Full Solutions
MATH101 April 2014

April 22, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

SOLUTION. In the integration by parts:

$$\int u dv = uv - \int v du,$$

put $u = \ln x$ and $dv = x dx$. Then, we have $du = \frac{1}{x}dx$ and $v = \frac{x^2}{2}$.
By applying the integration by parts, we obtain

$$\begin{aligned}
\int x \ln x = & \quad \frac{x^2}{2}\ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} dx \\
= & \quad \frac{x^2}{2}\ln x - \int \frac{x}{2} dx \\
= & \quad \frac{x^2}{2}\ln x - \frac{x^2}{4} + C.
\end{aligned}$$

Question 1 (b)

SOLUTION. To evaluate this integral, we use integration by substitution.
Let

$$\begin{aligned}
u = x^3 + 1 \quad \Rightarrow \quad du &= 3x^2 dx, \\
\frac{1}{3} du &= x^2 dx
\end{aligned}$$

Substituting this into the integral gives

$$\begin{aligned}
\int \frac{x^2}{(x^3+1)^{101}}\, dx &= \int \frac{1}{3} \frac{1}{u^{101}}\, du \\
&= \int \frac{u^{-101}}{3}\, du \\
&= \frac{1}{3} \int u^{-101}\, du + C \\
&= \frac{1}{3} \left(\frac{(u^{-101+1})}{-101+1} \right) + C \\
&= \frac{1}{3} \left(\frac{u^{-100}}{-100} \right) + C \\
&= \frac{1}{-300} u^{-100} + C
\end{aligned}$$

Substituting $u = x^3 + 1$ back into the result gives

$$\int \frac{x^2}{(x^3+1)^{101}}\, dx = -\frac{1}{300}(x^3+1)^{-100} + C$$

Question 1 (c)

SOLUTION. Using the trigonometric identity,

$$\begin{aligned}\int \cos^3 x \sin^4 x dx &= \int \cos^2 x \cos x \sin^4 x dx \\ &= \int (1-\sin^2 x)\sin^4 x(\cos x)dx \\ &= \int (\sin^4 x - \sin^6 x)(\cos x)dx.\end{aligned}$$

Let $u = \sin x$. Then, $du = \cos x dx$. Applying substitution,

$$\int (\sin^4 x - \sin^6 x)(\cos x)dx = \int (u^4 - u^6)du = \frac{u^5}{5} - \frac{u^7}{7} + C = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C.$$

Question 1 (d)

SOLUTION. Let $x = 2\sin\theta$. Then, $dx = 2\cos\theta d\theta$. By substitution,

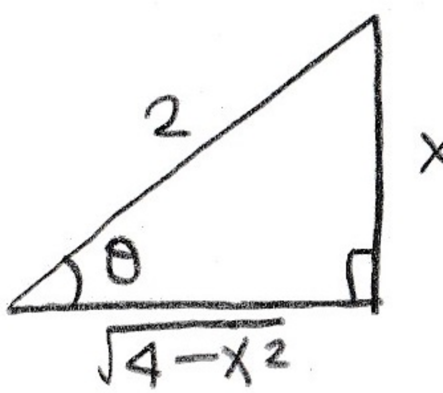
$$\begin{aligned}\int \sqrt{4-x^2}dx = &\int \sqrt{4-4\sin^2\theta}\cdot 2\cos\theta d\theta = \int \sqrt{4\cos^2\theta}\cdot 2\cos\theta d\theta \\ = &\int 2\cos\theta \cdot 2\cos\theta d\theta = \int 4\cos^2\theta d\theta.\end{aligned}$$

By using the following trigonometric identities:

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}d\theta \quad \text{and} \quad \sin 2\theta = 2\sin\theta\cos\theta$$

we obtain

$$\int 4\cos^2\theta d\theta = \int 2(1+\cos 2\theta)d\theta = 2\theta + \sin 2\theta + C = 2\theta + 2\sin\theta\cos\theta + C.$$



