x_2 + 0x_3 &= 0 \\ x_1 - \frac{1}{4} &= 0 \\ x_1 &= \frac{1}{4} \end{aligned}$$

and so Hartosh paints 1/4 rooms per hour (or 1 room every 4 hours). We have found the number of rooms that Hartosh, Mark, and Keiko can paint in an hour to be 1/4, 1/8, and 1/2 respectively.

Question B 02 (a)

SOLUTION. We want to find the eigenvector of

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

with eigenvalue $\lambda = 4$. We therefore want to find the vector $\mathbf{x} = [a, b, c]^T$ such that

$$(A - 4I)\mathbf{x} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \mathbf{0}$$

We therefore have three equations with three unknowns

$$\begin{aligned} -2a + b + c &= 0 \\ a - 2b + c &= 0 \\ a + b - 2c &= 0 \end{aligned}$$

We can solve the first equation for c to get

$$c = 2a - b$$

and can sub this into the second equation to get

$$\begin{aligned} a - 2b + 2a - b &= 0 \\ 3a - 3b &= 0 \\ b &= a. \end{aligned}$$

Using c=2a-b and b=a we conclude that c=a. From the last equation we get,

$$a + a - 2a = 0$$

which is satisfied for all a. Therefore, let a=1 without loss of generality. This implies that b=c=1 as well. Therefore the eigenvector for the eigenvalue 4 is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Of course, any multiple of this is also an eigenvector (expect multiplying by 0).

Question B 02 (b)

SOLUTION. We want to find two eigenvectors of

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

with eigenvalue $\lambda = 1$. Just like in B2(a) we want to find the vectors $\mathbf{x} = [a, b, c]^T$ such that

$$(A - 1I)\mathbf{x} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \mathbf{0}$$

We therefore have three equations with three unknowns

$$\begin{aligned} a + b + c &= 0 \\ a + b + c &= 0 \\ a + b + c &= 0 \end{aligned}$$

Notice that these are all the same equation, therefore we effectively only have one equation. Notice that if we row reduced the matrix $(A - 1I)\mathbf{x}$ we'd get two rows of all zeros, corresponding exactly to the situation of having repeated linear equations. This tells us that two of the variables will be free parameters. Contrast this to B2(a) where we had only one free parameter in the end. The emergence of two free parameters tells us that there will be two eigenvectors. We will take the two free parameters to be a and b, though you can pick any two of a,b, and c to be the free parameters.

First Eigenvector: a=1, b=0

To find the first eigenvector we will use set the first free parameter to 1 and the second to 0 (a=1,b=0). Note that you can pick any values you want (except for a=0,b=0 because that will lead to the eigenvector being zero).

With a=1 and b=0 we have

$$a+b+c=1+0+c=0$$

and therefore, c=-1. Therefore the first eigenvector for the eigenvalue 1 is

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Second Eigenvector: a=0, b=1
To find the second eigenvector we will use set the first free parameter to 0 and the second to 1 (a=0,b=1). Note that you can pick any values you want (except for a=0,b=0 because that will lead to the eigenvector being zero) as long as they are different from the first values you chose, otherwise you will get the same eigenvector.
With a=0 and b=1 we have

$$a+b+c=0+1+c=0$$

and once again, c=-1. Therefore the second eigenvector for the eigenvalue 1 is

$$\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

Any non-trivial linear combination of these two vectors is also an eigenvector, which you can easily verify.

Question B 03 (a)

Solution. If the walker starts in state 1, then the probability vector is given by
$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
(i.e: the walker is in state 1 with probability 1 and in state 2 with probability 0). To compute the probability that the walker is in state 2 after one step, simply compute Pv and look at the second entry:
$P\mathbf{v} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$
Thus, the probability of switching from state 1 to state 2 in one time step is 2/3.
(Equivalently, to answer this question you could look at the second entry in the first column of the matrix P).

