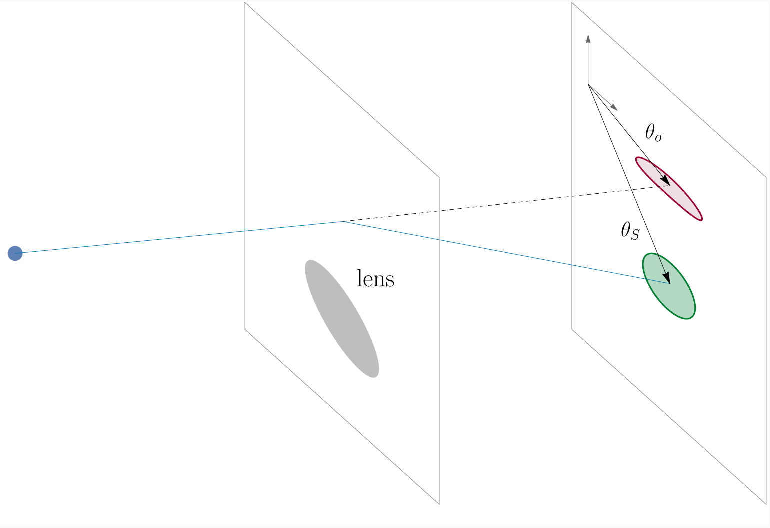
2. Weak lensing basics



θ_o
θ_S
lens

We use the perturbed metric in Newtonian Gauge

$$ds^2 = -(1+2\Psi)dt^2 + a^2(t)(1-2\Phi)\delta_{ij}dx^i dx^j.$$

With this convention, the weak field limit gives

$$\text{Geodesic equation:} \quad \ddot{\boldsymbol{x}} + H(t)\dot{\boldsymbol{x}} = -\frac{1}{a}\boldsymbol{\nabla}\Psi(\boldsymbol{x},t)$$

$$\text{Poisson equation:} \quad \frac{1}{a^2}\nabla^2\Phi(\boldsymbol{x},t) = 4\pi G\rho_m(t)\delta_m(\boldsymbol{x},t) \quad \text{(In GR)}$$

In the absence on anisotropic stresses (also assuming GR) $\Psi = \Phi$. e.g, this is the case of ΛCDM from the matter dominated epoch to today ($z \lesssim 100$),

Christoffel symbols:

$$\Gamma^0_{00} = \dot{\Psi}, \quad \Gamma^i_{00} = \frac{\partial\Psi}{\partial x_i}, \quad \Gamma^i_{j0} = \delta^i_j(H - \dot{\Phi}),$$

$$\Gamma^i_{00} = \delta_{ij}a^2\left[H + \dot{\Phi} - 2H(\Phi+\Psi)\right],$$

$$\Gamma^i_{jk} = \left(\delta_{jk}\frac{\partial}{\partial x_i} - \delta^i_j\frac{\partial}{\partial x^k} - \delta^i_k\frac{\partial}{\partial x^j}\right)\Phi.$$

$$\text{4-momentum of photons:} \quad g_{\mu\nu} P^\mu P^\nu = 0 \quad \text{with} \quad P^\mu = \frac{dx^\mu}{d\lambda}.$$

Hence, defining the generalized spatial momentum p,

$$p = g_{ij} P^i P^j,$$

and

$$P^0 = \frac{1}{\sqrt{1+2\Psi}} = p(1-\Psi).$$

This is the generalization to a perturbed FRW of the relativistic expression $E = p$. In the metric convention we are following, an overdense region has $\Psi < 0$. Therefore, photons loose energy (and redshift) as they move away from an overdense region.