Since $x = 2\sin\theta$, $\theta = \arcsin\left(\frac{x}{2}\right)$ and $\sin\theta = \frac{x}{2}$. Also, by using the below picture, $\cos\theta = \frac{\sqrt{4-x^2}}{2}$. Thus,

$$\int \sqrt{4-x^2}dx = 2\theta + 2\sin\theta\cos\theta + C = 2\arcsin\frac{x}{2} + 2\cdot\frac{x}{2}\cdot\frac{\sqrt{4-x^2}}{2} + C = 2\arcsin\frac{x}{2} + \frac{x\sqrt{4-x^2}}{2} + C.$$

Question 1 (e)

SOLUTION. First, we have $-2 < f''(x) < 0$, so $|f''(x)| < 2$ and thus $K = 2$ in the Trapezoid rule formula. Also, from the integral bounds, $1 \le x \le 4$, and so $a = 1$, and $b = 4$ in the formula.
Thus, using the Trapezoid Rule with these values,

$$|E_T| \le \frac{2(4-1)^3}{12n^2} = \frac{2\cdot 27}{12n^2} = \frac{9}{2n^2}.$$

Now, find n such that

$$|E_T| = \frac{9}{2n^2} < 10^{-3}.$$

This implies

$$\frac{2n^2}{9} > 10^3 \implies n^2 > \frac{9\cdot 10^3}{2} = 4500.$$

Thus, $n > 67$ works.

Question 1 (f)

SOLUTION. To use the integral test, we need to compute

$$\int_2^\infty \frac{1}{x(\ln x)^p}dx$$

Let $u = \ln x$, then $du = \frac{1}{x}dx$. With this change of variables, when $x = 2$ then $u = \ln 2$ and when $x = \infty$ then u does as well. Applying the substitution,

$$\int_2^\infty \frac{1}{x(\ln x)^p}dx = \int_{\ln 2}^\infty \frac{1}{u^p}du.$$

Any restrictions on p will come if we cannot substitute the infinite upper limit and get zero. We have the following for the indefinite integral of u^{-p}:

$$\int \frac{1}{u^p}\mathrm{d}u = \begin{cases} \frac{1}{1-p}\frac{1}{u^{p-1}}, & p > 1 \\ \ln u, & p = 1 \\ \frac{1}{1-p}u^{1-p}, & p < 1. \end{cases}$$

The only case in which plugging in and infinite limit will result in zero is for $p > 1$. Since the Right hand side of the equality converges whenever $p > 1$, by the integral test, the series converges for $p > 1$.

Question 1 (g)

SOLUTION. Applying the ratio test with

$$a_n = \frac{(x-2)^n}{n^2+1},$$

we find that

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)^2+1}\cdot\frac{n^2+1}{(x-2)^n}\right| \\
&= \lim_{n\to\infty}|x-2|\left|\frac{n^2+1}{n^2+2n+2}\right| \\
&= |x-2|\lim_{n\to\infty}\left|\frac{1+\frac{1}{n^2}}{1+\frac{2}{n}+\frac{2}{n^2}}\right| \\
&= |x-2|.
\end{aligned}$$

The radius of convergence is the bound on x values that allows the series to converge. Using the ratio test, the series converges if the ratio tends to a number smaller than 1. Hence the series $\sum_{n=0}^{\infty}\frac{(x-2)^n}{n^2+1}$ converges for $|x-2|<1$ by the ratio test, and so its radius of convergence is 1.

Question 1 (h)

SOLUTION. By geometric series, we have, for $-1<x<1$,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Multiplying both sides by x yields

$$\frac{x^3}{1-x} = x^3\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+3}$$

which is the power series we are looking for.

