# Full Solutions
# MATH307 April 2013

April 5, 2015

### How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

## Tips for Using Previous Exams to Study: Exam Simulation

*Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.*

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.



## Question 1 (a)

SOLUTION. The *matrix* or *operator norm* is defined as

$$\|A\| = \max_{x:\,\|x\|\neq 0} \frac{\|Ax\|}{\|x\|}$$

## Question 1 (c)

SOLUTION. Recall that $\left\|A\vec{x}\right\| \leqslant \left\|A\right\| \left\|\vec{b}\right\| = 2 \cdot 1 = 2$
and
$\left\|A\vec{x}\right\| = \left\|\begin{matrix} 3 \\ 0 \end{matrix}\right\| = 3$
3 is not smaller than 2, so this is impossible.

## Question 1 (d)

SOLUTION. Let $A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$
Observe that $U^{-1} = U^T$ and all entries are real, so U is an orthogonal matrix. This means that $\left\|U\right\| \left\|B\right\| = \left\|B\right\| = \left\|A\right\|$
$\left\|B\right\| = 2$
because the norm of a diagonal matrix is the entry with the largest magnitude.
and therefore, $\left\|A\right\| = 2$.
Next, notice that
$A^{-1} = B^{-1}U^{-1} = B^{-1}U$
Once again, since U is orthogonal,
$\left\|A^{-1}\right\| = \left\|B^{-1}\right\|$
As
$B^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$
we have that $\left\|B^{-1}\right\| = 1$. Thus
$\left\|A\right\| \left\|A^{-1}\right\| = 2 \cdot 1 = 2$
therefore, cond(A) = 2

## Question Section 201 06 (a)

SOLUTION. When $\alpha = 0$,
The stochastic matrix P is the sum of the outgoing links and the dangling links:

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1/6 \\ 1/2 & 0 & 0 & 0 & 0 & 1/6 \\ 0 & 1/2 & 0 & 0 & 0 & 1/6 \\ 1/2 & 0 & 0 & 0 & 0 & 1/6 \\ 0 & 1/2 & 0 & 1 & 0 & 1/6 \\ 0 & 0 & 1 & 0 & 1 & 1/6 \end{bmatrix}$$

Vertex 6 has no links leaving the site, so it is a dangling node. When the web surfer reaches a dangling node, we assume that any page on the internet can be chosen at random with equal probability. This is why all the entries in column 6 of matrix $P$ contain the value 1/6.

## Question 1 (b)

SOLUTION. cond(A) $= \|A\| \|A^{-1}\|$

The conditions number helps us find the maximum relative error, $\frac{\|\Delta\vec{x}\|}{\vec{x}}$, for the solution to $A\vec{x} = \vec{b}$
A large condition number for a matrix, A, means that small changes (errors) in $\vec{b}$ can result in large changes (errors) in $\vec{x}$.
We get this by considering

$$A\vec{x} = \vec{b}$$
$$\vec{\Delta x} = A^{-1}\vec{\Delta b}$$

then
$\left\|\vec{b}\right\| \|\vec{\Delta x}\| = \|A\vec{x}\| \left\|A^{-1}\vec{\Delta b}\right\|$
dividing by $\left\|\vec{b}\right\| \|\vec{x}\|$ on both sides produces
$\frac{\|\vec{\Delta x}\|}{\|\vec{x}\|} = \frac{\|A\vec{x}\|\|A^{-1}\vec{\Delta b}\|}{\|\vec{b}\|\|\vec{x}\|}$
since $\|A\vec{x}\| \leqslant \|A\| \left\|\vec{b}\right\|$
$\frac{\|\vec{\Delta x}\|}{\|\vec{x}\|} \leqslant \frac{\|A\|\|A^{-1}\|\|\vec{\Delta b}\|}{\|\vec{b}\|}$
since cond(A) $= \|A\| \|A^{-1}\|$
$\frac{\|\vec{\Delta x}\|}{\|\vec{x}\|} \leqslant cond(A)\frac{\|\vec{\Delta b}\|}{\|\vec{b}\|}$

## Question 2 (a)

SOLUTION. A hermitian matrix is a matrix $A$ such that $A = A^*$. A matrix that is its own conjugate transpose. All symmetric matrices with all real entries are hermitian.

