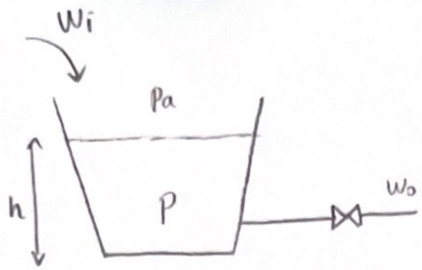
1)



input: w_i
output: h

$\Delta p = p - p_a \qquad w_o = k\sqrt{\Delta p} \qquad p = \rho g h + p_a$

a) Accumulation = Inflow - Outflow

$$\frac{dh}{dt} = \alpha\,(w_i - w_o)$$

α is a constant relating Δw with the rate of height change.

Assuming $\alpha = 1$,

$$\dot{h} = w_i - w_o = w_i - k\sqrt{\Delta p} = w_i - k\sqrt{p - p_a} = w_i - k\sqrt{\rho g h + \cancel{p_a} - \cancel{p_a}}$$

$$\rightarrow \boxed{\dot{h} = w_i - k\sqrt{\rho g h}}$$

b) $\Delta p = \rho g h$

$\Delta \dot{p} = \rho g \dot{h}$

Then, $\Delta \dot{p} = \rho g \left(w_i - k\underbrace{\sqrt{\rho g h}}_{\Delta p}\right)$

$$\boxed{\Delta \dot{p} = \rho g\,(w_i - k\sqrt{p})}$$

c) For constant r, we need $\dot{h} = 0$.

This means $w_i = w_o$, or $w_i - k\sqrt{\rho g h} = 0$

The input u_{ss} keeping output (h) at constant r,

$$\boxed{u_{ss} = k\sqrt{\rho g r}}$$

2) a. $\frac{dx_1}{dt} \cdot V = F x_{1f} - F x_1 + r_1 V$

$$\dot{x}_1 = \frac{F}{V} x_{1f} - \frac{F}{V} x_1 + r_1$$

$$\frac{dx_2}{dt} V = F x_{2f} - F x_2 - r_2 V$$

$$\dot{x}_2 = \frac{F}{V} x_{2f} - \frac{F}{V} x_2 - r_2$$

We can put $d = \frac{F}{V}$. We also know that $x_{1f} = 0$,

$r_1 = \mu(x_2) x_1$ & $r_2 = \frac{\mu(x_2) x_1}{Y}$

Then,

$$\dot{x}_1 = -d x_1 + \mu(x_2) x_1$$

$$\dot{x}_2 = d x_{2f} - d x_2 - \frac{\mu(x_2) x_1}{Y}$$

b) Equilibrium points mean $\dot{x}_1 = 0$, $\dot{x}_2 = 0$. Given $\mu(x_2) = \frac{\mu_m x_2}{k_m + x_2}$

$\rightarrow \dot{x}_1 = 0 \rightarrow \mu(x_2) x_1 = d x_1 \rightarrow \frac{\mu_m x_2 x_1}{k_m + x_2} = d x_1$

Keep in mind, $\underline{x_1 = 0}$ can be a part of an equilibrium point pair.

$\underline{\mu(x_2) = d}$, $d k_m + d x_2 = \mu_m x_2 \rightarrow \boxed{x_2 = \frac{d k_m}{\mu_m - d}}$

$\rightarrow \dot{x}_2 = 0$, $d x_{2f} - d x_2 - \frac{\overset{\rightarrow = d}{\mu(x_2)} x_1}{Y} = 0$ (*)

Then, $Y(x_{2f} - x_2) = x_1 \rightarrow \boxed{x_1 = Y\left(x_{2f} - \frac{d k_m}{\mu_m - d}\right)}$

pair 1

Also, from (*), for $x_1 = 0$, $x_2 = x_{2f}$ is a solution.