Geodesic equation $P^\mu \nabla_\mu P^\mu = 0$:

$$\frac{d^2 x^i}{d\lambda^2} = -\Gamma^i_{\mu\nu} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}.$$

The LHS is, remind $x^i = \chi\theta^i = \chi(\theta^1, \theta^2, \chi)$,

$$\begin{aligned}
\frac{d^2 x^i}{d\lambda^2} &= \frac{dt}{d\lambda} \frac{d\chi}{dt} \frac{d}{d\chi} \left[\frac{d\chi\theta^i}{d\chi} \frac{dt}{d\lambda} \frac{d\chi}{dt} \right] = -\frac{p}{a} \frac{d}{d\chi} \left[-\frac{p}{a} \frac{d\chi\theta^i}{d\chi} \right] \\
&= p^2 \frac{d}{d\chi} \left[\frac{1}{a^2} \frac{d}{d\chi} (\chi\theta^i) \right] \\
&= \frac{p^2}{a^2} \left[\frac{d^2}{d\chi^2} (\chi\theta^i) + 2aH \frac{d}{d\chi} (\chi\theta^i) \right]
\end{aligned}$$

where we used the definition of the radial comoving distance $\chi = \int_a^{a_0} dt/a(t)$ and $P^0 \equiv dt/d\lambda = p(1 - \Psi)$. We have assumed the deflection angle is small, and set $\theta \times \Psi = 0$. In the second line, we use the background evolution of the momentum $p \propto 1/a$, so at the lowest order ap is a constant that we can pull out of the derivative.

Using the Christoffel symbols we obtain

$$-\Gamma^i_{\mu\nu}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} = -p^2\left[-\frac{\partial}{\partial x_i}(\Psi+\Phi)+2\frac{H}{a}\frac{d}{d\chi}(\chi\theta^i)\right]$$

Hence, the geodesic equation becomes

$$\boxed{\frac{d^2}{d\chi^2}(\chi\theta_m) = -a^2\frac{\partial}{\partial x^m}(\Psi+\Phi)}$$

In the ΛCDM model at late times, the anisotropic stresses are neglibible, so $\Psi = \Phi$ and

$$\frac{d^2}{d\chi^2}(\chi\theta_m) = -2a^2\frac{\partial\Psi}{\partial x^m}$$

We can integrate twice to obtain the true source position $\theta_S^m = \theta^m(\chi)$, subject to the initial condition that the observed position is $\theta_O^m = \theta^m(\chi = 0)$

$$\theta_S^m = \theta^m(\chi) = \theta_O^m - \int_0^\chi d\chi' \, \frac{\chi - \chi'}{\chi} \frac{\partial}{\partial x^m} \Big[\Psi(\boldsymbol{x}) + \Phi(\boldsymbol{x}) \Big]$$

with $\boldsymbol{x} \equiv \boldsymbol{x}(\boldsymbol{\theta}(\chi'), \chi')$.

This is a non-linear relation between the observed and true source positions.

The first approximation considers $\boldsymbol{\theta}(\chi') = \boldsymbol{\theta}_O$ in the arguments of the gravitational potentials, corresponding to integrate the potential gradient along the unperturbed ray, which is called the Born approximation.

Defining the lensing potential ψ (from now on, we simplify the notation and name θ the observed position)

$$\psi(\boldsymbol{\theta}, \chi) = -\int_0^{\chi} \frac{d\chi'}{\chi'} \frac{\chi - \chi'}{\chi} \Big[\Psi(\boldsymbol{x}(\chi')) + \Phi(\boldsymbol{x}(\chi'))\Big],$$

the true and observed position are related by

$$\theta_S^m = \theta^m + \partial^m \psi(\theta),$$

where we used $\frac{\partial}{\partial x^m} = \frac{1}{\chi} \frac{\partial}{\partial \theta^m}$ inside the integral. And we denote $m, n = 1, 2$, so $\partial_m = \left(\frac{\partial}{\partial \theta^1}, \frac{\partial}{\partial \theta^2}\right)$.

This equation is valid for a single background source at, radial comoving distance χ, whose light travels toward us and deviates due to the foreground matter distribution

We are interested in small deflections of path light rays. In such a case, the relation between observed and true coordinates is linear:

$$A_{mn} \equiv \frac{\partial \theta_m^S}{\partial \theta^n} = I_{mn} + \partial_m \partial_n \psi \equiv \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix}.$$

Assuming small deflection angles we expand

$$\theta_m^S(\theta) = \underbrace{A_{mn}\theta_n}_{\text{Convergence + Shear}} + \underbrace{\frac{1}{2}D_{mns}\theta_n\theta_s}_{\text{Flexion}} + \cdots$$

$$\text{Weak lensing:} \qquad \boxed{\theta_m^S = A_{mn}\theta_n.}$$

That is, weak lensing is described by a linear map relating the observed and true positions of the sources.