Question 1 (i)

SOLUTION 1. By the formula for the Maclaurin series, the the fifth coefficient of the Maclaurin series for $f(x)$ is

$$c_5 = \frac{f^{(5)}(0)}{5!}.$$

Put $f(x)=e^{3x}$ and compute $f^{(5)}(x)$:

$$f(x)=e^{3x},\quad f'(x)=3e^{3x},\quad f^{(2)}(x)=3^2e^{3x},\quad f^{(3)}(x)=3^3e^{3x},\quad f^{(4)}(x)=3^4e^{3x},\quad f^{(5)}(x)=3^5e^{3x}.$$

Thus, we get $f^{(5)}(0)=3^5$ and the answer is

$$c_5 = \frac{f^{(5)}(0)}{5!} = \frac{3^5}{5!} = \frac{243}{120}.$$

Solution 2. Since the Maclaurin series for e^x

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

by plugging $3x$ in the position of x, we have the Maclaurin series for e^{3x}:

$$e^{3x} = \sum_{n=0}^{\infty} \frac{3^n x^n}{n!}.$$

Since the coefficient of the fifth degree term in this Maclaurin series is $\frac{3^5}{5!}$, the answer is $\frac{3^5}{5!} = \frac{243}{120}$.

Question 1 (j)

Solution. Note that f is increasing on some interval (a, b) if and only if $f'(x) > 0$ on (a, b). By the Fundamental theorem of Calculus, we have

$$f'(x) = 100(x^2 - 3x + 2)e^{-x^2}.$$

Since e^{-x^2} is always positive, we only need to check when the quadratic polynomial $x^2 - 3x + 2$ is positive. We can factor this to get

$$x^2 - 3x + 2 = (x - 2)(x - 1).$$

This will be positive when both factors are positive or both factors are negative. Both factors are positive when $x > 2$ and both factors are negative when $x < 1$. Therefore the function is increasing when $x < 1$ or $x > 2$.

Question 1 (k)

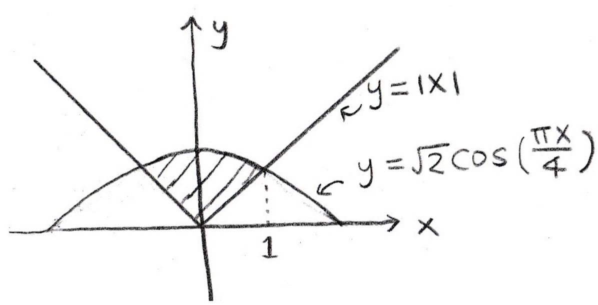
Solution. Since the power series representation of $\cos x$ and e^x are

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots$$
$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots,$$

we have

$$
\begin{aligned}
\lim_{x\to 0}\frac{1-\cos x}{1+x-e^x} &= \lim_{x\to 0}\frac{1-\left(1-\frac{x^2}{2}+\frac{x^4}{24}+\cdots\right)}{1+x-\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots\right)}\\
&= \lim_{x\to 0}\frac{\frac{x^2}{2}-\frac{x^4}{24}+\cdots}{-\frac{x^2}{2}-\frac{x^3}{6}+\cdots}\\
&= \lim_{x\to 0}\frac{x^2\left(\frac{1}{2}-\frac{x^2}{24}+\cdots\right)}{x^2\left(-\frac{1}{2}-\frac{x}{6}+\cdots\right)}\\
&= \lim_{x\to 0}\frac{\frac{1}{2}-\frac{x^2}{24}+\cdots}{-\frac{1}{2}-\frac{x}{6}+\cdots}\\
&= -1
\end{aligned}
$$

Question 2



SOLUTION 1. The given region is the following:
According to the picture above, the region has symmetry with respect to the y-axis (i.e. it is the same on the left and right). Thus, it's enough to compute the area when $x \geq 0$ or $x < 0$ and then double it. We will solve for $x > 0$.
We have to find the boundary of the region which is where the curves intersect. Since we will find the area when $x \geq 0$, it's enough to find the intersection point of two graphs on the first quadrant

$$y = x$$

and $y = \sqrt{2}\cos(\frac{\pi x}{4})$.

$$
\begin{aligned}
x &= \sqrt{2}\cos\left(\frac{\pi x}{4}\right)\\
\frac{x}{\sqrt{2}} &= \cos\left(\frac{\pi x}{4}\right)
\end{aligned}
$$