Question B 03 (b)

Solution. To determine the eigenvectors, we must solve the equation

$$(P-\lambda_k I)\mathbf{v}_k = 0$$

where λ_k are the eigenvalues and $\mathbf{v}_k$ are the eigenvectors.
$\lambda_1 = 1$

$$\begin{aligned}(P-I)\mathbf{v}_1 &= \begin{bmatrix} -2/3 & 1/4 \\ 2/3 & -1/4 \end{bmatrix}\mathbf{v}_1 = 0 \\ \rightarrow \quad \mathbf{v}_1 &= \begin{bmatrix} 1/4 \\ 2/3 \end{bmatrix}\end{aligned}$$

(Note that we are not normalizing the eigenvectors. It is not necessary to do so at this point.)
$\lambda_2 = 1/12$

$$(P - \frac{1}{12}I)\mathbf{v}_2 = \begin{bmatrix} 1/4 & 1/4 \\ 2/3 & 2/3 \end{bmatrix}\mathbf{v}_2 = 0$$
$$\rightarrow \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Question B 03 (c)

SOLUTION 1. We know that the probability vector after n steps is determined by multiplying the initial state by the matrix P, $n$times.
We begin by expressing the initial probability vector in terms of the eigenvectors of P. i.e:

$$c_1\begin{bmatrix} 1/4 \\ 2/3 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1/4 & 1 \\ 2/3 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1/4 & 1 \\ 11/12 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 8/11 \\ 12/11 \end{bmatrix}$$

Thus $c_1 = 12/11$, $c_2 = 8/11$.
Now that we know the values of the constants, we will exploit the relationship

$$P\mathbf{v}_k = \lambda_k\mathbf{v}_k \quad \rightarrow \quad P^n\mathbf{v}_k = \lambda_k^n\mathbf{v}_k,$$

which holds for eigenvectors $\mathbf{v}_k$.
Using our answer from part b), we have the following

$$\begin{aligned} P^n\begin{bmatrix} 1 \\ 0 \end{bmatrix} &= P^n(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) \\ &= c_1\lambda_1^n\mathbf{v}_1 + c_2\lambda_2^n\mathbf{v}_2 \\ &= \frac{12}{11}\begin{bmatrix} 1/4 \\ 2/3 \end{bmatrix} + \frac{8}{11}\left(\frac{1}{12}\right)^n\begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \frac{1}{11}\begin{bmatrix} 3 + 8\left(\frac{1}{12}\right)^n \\ 8 - 8\left(\frac{1}{12}\right)^n \end{bmatrix} \end{aligned}$$

Therefore, assuming that the walker starts in state 1, the probability that the walker is in state 1 after n steps is

$$\frac{1}{11}\left(3 + 8\left(\frac{1}{12}\right)^n\right)$$

SOLUTION 2. You can also solve this directly when you diagonalize P:

$$P = MDM^{-1}$$

where M is the matrix with the eigenvectors of P as its columns, and D is a diagonal matrix with the corresponding eigenvalues on its diagonal. The catch is that then

$$P^n = MD^nM^{-1}$$

and that powers of a diagonal matrix D are very easy to calculate.
So let's do the work: Recall that and an eigenvector to the eigenvalue **1** is $v = \begin{bmatrix} 1/4 \\ 2/3 \end{bmatrix}$, and an eigenvector to the eigenvalue **1/12** is $w = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Possible choices of M and D are therefore

$$M = \begin{bmatrix} 1/4 & 1 \\ 2/3 & -1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1/12 \end{bmatrix}$$

or

$$M = \begin{bmatrix} 1 & 1/4 \\ -1 & 2/3 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1/12 & 0 \\ 0 & 1 \end{bmatrix}$$

Multiplying any column of M (but not of D) by a non-zero constant is also possible. We choose the former matrices M and D.
Next step is to find M^{-1}, which is particularly easy for a *2*x*2* matrix since we use the formula

