Driven Series RLC Circuit

Laplace solution of non-homogeneous 2nd order differential equation

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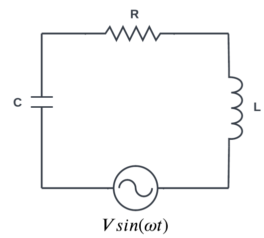

Series RLC Circuit with sinusoidal driving function

The Kirchhoff voltage equation for the series RLC circuit with a driving function, $f(t)$, is given by:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = f(t) \tag{1}$$

Take the derivative of Equation 1 to remove the integral:

$$\frac{di^2}{dt^2} + \lambda\frac{di}{dt} + \omega_0^2 i = \frac{1}{L}f'(t) \tag{2}$$

where,

$$\lambda = \frac{R}{L} \qquad \omega_0 = \frac{1}{\sqrt{LC}} \tag{3}$$

For a sinusoidal driving function:

$$f(t) = V sin(\omega t) \tag{4}$$

Applying the driving function, Equation 2 becomes:

$$\frac{di^2}{dt^2} + \lambda\frac{di}{dt} + \omega_0^2 i = \frac{V\omega}{L}cos(t) \tag{5}$$

The solution to Equation 5 comprises a steady state response and a transient response.

$$i(t) = i(t)_{transient} + i(t)_{steady\_state} \tag{6}$$

The transient response has three possibilities depending on whether the circuit is under-damped, over-damped, or critically-damped. The steady-state and transient responses are derived below.

Derivation of the solution

The initial conditions required to solve the differential equation are given by:

$$i(t)\Big|_{t=0} = 0 \tag{7}$$

$$i(t)' = \frac{di}{dt}\Big|_{t=0} = 0 \tag{8}$$

The initial condition in Equation 8 is derived from the Kirchhoff voltage equation. At $t = 0$, the capacitor, C, has an initial charge of $Q = 0$, yielding a capacitor voltage of $V_C = \frac{Q}{C} = 0$. Also, since $i(0) = 0$ (Equation 7), the voltage across the resistor, R, is $V_R = iR = 0$. The driving voltage at $t = 0$ is $f(t) = V sin(\omega t) = 0$. Therefore, the voltage equation around the circuit reduces to $V_L = L\frac{di}{dt} = 0$, where V_L is the voltage across the inductor, L. Since $L > 0$, $\frac{di}{dt} = 0$.

Take the Laplace transform of Equation 5:

$$s^2 I(s) - si(o) - si'(0) + \lambda sI(s) - \lambda si(0) + \omega_0^2 I(s) = \frac{V\omega}{L}\frac{s}{s^2+\omega^2} \tag{9}$$

Apply the initial conditions:

$$s^2 I(s) + \lambda sI(s) + \omega_0^2 I(s) = \frac{V\omega}{L}\frac{s}{s^2+\omega^2} \tag{10}$$

Solve for $I(s)$:

$$I(s) = \frac{V\omega}{L}\frac{s}{(s^2+\omega^2)(s^2+\lambda s+\omega_0^2)} \tag{11}$$

Apply partial fraction decomposition:

$$I(s) = \frac{V\omega}{L}\Big(\frac{As+B}{s^2+\omega^2} + \frac{Cs+D}{s^2+\lambda s+\omega_0^2}\Big) \tag{12}$$

$$= \frac{V\omega}{L}\left(\frac{(As+B)(s^2+\lambda s+\omega_0^2)+(Cs+D)(s^2+\omega^2)}{(s^2+\omega^2)(s^2+\lambda s+\omega_0^2)}\right) \tag{13}$$

Set the numerators of Equation 11 and Equation 13 equal, expand, and collect on similar powers of s:

$$s = (A+C)s^3 + (A\lambda+B+D)s^2 + (A\omega_0^2+B\lambda+C\omega^2)s + B\omega_0^2 + D\omega^2 \tag{14}$$

Set the coefficients of similar powers of s equal:

$$s^3: \qquad 0 = A + C \tag{15}$$

$$s^2: \qquad 0 = A\lambda + B + D \tag{16}$$

$$s^1: \qquad 1 = A\omega_0^2 + B\lambda + C\omega^2 \tag{17}$$

$$s^0: \qquad 0 = B\omega_0^2 + D\omega^2 \tag{18}$$

Determine A, B, C, and D from the simultaneous equations, s^3, s^2, s^1, and s^0:

$$A = \frac{-\omega^2+\omega_0^2}{Den} \qquad B = \frac{\lambda\omega^2}{Den} \qquad C = \frac{\omega^2-\omega_0^2}{Den} \qquad D = \frac{-\lambda\omega_0^2}{Den} \tag{19}$$

where,

$$Den = (\omega^2-\omega_0^2)^2 + \lambda^2\omega^2 \tag{20}$$

Apply A, B, C, and D to Equation 12:

$$I(s) = \frac{V\omega}{L\cdot Den}\left(\frac{s(-\omega^2+\omega_0^2)+\lambda\omega^2}{s^2+\omega^2} + \frac{s(\omega^2-\omega_0^2)-\lambda\omega_0^2}{s^2+\lambda s+\omega_0^2}\right) \tag{21}$$