So, the other equilibrium point is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_{2f} \end{bmatrix} \rightarrow \text{pair 2}$$

c) $F_1 = \dot{x}_1 = -d x_1 + \dfrac{\mu_m x_2 x_1}{k_m + x_2}$

$$F_2 = \dot{x}_2 = d x_{2f} - d x_2 - \frac{\mu_m x_2 x_1}{(k_m + x_2) Y}$$

$$J = \begin{bmatrix} \dfrac{\partial F_1}{\partial x_1} & \dfrac{\partial F_1}{\partial x_2} \\ \dfrac{\partial F_2}{\partial x_1} & \dfrac{\partial F_2}{\partial x_2} \end{bmatrix}$$

Constructing the matrix,

$$J = \begin{bmatrix} -d + \dfrac{\mu_m x_2}{k_m + x_2} & \dfrac{x_1 \mu_m k_m}{(k_m + x_2)^2} \\ \dfrac{-\mu_m x_2}{(k_m + x_2) Y} & -d + \dfrac{x_1 \mu_m k_m}{Y (k_m + x_2)^2} \end{bmatrix}$$

for pair 1, lets insert x_1 & x_2 values to find numerical J_1^{op}

$$J_{11} = -d + \frac{\cancel{\mu_m} d\, \cancel{k_m}}{\cancel{k_m \mu_m} - d\cancel{k_m} + d\cancel{k_m}} = 0$$

$$J_{21} = \frac{-d}{Y}$$

$$J_{12} = \mu_m \cancel{k_m} \cdot Y \left(x_{2f} - \frac{d k_m}{\mu_m - d} \right) \cdot \frac{1}{(\mu_m k_m)^2} = \frac{Y}{\mu_m k_m} \left(x_{2f} - \frac{d k_m}{\mu_m - d} \right)$$

$$J_{22} = -d + \frac{1}{\mu_m k_m} \left(x_{2f} - \frac{d k_m}{\mu_m - d} \right)$$

So,

$$J_1 = \begin{bmatrix} 0 & \dfrac{Y}{\mu_m k_m} \left(x_{2f} - \dfrac{d k_m}{\mu_m - d} \right) \\ -\dfrac{d}{Y} & -d + \dfrac{1}{\mu_m k_m} \left(x_{2f} - \dfrac{d k_m}{\mu_m - d} \right) \end{bmatrix}$$

$$X_1^{op} = \begin{bmatrix} Y\left(k_{2f} - \frac{d k_m}{\mu_m - d}\right) \\ \frac{d k_m}{\mu_m - d} \end{bmatrix}$$

$$z_1 = X - X_1^{op} = \begin{bmatrix} x_1 - Y\left(k_{2f} - \frac{d k_m}{\mu_m - d}\right) \\ x_2 - \frac{d k_m}{\mu_m - d} \end{bmatrix}$$

$$\dot{z}_1 = J_1 z_1$$

for pair 2,

$$x_2^{op} = \begin{bmatrix} 0 \\ x_{2f} \end{bmatrix}$$

$$J_2 = \begin{bmatrix} -d + \frac{\mu_m x_{2f}}{k_m + x_{2f}} & 0 \\ \frac{-\mu_m x_{2f}}{(k_m + x_{2f}) Y} & -d \end{bmatrix}$$

$$z_2 = X - x_2^{op} = \begin{bmatrix} x_1 \\ x_2 - x_{2f} \end{bmatrix}$$

$$\dot{z}_2 = J_2 z_2$$

d) $\mu = \dfrac{\mu_m x_2}{k_m + x_2 + k_1 x_2^2}$ $d < \max_{x_2 > 0}(\mu(x_2))$

Again, to find equilibrium points, $\dot{x}_1 = 0$, $\dot{x}_2 = 0$

* $\dot{x}_1 = 0 \rightarrow \mu(x_2)x_1 = d\,x_1$ → keeping in mind $x_1 = 0$ is possible,

$\mu(x_2) = d \rightarrow d = \dfrac{\mu_m x_2}{k_m + x_2 + k_1 x_2^2}$
(*)