This may look difficult to solve by hand and in general it is. However, we have a few special values of cosine that we know and one of them is $\pi/4$ so it seems reasonable to check if $x = 1$ works. Substituting $x = 1$, $\frac{1}{\sqrt{2}} = \cos(\frac{\pi}{4})$, so the equality holds and our guess works. Thus, the two graphs on the first quadrant intersects at $x = 1$ and the region of interest is on $0 \leq x \leq 1$.
Compute the area. Denote the area of the region when $x \geq 0$ by A_+. Then since the trig function takes on larger values over the region, we have

$$
\begin{aligned}
A_+ = \int_0^1 \sqrt{2}\cos(\frac{\pi x}{4}) - x dx &= \left[\frac{4\sqrt{2}}{\pi}\sin(\frac{\pi x}{4}) - \frac{x^2}{2}\right]_{x=0}^{x=1} \\
&= \frac{4}{\pi}\cdot\sqrt{2}\cdot\frac{1}{\sqrt{2}} - \frac{1}{2} \\
&= \frac{4}{\pi} - \frac{1}{2}.
\end{aligned}
$$

Since the area of the whole region in the problem is the double of A_+ (by the reasons discussed above), the answer is $\frac{8}{\pi} - 1$.

Solution 2. This solution doesn't use the symmetry of the region.

Find the intersection points.

Since we have

$$
|x| = \sqrt{2}\cos(\frac{\pi x}{4}) \Leftrightarrow \begin{cases} x = \sqrt{2}\cos(\frac{\pi x}{4}) & x \geq 0 \\ -x = \sqrt{2}\cos(\frac{\pi x}{4}) & x < 0 \end{cases} \Leftrightarrow \begin{cases} \frac{x}{\sqrt{2}} = \cos(\frac{\pi x}{4}) & x \geq 0 \\ -\frac{x}{\sqrt{2}} = \cos(\frac{\pi x}{4}) & x < 0 \end{cases}
$$

by inspection, at $x = 1$, $\frac{1}{\sqrt{2}} = \cos(\frac{\pi}{4})$, so the first equality holds. Also, at $x = -1$, $-\frac{-1}{\sqrt{2}} = \frac{1}{\sqrt{2}} = \cos(\frac{\pi}{4})$, so the second equality holds. Thus, the two graph intersect at $x = \pm 1$.

Compute the area of the region.

$$
\begin{aligned}
\text{Area} &= \int_{-1}^1 \sqrt{2}\cos(\frac{\pi x}{4}) - |x| dx \\
&= \int_{-1}^0 \sqrt{2}\cos(\frac{\pi x}{4}) - |x| dx + \int_0^1 \sqrt{2}\cos(\frac{\pi x}{4}) - |x| dx \\
&= \int_{-1}^0 \sqrt{2}\cos(\frac{\pi x}{4}) + x dx + \int_0^1 \sqrt{2}\cos(\frac{\pi x}{4}) - x dx \\
&= \left[\frac{4\sqrt{2}}{\pi}\sin(\frac{\pi x}{4}) + \frac{x^2}{2}\right]_{x=-1}^{x=0} + \left[\frac{4\sqrt{2}}{\pi}\sin(\frac{\pi x}{4}) - \frac{x^2}{2}\right]_{x=0}^{x=1} \\
&= \frac{4}{\pi}\cdot\sqrt{2}\cdot\frac{1}{\sqrt{2}} - \frac{1}{2} + \frac{4}{\pi}\cdot\sqrt{2}\cdot\frac{1}{\sqrt{2}} - \frac{1}{2} \\
&= \frac{8}{\pi} - 1.
\end{aligned}
$$

Question 3

Solution. The given region is the following:thumb|center|upright=2.0.