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Therefore, we obtain

$$\begin{aligned} P &= MDM^{-1} \\ \begin{bmatrix} 1/3 & 1/4 \\ 2/3 & 3/4 \end{bmatrix} &= \begin{bmatrix} 1/4 & 1 \\ 2/3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/12 \end{bmatrix} \frac{-12}{11} \begin{bmatrix} -1 & -1 \\ -2/3 & 1/4 \end{bmatrix} \end{aligned}$$

Therefore

$$\begin{aligned} P^n &= MD^nM^{-1} \\ &= \begin{bmatrix} 1/4 & 1 \\ 2/3 & -1 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & (1/12)^n \end{bmatrix} \frac{-12}{11} \begin{bmatrix} -1 & -1 \\ -2/3 & 1/4 \end{bmatrix} \\ &= \frac{-12}{11} \begin{bmatrix} -\frac{1}{4} - \frac{2}{3}(\frac{1}{12})^n & -\frac{1}{4} + \frac{1}{4}(\frac{1}{12})^n \\ -\frac{2}{3} + \frac{2}{3}(\frac{1}{12})^n & -\frac{2}{3} - \frac{1}{4}(\frac{1}{12})^n \end{bmatrix} \\ &= \frac{1}{11} \begin{bmatrix} 3 + 8(\frac{1}{12})^n & 3 - 3(\frac{1}{12})^n \\ 8 - 8(\frac{1}{12})^n & 8 + 3(\frac{1}{12})^n \end{bmatrix} \end{aligned}$$

Hence, finally, we can conclude that

$$\begin{aligned} P^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \frac{1}{11} \begin{bmatrix} 3 + 8(\frac{1}{12})^n & 3 - 3(\frac{1}{12})^n \\ 8 - 8(\frac{1}{12})^n & 8 + 3(\frac{1}{12})^n \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{11} \begin{bmatrix} 3 + 8(\frac{1}{12})^n \\ 8 - 8(\frac{1}{12})^n \end{bmatrix} \end{aligned}$$

which means that the probability of being at stage *1* after n steps is

$$\frac{1}{11}\left(3+8\left(\frac{1}{12}\right)^n\right)$$

Question B 03 (d)

Solution. Using our answer from c), we can study the limit where n goes to infinity. In this case the probability vector approaches

$$\hat{\mathbf{v}} = \lim_{n\to\infty} P^n\mathbf{v} = \begin{bmatrix} 3/11 \\ 8/11 \end{bmatrix}.$$

Hence, it is more likely, that the walker will be be in state 2 after a large number of steps, since the 2nd entry in the vector is larger than the first entry.

Question B 04 (a)

Solution. Since the last two rows are zeros and the augmented part is zero we have two variables that are free parameters. Since the first row tells us that x_2=0 then we have that x_1 and x_3 are free. Therefore the two null vectors are

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

so therefore the solution is

$$\mathbf{x} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and there are an infinite number of solutions.

Question B 04 (b)

Solution. From the matrix,

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

we see that the third row is all zeros (including the augmented part) and so one of the variables is a free parameter. From the second row we see that x_3=0. Let the free parameter be x_2. Then from the first row we see that x_1=-2x_2. Therefore we have a null vector,

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

and the solution is

$$\mathbf{x} = t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$$

There are an infinite number of solutions.

Question B 04 (c)

SOLUTION. From the matrix,

$$\left[\begin{array}{cccc|c} 1 & 0 & 4 & 5 & 1 \\ 0 & 1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

we see that the last row is all zeros (including the augmented part) and so we have a free parameter. Since there are more columns than rows to this matrix (the matrix is rank deficient) then there is a second free parameter. Let the free parameters be x_3 and x_4. Then from the second row we have

$$\begin{aligned} x_2 + x_3 + 3x_4 &= 2 \\ x_2 &= 2 - x_3 - 3x_4 \end{aligned}$$

and from the first row we have

$$\begin{aligned} x_1 + 4x_3 + 5x_4 &= 1 \\ x_1 &= 1 - 4x_3 - 5x_4. \end{aligned}$$

Therefore we have two null vectors

$$\begin{bmatrix} -4 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

as well as a particular solution

$$\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

so that the solution is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

There are an infinite number of solutions.