$\Delta = b^2 - 4ac$
$x_{1,2} = \dfrac{-b \mp \sqrt{\Delta}}{2a}$

$$d k_m + d x_2 + d k_1 x_2^2 = \mu_m x_2$$

$$x_2^2 (d k_1) + x_2 (d - \mu_m) + d k_m = 0$$

$$\Delta = (d - \mu_m)^2 - 4 d k_1 d k_m$$

$$\boxed{x_{2_{1,2}}^{op} = \frac{-(d-\mu_m) \mp \sqrt{\Delta}}{2dk_1} = \frac{-(d-\mu_m) \mp \sqrt{(d-\mu_m)^2 - 4d^2 k_1 k_m}}{2dk_1}}$$

* $\dot{x}_2 = 0 \rightarrow d x_{2f} - d x_2 - \dfrac{\mu(x_2)\, x_1}{Y} = 0$ $\mu(x_2) = d$ from (*)

Then, $(x_{2f} - x_2) \cdot Y = x_1$

$$\boxed{x_1^{op} = Y\left(x_{2f} + \frac{(d-\mu_m) \mp \sqrt{(d-\mu_m)^2 - 4d^2 k_1 k_m}}{2dk_1}\right)}$$

Also, for $x_1 = 0$,
$x_{2f} - x_2 = 0$, $x_2 = x_{2f}$ $\Big\}$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_{2f} \end{bmatrix}$ is another equilibrium point.

Q3)

a) Created simulink model can be seen below, moreover it can be found on our GitHub repo.

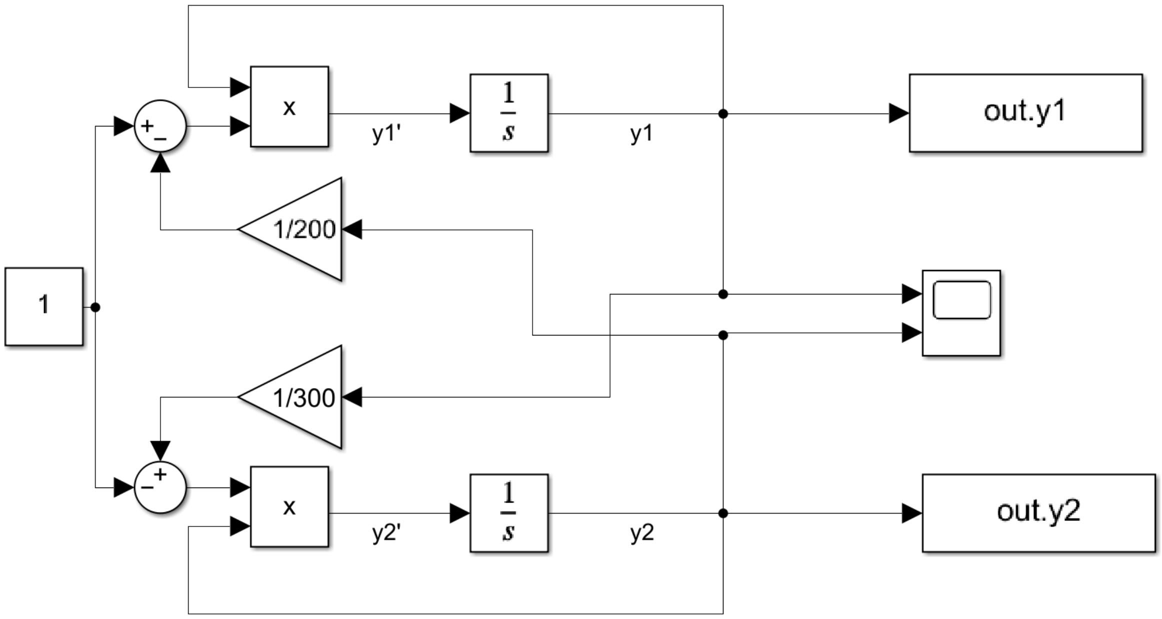


Figure X. Created simulink model with given values

b) Output of the system for 25 seconds can be seen below.

