

# Neural Network Operations and Susuki-Trotter evolution of Neural Network States

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It was recently proposed to leverage the representational power of artificial neural networks, in particular Restricted Boltzmann Machines, in order to model complex quantum states of many-body systems [Science, 355(6325), 2017]. States represented in this way, called Neural Network States (NNSs), were shown to display interesting properties like the ability to efficiently capture long-range quantum correlations. However, identifying an optimal neural network representation of a given state might be challenging, and so far this problem has been addressed with stochastic optimization techniques. In this work we explore a different direction. We study how the action of elementary quantum operations modifies NNSs. We parametrize a family of many body quantum operations that can be directly applied to states represented by Unrestricted Boltzmann Machines, by just adding hidden nodes and updating the network parameters. We show that this parametrization contains a set of universal quantum gates, from which it follows that the state prepared by any quantum circuit can be expressed as a Neural Network State with a number of hidden nodes that grows linearly with the number of elementary operations in the circuit. This is a powerful representation theorem (which was recently obtained with different methods) but that is not directly useful, since there is no general and efficient way to extract information from this unrestricted description of quantum states. To circumvent this problem, we propose a step-wise procedure based on the projection of *Unrestricted* quantum states to *Restricted* quantum states. In turn, two approximate methods to perform this projection are discussed. In this way, we show that it is in principle possible to approximately optimize or evolve Neural Network States without relying on stochastic methods such as Variational Monte Carlo, which are computationally expensive.

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## I. INTRODUCTION

As is well known, the description of general quantum states of composite systems requires an amount of information that grows exponentially with the number of subsystems. This simple fact is one of the reasons why general quantum systems are hard to simulate with ordinary computers. A possible workaround for this problem is to abandon the desire of describing arbitrary quantum states, and only concentrate on a manifold of physically meaningful states[1]. A prominent example along this line is given by the Matrix Product States (MPSs)[2]<sup>1</sup>. Here, the physically meaningful states that are addressed are the low energy states of gapped Hamiltonians with local interactions. In one dimension, it is known that those states satisfy an entanglement area law, and MPSs are a sufficiently general class of states compatible with such law [3–7]. This is also a limitation for MPSs, since they are then not sufficient to efficiently capture the rich physics close to quantum critical points, where the gap typically closes and the quantum correlations no longer obey an area law[8, 9]. This is also the case for systems

with long range interactions[10–13].

Recently, a new family of states was proposed by Carleo and Troyer[14] to deal with long range quantum correlations in many-body systems: the so-called Neural Network States (NNSs) or Neural Quantum States (NQSs). The main idea behind this proposal is to treat the wave function as a functional that maps configurations of lattice spin systems (states of a given computational basis) to complex numbers (probability amplitudes). As the name suggests, a neural network architecture is used to model this mapping. In particular, the neural networks employed in [14] are Restricted Boltzmann Machines (RBMs). An RBM is defined in terms of a network of hidden and visible nodes, with weighted connections between these two groups, which thus form a bipartite graph (see Figure 1). An Ising-like energy functional is assigned to the network, and the distribution realized by it is specified by the Boltzmann factor corresponding to that energy, often conditioned on some configuration of either the visible or both hidden and visible parts of the network. The ‘programming’ or ‘training’ of the network consists in adjusting the weights such that the network matches, or well approximates, a target distribution over the visible nodes, which is often specified only via a set of samples drawn from it.

In order to model a quantum many body state each visible node is associated with a subsystem, for exam-

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<sup>1</sup> Here, we assume a restricted bond dimension of the MPS, such that the total number of parameters which need to be specified is polynomial in the number of subsystems. If the bond dimension is not restricted, any quantum state can be cast as an MPS.

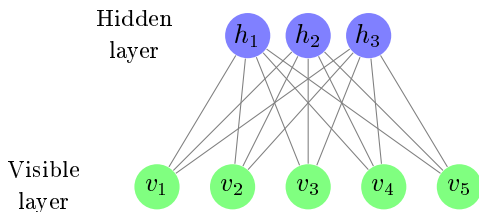


FIG. 1: Restricted Boltzmann Machine with  $N = 5$  visible and  $M = 3$  hidden nodes. When modelling a quantum state, each of the visible nodes  $v_1, \dots, v_N$  represents, for instance, a spin  $s_j$  on a lattice. The number of hidden neurons determines the power that the network has to represent distributions over the visible nodes.

ple a spin on a lattice, and the expressive power<sup>2</sup>, of the machine increases with the number of hidden nodes considered.