Question B 04 (d)

SOLUTION. Since the reduced row form of the matrix leads to the identity we know that there is one unique solution and it is the values in the augmented part of the matrix. Therefore $x_1 = 5$, $x_2 =$ -1, and $x_3 = 2$.

Question B 04 (e)

SOLUTION. For the matrix,

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 1 \\ 0 & 1 & 2 & -2 & 2 \end{array}\right]$$

we have two more columns than rows (the matrix is rank deficient) and so we have two free variables. Let x_3 and x_4 be those variables. Then from row 2 we have,

$$\begin{aligned} x_2 + 2x_3 - 2x_4 &= 2 \\ x_2 &= 2 - 2x_3 + 2x_4 \end{aligned}$$

and from row 1 we have,

$$\begin{aligned} x_1 + 2x_3 - x_4 &= 1 \\ x_1 &= 1 - 2x_3 + x_4. \end{aligned}$$

Therefore we have two null vectors,

$$\begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

as well as a particular solution

$$\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore the solution is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

There are an infinite number of solutions.

Question B 05 (a)

SOLUTION. There are 3 sections in this page but only the first section is "what is supposed to be written" in the exam.

Solution

Since eigenvalues and eigenvectors occur in conjugate pairs for the real matrix A, the other pair is:

$$\begin{aligned} \lambda_2 &= 1 - 2i \\ \mathbf{k}_2 &= \begin{bmatrix} 2 - 3i \\ 1 \end{bmatrix} \end{aligned}$$

The general solution is then:

$$\begin{aligned} y &= C_1 \mathbf{k}_1 \exp \lambda_1 t + C_2 \mathbf{k}_2 \exp \lambda_2 t \\ &= C_1 \begin{bmatrix} 2 + 3i \\ 1 \end{bmatrix} \exp((1 + 2i)t) + C_2 \begin{bmatrix} 2 - 3i \\ 1 \end{bmatrix} \exp((1 - 2i)t) \end{aligned}$$

We delay showing how to write it into real form to part (b).

Motivation

To see a reason why this is so. Let us recall that e^{at} solves $x' = ax$ which is the single equation analogue of the given system. From this, we can make a guess that a constant vector times an exponential function might solve our system. More precisely, we substitute the guess $\mathbf{v}e^{ct}$ into the system to get (both $\mathbf{v}, c$ are to be determined):

$$\begin{aligned} (\mathbf{v}e^{ct})' &= A\mathbf{v}e^{ct} \\ c\mathbf{v}e^{ct} &= A\mathbf{v}e^{ct} \\ c\mathbf{v} &= A\mathbf{v} \end{aligned}$$

(The cancellation in the last step is because the exponential function is always positive). So it turns out for our guess to work, c and $\mathbf{v}$ as a pair must solve the eigenvalue problem $A\mathbf{v} = c\mathbf{v}$, i.e. they must be a pair of eigenvalue and eigenvector of A!

Now, from the two pairs of eigenvalues and eigenvectors we know, we have the following solutions:

$$\mathbf{y}_1 = \mathbf{k}_1 e^{\lambda_1 t}$$

$$\mathbf{y}_2 = \mathbf{k}_2 e^{\lambda_2 t}$$

To be precise, we should say they are just *particular* solutions of the problem. To obtain the formula of the general solution, we observe that the differential equation system is linear and so a linear combination of $\mathbf{y}_1$ and $\mathbf{y}_2$, i.e.

$$\mathbf{y} = C_1\mathbf{y}_1 + C_2\mathbf{y}_2$$

solves the system too:

$$\begin{aligned}
\mathbf{y} &= (C_1\mathbf{y}_1 + C_2\mathbf{y}_2)' \\
&= C_1\mathbf{y}_1' + C_2\mathbf{y}_2' \\
&= C_1A\mathbf{y}_1 + C_2A\mathbf{y}_2 \\
&= A\left(C_1\mathbf{y}_1 + C_2\mathbf{y}_2\right) \\
&= A\mathbf{y}
\end{aligned}$$

It turns out this represents all possible solutions.

