MIDDLE EAST TECHNICAL UNIVERSITY

ELECTRICAL AND ELECTRONICS ENGINEERING

EE407 Homework 2

Prepared by

Rafet KAVAK - Ersin KESKİN

2166783 - 2166817

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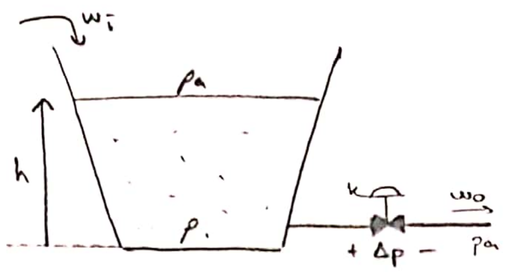
Q1: A Hydraulic System



ρ: Density of the liquid

p: Absolute pressure

p_a: Atmosphere pressure (Assumed constant)

g: Gravitational acceleration

h: Liquid height

w_i = Liquid inflow rate

w_o = Liquid outflow rate

Control input: w_i

Control output: h

$p = \rho g h + p_a$

$\Delta p = p - p_a = \rho g h$

$w_o = k\sqrt{\Delta p} = k\sqrt{\rho g h}$

a) Let's assume average area of the tank is A, then the volume is

$V(t) = A \cdot h(t)$

Then, we can represent volume changing as,

$$\frac{dV(t)}{dt} = w_i(t) - w_o(t)$$

$$A\frac{dh(t)}{dt} = w_i(t) - k\sqrt{\rho g h(t)} \Rightarrow \boxed{\frac{dh(t)}{dt} = \frac{w_i(t)}{A} - \frac{k}{A}\sqrt{\rho g h(t)}}$$

b) $\Delta p(t) = \rho g h(t)$

$$\frac{d\Delta p(t)}{dt} = \rho g \frac{dh(t)}{dt}$$

$$\frac{d\Delta p(t)}{dt} = \rho g \left[\frac{w_i(t)}{A} - \frac{k}{A}\sqrt{\underbrace{\rho g h(t)}_{\Delta p(t)}}\right]$$

$$\boxed{\frac{d\Delta p(t)}{dt} = \frac{\rho g}{A}\left(w_i(t) - k\sqrt{\Delta p(t)}\right)}$$

c) Control output : $h(t) \Rightarrow \boxed{h = r}$ Control input : $w_1(t) \Rightarrow \boxed{w_1(t) = u_{ss}}$

$$\frac{dh(t)}{dt} = \frac{w_1(t)}{A} - \frac{k}{A}\sqrt{\rho g h(t)}$$

$$0 = \frac{u_{ss}}{A} - \frac{k}{A}\sqrt{\rho g r}$$

$$\boxed{u_{ss} = k\sqrt{\rho g r}}$$

Q-2 [A biochemical reactor]

V : constant
x_1 : concentration of biomass cells kg/m^3 /
x_2 : concentration of substrate kg/m^3
x_{1f} : concentration in the feed stream $\rightarrow x_{1f} = 0$ (Assumed)
x_{2f} : concentration in the feed stream
r_1 : rate of biomass cell generation $kg/m^3 \cdot s$
r_2 : rate of substrate consumption $kg/m^3 \cdot s$
F : flow rate m^3/s

Rate of Biomass Accumulation = In flow rate − Out flow rate + Generation rate (1)
Rate of Substrate Accumulation = In flow − Out flow − Consumption (2)

$r_1 = \mu x_1$ (3)
$\mu = f(x_2)$ (4)
$r_1 = Y \cdot r_2$, Y : constant (5) .

(1) $\frac{d}{dt} x_1 = 0 - x_1 \cdot \frac{F}{V} + r_1$, $\frac{d}{dt} x_1 = -x_1 \cdot d + x_1 \cdot \mu$

(2) $\frac{d}{dt} x_2 = x_{2f} \cdot \frac{F}{V} - x_2 \cdot \frac{F}{V} - r_2$, $\frac{d}{dt} x_2 = x_{2f} d - x_2 d - x_1 \cdot \frac{\mu}{Y}$

a)
$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -d+\mu & 0 \\ -\mu/Y & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ d \end{bmatrix} \cdot x_{2f}$$

b) $\frac{d}{dt} x_1 = 0$ $\frac{d}{dt} x_2 = 0$

$$\frac{d}{dt} x_1 = -x_1 \cdot d + x_1 \cdot \mu_m \frac{x_2}{k_m + x_2} = 0$$

$\hookrightarrow$ $-x_1 \cdot d \cdot k_m - x_1 \cdot x_2 \cdot d + x_1 \cdot x_2 \cdot \mu_m = 0$

$$x_1\left(-d\cdot k_m + x_2(\mu_m - d)\right) = 0 \quad \rightsquigarrow \quad x_1 = 0 \quad , \quad x_2 = \frac{d\cdot k_m}{\mu_m - d}$$