## Question 2 (b)

SOLUTION. True, because:
$\lambda_2\langle x_1, x_2\rangle = \langle x_1, \lambda_2 x_2\rangle = \langle x_1, Ax_2\rangle = \langle A^* x_1, x_2\rangle = \langle Ax_1, x_2\rangle$
Because the matrix is hermitian
$= \langle \lambda_1 x_1, x_2\rangle = \overline{\lambda_1}\langle x_1, x_2\rangle = \lambda_1\langle x_1, x_2\rangle$
Because hermitian matrices have real eigenvalues. So
$\lambda_1\langle x_1, x_2\rangle = \lambda_2\langle x_1, x_2\rangle$.
Given that $\lambda_1 \neq \lambda_2$, this can only be true if $\langle x_1, x_2\rangle = 0$

## Question 2 (c)

SOLUTION. False, because $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has a repeated eigenvalue of 1 and is diagonalized by $I$

## Question 2 (d)

SOLUTION. A stochastic matrix is a square matrix with no negative entries, and the columns sum to 1.

## Question 2 (e)

SOLUTION. -> The largest eigenvalue is 1
-> The eigenvector of the eigenvalue of 1 can be scaled to have no negative entries

## Question 2 (f)

SOLUTION. -> The largest eigenvalue is 1
-> All other eigenvalues are less than 1
-> The eigenvector of the eigenvalue of 1 can be scaled to have all positive entries

## Question 3 (a)

SOLUTION. Given

$$A$$

is a real symmetric matrix.
Assuming: The largest eigenvalue of $A$ is not repeated.
1) Take a random vector $x_0$.
2) Take $x_{n+1} = Ax_n$
3) Normalize $x$
4) If $x_n$ is not close enough to $x_{n-1}$ (ignoring sign flips), go to 2) ('close enough' being REALLY close)
$\rightarrow$ $x$ is now an approximation of the eigenvector of the largest eigenvalue
5) $\lambda = \langle x, Ax \rangle$ ($x$ was normalized earlier).

## Question 3 (b)

SOLUTION. Given: A is a real symmetric matrix.
Assuming: The eigenvalue of A closest to 2 is not repeated.
$\rightarrow$ The eigenvalues of $(A - sI)^{-1}$ are $(\lambda - s)^{-1}$.
$\rightarrow$ The eigenvectors of $(A - sI)^{-1}$ are the eigenvectors of $A$.
1) Take a random vector $x_0$.
2) Take $x_{n+1} = (A - 2I)^{-1}x_n$
3) Normalize x
4) If $x_n$ is not close enough to $x_{n-1}$ (ignoring sign flips), go to 2) ('close enough' being REALLY close)
$\rightarrow$ x is now an approximation of the eigenvector of the eigenvalue closest to 2
5) $\lambda = \langle x, Ax \rangle$ (x was normalized earlier).

## Question 3 (c)

SOLUTION. x=rand(1000,1); % Make x a random vector
x=x/norm(x); % Normalize x
for n=1:N
x=(A-2*eye(1000))\x; % The next iteration of x is (A-2I)/x
x=x/norm(x); % Normalize x
end
lambda=dot(x,A*x)

## Question 4 (a)

**SOLUTION.** We are given $\begin{bmatrix}1 & 2 & 4\\ 1 & -1 & 1\\ 1 & -1 & 1\end{bmatrix}$ and its reduced row echelon form $\begin{bmatrix}1 & 0 & 2\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}$

If we take the original matrix $\begin{bmatrix}1 & 2 & 4\\ 1 & -1 & 1\\ 1 & -1 & 1\end{bmatrix}$ as A, then $A^TA$ is not invertible, so we cannot obtain P. So, we need to find another matrix.

