Time Series and Forecasting
Stationarity and Dependence Measures

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Formal definition of Time Series-Stochastic Processes

- A **Stochastic Process** is a family of random variables indexed by $t \in T$.
- **recall** A random variable (real valued) is a mapping from sample space to real line
- notation:
  - discrete parameter stochastic process X_t, $t = \ldots, -2, -1, 0, 1, 2, \ldots$
  - continuous parameter stochastic process $X(t)$, $-\infty < t < \infty$
- A **Time Series** is a stochastic process indexed by time, that is a sequence of random variables indexed by time. The stochastic process is observed at regularly spaced intervals and those observations become the time series.

Measures of Dependence

We now consider some functions to help measuring and characterizing the dependence in the time series. We start with populational measures and then talk about sample counterparts.

- **Mean function** $\mu_t = E(X_t)$
- Autocovariance funtion

$$\gamma_X(s,t) = cov(X_s, X_t) = E\left[(X_s - \mu_s)(X_t - \mu_t)\right]$$

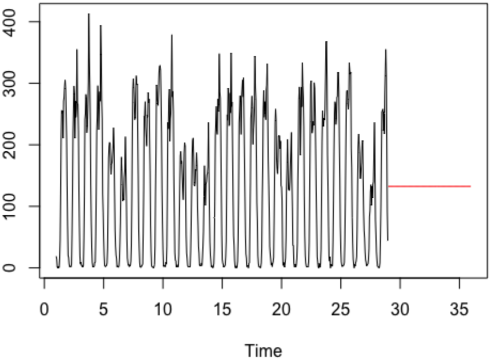
- **Autocorrelation function ACF**

$$\rho(s,t) = \frac{\gamma(s,t)}{\sqrt{\gamma(t,t)\gamma(s,s)}}$$

Stationarity

- The above definitions are *population* definitions that are computed for specific models/time series processes, the Data Generating Process, DGP
- Usually we do not know the process. We have data and we want to compute these dependence measures for the observed time series- sample measures
- We estimate the mean and the other dependence measures from our data

Mean of the number of calls



Stationarity

- Think about computing the mean of the time series: it only makes sense if the observations come from variables with the same mean,

$$\overline{X} = \frac{1}{n}\sum_{t=1}^{n} X_t$$

- To compute ACF, say at lag 1, we need to compute the correlation between all pairs (X_t, X_{t-1}) and it only makes sense if that correlation is the same over time

Stationary Time Series

- The computation (estimation) of the dependence measures requires some regularity on the behaviour of the time series so that averaging makes sense
- This leads to the need of introducing some assumptions about the behaviour of the time series

Stationary Time Series

- A time series is called strictly stationary if $F(X_{t_1}, X_{t_2}, \ldots, X_{t_k}) = F(X_{t_1+s}, X_{t_2+s}, \ldots, X_{t_k+s})$ for all $t_1, \ldots, t_k$ and s. Too difficult to check in practice
- A time series is called **Weakly Stationary** or Covariance Stationary if
  1. $E(X_t) = \mu$, a finite constant independent of t
  2. $var(X_t) = E[(X_t - \mu)^2] = \sigma_X^2$, a finite constant independent of t
  3. $cov(X_t, X_{t+\tau}) = \gamma_X(\tau)$, a finite constant that can depend on τ **but not on** t

     the **autocovariance function, FACOV**, τ is known as *lag*
  4. $$corr(X_t, X_{t+\tau}) = \frac{cov(X_t, X_{t+\tau})}{\sqrt{Var(X_t)Var(X_{t+\tau})}} = \frac{\gamma(\tau)}{\gamma(0)} = \rho_\tau$$

     a constant that can depend on τ, known as *lag* but not on t
  5. $\rho_\tau, \tau = \ldots, -2, -1, 0, 1, 2, \ldots$ is the **autocorrelation function, ACF**
  6. ρ_τ, measures the linear dependence between X_t and $X_{t+\tau}$