Weak lensing assumes that the value the derivatives of the lensing potential $\partial_m \partial_n \psi$ do not change along the source surface (e.g., through a galaxy subtended solid angle), otherwise we would have to account for $D_{mns} = \partial_m \partial_n \partial_s \psi$.

Weak lensing:

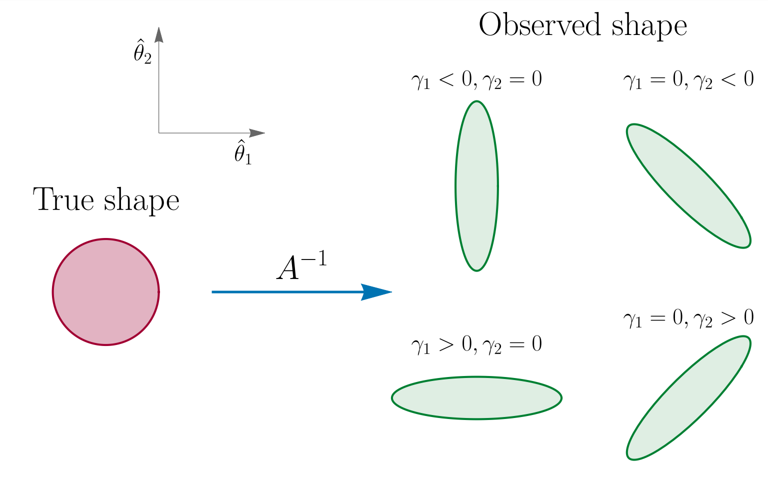
$$\begin{pmatrix} \theta_1^S \\ \theta_2^S \end{pmatrix} = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} + \begin{pmatrix} -\kappa - \gamma_1 & -\gamma_2 \\ -\gamma_2 & -\kappa + \gamma_1 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}.$$

Using $\theta_i^S = \theta_i + (\partial_i \partial_j \psi)\, \theta^j + \cdots$, with ψ the lensing potential

$$\kappa = -\frac{1}{2}(\partial_1\partial_1 + \partial_2\partial_2)\psi = -\frac{1}{2}\partial^2\psi$$
$$\gamma_1 = -\frac{1}{2}(\partial_1\partial_1 - \partial_2\partial_2)\psi, \qquad \gamma_2 = -\partial_1\partial_2\psi$$

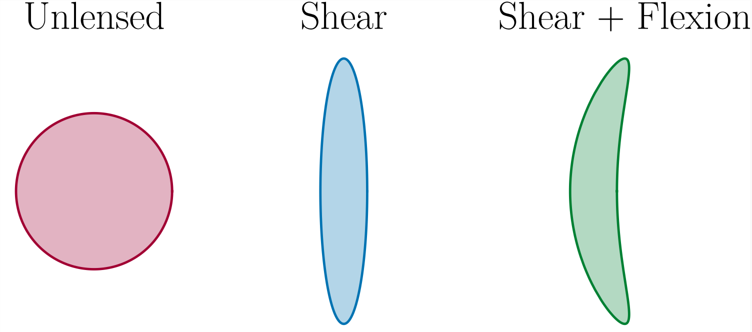
The inverse Jacobian A^{-1} describes the local mapping of the source light distribution to image coordinates.

The convergence κ is an isotropic increase or decrease of the observed size of a source image. The shear (γ_1, γ_2), the trace-free part of A, quantifies an anisotropic stretching, turning a circular into an elliptical light distribution.