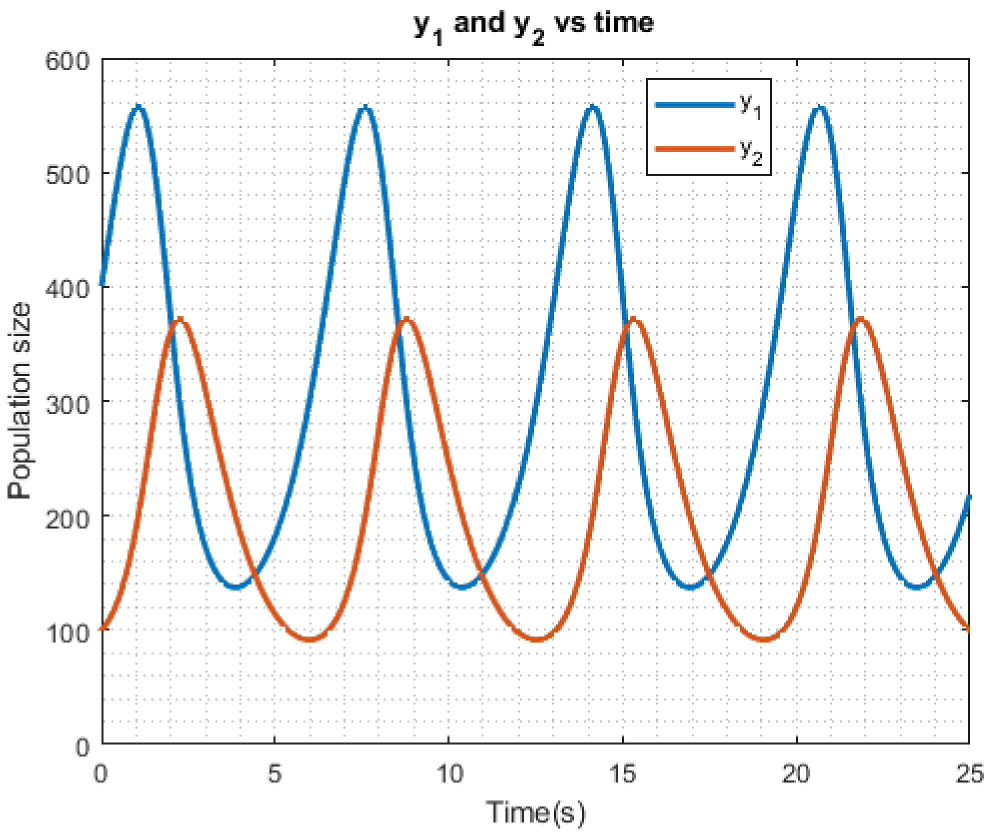


Figure X. Output of the system for 25 second with given initial conditions

c) As can be seen from the figure in part b, the system is showing fully oscillatory behavior. Oscillations are periodic, and phase shifted from each other. They look like distorted sinusoids. In real life RLC circuits could have the similar response or many other things.

d)

e) At the equilibrium point

$$\frac{dy_1}{dt} = \frac{dy_2}{dt} = 0$$

This means that size of both populations remain constant, by putting this condition to given equations, we can obtain that,

$y_2 = n_2$ and $y_1 = n_1$

Since we have given $n_1 = 300\ and\ n_2 = 200$, we know that

$y_{1,equilibrium} = 300\ and\ y_{2,equilibrium} = 200$

Since we asked to choose the points close to the equilibrium point, we are selecting initial conditions as, $y_{1,initial} = 295\ and\ y_{2,initial} = 205$.