Since column 1 and column 2 are pivot columns, we take corresponding columns of A.

Those are $\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$ and $\begin{bmatrix}2\\ -1\\ -1\end{bmatrix}$, and they are linearly independent.

So, we define a new matrix $A = \begin{bmatrix}1 & 2\\ 1 & -1\\ 1 & -1\end{bmatrix}$ $A^T = \begin{bmatrix}1 & 1 & 1\\ 2 & -1 & -1\end{bmatrix}$

Projection matrix is given by $P = A(A^TA)^{-1}A^T$

So, if we substitute $A$ and $A^T$, we get

$$P = A(A^TA)^{-1}A^T = \begin{bmatrix}1 & 2\\ 1 & -1\\ 1 & -1\end{bmatrix}\left(\begin{bmatrix}1 & 1 & 1\\ 2 & -1 & -1\end{bmatrix}\begin{bmatrix}1 & 2\\ 1 & -1\\ 1 & -1\end{bmatrix}\right)^{-1}\begin{bmatrix}1 & 1 & 1\\ 2 & -1 & -1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & \frac{1}{2} & \frac{1}{2}\\ 0 & \frac{1}{2} & \frac{1}{2}\end{bmatrix}$$

## Question 4 (b)

**SOLUTION.** Vector in S closest to $\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$ means minimizing $\|Ax - b\|$.

This will happen when $Ax$ is the projection of b onto $R(A)$, that is, $Ax = Pb$

Thus, MATLAB commands are following

> P=[1 0 0; 0 0.5 0.5; 0 0.5 0.5]

> b=[1; 0; 0]

>P*b

I got P in (a) as follows.

We are given $\begin{bmatrix}1 & 2 & 4\\ 1 & -1 & 1\\ 1 & -1 & 1\end{bmatrix}$ and its reduced row echelon form $\begin{bmatrix}1 & 0 & 2\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}$

Since column 1 and column 2 are pivot columns, we take corresponding columns of A.

Those are $\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$ and $\begin{bmatrix}2\\ -1\\ -1\end{bmatrix}$, and they are linearly independent.

So, we define a new matrix $A = \begin{bmatrix}1 & 2\\ 1 & -1\\ 1 & -1\end{bmatrix}$ $A^T = \begin{bmatrix}1 & 1 & 1\\ 2 & -1 & -1\end{bmatrix}$

Projection matrix is given by $P = A(A^TA)^{-1}A^T$

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$$P = A(A^TA)^{-1}A^T = \begin{bmatrix}1 & 2\\ 1 & -1\\ 1 & -1\end{bmatrix}\left(\begin{bmatrix}1 & 1 & 1\\ 2 & -1 & -1\end{bmatrix}\begin{bmatrix}1 & 2\\ 1 & -1\\ 1 & -1\end{bmatrix}\right)^{-1}\begin{bmatrix}1 & 1 & 1\\ 2 & -1 & -1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & \frac{1}{2} & \frac{1}{2}\\ 0 & \frac{1}{2} & \frac{1}{2}\end{bmatrix}$$

## Question Section 201 05 (a)

**SOLUTION.** Write the incidence matrix $D$ and Laplacian matrix $L$. Also show that (for any graph) $N(L) = N(D)$. Is it true that $R(D^T) = R(L)$?

Answer:

Reading off the graph (where -1 is info leaving node and +1 is info entering node):

$$D = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

and can either compute L by } L = DˆTD \text{ or get it directly from the graph using the following method:
$L_{i,i} =$ number of other nodes connected to the $i^{th}$ node
$L_{i,j} = -1$if the two nodes are connected, 0 otherwise

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & 0 & 0 & -1 \\ -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 0 & -1 & 2 \end{bmatrix}$$

