2) a. $\frac{dx_1}{dt} \cdot V = F x_{1f} - F x_1 + r_1 V$

$$\dot{x}_1 = \frac{F}{V} x_{1f} - \frac{F}{V} x_1 + r_1$$

$$\frac{dx_2}{dt} V = F x_{2f} - F x_2 - r_2 V$$

$$\dot{x}_2 = \frac{F}{V} x_{2f} - \frac{F}{V} x_2 - r_2$$

We can put $d = \frac{F}{V}$. We also know that $x_{1f} = 0$,

$r_1 = \mu(x_2) x_1$ & $r_2 = \frac{\mu(x_2) x_1}{Y}$

Then,

$$\dot{x}_1 = -d x_1 + \mu(x_2) x_1$$

$$\dot{x}_2 = d x_{2f} - d x_2 - \frac{\mu(x_2) x_1}{Y}$$

b) Equilibrium points mean $\dot{x}_1 = 0$, $\dot{x}_2 = 0$. Given $\mu(x_2) = \frac{\mu_m x_2}{k_m + x_2}$

$\rightarrow \dot{x}_1 = 0 \rightarrow \mu(x_2) x_1 = d x_1 \rightarrow \frac{\mu_m x_2 x_1}{k_m + x_2} = d x_1$

Keep in mind, $\underline{x_1 = 0}$ can be a part of an equilibrium point pair.

$\underline{\mu(x_2) = d}$, $d k_m + d x_2 = \mu_m x_2 \rightarrow \boxed{x_2 = \frac{d k_m}{\mu_m - d}}$

$\rightarrow \dot{x}_2 = 0$, $d x_{2f} - d x_2 - \frac{\overset{\rightarrow = d}{\mu(x_2)} x_1}{Y} = 0$ (*)

Then, $Y(x_{2f} - x_2) = x_1 \rightarrow \boxed{x_1 = Y\left(x_{2f} - \frac{d k_m}{\mu_m - d}\right)}$ pair 1

Also, from (*), for $x_1 = 0$, $x_2 = x_{2f}$ is a solution.

So, the other equilibrium point is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_{2f} \end{bmatrix} \rightarrow \text{pair 2}$$

c) $F_1 = \dot{x}_1 = -d x_1 + \dfrac{\mu_m x_2 x_1}{k_m + x_2}$

$$F_2 = \dot{x}_2 = d x_{2f} - d x_2 - \frac{\mu_m x_2 x_1}{(k_m + x_2) Y}$$

$$J = \begin{bmatrix} \dfrac{\partial F_1}{\partial x_1} & \dfrac{\partial F_1}{\partial x_2} \\ \dfrac{\partial F_2}{\partial x_1} & \dfrac{\partial F_2}{\partial x_2} \end{bmatrix}$$

Constructing the matrix,

$$J = \begin{bmatrix} -d + \dfrac{\mu_m x_2}{k_m + x_2} & \dfrac{x_1 \mu_m k_m}{(k_m + x_2)^2} \\ \dfrac{-\mu_m \dot{x}_2}{(k_m + x_2) Y} & -d + \dfrac{x_1 \mu_m k_m}{Y (k_m + x_2)^2} \end{bmatrix}$$

for pair 1, lets insert x_1 & x_2 values to find numerical J_1^{op}

$$J_{11} = -d + \frac{\cancel{\mu_m} d k_m}{k_m \cancel{\mu_m} - d \cancel{k_m} + d \cancel{k_m}} = 0$$

$$J_{21} = \frac{-d}{Y}$$

$$J_{12} = \mu_m \cancel{k_m} \cdot Y \left(x_{2f} - \frac{d k_m}{\mu_m - d} \right) \cdot \frac{1}{(\mu_m k_m)^2} = \frac{Y}{\mu_m k_m} \left(x_{2f} - \frac{d k_m}{\mu_m - d} \right)$$

$$J_{22} = -d + \frac{1}{\mu_m k_m} \left(x_{2f} - \frac{d k_m}{\mu_m - d} \right)$$

So,

$$J_1 = \begin{bmatrix} 0 & \dfrac{Y}{\mu_m k_m} \left(x_{2f} - \dfrac{d k_m}{\mu_m - d} \right) \\ -\dfrac{d}{Y} & -d + \dfrac{1}{\mu_m k_m} \left(x_{2f} - \dfrac{d k_m}{\mu_m - d} \right) \end{bmatrix}$$

$$X_1^{op} = \begin{bmatrix} Y\left(k_{2f} - \dfrac{d k_m}{\mu_m - d}\right) \\ \dfrac{d k_m}{\mu_m - d} \end{bmatrix}$$

$$Z_1 = X - X_1^{op} = \begin{bmatrix} x_1 - Y\left(k_{2f} - \dfrac{d k_m}{\mu_m - d}\right) \\ x_2 - \dfrac{d k_m}{\mu_m - d} \end{bmatrix}$$

$$\dot{Z}_1 = J_1 Z_1$$

for pair 2,

$$X_2^{op} = \begin{bmatrix} 0 \\ X_{2f} \end{bmatrix}$$

$$J_2 = \begin{bmatrix} -d + \dfrac{\mu_m X_{2f}}{k_m + X_{2f}} & 0 \\ \dfrac{-\mu_m X_{2f}}{(k_m + X_{2f}) Y} & -d \end{bmatrix}$$

$$Z_2 = X - X_2^{op} = \begin{bmatrix} x_1 \\ x_2 - x_{2f} \end{bmatrix}$$

$$\dot{Z}_2 = J_2 Z_2$$

d) $\mu = \dfrac{\mu_m x_2}{k_m + x_2 + k_1 x_2^2}$ $\qquad d < \max_{x_2 > 0} (\mu(x_2))$

Again, to find equilibrium points, $\dot{x}_1 = 0$, $\dot{x}_2 = 0$

* $\dot{x}_1 = 0 \rightarrow \mu(x_2) x_1 = d\, x_1 \rightarrow$ keeping in mind $x_1 = 0$ is possible,

$\underset{(*)}{\mu(x_2) = d} \rightarrow d = \dfrac{\mu_m x_2}{k_m + x_2 + k_1 x_2^2}$

$\Delta = b^2 - 4ac$

$x_{1,2} = \dfrac{-b \mp \sqrt{\Delta}}{2a}$

$$d k_m + d x_2 + d k_1 x_2^2 = \mu_m x_2$$

$$x_2^2 (d k_1) + x_2 (d - \mu_m) + d k_m = 0$$

$$\Delta = (d - \mu_m)^2 - 4 d k_1 d k_m$$

$$\boxed{x_{2_{1,2}}^{op} = \frac{-(d-\mu_m) \mp \sqrt{\Delta}}{2 d k_1} = \frac{-(d-\mu_m) \mp \sqrt{(d-\mu_m)^2 - 4 d^2 k_1 k_m}}{2 d k_1}}$$

* $\dot{x}_2 = 0 \rightarrow d x_{2f} - d x_2 - \dfrac{\mu(x_2) x_1}{Y} = 0$ $\qquad \mu(x_2) = d$ from $(*)$

Then, $(x_{2f} - x_2) \cdot Y = x_1$

$$\boxed{x_1^{op} = Y \left(x_{2f} + \frac{(d-\mu_m) \mp \sqrt{(d-\mu_m)^2 - 4 d^2 k_1 k_m}}{2 d k_1} \right)}$$

Also, for $x_1 = 0$,

$x_{2f} - x_2 = 0$, $x_2 = x_{2f}$ $\Big\}$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_{2f} \end{bmatrix}$ is another equilibrium point.