- $x_2 = \dfrac{d.k_m}{\mu_m - d}$

$$\frac{d}{dt}x_2 = x_{2f}\cdot d - x_2\cdot d - x_1\cdot\frac{\mu_m\cdot x_2}{(k_m + x_2)\,Y} = 0$$

$$\hookrightarrow \quad x_{2f}\cdot d - \frac{d^2 k_m}{\mu_m - d} - x_1\cdot\frac{d\cdot k_m\cdot\mu_m}{(\mu_m - d)\left(k_m + \dfrac{d k_m}{\mu_m - d}\right)\cdot Y} = 0$$

$$x_{2f}\cdot d - \frac{d^2 k_m}{\mu_m - d} - x_1\cdot\frac{d\cdot k_m\cdot\mu_m}{k_m\cdot\mu_m\cdot Y} = 0$$

$$x_1 = x_{2f}\cdot Y - \frac{k_m\cdot d\cdot Y}{\mu_m - d}$$

- $x_1 = 0$

$$\frac{d}{dt}x_2 = x_{2f}\cdot d - x_2 d = 0$$

$$x_2 = x_{2f}$$

Equilibrium Points :

$$\left.\begin{aligned} x_1 &= \frac{Y\left(x_{2f}(\mu_m - d) - k_m d\right)}{\mu_m - d} & x_2 &= \frac{d\,k_m}{\mu_m - d}\end{aligned}\right\} \text{set } 1$$

$$\left.\begin{aligned} x_1 &= 0 & x_2 &= x_{2f}\end{aligned}\right\} \text{set } 2$$

c) Neglecting the efect of Higher Order Terms on linearization ;

$$f_{lin}(x_1, x_2) \simeq f(x_1^{op}, x_2^{op}) + (x_1 - x_1^{op})\cdot\left[\frac{\partial f(x_1,x_2)}{\partial x_1}\right]_{\substack{x_1\to x_1^{op}\\ x_2\to x_2^{op}}} + (x_2 - x_2^{op})\cdot\left[\frac{\partial f(x_1,x_2)}{\partial x_2}\right]_{\substack{x_1\to x_1^{op}\\ x_2\to x_2^{op}}}$$

① $$\frac{d}{dt}x_1 = -x_1\cdot d + x_1\cdot x_2\cdot\frac{\mu_m}{k_m + x_2} = x_1\left(-d + \frac{x_2\,\mu_m}{k_m + x_2}\right)$$

1

$$\frac{d}{dt}x_1 = \frac{Y\left(x_{2f}(\mu_m - d) - k_m d\right)}{\mu_m - d}\cdot\left(-d + \frac{\dfrac{d\,k_m\,\mu_m}{\mu_m - d}}{k_m + \dfrac{d\,k_m}{\mu_m - d}}\right)$$

$$+ \left(x_1 - \frac{Y\left(x_{2f}(\mu_m - d) - k_m d\right)}{\mu_m - d}\right)\cdot\left[-d + x_2\cdot\frac{\mu_m}{k_m + x_2}\right]_{x_2\to x_2^{op}}$$

$$+ \left(x_2 - \frac{d\,k_m}{\mu_m - d}\right)\cdot\left[x_1\cdot\mu_m\cdot\frac{(k_m + x_2) - x_2}{(k_m + x_2)^2}\right]_{\substack{x_2\to x_2^{op}\\ x_1\to x_1^{op}}}$$

$$\frac{d}{dt}x_1 = \frac{y(x_{2f}(\mu_m-d)-k_m d)}{\mu_m-d}\cdot(-d+d) \quad \Big\} 0$$

$$+ \left(x_1 - \frac{y(x_{2f}(\mu_m-d)-k_m d)}{\mu_m-d}\right)\cdot\left(-d + \frac{d\,k_m}{\mu_m-d}\cdot\frac{\mu_m}{k_m+\frac{d\,k_m}{\mu_m-d}}\right)$$

$$+ \left(x_2 - \frac{d\,k_m}{\mu_m-d}\right)\cdot\left(\frac{y(x_{2f}(\mu_m-d)-k_m d)}{\mu_m-d}\cdot\mu_m\cdot\frac{k_m}{\left(\frac{\mu k_m}{\mu_m-d}\right)^2}\right)$$