We can find the bounds of the region by finding the intersection points of two graphs. By substituting $y = \frac{1}{x}$ into $3x + 3y = 10$, we have

$$
\begin{aligned}
3x + \frac{3}{x} &= 10 \\
3x^2 + 3 &= 10x \\
3x^2 - 10x + 3 &= 0 \\
(3x - 1)(x - 3) &= 0.
\end{aligned}
$$

Thus, at $x = \frac{1}{3}$ and $x = 3$, the two functions intersect and therefore the region to generate volume is bounded on $1/3 \leq x \leq 3$. We will compute the volume of the region by determining the volume generated by the wider function $(3x + 3y = 10)$ on its own and subtracting the volume generated by the narrower function $(y = 1/x)$. For a general function given by $y = f(x)$ on $a \leq x \leq b$, we think of the volume generated around the x-axis by integrating small cylinders with height dx and radius $r = f(x)$. The volume generated by this is

$$V = \int_a^b \pi f(x)^2 \mathrm{d}x.$$

For the function $3x + 3y = 10$, we need to write this as $y = f(x)$ to get $y = 10/3 - x$. The volume generated by this first function is therefore

$$V_1 = \pi \int_{\frac{1}{3}}^{3} \left(\frac{10}{3} - x\right)^2 \mathrm{d}x = \pi \int_{\frac{1}{3}}^{3} \frac{100}{9} - \frac{20}{3}x + x^2 \mathrm{d}x = \pi \left[\frac{100}{9}x - \frac{10}{3}x^2 + \frac{1}{3}x^3\right]_{\frac{1}{3}}^{3} = \frac{728}{81}\pi.$$

For the second function $y = 1/x$ the volume generated is

$$V_2 = \pi \int_{\frac{1}{3}}^{3} \left(\frac{1}{x}\right)^2 \mathrm{d}x = -\pi \left[\frac{1}{x}\right]_{\frac{1}{3}}^{3} = \frac{8}{3}\pi.$$

Therefore, the volume generated by the region between these two functions is

$$V = V_1 - V_2 = \frac{512}{81}\pi.$$

Question 4

Solution. Before beginning, note that all of the units are compatible with each other (i.e. metres go with cubic metres) and so we can just use numbers. However, in general, make sure you convert to the correct units before solving the problem.
First, if we describe the given situation on the graph, we get thumb|center|upright=2.
The formula for work is

$$dW = ahdm$$

with dm a small mass, h the distance from the mass to where we want the water to go, and a, the acceleration to move the mass.
Note that the mass of water and the distance from the water surface to the top of the tank are changed, as we pump the water. Thus, let′s introduce a variable y which represents the depth of the water as measured from the bottom of the tank. (See the above picture). The goal then is to come up with an integral involving this variable (i.e. we want acceleration, distance out of the tank, and mass to be written in terms of y).
First, since the only force on the water we have to overcome is from gravity, the acceleration is constant and given by Earth's gravitational field, $a = g = 9.8$.
Next, since we're dealing with a fluid, it is often easier to use the density instead of mass directly, i.e. we can relate dm to the volume it takes up dV with the fluid density ρ, via,

$$dm = \rho dV.$$

At a fixed depth, y, it will be the same amount of work to remove a single drop as it will the whole disk of water at this depth. Denote the radius of the circle at depth y by $R = R(y)$ as in the picture. Then, R satisfies

$$R(y) = \sqrt{3^2 - (3-y)^2}.$$

Therefore, the amount of volume we need to remove at depth y is the small cylinder made from the disk of radius $R(y)$ with a small amount of depth, dy,

$$dV = \pi R(y)^2 dy = \pi(9 - (3-y)^2)dy = \pi(6y - y^2)dy.$$

Finally, to setup the work, we need the height it takes to get the water out of the tank **as a function of** y. This last step is important and often the most overlooked since at first glance it may seem like this height is the same as y itself since they both measure distance. However, remember that y is the depth measured from the bottom of the tank but we need to get the water out of the **top** of the tank. From the picture we see that the distance from the top of the tank to the bottom of the tank is 7. As the depth increases the distance to the top of the tank decreases as

$$h = 7 - y.$$

Plugging everything together and using from the problem that $\rho = 1000$,

$$dW = (9.8)(7-y) \cdot 1000\pi(6y - y^2)dy.$$

For the work required to pump all of the water in the tank, the range for y is from 0 to 3, which implies

$$\begin{aligned}
W &= \int_0^3 (9.8)(7-y) \cdot 1000\pi(6y-y^2)dy = 9800\pi \int (7-y)(6y-y^2)dy \\
&= 9800\pi \int_0^3 y^3 - 13y^2 + 42y dy = 9800\pi \left[\frac{y^4}{4} - 13\frac{y^3}{3} + 21y^2\right]_{y=0}^{y=3} \\
&= 9800\pi \left(\frac{3^4}{4} - 13\frac{3^3}{3} + 21 \cdot 3^2\right). \\
&= 904050\pi
\end{aligned}$$