As for nullspaces of L and D,
$L = D^T RD = D^T D$
so if $D\underline{v} = \underline{0}$
$L\underline{v} = d^T D\underline{v}$
$= D^T(\underline{0})$
$= \underline{0}$
$\therefore N(L) = N(D)$
Also:
$R(D^T) = N(D)^{\perp} = N(L)^{\perp}$ by orthogonality relations
$= R(L^T)$
$= R(L)$ since L is symmetric
$\therefore R(D^T) = R(L)$

## Question Section 201 05 (b)

SOLUTION. Reading off graph (using -1 indicated opposite direction as shown, +1 to indicate same direction as shown) (each row corresponds to an edge)

$$\vec{v_1} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

These form a complete basis in $N(D^T)$ if the $dim(N(D^T)) = 2$. So (if $D$ has $n$ rows and $m$ columns): $dim(N(D)) = 1$ (# of connected circuits). Therefore, $r(D) = m - dim(N(D)) = 6 - 1 = 5$ (by Rank-Nullity theorem). Also by Rank-Nullity theorem: $dim(N(D^T)) = n - r(D) = 7 - 5 = 2$ and hence these two loop current vectors form a basis in $N(D^T)$ and any other loop current can be expressed using a linear combination of these two basis vectors.

## Question Section 201 05 (c)

Solution.

$$\vec{v} = \begin{bmatrix} b_1 \\ b_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix}$$

Here every entry is the voltage at the corresponding node. $b_1$ and $b_2$ are the voltages held by the battery connected at nodes 1 and 2.

$$\vec{J} = \begin{bmatrix} c \\ -c \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Here every entry is the current flowing into/out of the corresponding node. $c$ is the current coming out of node 1 from the battery, -$c$ is the current coming out of node 2 from the battery.
To find $R_{eff}$ remember Ohm's law: $V$=$IR$. Hence

$$R_{\text{eff}} = \frac{b_2 - b_1}{c}$$

Notice that $c$ is determined by the resistance of the rest of the circuit. If $c$ is unknown, the effective resistance can be computed using Schur′s Complement. Remember:

$$S = \frac{1}{R_{\text{eff}}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

## Question Section 201 06 (b)

Solution. The stochastic matrix for a damped system is

$$\begin{aligned} S &= (1-\alpha)Q + \alpha P \\ &= \frac{1}{2}Q + \frac{1}{2}P \\ &= \begin{bmatrix} 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/6 \\ 1/3 & 1/12 & 1/12 & 1/12 & 1/12 & 1/6 \\ 1/12 & 1/3 & 1/12 & 1/12 & 1/12 & 1/6 \\ 1/3 & 1/12 & 1/12 & 1/12 & 1/12 & 1/6 \\ 1/12 & 1/3 & 1/12 & 7/12 & 1/12 & 1/6 \\ 1/12 & 1/12 & 7/12 & 1/12 & 7/12 & 1/6 \end{bmatrix} \end{aligned}$$

When the damping factor $\alpha$ tends to zero, the eigenvalues of $S$ tend to $Q$. Because $Q$ projects onto a 1 dimension subspace, there is one eigenvalue equal to 1, and the rest are zero.

## Question Section 201 06 (c)

 

Solution.

$$\begin{aligned}
\underline{x_1} &= S\underline{x_0} \\
&= \begin{bmatrix} 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 1/6 \\ 1/3 & 1/12 & 1/12 & 1/12 & 1/12 & 1/6 \\ 1/12 & 1/3 & 1/12 & 1/12 & 1/12 & 1/6 \\ 1/3 & 1/12 & 1/12 & 1/12 & 1/12 & 1/6 \\ 1/12 & 1/3 & 1/12 & 7/12 & 1/12 & 1/6 \\ 1/12 & 1/12 & 7/12 & 1/12 & 7/12 & 1/6 \end{bmatrix} \begin{bmatrix} 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \end{bmatrix} \\
&= (1/72) \begin{bmatrix} 7 \\ 10 \\ 10 \\ 10 \\ 16 \\ 19 \end{bmatrix}
\end{aligned}$$

From this solution, it is clear that the page with the highest rank after one step is page 6.