$$= \left(x_1 - \frac{y(x_{2f}(\mu_m-d)-k_m d)}{\mu_m-d}\right)\cdot(-d+d) \quad \Big\} 0$$

$$+ \frac{x_2(\mu_m-d)-d\cdot k_m}{\mu_m-d}\cdot\frac{y(x_{2f}(\mu_m-d)-k_m\cdot d)}{\mu_m-d}\cdot\mu_m\cdot\frac{(\mu_m-d)^2}{\mu_m\cdot k_m}$$

$$= (x_2(\mu_m-d)-d\cdot k_m)\cdot y\,(x_{2f}(\mu_m-d)-k_m\cdot d)$$

$$\hookrightarrow \frac{d}{dt}x_1 = x_2\cdot(\mu_m-d)\cdot y\cdot(x_{2f}(\mu_m-d)-k_m d) - d\cdot k_m\cdot y\cdot(x_{2f}(\mu_m-d)-k_m\cdot d)$$

$$\frac{d}{dt}x_2 = x_{2f}d - x_2 d - x_1\cdot\frac{\mu_m\cdot x_2}{(k_m+x_2)\,y}$$

$\hookrightarrow$

$$\frac{d}{dt}x_2 = x_{2f}\cdot d - \frac{d^2 k_m}{\mu_m-d} - \frac{x_{2f}(\mu_m-d)-k_m d}{\mu_m-d}\cdot\mu_m\cdot\frac{\frac{d\,k_m}{\mu_m-d}}{\frac{\mu_m k_m}{\mu_m-d}}$$

$$+ \left(x_1 - \frac{y(x_{2f}(\mu_m-d)-k_m\cdot d)}{\mu_m-d}\right)\cdot\left[-\frac{1}{y}\cdot\mu_m\cdot\frac{x_2}{k_m+x_2}\right]_{x_2\to x_2^{op}}$$

$$+ \left(x_2 - \frac{d\,k_m}{\mu_m-d}\right)\cdot\left[-d - x_1\cdot\mu_m\cdot\frac{k_m}{(k_m+x_2)^2}\right]_{\substack{x_1^{op}\to x_1\\ x_2^{op}\to x_2}}$$

$$= x_{2f}\cdot d - \frac{d^2 k_m}{\mu_m-d} - \frac{x_{2f}(\mu_m-d)\cdot d - k_m d^2}{\mu_m-d}$$

$$+ \left(x_1 - \frac{y(x_{2f}(\mu_m-d)-k_m\cdot d)}{\mu_m-d}\right)\cdot\left(-\frac{1}{y}\cdot d\right)$$

$$+ \left(x_2 - \frac{d\,k_m}{\mu_m-d}\right)\cdot\left(-d - \frac{y(x_{2f}(\mu_m-d)-k_m\cdot d)}{\mu_m-d}\cdot\frac{(\mu_m-d)^2}{k_m\,\mu_m}\right)$$

$$= \left.\frac{X_{2f}\cdot d\,(\mu_m-d) - d^2 k_m}{\mu_m\, d} - \frac{X_{2f}\,(\mu_m-d)\cdot d - k_m\cdot d^2}{\mu_m-d}\right\}0$$

$$+\ X_1\cdot\left(-\frac{d}{Y}\right) + \frac{X_{2f}\,(\mu_m-d)\cdot d - k_m\cdot d^2}{\mu_m-d}$$

$$+\ X_2\cdot\left(-d - \frac{Y\,(X_{2f}(\mu_m-d) - k_m\cdot d)\,(\mu_m-d)}{k_m\,\mu_m}\right) + \frac{k_m\, d^2}{\mu_m-d} + \frac{d\cdot Y\cdot(X_{2f}(\mu_m-d) - k_m d)}{\mu_m}$$

$$= X_1\cdot\left(-\frac{d}{Y}\right) + X_2\cdot\left(-d - \frac{Y\,(X_{2f}(\mu_m-d) - k_m d)\,(\mu_m-d)}{k_m\,\mu_m}\right) + X_{2f}\cdot d + \frac{d\cdot Y\,(X_{2f}(\mu_m-d) - k_m d)}{\mu_m}$$