Steady-state response

The steady-state response is derived from the left-hand term of Equation 21:

$$I(s)_{steady\_state} = \frac{V\omega}{L \cdot Den}\left(\frac{s(-\omega^2 + \omega_0^2) + \lambda\omega^2}{s^2 + \omega^2}\right) \tag{22}$$

$$= \frac{V\omega}{L \cdot Den}\left(\frac{-s\omega^2}{s^2 + \omega^2} + \frac{s\omega_0^2}{s^2 + \omega^2} + \frac{\lambda\omega^2}{s^2 + \omega^2}\right) \tag{23}$$

Take the inverse Laplace transform of Equation 23:

$$i(t)_{steady\_state} = \frac{V\omega}{L \cdot Den}\Big(-\omega^2 cos(\omega t) + \omega_0^2 cos(\omega t) + \lambda\omega sin(\omega t)\Big) \tag{24}$$

$$= \frac{V\omega}{L \cdot Den}\Big((\omega_0^2 - \omega^2)cos(\omega t) + \lambda\omega sin(\omega t)\Big) \tag{25}$$

Transient response

The transient response is derived from the right-hand term of Equation 21:

$$I(s)_{transient} = \frac{V\omega}{L \cdot Den}\left(\frac{s(\omega^2 - \omega_0^2) - \lambda\omega_0^2}{s^2 + \lambda s + \omega_0^2}\right) \tag{26}$$

$$= \frac{V\omega}{L \cdot Den}\left(\frac{s(\omega^2 - \omega_0^2) - \lambda\omega_0^2}{(s + \frac{\lambda}{2} + \mu)(s + \frac{\lambda}{2} - \mu)}\right) \tag{27}$$

where,

$$\mu = \sqrt{\left|\left(\frac{\lambda}{2}\right)^2 - \omega_0^2\right|} \tag{28}$$

under-damped case

For the under-damped case, where $\left(\frac{\lambda}{2}\right)^2 - \omega_0^2 < 0$, Equation 27 becomes:

(note: $j = \sqrt{-1}$)

$$I(s)_{transient} = \frac{V\omega}{L \cdot Den}\left(\frac{s(\omega^2 - \omega_0^2) - \lambda\omega_0^2}{(s + \frac{\lambda}{2} + j\mu)(s + \frac{\lambda}{2} - j\mu)}\right) \tag{29}$$

Apply partial fraction decomposition:

$$I(s)_{transient} = \frac{V\omega}{L \cdot Den}\left(\frac{G}{s+\frac{\lambda}{2}+j\mu} + \frac{H}{s+\frac{\lambda}{2}-j\mu}\right) \tag{30}$$

$$= \frac{V\omega}{L \cdot Den}\left(\frac{G(s+\frac{\lambda}{2}-j\mu)+H(s+\frac{\lambda}{2}+j\mu)}{(s+\frac{\lambda}{2}+j\mu)(s+\frac{\lambda}{2}-j\mu)}\right) \tag{31}$$

Set the numerators of Equation 29 and Equation 31 equal, expand, and collect on similar powers of s:

$$s(\omega^2-\omega_0^2)-\lambda\omega_0^2 = G(s+\frac{\lambda}{2}-j\mu)+H(s+\frac{\lambda}{2}+j\mu) \tag{32}$$

$$s(\omega^2-\omega_0^2)-\lambda\omega_0^2 = s(G+H)+G(\frac{\lambda}{2}-j\mu)+H(\frac{\lambda}{2}+j\mu) \tag{33}$$

Set the coefficients of similar powers of s equal:

$$s^1: \qquad \omega^2-\omega_0^2 = G+H \tag{34}$$

$$s^0: \qquad -\lambda\omega_0^2 = G(\frac{\lambda}{2}-j\mu)+H(\frac{\lambda}{2}+j\mu) \tag{35}$$

Determine G and H from simultaneous equations s^1 and s^0:

$$G = -jP+Q \tag{36}$$

$$H = jP+Q \tag{37}$$

where,

$$P = \frac{\lambda}{4\mu}(\omega^2+\omega_0^2) \tag{38}$$

$$Q = \frac{1}{2}(\omega^2-\omega_0^2) \tag{39}$$

Apply G and H to Equation 30:

$$I(s)_{transient} = \frac{V\omega}{L \cdot Den}\left(\frac{-jP+Q}{s+\frac{\lambda}{2}+j\mu} + \frac{jP+Q}{s+\frac{\lambda}{2}-j\mu}\right) \tag{40}$$