Extended Reading

The following outlines the general method to deduce the general solution systematically (in the syllabus of Math 215/255). The key is to see that the existence of the two pairs of eigenvalues and eigenvectors of A allows us to "diagonalize" the matrix A.
In fact, once we have found all the eigenvalues and corresponding eigenvectors, if the number of eigenvectors equals the rank of the matrix A (in this case, it is *2*), then we can rewrite A into the following form (known as the diagonalization of A):

$$A = VDV^{-1}$$

where

$$V = \begin{bmatrix} \mathbf{k}_1 & \mathbf{k}_2 \end{bmatrix} = \begin{bmatrix} 2+3i & 2-3i \\ 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1+2i & 0 \\ 0 & 1-2i \end{bmatrix}.$$

To solve the given linear system of differential equations, we proceed as follows exploiting the diagonalizability of A.
Left multiplying V^{-1} on both sides of the linear system gives:

$$V^{-1}\mathbf{y}' = DV^{-1}\mathbf{y}$$

So if we let

$$\mathbf{x} = V^{-1}\mathbf{y}$$

Then, we get a simple decoupled system for $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ (see note):

$$\begin{cases} x_1' & = \lambda_1 x_1 \\ x_2' & = \lambda_2 x_2 \end{cases}$$

The general solution must then be:

$$\mathbf{x} = \begin{bmatrix} C_1 \exp \lambda_1 t \\ C_2 \exp \lambda_2 t \end{bmatrix} = \begin{bmatrix} C_1 \exp (1+2i)t \\ C_2 \exp (1-2i)t) \end{bmatrix}$$

:
Therefore,

$$\begin{aligned} \mathbf{y} &= V\mathbf{x} \\ &= \begin{bmatrix} \mathbf{k}_1 & \mathbf{k}_2 \end{bmatrix} \begin{bmatrix} C_1 \exp \lambda_1 t \\ C_2 \exp \lambda_2 t \end{bmatrix} \\ &= C_1 \mathbf{k}_1 \exp \lambda_1 t + C_2 \mathbf{k}_2 \exp \lambda_2 t \\ &= C_1 \begin{bmatrix} 2+3i \\ 1 \end{bmatrix} \exp (1+2i)t + C_2 \begin{bmatrix} 2-3i \\ 1 \end{bmatrix} \exp (1-2i)t \end{aligned}$$

where C_1, C_2 are complex constants.
Note: The word *decoupled* means the equations can be regarded separately as single equations themselves, i.e. this *system* is just a collection of single equations whose solutions do not affect each other.

Question B 05 (b)

SOLUTION. Continuing from the hint, to choose real-valued solutions, we choose $C_2 = \bar{C}_1$, so that the general solution becomes:

$$\mathbf{y} = C_1\mathbf{y_1} + C_2\mathbf{y_2} = C_1\mathbf{y_1} + \overline{C_1\mathbf{y_1}} = 2\mathrm{Re}\left[C_1\mathbf{y_1}\right]$$

Recall that

$$\mathbf{y_1} = \mathbf{k_1} \exp \lambda_1 t = \begin{bmatrix} 2+3i \\ 1 \end{bmatrix} \exp (1+2i)t$$

Since

$$\mathrm{Re}\left[\mathbf{y_1}\right] = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t \cos 2t + \begin{bmatrix} -3 \\ 0 \end{bmatrix} e^t \sin 2t$$

$$\mathrm{Im}\left[\mathbf{y_1}\right] = \begin{bmatrix} 3 \\ 0 \end{bmatrix} e^t \cos 2t + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t \sin 2t$$