②

$$\frac{d}{dt}X_1 = X_1\left(-d + \frac{X_2\,\mu_m}{k_m+X_2}\right)$$

$$\frac{d}{dt}X_1 = 0 + (X_1-0)\cdot\left[\frac{X_{2f}\,\mu_m}{k_m+X_{2f}}\right] + (X_2-X_{2f})\cdot\left[\frac{\partial}{\partial X_2}\left(0\cdot\left(-d+\frac{X_2\mu_m}{k_m+X_2}\right)\right)\right]_{X_2\to X_2^{op}}$$

$$= X_1\cdot\frac{X_{2f}\,\mu_m}{k_m+X_{2f}}$$

$$\frac{d}{dt}X_2 = X_{2f}\cdot d - X_2\cdot d - X_1\cdot\frac{\mu_m\cdot X_2}{(k_m+X_2)\cdot Y}$$

$$\frac{d}{dt}X_2 = \left(X_{2f}\cdot d - X_{2f}\cdot d - 0\right) + (X_1-0)\cdot\left[-\frac{\mu_m}{Y}\,\frac{X_2}{k_m+X_2}\right]_{X_2\to X_2^{op}}$$

$$+\ (X_2 - X_{2f})\cdot\left(-d - 0\cdot\left[\frac{\partial}{\partial X_2}\left(\frac{\mu_m X_2}{(k_m+X_2)\,Y}\right)\right]_{X_2\to X_2^{op}}\right)$$

$$= -\frac{\mu_m X_{2f}}{Y(k_m+X_{2f})}\cdot X_1 - d\cdot X_2 + X_{2f}\cdot d$$

Thus,

① : $$\frac{d}{dt}x_1 = x_2 \cdot (\mu_m - d) \cdot y \cdot [x_{2f}(\mu_m - d) - k_m d] - d \cdot k_m \cdot y \cdot [x_{2f}(\mu_m - d) - k_m d]$$

$$\frac{d}{dt}x_2 = -x_1 \cdot \frac{d}{y} - x_2 \cdot \left(d + \frac{y(x_{2f}(\mu_m - d) - k_m d)(\mu_m - d)}{k_m \cdot \mu_m} \right) + x_{2f} \cdot d + \frac{d \cdot y \cdot (x_{2f}(\mu_m - d) - k_m d)}{\mu_m}$$

② : $$\frac{d}{dt}x_1 = x_1 \cdot \frac{x_{2f}\,\mu_m}{k_m + x_{2f}}$$

$$\frac{d}{dt}x_2 = -x_1 \cdot \frac{x_{2f}\,\mu_m}{y(k_m + x_{2f})} - x_2 \cdot d + x_{2f} \cdot d$$

d) $\frac{d}{dt} x_1 = -x_1 \cdot d + x_1 \cdot \frac{\mu_m x_2}{k_m + x_2 + k_1 \cdot x_2^2} = 0$

$\hookrightarrow \quad -d k_m \cdot x_1 - d \cdot x_1 \cdot x_2 - d \cdot k_1 \cdot x_1 \cdot x_2^2 + \mu_m \cdot x_1 \cdot x_2 = 0$

$x_1 \left(-d k_m + (\mu_m - d) x_2 - d k_1 x_2^2 \right) = 0 \qquad (x_1 = 0)$

or $\rightarrow \quad x_2^2 + \frac{d - \mu_m}{d k_1} x_2 + \frac{k_m}{k_1} = 0$

$$x_{2,\text{zero}1} = \frac{\frac{\mu_m - d}{d k_1} + \sqrt{\left(\frac{\mu_m - d}{d k_1}\right)^2 - \frac{4 k_m}{k_1}}}{2}$$

$$= \frac{\mu_m - d}{2 d \cdot k_1} + \frac{1}{2} \sqrt{\mu_m^2 + d(d - 2\mu_m - 4 k_m)} \cdot \frac{1}{\sqrt{dk}}$$

$$x_{2,\text{zero}2} = \frac{\mu_m - d}{2 \cdot d \cdot k_1} - \frac{1}{2} \sqrt{\mu_m^2 + d(d - 2\mu_m - 4 k_m)} \cdot \frac{1}{\sqrt{dk}}$$

$$\frac{d}{dt} x_2 = x_{2f} \cdot d - x_2 \cdot d - x_1 \cdot \frac{\mu_m x_2}{(k_m + x_2 + k_1 \cdot x_2^2) Y} = 0$$

$$\hookrightarrow \quad x_1 = \frac{Y \cdot x_{2f} \cdot d \cdot (k_m + x_2 + k_1 x_2^2)}{\mu_m x_2} - \frac{Y \cdot d \cdot (k_m + x_2 + k_1 \cdot x_2^2)}{\mu_m} = F(x_2)$$

set 1 : $x_1 = F(x_{2,\text{zero}1}) \qquad x_2 = x_{2,\text{zero}1}$

set 2 : $x_1 = F(x_{2,\text{zero}2}) \qquad x_2 = x_{2,\text{zero}2}$

Q3)

a.