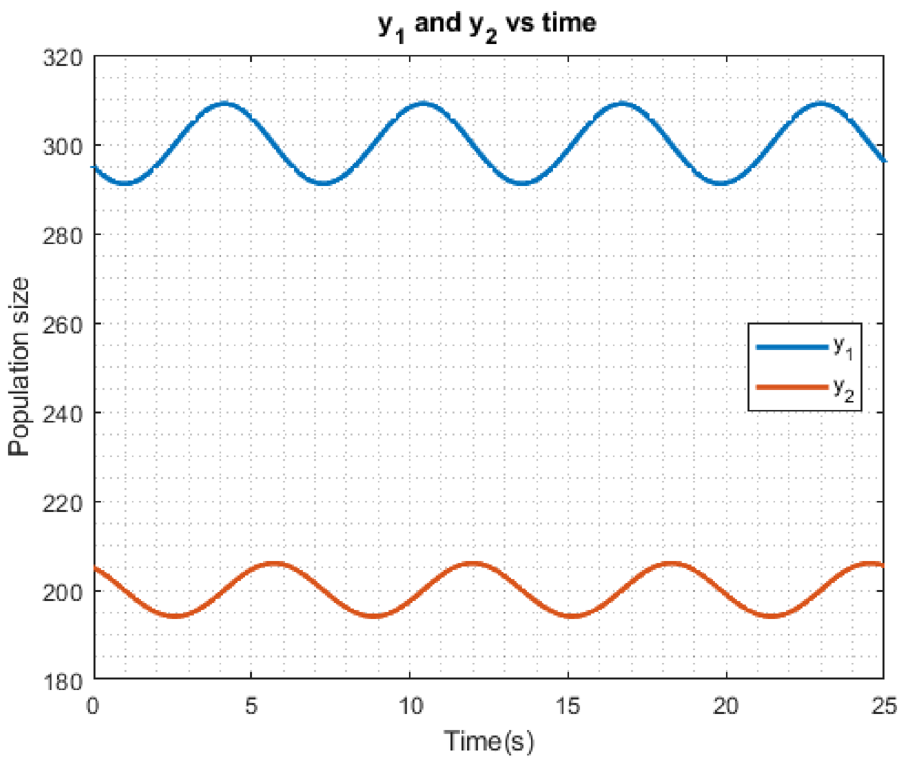


Figure X. Output of the system for 25 second with given initial conditions near equilibrium point

As can be seen from the figure above, when we give initial conditions near the equilibrium point, size of the oscillations are reduced, shape is closer to the sinusoid and the period is slightly increased. If we choose initial conditions directly at the equilibrium point, then size of the populations will be a flat line.

f) Modified model can be seen below.