Take the inverse Laplace transform of Equation 40:

$$i(t)_{transient} = \frac{V\omega}{L \cdot Den}\Big(-jP\, e^{-(\frac{\lambda}{2}+j\mu)t} + Q\, e^{-(\frac{\lambda}{2}+j\mu)t} + jP\, e^{-(\frac{\lambda}{2}-j\mu)t} + Q\, e^{-(\frac{\lambda}{2}-j\mu)t}\Big) \tag{41}$$

$$= \frac{V\omega}{L \cdot Den}\, e^{-\frac{\lambda}{2}t}\,\Big(jP\Big(-e^{-j\mu t} + e^{j\mu t}\Big) + Q\Big(e^{-j\mu t} + e^{j\mu t}\Big)\Big) \tag{42}$$

$$= \frac{V\omega}{L \cdot Den}\, e^{-\frac{\lambda}{2}t}\,\Big(-2P\Big(\frac{-e^{-j\mu t} + e^{j\mu t}}{2j}\Big) + 2Q\Big(\frac{e^{-j\mu t} + e^{j\mu t}}{2}\Big)\Big) \tag{43}$$

$$i(t)_{transient} = \frac{V\omega}{L \cdot Den}\, e^{-\frac{\lambda}{2}t}\,\Big(-2P\, sin(\mu t) + 2Q\, cos(\mu t)\Big) \tag{44}$$

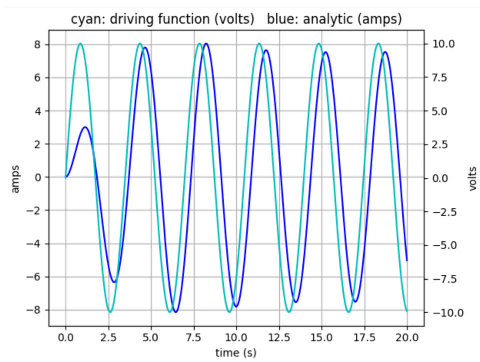

Figure 2: **under-damped transient + steady-state**
$R = 1.0$, $L = 1.51$, $C = 0.3$, $\omega = 1.8$, $V = 10$

over-damped case

For the over-damped case, where $\left(\frac{\lambda}{2}\right)^2 - \omega_0^2 > 0$, Equation 27 becomes:

$$I(s)_{transient} = \frac{V\omega}{L \cdot Den}\left(\frac{s(\omega^2 - \omega_0^2) - \lambda\omega_0^2}{(s + \frac{\lambda}{2} + \mu)(s + \frac{\lambda}{2} - \mu)}\right) \tag{45}$$

Apply partial fraction decomposition:

$$I(s)_{transient} = \frac{V\omega}{L \cdot Den}\left(\frac{G}{s + \frac{\lambda}{2} + \mu} + \frac{H}{s + \frac{\lambda}{2} - \mu}\right) \tag{46}$$

$$= \frac{V\omega}{L \cdot Den}\left(\frac{G(s + \frac{\lambda}{2} - \mu) + H(s + \frac{\lambda}{2} + \mu)}{(s + \frac{\lambda}{2} + \mu)(s + \frac{\lambda}{2} - \mu)}\right) \tag{47}$$

Set the numerators of Equation 45 and Equation 47 equal, expand, and collect on similar powers of s:

$$s(\omega^2 - \omega_0^2) - \lambda\omega_0^2 = G(s + \frac{\lambda}{2} - \mu) + H(s + \frac{\lambda}{2} + \mu) \tag{48}$$

$$s(\omega^2 - \omega_0^2) - \lambda\omega_0^2 = s(G + H) + G(\frac{\lambda}{2} - \mu) + H(\frac{\lambda}{2} + \mu) \tag{49}$$

Set the coefficients of similar powers of s equal:

$$s^1: \qquad \omega^2 - \omega_0^2 = G + H \tag{50}$$

$$s^0: \qquad -\lambda\omega_0^2 = G(\frac{\lambda}{2} - \mu) + H(\frac{\lambda}{2} + \mu) \tag{51}$$

Determine G and H from simultaneous equations s^1 and s^0:

$$G = P + Q \tag{52}$$

$$H = -P + Q \tag{53}$$

where,

$$P = \frac{\lambda}{4\mu}(\omega^2 + \omega_0^2) \tag{54}$$

$$Q = \frac{1}{2}(\omega^2 - \omega_0^2) \tag{55}$$

Apply G and H to Equation 46:

$$I(s)_{transient} = \frac{V\omega}{L \cdot Den}\left(\frac{P + Q}{s + \frac{\lambda}{2} + \mu} + \frac{-P + Q}{s + \frac{\lambda}{2} - \mu}\right) \tag{56}$$