Question 5

SOLUTION. First, the given region is the following: thumb|center|upright=2.0
On the region of interest $0 \le x \le \frac{\pi}{4}$, we have $\cos(x) \ge \sin(x)$. By the formula for the centroid, we have

$$\bar{x} = \frac{1}{A}\int_0^{\frac{\pi}{4}} x(\cos x - \sin x)dx, \quad \text{and} \quad \bar{y} = \frac{1}{2A}\int_0^{\frac{\pi}{4}} \cos^2 x - \sin^2 x dx$$

where A is the area of the region which we will compute first:

$$A = \int_0^{\frac{\pi}{4}} \cos x - \sin x dx = [\sin x + \cos x]_0^{\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 = \sqrt{2} - 1.$$

Next we will compute the integral for the x portion of the centroid,

$$\int_0^{\frac{\pi}{4}} x(\cos x - \sin x)\mathrm{d}x = \int_0^{\frac{\pi}{4}} x\cos x\mathrm{d}x - \int_0^{\frac{\pi}{4}} x\sin x\mathrm{d}x.$$

We integrate each term by parts. For the first integral,

$$\int_0^{\frac{\pi}{4}} x\cos x\mathrm{d}x = [x\sin x]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sin x\mathrm{d}x = [x\sin x]_0^{\frac{\pi}{4}} + [\cos x]_0^{\frac{\pi}{4}} = \frac{\pi}{4}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1.$$

For the second integral,

$$\int_0^{\frac{\pi}{4}} x\sin x\mathrm{d}x = -\,[x\cos x]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \cos x\mathrm{d}x = -\,[x\cos x]_0^{\frac{\pi}{4}} + [\sin x]_0^{\frac{\pi}{4}} = -\frac{\pi}{4}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}.$$

subtracting the two integrals we have

$$\int_0^{\frac{\pi}{4}} x(\cos x - \sin x)\mathrm{d}x = \frac{\pi}{2\sqrt{2}} - 1.$$

Next we need to consider the integral for the y component of the centroid,

$$\int_0^{\frac{\pi}{4}} \cos^2 x - \sin^2 x\mathrm{d}x = \int_0^{\frac{\pi}{4}} \cos 2x\mathrm{d}x,$$

where we have used the trigonometric identity $\cos^2 x - \sin^2 x = \cos 2x$. Therefore we have

$$\int_0^{\frac{\pi}{4}} \cos 2x\mathrm{d}x = \left[\frac{1}{2} - 0\right] = \frac{1}{2}.$$

Thus, the centroid $(\bar{x}, \bar{y})$ is

$$\bar{x} = \frac{1}{A}\int_0^{\frac{\pi}{4}} x(\cos x - \sin x)dx, = \frac{1}{\sqrt{2}-1}\left(\frac{\pi}{2\sqrt{2}} - 1\right)$$

and

$$\bar{y} = \frac{1}{2A}\int_0^{\frac{\pi}{4}} \cos^2 x - \sin^2 x dx = \frac{1}{2(\sqrt{2}-1)}\cdot\frac{1}{2} = \frac{1}{4(\sqrt{2}-1)}.$$

Question 6

SOLUTION. By the separation of variables,

$$\begin{aligned}
x\frac{dy}{dx} &= y^2 - y \\
\frac{dy}{y^2 - y} &= \frac{dx}{x} \\
\int \frac{1}{y^2 - y}dy &= \int \frac{1}{x}dx \\
\int \frac{1}{y(y-1)}dy &= \ln|x| + C
\end{aligned}$$