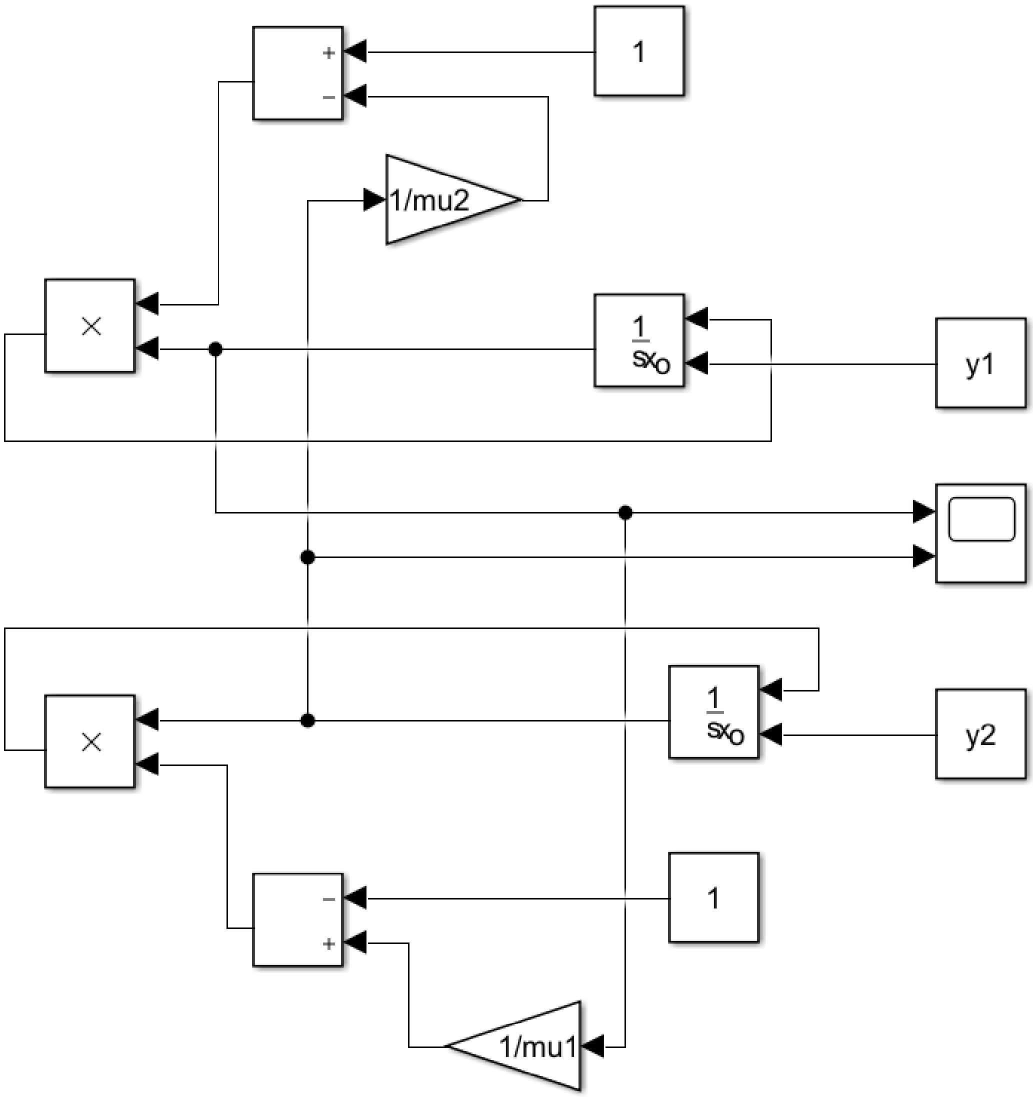


Figure 1: Simulink model for the ecosystems

b.