Properties of the ACF

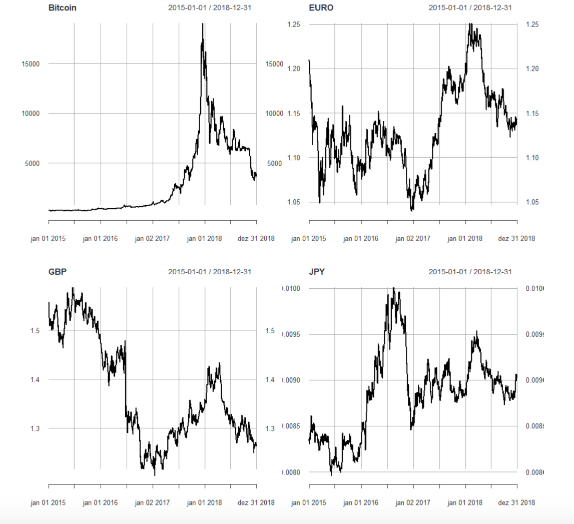
- $\rho_0 = 1.$
- $\rho_\tau = \rho_{-\tau}$, symmetric function
- $|\rho_\tau| \leq 1$
- If X_t is IID the sample autocorrelations $\hat{\rho}_\tau,\ \tau > 0$ are approximately $N(0, 1/n)$
- An autocorrelation $\hat{\rho}_\tau$, (partial autocorrelation, $\hat{\phi}_k$) is significant if $|\hat{\rho}_\tau| > 2/\sqrt{n}$ (Bartlett test)

Stationary Time Series

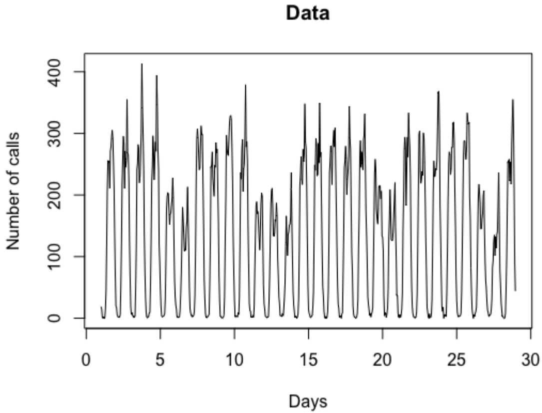
A **stationary series** is:

- roughly horizontal
- constant variance
- no patterns predictable in the long term

Stationary ?


Bitcoin
2015-01-01 / 2018-12-31
EURO
2015-01-01 / 2018-12-31
GBP
2015-01-01 / 2018-12-31
JPY
2015-01-01 / 2018-12-31
jan 01 2015
jan 01 2016
jan 02 2017
jan 01 2018
dez 31 2018

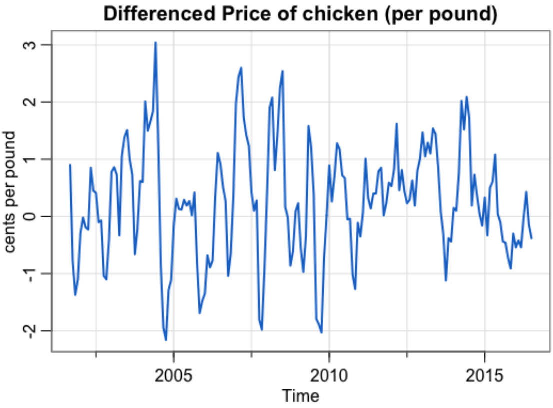
Stationary?



Stationary?

Price of chicken (per pound)
cents per pound
60
70
80
90
100
110
2005
2010
2015
Time

Stationary?


Differenced Price of chicken (per pound)
cents per pound
3
2
1
0
-1
-2
2005
2010
2015
Time

Which are stationary?