For the y integral, we can decompose

$$\frac{1}{y(y-1)} = \frac{1}{y-1} - \frac{1}{y},$$

and therefore, we have,

$$\begin{aligned}
\int \frac{1}{y-1} - \frac{1}{y} dy &= \ln|x| + C \\
\ln|y-1| - \ln|y| &= \ln|x| + C \\
\ln\left|\frac{y-1}{y}\right| &= \ln|x| + C.
\end{aligned}$$

This implies

$$\left|\frac{y-1}{y}\right| = e^C|x|.$$

The initial condition is $y(1) = -1$, and so we get $2 = e^C$. Thus,

$$\left|\frac{y-1}{y}\right| = 2|x|.$$

Furthermore, when we use the initial data both quantities are positive and so we can drop the absolute value signs:

$$\begin{aligned}
\frac{y-1}{y} &= 2x \\
1 - \frac{1}{y} &= 2x \\
\frac{1}{y} &= 1 - 2x \\
y &= \frac{1}{1-2x}.
\end{aligned}$$

Question 7 (a)

SOLUTION. Note that $\frac{1}{\sqrt{n^2+1}}$ is similar to $\frac{1}{\sqrt{n^2}} = \frac{1}{n}$ for large n. The series of $1/n$ diverges and so, we expect that the series we are interested in diverges as well. It will be helpful if we can use the comparison test to compare to the $1/n$ series. To do this we need to find a sequence related to $1/n$ that is greater than the sequence we have.
Notice that for $n \geq 1$,

$$\sqrt{n^2+1} \leq \sqrt{n^2+n^2} = \sqrt{2}n.$$

If we invert this then the inequality changes,

$$\frac{1}{\sqrt{n^2+1}} \geq \frac{1}{\sqrt{2}n}.$$

The left of this expression is the set of terms we wish to sum and on the right is a sequence involving $1/n$. The terms we wish to sum are larger than the terms that we know diverge. This is precisely what we wanted and therefore, if we compare the series then

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}} \geq \sum_{n=1}^{\infty} \frac{1}{\sqrt{2}n} = \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} \frac{1}{n}$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges then, by the comparison test, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ diverges as well.

Question 7 (b)

Solution. First, if we evaluate a few terms we see that $\cos(\pi) = -1$, $\cos(2\pi) = 1$, $\cos(3\pi) = -1$, $\cdots$, $\cos(n\pi) = (-1)^n$ and therefore the terms alternate. Thus, the given series is alternating series,

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{2^n}.$$

Since $\lim_{n\to\infty} \frac{n}{2^n} = 0$ and a_n is decreasing for $n \geq 3$, by the alternating series test, the series $\sum_{n=1}^{\infty} \frac{n\cos(n\pi)}{2^n}$ converges.

Question 8

Solution. By the Taylor series formula (taking $a = 2$),

$$f(x) = \sum_{n=0}^{\infty} c_n (x-2)^n$$

where

$$c_0 = f(2) \quad \text{and} \quad c_n = \frac{f^{(n)}(2)}{n!}.$$

First, $c_0 = f(2) = \ln 2$.
Next, if we take a copule derivatives,

$$f^{(1)}(x) = \frac{1}{x}, \quad f^{(2)}(x) = -\frac{1}{x^2}, \quad f^{(3)}(x) = \frac{2}{x^3}, \quad f^{(4)}(x) = -\frac{6}{x^4},$$

we can find the following patterns: the sign keeps changing, the power of x in denominator of $f^{(n)}(x)$ is n, and the number in numerator of $f^{(n)}(x)$ is $(n-1)!$. Thus, we can get a general formula for $f^{(n)}(x)$:

$$f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^n},$$

which implies

$$f^{(n)}(2) = \frac{(-1)^{n+1}(n-1)!}{2^n}$$

and

$$c_n = \frac{(-1)^{n+1}(n-1)!}{2^n n!} = \frac{(-1)^{n+1}}{n2^n}.$$

Thus, we have the Taylor series:

$$\ln x = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n}(x-2)^n.$$

For the interval of convergence, we will use the ratio test. First, compute

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+2}}{(n+1)2^{n+1}}(x-2)^{n+1}}{\frac{(-1)^{n+1}}{n2^n}(x-2)^n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(x-2)^{n+1}}{(n+1)2^{n+1}}\frac{n2^n}{(-1)^{n+1}(x-2)^n}\right| = \lim_{n\to\infty}\frac{n}{2(n+1)}\cdots|x-2| = \frac{|x-2|}{2}.$$