Observed shape
$\hat{\theta}_2$
$\hat{\theta}_1$
$\gamma_1 < 0, \gamma_2 = 0$
$\gamma_1 = 0, \gamma_2 < 0$
True shape
A^{-1}
$\gamma_1 = 0, \gamma_2 > 0$
$\gamma_1 > 0, \gamma_2 = 0$

$$\boldsymbol{\theta}_s = \boldsymbol{A}\,\boldsymbol{\theta} + \frac{1}{2}\boldsymbol{\theta}^T \boldsymbol{D}\,\boldsymbol{\theta}$$



In a right-handed orthonormal coordinate basis $(\hat{\theta}_1, \hat{\theta}_2, \hat{z} = \hat{\boldsymbol{n}})$

$$A_{mn} = \begin{pmatrix} 1-\kappa & 0 \\ 0 & 1-\kappa \end{pmatrix} + \begin{pmatrix} -\gamma_1 & -\gamma_2 \\ -\gamma_2 & \gamma_1 \end{pmatrix} \equiv (1-\kappa) I_{mn} + \Gamma_{mn}.$$

Notice that if the convergence is not observable what we actually measure is the reduced shear $g \equiv \gamma/(1-\kappa) \simeq \gamma$.

Against a rotation by an angle φ, Γ_{mn} transforms as

$$\Gamma'_{mn} = (R\Gamma R^{-1})_{mn} = \begin{pmatrix} -\gamma_1 \cos(2\varphi) + \gamma_2 \sin(2\varphi) & -\gamma_2 \cos(2\varphi) - \gamma_1 \sin(2\varphi) \\ -\gamma_2 \cos(2\varphi) - \gamma_1 \sin(2\varphi) & +\gamma_1 \cos(2\varphi) - \gamma_2 \sin(2\varphi) \end{pmatrix}$$
$$= \begin{pmatrix} -\gamma'_1 & -\gamma'_2 \\ -\gamma'_2 & \gamma'_1 \end{pmatrix}$$

That is, the shear components γ_1 and γ_2 transform between themselves against the rotation of coordinates $R(\varphi) : (\hat{\theta}_1, \hat{\theta}_2) \to (\hat{\theta}'_1, \hat{\theta}'_2)$ as

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \to \begin{pmatrix} \gamma'_1 \\ \gamma'_2 \end{pmatrix} = \begin{pmatrix} \gamma_1 \cos(2\varphi) - \gamma_2 \sin(2\varphi) \\ \gamma_2 \cos(2\varphi) + \gamma_1 \sin(2\varphi) \end{pmatrix},$$

that is, they transform with the double angle.

Define the helicity basis

$$\boldsymbol{\epsilon}_{\pm} = \frac{1}{\sqrt{2}}(\hat{\theta}_1 \pm i\hat{\theta}_2).$$

We use the components

$$\gamma \equiv \Gamma_{++} = \Gamma_{mn}\boldsymbol{\epsilon}_{+}^{m}\boldsymbol{\epsilon}_{+}^{n} = \gamma_1 + i\gamma_2$$
$$\bar{\gamma} \equiv \Gamma_{--} = \Gamma_{mn}\boldsymbol{\epsilon}_{-}^{m}\boldsymbol{\epsilon}_{-}^{n} = \gamma_1 - i\gamma_2$$

and $\Gamma_{+-} = \Gamma_{-+} = 0$.

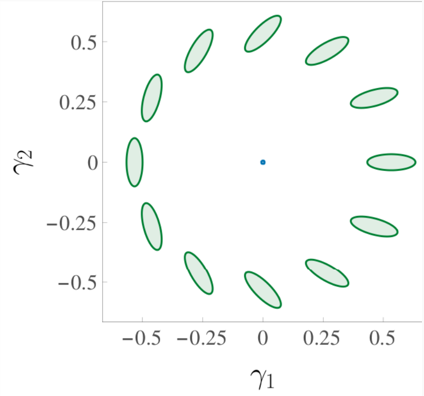
With this, the transformation becomes

$$\gamma_1' \pm i\gamma_2' = e^{\pm 2i\varphi}(\gamma_1 \pm i\gamma_2),$$

or, equivalently

$$\gamma \quad \rightarrow \quad \gamma' = e^{2i\varphi}\gamma.$$

That is, γ and $\bar{\gamma}$ transform as spin-2 functions against rotation.