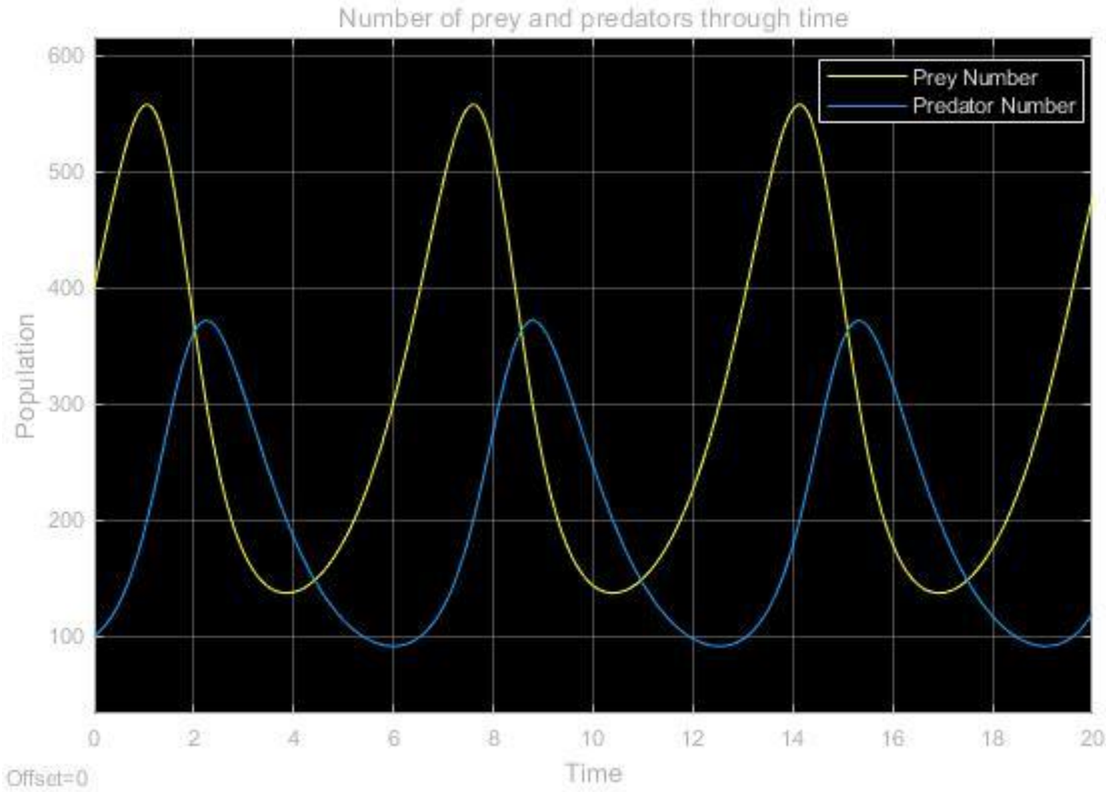


Figure 2: Simulation of the ecosystem model (y1,initial=400, y2,initial=100, mu1=300, mu2=200)

c.

The chosen set of parameters resulted in an operating point where the population of the inhabitants oscillates periodically where the period corresponds approximately to 6 times the sampling interval. As it can be seen in the initial response characteristics of the populations, when the number of preys are greater than the number of predators provided that it has a positive slope; this means there is an increasing amount of food for the predators which results in an increase in the number of predators. On the other hand, when the number of predators increases, the food supply decreases accordingly. Consequently, the population of the predator lags in frequency with respect to the population of prey. A similar behavior can be observed in the blood glycogen level and the number of parasites. The parasite population in the blood keeps pace of the glycogen level with an amount of lag.

e.

$$D(y_1) = (1-y_2/\eta_2)\, y_1 \qquad (1)$$

$$D(y_2) = -(1-y_1/\eta_1)\, y_2 \qquad (2)$$

A non-zero equilibrium is observed when y_1 and y_2 are equal to η_1 and η_2 respectively as the population of the inhabitants does not change. When the initial values of populations are around the equilibrium point but do not have the exact value, the oscillations are again observed as demonstrated in Figure 3. Notice that, the period of the oscillations is not altered; however, the characteristics of each plot behaves more like a sinusoidal. The reason behind this can be explained by small signal approximation. That is, as deviation from the chosen equilibrium point is decreased, a more convergent characteristics to a perfect sinusoid at the output is achieved.