Take the inverse Laplace transform of Equation 56:

$$i(t)_{transient} = \frac{V\omega}{L \cdot Den}\, e^{-\frac{\lambda}{2}t}\left(P\Big(e^{-\mu t} - e^{\mu t}\Big) + Q\Big(e^{-\mu t} + e^{\mu t}\Big)\right) \tag{57}$$

$$i(t)_{transient} = \frac{V\omega}{L \cdot Den}\, e^{-\frac{\lambda}{2}t}\left(\Big(P + Q\Big)e^{-\mu t} + \Big(Q - P\Big)e^{\mu t}\right) \tag{58}$$

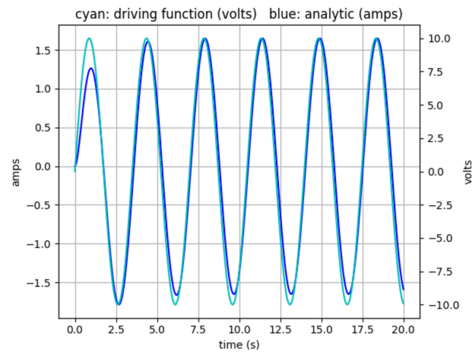

Figure 3: **over-damped transient + steady-state**
$R = 6.0$, $L = 1.51$, $C = 0.3$, $\omega = 1.8$, $V = 10$

critically-damped case

For the critically-damped case, where $\left(\frac{\lambda}{2}\right)^2 - \omega_0^2 = 0$ and $\mu = 0$, Equation 27 becomes:

$$I(s)_{transient} = \frac{V\omega}{L \cdot Den}\left(\frac{s(\omega^2 - \omega_0^2) - \lambda\omega_0^2}{(s + \frac{\lambda}{2})(s + \frac{\lambda}{2})}\right) \tag{59}$$

Apply partial fraction decomposition:

$$I(s)_{transient} = \frac{V\omega}{L \cdot Den}\left(\frac{G}{(s + \frac{\lambda}{2})} + \frac{H}{(s + \frac{\lambda}{2})^2}\right) \tag{60}$$

$$= \frac{V\omega}{L \cdot Den}\left(\frac{G(s + \frac{\lambda}{2}) + H}{(s + \frac{\lambda}{2})^2}\right) \tag{61}$$

Set the numerators of Equation 59 and Equation 61 equal, expand, and collect on similar powers of s:

$$s(\omega^2 - \omega_0^2) - \lambda\omega_0^2 = G(s + \frac{\lambda}{2}) + H \tag{62}$$

$$s(\omega^2 - \omega_0^2) - \lambda\omega_0^2 = sG + G\frac{\lambda}{2} + H \tag{63}$$

Set the coefficients of similar powers of s equal:

$$s^1: \quad \omega^2 - \omega_0^2 = G \tag{64}$$

$$s^0: \quad -\lambda\omega_0^2 = G\frac{\lambda}{2} + H \tag{65}$$

Determine G and H from simultaneous equations s^1 and s^0:

$$G = \omega^2 - \omega_0^2 \tag{66}$$

$$H = -\frac{\lambda}{2}(\omega^2 + \omega_0^2) \tag{67}$$

Apply G and H to Equation 60:

$$I(s)_{transient} = \frac{V\omega}{L \cdot Den}\left(\frac{\omega^2 - \omega_0^2}{(s + \frac{\lambda}{2})} - \frac{\frac{\lambda}{2}(\omega^2 + \omega_0^2)}{(s + \frac{\lambda}{2})^2}\right) \tag{68}$$

Take the inverse Laplace transform of Equation 68:

$$i(t)_{transient} = \frac{V\omega}{L \cdot Den}\left((\omega^2 - \omega_0^2)\, e^{-\frac{\lambda}{2}t} - \frac{\lambda}{2}t\,(\omega^2 + \omega_0^2)\, e^{-\frac{\lambda}{2}t}\right) \tag{69}$$

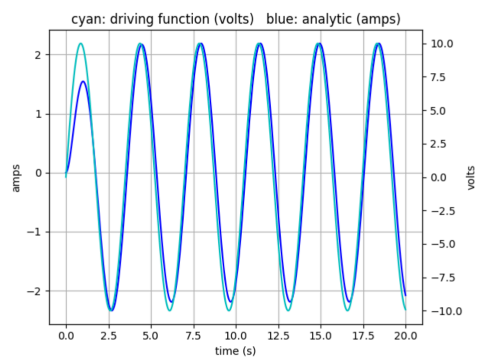

Figure 4: **critically-damped transient + steady-state**
$R = 6.0$, $L = 1.51$, $C = 0.3$, $\omega = 1.8$, $V = 10$