The series converges when this ratio is smaller than 1 and so for x in $|x-2| < 2$, the series converges and for x in $|x-2| > 2$ the series diverges. We still have to check the endpoints, that is when $|x-2| = 2$. This occurs at $x = 0$ and $x = 4$. At $x = 0$, the series becomes,

$$\ln x = \ln 2 + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n2^n}(-2)^n = \ln 2 + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-1)^n}{n} = \ln 2 + \sum_{n=1}^{\infty}\frac{(-1)}{n}.$$

We know that this series diverges and therefore the Taylor series does not converge at $x = 0$. Note that we expect this to happen because $\lim_{x\to 0}\ln(x) = -\infty$. At $x = 4$, the series becomes,

$$\ln x = \ln 2 + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n2^n}(2)^n = \ln 2 + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}.$$

This is an alternating series with decreasing terms that tend to zero and therefore by the alternating series test, converges. Once again, this makes sense since $\ln(4)$ has a finite value. Combining everything together, the interval of convergence for the series is $x \in (0, 4]$.

Question 9 (a)

SOLUTION. We know that when a geometric series converges,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

A geometric series converges when $-1 < x < 1$ which is what we have in the problem. If we differentiate both sides:

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\sum_{n=0}^{\infty}x^n = \sum_{n=0}^{\infty}nx^{n-1}.$$

To get the extra factor of x, multiply both sides by x:

$$\frac{x}{(1-x)^2} = \sum_{n=0}^{\infty}nx^n.$$

This multiplication is okay to do since the sum changes with n and not x. This same logic is why we were not allowed to start by multiplying both sides by n.
We have shown the series relationship holds.

Question 9 (b)

SOLUTION. Start with the formula we derived in (a). If we differentiate this formula with respect to x,

$$\begin{aligned}\sum_{n=0}^{\infty} n^2 x^{n-1} &= \frac{d}{dx}\sum_{n=0}^{\infty} n x^n \\ &= \frac{d}{dx}\left(\frac{x}{(1-x)^2}\right) \\ &= \frac{1}{(1-x)^2} + \frac{2x}{(1-x)^3} \\ &= \frac{1+x}{(1-x)^3}.\end{aligned}$$

If we multiply both sides by x:

$$\sum_{n=0}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}$$

then we have the series we are interested in. We note that the two expression are only equivalent if the series converges. We will check the radius of convergence by using the ratio test.

$$\lim_{n\to\infty}\left|\frac{(n+1)^2 x^n}{n^2 x^{n-1}}\right| = \left(\lim_{n\to\infty}\left|\frac{(n+1)^2}{n^2}\right|\right)|x| = |x|.$$

By the ratio test, if $|x| < 1$, the series converges and if $|x| > 1$, the series diverges. This is unsurprising since we already know this is required for the geometric series in (a). We have to check the boundaries independently. If $x = 1$, the expression

$$\frac{x(1+x)}{(1-x)^3}$$

fails to exist and so the series must diverge. If $x = -1$, this expression is finite but that does **not** mean the series converges. We have to look at the series itself which becomes

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n n^2 &= -1^2 + 2^2 - 3^2 + 4^2 - 5^2 + 6^2 + \cdots = (-1^2 + 2^2) + (-3^2 + 4^2) + (-5^2 + 6^2) + \cdots \\ &= (2-1)(1+2) + (3-2)(3+4) + (6-5)(5+6) + \cdots = \sum_{n=0}^{\infty} n,\end{aligned}$$

and hence diverges. Thus, the series only converges for $x \in (-1, 1)$.

Good Luck for your exams!