## Question Section 201 06 (d)

Solution. >> [V D] = eig(S)
Where $V$ is the matrix of normalized eigenvectors, and $D$ is the diagonal matrix of eigenvalues.
It can be assumed that $\lambda_1 = 1$ is the first entry in $D$, the corresponding eigenvector is the first column of matrix $V$. To get the ranking we scale so that the sum is equal to 1.
>> V(:,1) / sum(V(:,1))

## Question Section 201 07 (a)

Solution. If we define

$$U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

Matrices $U$ and $V$ are orthogonal matrices because their columns form a basis of orthonormal vectors, all the entries are real, and the sum of the column entries are equal to 1.
Matrix $\Sigma$ is a diagonal matrix.
A property of orthogonal matrices is that they have no effect on the length of a norm.
$||A|| = ||\Sigma|| = \sigma_1$
The norm of matrix $A$ is equal to the largest value on the diagonal of matrix $\Sigma$.
The rank of $A$ is equal to the number of pivot columns in the diagonal matrix $\Sigma$:
$\dim(A) = 2$

## Question Section 201 07 (b)

Solution. The equation for the singular value matrix decomposition is
$A = U\Sigma V^* = U\Sigma V^T$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{(2)}} & 0 & -\frac{1}{\sqrt{(2)}} \\ \frac{1}{\sqrt{(2)}} & 0 & \frac{1}{\sqrt{(2)}} \\ 0 & 1 & 0 \end{bmatrix}$$

It can be seen that the third column of V is an orthogonal basis for N(A), and both columns of U form the basis for R(A).

## Question Section 201 07 (c)

Solution. The eigenvalues and eigenvectors for $A^*A$:
$A^*A = V\Sigma^T\Sigma V^T$
Since $\Sigma^T = \Sigma$,
$\Sigma^T\Sigma = \Sigma^2$
Which means that our eigenvalues are $\sigma_1^2,\ \sigma_2^2$, and 0
and the eigenvectors are the columns of V.
The eigenvalues and eigenvectors for $AA^*$:
$AA^* = U\Sigma^2U^T$
From this equation, we can see that our eigenvalues are the same as the last. However, the corresponding eigenvectors are now the columns of U.

## Question Section 202 05 (a)

Solution. First locate the pivot columns, which are columns 1 and 3.
Then, the basis for R(A) are the columns 1 and 3 in matrix A,
$\begin{bmatrix}1\\2\\3\end{bmatrix}\ \begin{bmatrix}0\\1\\1\end{bmatrix}$

## Question Section 202 05 (b)

Solution. To do this, we identify $x_2$ and $x_4$ being free variables.
And use the equation A$\vec{x} = \vec{0}$ where $\vec{x} = [x_1\ x_2\ x_3\ x_4]]^T$
hence forming the following equations from the rref(A)
$x_1$ = - $x_2$ - $x_4$
$x_2 = x_2$
$x_3$ = - $x_4$
$x_4 = x_4$.
therefore $\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = x_2\begin{bmatrix}-1\\1\\0\\0\end{bmatrix} + x_4\begin{bmatrix}-1\\0\\-1\\1\end{bmatrix}$
If we let $x_2 = x_4 = 1$ for example, then N(A) =span( $\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\-1\\1\end{bmatrix}$)

## Question Section 202 05 (c)

Solution. This is equivalent to the pivot rows of rref(A) since the range or columns space of $A^T$ is equal to row space of A. The rows are written as columns.
Therefore the basis for R ($A^T$) = $\begin{bmatrix}1\\1\\0\\1\end{bmatrix}$ and $\begin{bmatrix}0\\0\\1\\1\end{bmatrix}$.