Convergence as a projected density

$$\begin{aligned}
\kappa(\boldsymbol{\theta}, \chi) = -\frac{1}{2}\partial^2\psi &= \int_0^\chi \frac{d\chi'}{\chi'} \frac{\chi - \chi'}{\chi} \partial^2\Phi \\
&= \int_0^\chi d\chi' \, \chi' \frac{\chi - \chi'}{\chi} \nabla_{\boldsymbol{x}}^2 \Phi \\
&= \frac{3}{2}\Omega_m H_0^2 \int_0^\chi \frac{d\chi'}{a(\chi')} \chi' \frac{\chi - \chi'}{\chi} \delta(\chi'\boldsymbol{\theta}, \chi'),
\end{aligned}$$

where we used $\partial^2 = \chi^2 \nabla_{\boldsymbol{x}}^2$ in the third equality, and the Poisson equation to relate the gravitational potential Φ with the overdensity field δ.

Notice that the geometrical factor $\chi'(\chi - \chi')$ is a parabola with maximum at $\chi' = \chi/2$. Hence, structures at half the distance between the source and the observers are more efficient to produce lensing distortions.

The total convergence from a population of source galaxies is obtained by weighting $\kappa(\theta, \chi)$ expression with the galaxy probability distribution $W_g(\chi)$:

$$\begin{aligned}
\kappa(\boldsymbol{\theta}) &= \int_0^\infty d\chi W_g(\chi)\kappa(\boldsymbol{\theta}, \chi) \\
&= \frac{3}{2}\Omega_m H_0^2 \int_0^\infty d\chi W_g(\chi) \int_0^\chi d\chi' \frac{1}{a(\chi')}\chi' \frac{\chi - \chi'}{\chi}\delta(\boldsymbol{\theta}, \chi') \\
&= \frac{3}{2}\Omega_m H_0^2 \int_0^\infty d\chi' \int_{\chi'}^\infty d\chi\, W_g(\chi) \frac{1}{a(\chi')}\chi' \frac{\chi - \chi'}{\chi}\delta(\boldsymbol{\theta}, \chi') \\
&= \int_0^\infty d\chi' q(\chi')\delta(\chi'\boldsymbol{\theta}, \chi')
\end{aligned}$$

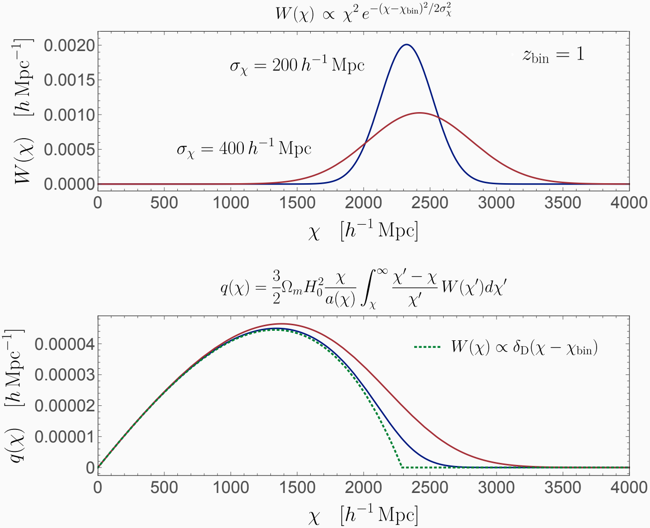
with the lens efficiency q defined as

$$q(\chi) = \frac{3}{2}\Omega_m H_0^2 \frac{\chi}{a(\chi)} \int_\chi^\infty d\tilde{\chi}\, W_g(\tilde{\chi}) \frac{\tilde{\chi} - \chi}{\tilde{\chi}}$$

The convergence becomes a linear measure of the total matter density, projected along the line of sight and weighted by the source galaxy distribution W_g.

q yields the lensing strength at a distance χ of the combined background galaxy distribution.