The general form of real-valued solutions is:

$$
\begin{aligned}
\mathbf{y} &= D_1 \mathrm{Re}\left[\mathbf{y_1}\right] + D_2 \mathrm{Im}\left[\mathbf{y_1}\right] \\
&= D_1\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t \cos 2t + \begin{bmatrix} -3 \\ 0 \end{bmatrix} e^t \sin 2t\right) \\
&+ D_2\left(\begin{bmatrix} 3 \\ 0 \end{bmatrix} e^t \cos 2t + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t \sin 2t\right)
\end{aligned}
$$

where D_1 and D_2 are real constants.
Their relationship with the previous constants are $D_1 = 2\mathrm{Re}\left[C_1\right]$ and $D_2 = -2\mathrm{Im}\left[C_1\right] = e^t$
Now if we impose the initial condition $\mathbf{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we get

$$D_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + D_2 \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Thus, $D_1 = 0$ and $D_2 = \frac{1}{3}$, and the solution of the initial value problem is

$$\frac{1}{3}\left(\begin{bmatrix} 3 \\ 0 \end{bmatrix} e^t \cos 2t + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t \sin 2t\right) = \frac{e^t}{3}\begin{bmatrix} 3\cos 2t + 2\sin 2t \\ \sin 2t \end{bmatrix}$$

Alternatively, one can solve directly from

$$\mathbf{y} = 2\mathrm{Re}\left[C_1 \mathbf{y_1}\right]$$

And impose the initial condition $\mathbf{y}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ direcly to find that $C_1 = -\frac{i}{6}$.
Then compute the real part to get the same result as above. The amount of algebra involved will be slightly less.

Question B 06 (a)

SOLUTION. The area of T is half of the area of **a** parallelogram with vertices *(0, 1, 2)*, *(1, 1, 5)* and *(-1, 2, 2)*. Meanwhile, the area of a parallelogram is the magnitude of the cross product of vectors of two adjacent sides.
A convenient choice of vectors representing two adjacent sides are the ones with common vertex *(0, 1, 2)*. They are *(1, 1, 5) - (0, 1, 2) = (1, 0, 3)* and *(-1, 2, 2) - (0, 1, 2) = (-1, 1, 0)*
Hence,

$$T = \frac{1}{2}\left|(1,0,3) \times (-1,1,0)\right| = 1/2\left|(-3,-3,1)\right| = \frac{\sqrt{19}}{2}$$

Question B 06 (b)

SOLUTION. First pick any two linearly independent vectors lying in P (meaning the heads and tails of the vectors can be drawn in the plane P). A convenient choice will be two sides of the triangle T: *(1, 1, 5) - (0, 1, 2) = (1, 0, 3)* and *(-1, 2, 2) - (0, 1, 2) = (-1, 1, 0)*.
A normal vector to the plane is given by the cross product of these two vectors (which is calculated in part (a) already):

$$\vec{N} = (1,0,3) \times (-1,1,0) = (-3,-3,1)$$

The above vector gives the normal direction.

Question B 06 (c)

SOLUTION. First pick a known vector on P. A convenient choice will be *(0, 1, 2)* (other vertices will work too).
Let *(x, y, z)* be an arbitrary vector on P (this represents all possible vectors on P, drawn with tail at origin and head on P), then the difference *(x, y, z)* - *(0, 1, 2)* gives a vector lying in P.
This vector must then be perpendicular to the normal vector, which gives the equation:

$$[(x,y,z) - (0,1,2)] \cdot (-3,-3,1) = 0$$

So

$$-3x - 3y + z = (0,1,2) \cdot (-3,-3,1) = -1$$

Dividing both sides by -3 gives $b = 1, c = -\frac{1}{3}, d = \frac{1}{3}$

Question B 06 (d)

SOLUTION. Plug in *(0, 0,-1)* into the equation in part (c), i.e. put x = *0*, y = *0*, z =*-1* into the equation

$$x + y - \frac{1}{3}z = \frac{1}{3}$$

yields $0 + 0 - (-\frac{1}{3}) = \frac{1}{3}$.
Thus *(0, 0,-1)* is on P.