## Question Section 202 05 (d)

Solution. From part a) the rank of A (also written as r(A) is 2. (number of pivot columns)
From the nullity theorem, dim(N($A^T$)) = n-r(A)

but n = 3
r(A) = 2
hence dim(N($A^T$)) = 3-2 = 1

## Question Section 202 06 (a)

SOLUTION. Since $e^{i2\pi nx}$ is an orthonormal basis for any function in the Hilbert Space we can use the inner product to find the coefficients:

$$\left\langle e^{i2\pi mx}, f(x)\right\rangle = \sum_{n=-\infty}^{\infty} c_n \left\langle e^{i2\pi mx}, e^{i2\pi nx}\right\rangle = c_m L$$

Therefore we get:

$$c_n = \frac{1}{L}\int_a^b e^{-2\pi inx/L} f(x)dx$$

Computing this for the given function

$$\begin{aligned}
c_0 &= \frac{1}{1}[\int_0^{1/2} e^{-i2\pi(0)x}dx + \int_{1/2}^{1} e^{-i2\pi(0)x}(0)dx] = \frac{1}{2} \\
c_n &= \frac{1}{1}[\int_0^{1/2} e^{-i2\pi nx}dx + \int_{1/2}^{1} e^{-i2\pi nx}(0)dx] \\
&= \frac{-1}{i2\pi n} e^{-i2\pi nx}\Big|_0^{1/2} = \frac{-1}{i2\pi n}[e^{-i\pi n} - e^0] \\
&= \frac{i}{2\pi n}[(-1)^n - 1]
\end{aligned}$$

$$c_n = \begin{cases} \frac{1}{2} & \text{if } n = 0 \\ 0 & \text{if } n \text{ is even, } n \neq 0 \\ \frac{-i}{\pi n} & \text{if } n \text{ is odd} \end{cases}$$

## Question Section 202 06 (b)

SOLUTION. From part (a) the coefficients of the Fourier series were found to be:

$$\begin{aligned}
c_0 &= \frac{1}{2} \\
c_n &= \begin{cases} 0 & \text{if } n \text{ is even, } n \neq 0 \\ \frac{-i}{\pi n} & \text{if } n \text{ is odd} \end{cases}
\end{aligned}$$

This yields

$$f(x) = c_0 + \sum_{n \text{ odd, } n=-\infty}^{\infty} \frac{-i}{\pi n} e^{2\pi inx}$$

Parseval′s Theorem states:

$$\int_a^b |f(x)|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Computing the left side:

$$\text{LHS} = \int_a^b |f(x)|^2 = \int_0^{1/2} 1^2 = \frac{1}{2}$$

Computing the right side:

$$\text{RHS} = \sum_{n=-\infty}^{\infty} |c_n|^2 = (\frac{1}{2})^2 + \sum_{n \text{ odd, } n=1}^{\infty} \frac{1}{\pi^2 n^2} + \sum_{n \text{ odd, } n=1}^{\infty} \frac{1}{\pi^2(-n)^2}$$

Here the sum of odds from $-\infty$ to $\infty$ was split into two sums from 1 to $\infty$ and -1 to $-\infty$. It can be observed that the second sum is the same as the first where $n = -n$ and since $(-n)^2 = n^2$, these two sums are equivalent.

$$\text{RHS} = (\frac{1}{2})^2 + 2(\sum_{n \text{ odd, } n=1}^{\infty} \frac{1}{\pi^2 n^2})$$

To get the expression that we want we can make a substitution of variables to remove the odd restriction in our sum. Let $n = 2k+1$. It can be seen that for *k=0,1,2,3,...*, $n$ will always be odd.
Note: Making this substitution will change summation range.
At $n = 1, 1 = 2k + 1 \Rightarrow k = 0$. So the new summation range will be $k = 0$ to $\infty$.

$$\text{RHS} = \frac{1}{4} + 2(\sum_{k=0}^{\infty} \frac{1}{\pi^2(2k+1)^2})$$

Using the left side that was computed above we get:

$$\frac{1}{2} = \frac{1}{4} + \frac{2}{\pi^2}(\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2})$$

Rearranging this to get the final answer:

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$

## Question Section 202 07 (a)

SOLUTION. Since we want to interpolate with cubic splines:

$$p_1(x) = a_1 x^3 + b_1 x^2 + c_1 x + d_1$$
$$p_2(x) = a_2(x-1)^3 + b_2(x-1)^2 + c_2(x-1) + d_2$$

## Question Section 202 07 (b)

SOLUTION. We want

$$\begin{aligned}(0,1) &\Rightarrow p_1(0) = 1\\ (1,0) &\Rightarrow p_1(1) = 0\\ (1,0) &\Rightarrow p_2(1) = 0\\ (2,2) &\Rightarrow p_2(2) = 2\end{aligned}$$

These translate to:

$$\begin{aligned}1 &= 0 + 0 + 0 + d_1\\ 0 &= a_1 + b_1 + c_1 + d_1\\ 0 &= 0 + 0 + 0 + d_2\\ 2 &= a_2 + b_2 + c_2 + d_2\end{aligned}$$

## Question Section 202 07 (c)

SOLUTION. The only interior point is $(1, 0)$. This condition can be written as:

$$\begin{aligned}p_1'(1) &= p_2'(1)\\ p_2''(1) &= p_2'(1)\end{aligned}$$

with
$p_1'(x) = 3a_1x^2 + 2b_1x + c_1 \Rightarrow p_1'(1) = 3a_1 + 2b_1 + c_1$
$p_2'(x) = 3a_2(x-1)^2 + 2b_2(x-1) + c_2 \Rightarrow p_2'(1) = 0 + 0 + c_2$
and the second derivatives:
$p_1''(x) = 6a_1x + 2b_1 \Rightarrow p_1''(1) = 6a_1 + 2b_1$
$p_2''(x) = 6a_2(x-1) + 2b_2 \Rightarrow p_2''(1) = 2b_2$
The equations are therefore:

$$\begin{aligned}3a_1 + 2b_1 + c_1 &= c_2\\ 6a_1 + 2b_1 &= 2b_2\end{aligned}$$

## Question Section 202 07 (d)

SOLUTION. The two endpoints are $(0, 1)$ and $(2, 2)$. This condition can be written as:

$$\begin{aligned}p_1''(0) &= 0\\ p_2''(2) &= 0\end{aligned}$$

The first derivatives:

$$\begin{aligned}p_1'(x) &= 3a_1x^2 + 2b_1x + c_1\\ p_2'(x) &= 3a_2(x-1)^2 + 2b_2(x-1) + c_2\end{aligned}$$

and the second derivatives:

$$p_1''(x) = 6a_1x + 2b_1 \Rightarrow p_1''(0) = 2b_1$$
$$p_2''(x) = 6a_2(x-1) + 2b_2 \Rightarrow p_2''(2) = 6a_2 + 2b_2$$

Therefore the equations are:

$$\begin{aligned} 2b_1 &= 0 \\ 6a_2 + 2b_2 &= 0 \end{aligned}$$

## Question Section 202 07 (e)

**Solution.** The equations are:

$$\begin{aligned} d_1 &= 1 \\ a_1 + b_1 + c_1 + d_1 &= 0 \\ d_2 &= 0 \\ a_2 + b_2 + c_2 + d_2 &= 2 \\ 3a_1 + 2b_1 + c_1 - c_2 &= 0 \\ 2b_1 &= 0 \\ 6a_2 + 2b_2 &= 0 \\ 6a_1 + 2b_1 - 2b_2 &= 0 \end{aligned}$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 2 & 0 & 0 \\ 6 & 2 & 0 & 0 & 0 & -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \\ a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Matlab:

```
   A=[0,0,0,1,0,0,0,0; 1,1,1,1,0,0,0,0; 0,0,0,0,0,0,0,1; ...
0,0,0,0,1,1,1,1; 3,2,1,0,0,0,-1,0; 0,2,0,0,0,0,0,0; ...
0,0,0,0,6,2,0,0; 6,2,0,0,0,-2,0,0];
b=[1,0,0,2,0,0,0,0]';
a=A\b;
```

# Good Luck for your exams!