There are different conventions/definition for the lens efficiency



$W(\chi) \propto \chi^2 e^{-(\chi-\chi_{\rm bin})^2/2\sigma_\chi^2}$
$W(\chi)\quad [h\,\mathrm{Mpc}^{-1}]$
$\sigma_\chi = 200\,h^{-1}\,\mathrm{Mpc}$
$\sigma_\chi = 400\,h^{-1}\,\mathrm{Mpc}$
$z_{\rm bin} = 1$
0.0000
0.0005
0.0010
0.0015
0.0020
0
500
1000
1500
2000
2500
3000
3500
4000
$\chi\quad [h^{-1}\,\mathrm{Mpc}]$
$q(\chi) = \frac{3}{2}\Omega_m H_0^2 \frac{\chi}{a(\chi)} \int_\chi^\infty \frac{\chi' - \chi}{\chi'} W(\chi') d\chi'$
$q(\chi)\quad [h\,\mathrm{Mpc}^{-1}]$
$W(\chi) \propto \delta_{\rm D}(\chi - \chi_{\rm bin})$
0
0.00001
0.00002
0.00003
0.00004
0
500
1000
1500
2000
2500
3000
3500
4000
$\chi\quad [h^{-1}\,\mathrm{Mpc}]$

The convergence angular power spectrum is computed through the definition,

$$\langle \kappa(\boldsymbol{\ell})\kappa(\boldsymbol{\ell}')\rangle = (2\pi)^2 \delta_{\mathrm{D}}(\boldsymbol{\ell}+\boldsymbol{\ell}') C_\kappa(\ell),$$

under the Limber/flat sky approximation, we have

$$\boxed{C_\kappa(\ell) = \int_0^\infty \frac{d\chi}{\chi^2}\, q^2(\chi) P_\delta\left(\frac{\ell+1/2}{\chi}, \chi\right).}$$

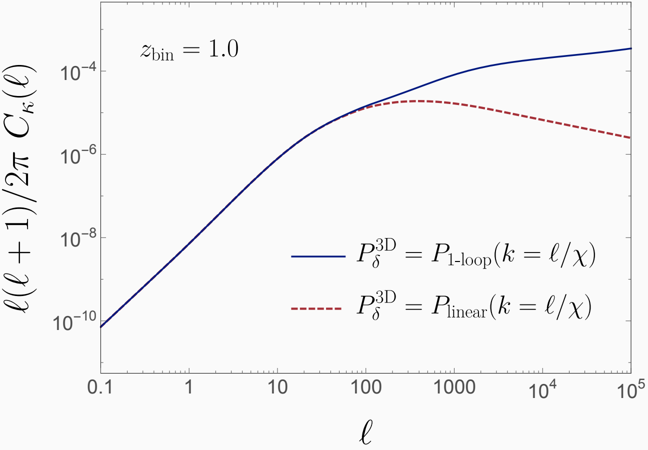
Compare it to the galaxy angular power spectrum in

$$C_g(\ell) = \int_0^\infty \frac{d\chi}{\chi^2}\, W_g^2(\chi) P_g\left(\frac{\ell+1/2}{\chi}, \chi\right)$$

$W_g(\chi) \longrightarrow q(\chi)$, but more important $P_g(\chi) \longrightarrow P_\delta(\chi)$. That is, weak lensing probes the whole power spectrum, including the dark matter.

Convergence angular power spectrum

$$C_\kappa(\ell) = \int_0^\infty \frac{d\chi}{\chi^2}\, q^2(\chi) P_\delta\Big(\frac{\ell+1/2}{\chi}, \chi\Big).$$



Shear power spectra

We comeback to the equations relating the components of $\boldsymbol{A}$ with the lensing potential ψ_l:

$$\kappa = -\frac{1}{2}(\partial_1\partial_1 + \partial_2\partial_2)\psi_l = -\frac{1}{2}\nabla^2\psi_l$$
$$\gamma_1 = -\frac{1}{2}(\partial_1\partial_1 - \partial_2\partial_2)\psi_l, \quad \gamma_2 = -\partial_1\partial_2\psi_l$$

Configuration space: $\gamma(\boldsymbol{\theta}) = (\partial_1 + i\partial_2)^2\nabla^{-2}\kappa(\boldsymbol{\theta})$

Fourier space: $\gamma(\boldsymbol{\ell}) = \dfrac{(\ell_1 + i\ell_2)^2}{\ell^2}\kappa(\boldsymbol{\ell})$