Question B 06 (e)

SOLUTION 1. The answer will be that *(0, 0, 1)* is outside of T. There can be many different ways to show this, and it takes a correct formulation of a *decisive* criterion and then patience and carefulness to work till the end.
For the subsequent paragraphs, let us denote $\vec{OA} = (0,1,2), \vec{OB} = (-1,2,2), \vec{OC} = (1,1,5)$ for brevity.
This solution explores the parametrization method. This method is somewhat a more sophisticated but it works beautifully.
Starting from point A, we can travel to B and C by the vectors

$$\vec{AB} = \vec{OB} - \vec{OA} = (-1,2,2) - (0,1,2) = (-1,1,0)$$

and

$$\vec{AC} = \vec{OC} - \vec{OA} = (1,1,5) - (0,1,2) = (1,0,3)$$

respectively.
Therefore, the expressions

$$\vec{OA} + t\vec{AB}$$

and

$$\vec{OA} + s\vec{AC}$$

with $0 \leq s, t \leq 1$, represents all points from A to B and A to C respectively.
It turns out we can combine them and write:

$$\vec{OA} + t\vec{AB} + s\vec{AC} = (0, 1, 2) + t(-1, 1, 0) + s(1, 0, 3) = (s - t, 1 + t, 2 + 3s)$$

with the **extra** requirement that $0 \leq s + t \leq 1$ to describe all points in the triangle T.
Now $(s - t, 1 + t, 2 + 3s) = (0, 0, -1)$ implies $s = t = -1$ Thus, it's not in T.
Remark 1: *(0, 0, -1)* is not even a point in the parallelogram

$$(s - t, 1 + t, 2 + 3s)$$

with $0 \leq s, t \leq 1$)
Remark 2: The fact that it was solvable for s and t shows that the point in the plane P. If it weren't in the plane, e.g. the point *(0, 0, 1)*, it wouldn't be solvable and some inconsistencies in the linear equations will result.

SOLUTION 2. The answer will be that *(0, 0, -1)* is outside of T. There are many different possible ways to show this, but they all take a correct formulation of a *decisive* criterion and then patience and carefulness to work till the end.
For this solution, let's explore some methods using basic geometry. We will hardly use the concept of vectors, and so for the subsequent paragraphs, let us denote $A = (0, 1, 2)$, $B = (-1, 2, 2)$, $C = (1, 1, 5)$, and $D = (0, 0, -1)$ for brevity.
This are a few ways to formulate a criterion to determine whether D = *(0, 0, -1)* is inside or outside T. Here we suggest 2 possible criteria.
In both cases, we consider the triangles $\triangle ABD, \triangle BCD, \triangle CAD$. There are two qualitatively different pictures for D inside or outside T.
Need to include the sketch here
Criterion 1 - Comparing Lengths
If D is outside T, then at least one of the lengths AD, BD or CD is strictly greater than the largest of AB, BC, CA.
Conversely if D is inside T, then all the lengths AD, BD or CD is less than or equal to the largest of AB, BC, CA.
Now, we calculate:

$$AB = |(-1, 1, 0)| = \sqrt{2}$$

$$BC = |(2, -1, 3)| = \sqrt{14}$$

$$CA = |(1, 0, 3)| = \sqrt{10}$$

But since

$$DC = |(1,1,6)| = \sqrt{38},$$

D is outside of T.
Criterion 2 - Compare Areas
If D is outside T, then the sum of areas of $\triangle ABD, \triangle BCD, \triangle CAD$ is strictly greater than that of $\triangle ABC$
Conversely if D is inside T, then equality holds instead.
We already knew that area of $T = \triangle ABC$ is $\sqrt{19}/2$. Let's calculate the area of $\triangle BCD$

$$\begin{aligned}
\text{Area}(\triangle BCD) &= \frac{1}{2}\left|\vec{BD} \times \vec{CD}\right| \\
&= \frac{1}{2}\left|(-1,2,3) \times (1,1,6)\right| \\
&= \frac{1}{2}|(9,9,-3)| \\
&= \frac{3\sqrt{19}}{2}
\end{aligned}$$

So, without calculating the other areas, we can be sure that D is outside of T.

Good Luck for your exams!