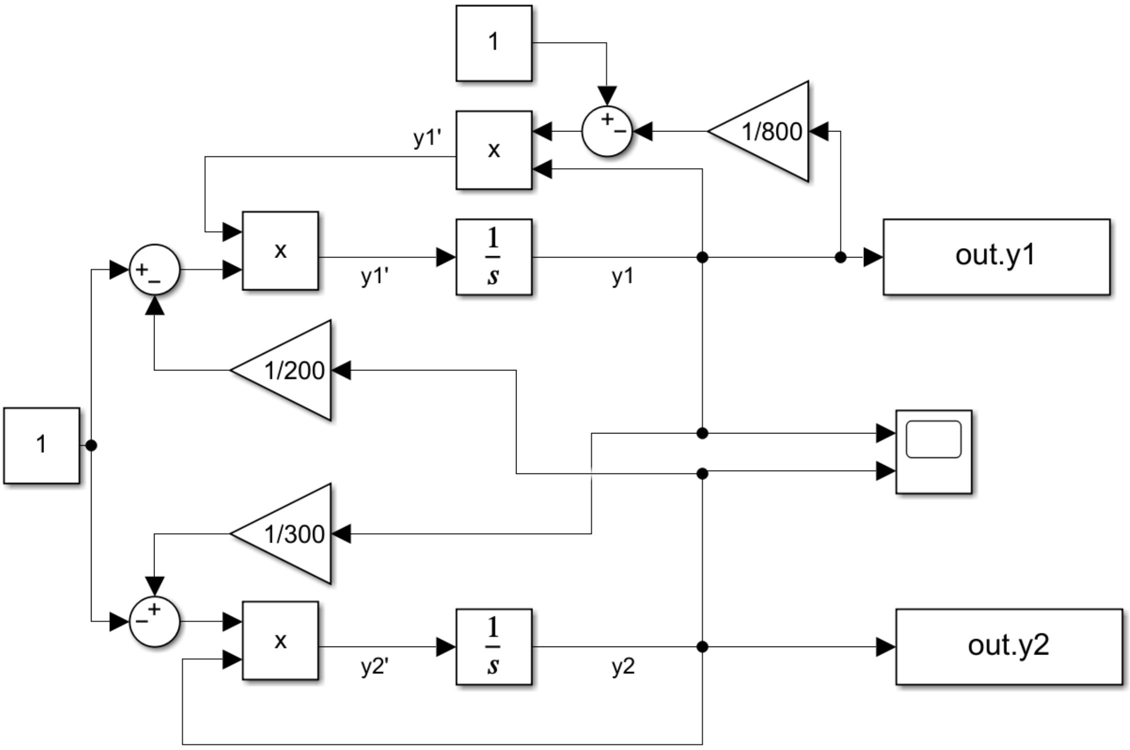


Figure X. Modified model for part f

g)

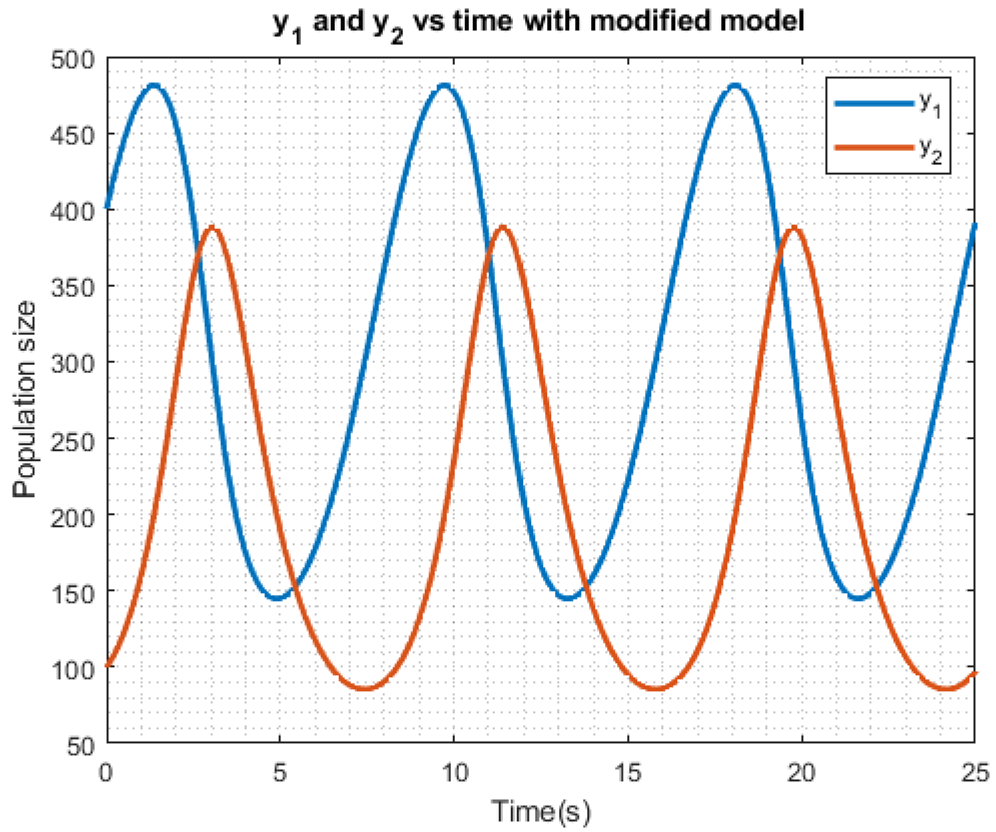


Figure X. Output of modified system with given initial conditions

Response of the system with modified simulink model shown above, as can be seen from the figure, size of the population look like sinusoids. It still shows periodic oscillatory behavior, however even though the initial conditions are the same, the period of the oscillations has changed, we can see that the period of oscillations are larger than the non-modified case(frequency of oscillations decreased). Peak to peak values of the y1(prey population) is smaller, however the peak to peak values of the y2(predator population) is larger.