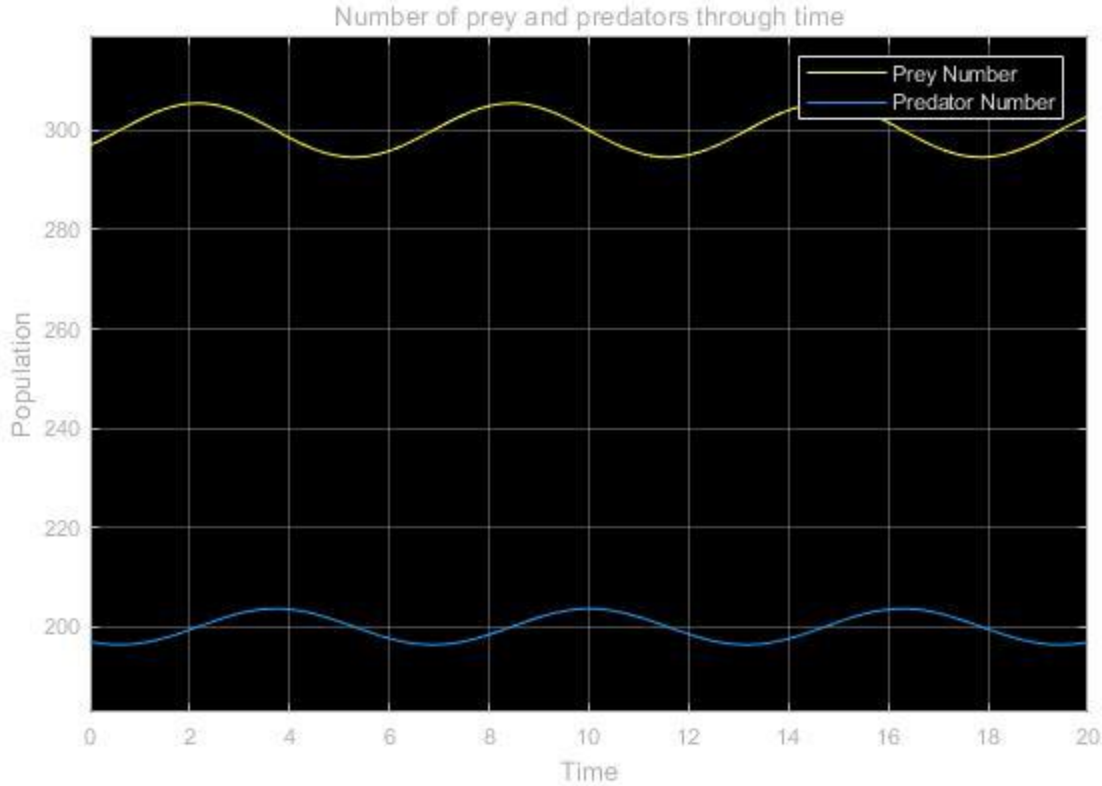


Figure 3: Simulation of the ecosystem model (y1,initial=297, y2,initial=197, mu1=300, mu2=200)

f.

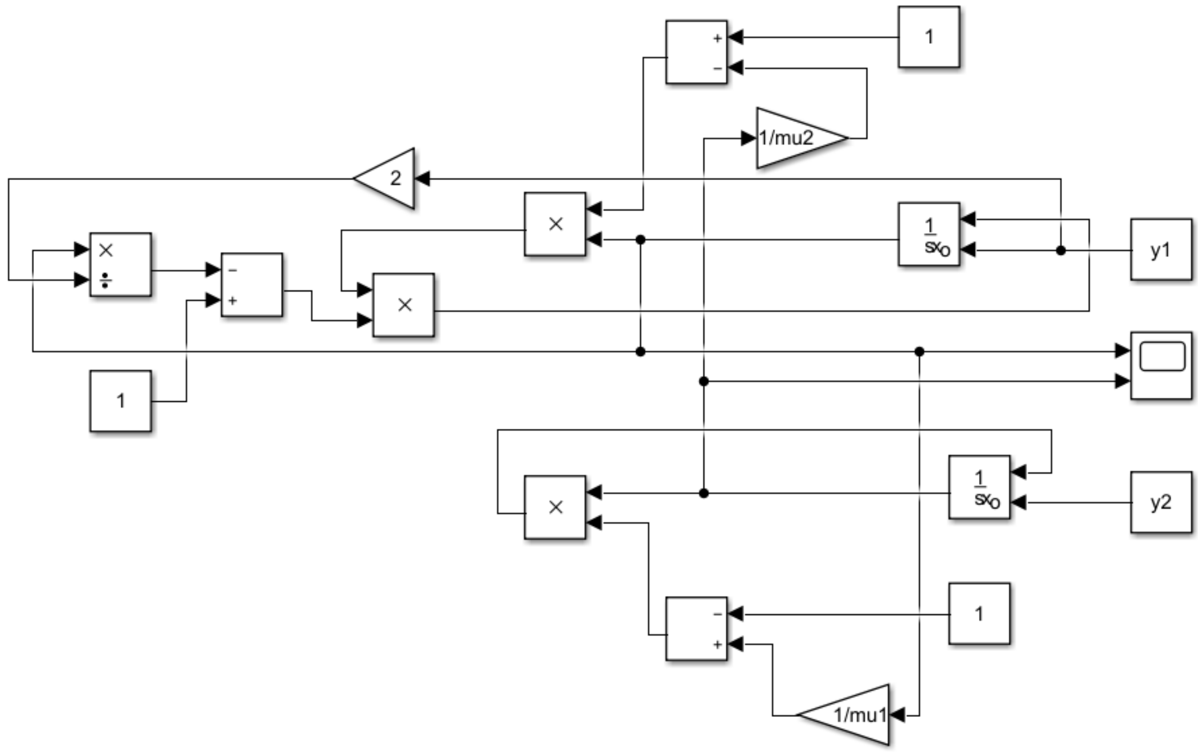


Figure 4: Simulink model for the ecosystems after imposing growth limiting for prey population

g.

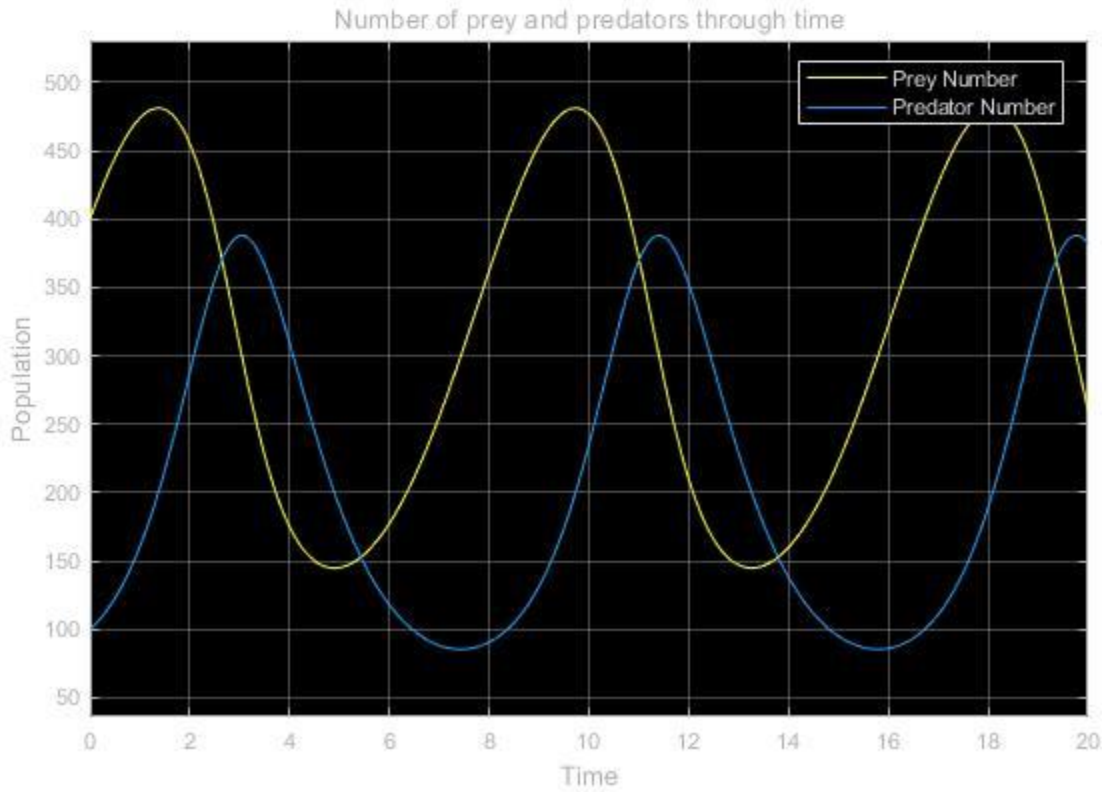


Figure 5: Simulation of the ecosystem model after imposing growth limiting for prey population (y1,initial=400, y2,initial=100, mu1=300, mu2=200)

Notice that, even after adding a term to limit the growth of prey population, the basic population characteristics are not altered. The population of the predators still keeps pace with the population of the preys with a lag. Moreover, the corresponding behaviors are also periodic, however, the period duration is now increased to an amount of approximately 8 times the unit sampling period. This can be an emerging model for an ecosystem where the population of the preys is dependent not only the population of the predators but also a secondary limiting factor such as food or a finite capacity of the habitat for the preys.