The family of states thus obtained with efficient NNSs, that is, those utilizing only a polynomial number of hidden nodes was later shown to be able to efficiently describe volume-law entanglement[13]<sup>3</sup>. The relationship between NNSs and MPSs was explored in [15], showing that, in general, in order to exactly represent a given NNS as a MPS, an exponentially large bond dimension is required for the latter. Thus, it was shown that MPSs cannot efficiently describe general NNSs based on RBMs (RBM-NNSs). In subsequent works, it was shown that RBM-NNSs are also related to other previously known families such as String-Bond States[16], or arbitrary graph states[17].

So far, NNSs were mainly employed as a variational Ansatz, either to minimize the energy of a model Hamiltonian, to evolve a state over time, or for quantum state tomography[13, 14, 16, 18]. In these cases the neural network representing the state was optimized by usual Variational Monte Carlo (VMC) techniques. This is computationally expensive, since at each iteration in the optimization it is necessary to stochastically estimate the gradient of an objective function with respect to the network parameters[14]. In this work we explore a different direction. Our main aim is to find a method to evolve a given NNS, on the level of the representation and in a controlled way, without requiring stochastic sampling and estimation. We begin by investigating how to update the parameters of a given NNS to take into account the action of simple physical processes. Thus, we pose the following question: is there any family of (non-trivial)

elementary quantum operations (*i.e.* unitary gates) that can be applied to a NNS state, in such a way that the resulting state is also efficiently representable as a NNS, and such that the update can be performed on the level of the NN representation itself? How can this family be parametrized? An answer to this question would shed further light on the properties and limitations of NNSs, and could guide the development of efficient numerical algorithms to evolve and optimize NNSs, without the need to rely exclusively on stochastic methods. In other words, what we are asking for is a collection of efficient rewriting rules, which can approximate the evolution of quantum states, on the level of the graphs representing them.

As it turns out, even very simple one-body unitary operations take a general RBM-NNS to a new state that is naturally represented by *Unrestricted* Boltzmann Machines (UBMs), for which connections among the hidden nodes are allowed. We point out that this does not mean that applying a single-body unitary to a RBM-NNS necessarily renders this state into a new one without an efficient RBM representation. However, an extremely simple update rule can be identified if the restriction on the connections in the hidden layer is relaxed. We will refer to states described by UBMs as UBM-NNSs. For this more general representation, there exists a family of non-trivial operations whose action can be easily represented by simple update rules. Specifically, we identify a family of  $K$ -body operations that can be applied to any UBM-NNS by only adding  $K$  hidden nodes to the UBM, followed by a simple update of the network parameters. This family contains universal sets of gates, so in this way, we also show that the quantum state prepared by any quantum circuit can be expressed as a UBM-NNS with a number of hidden units that increases linearly with the number of elementary operations in the circuit, provided that the initial state is also a UBM-NNS. These results are compatible to those obtained in [17], where the representational power of *Deep* Boltzmann Machines was explored and compared to the shallow or restricted case<sup>4</sup>. However, our methods are different and provide new insights. Although the mentioned results are interesting and show the power of Boltzmann Machines to represent quantum states, they are not directly useful. The reason is that, in contrast to RBM-NNSs, there is no accurate and efficient way to extract information out of a UBM-based description of a quantum state. To explain this we compare the problem of sampling a UBM-NNS to standard Quantum Monte Carlo techniques based on path integrals where a classical model, ‘*dual*’ to a quantum model of interest, is sampled[19–21].

In fact, the main advantage of RBMs is that they can be sampled efficiently (since the quotients between probability amplitudes can be readily computed). Of

<sup>2</sup> By expressive power here we mean the size and complexity of the set of distributions which can be realized over the visible nodes, by marginalizing over the hidden nodes.

<sup>3</sup> It is well-known that RBMs (and NNSs) can represent any distribution and/or state, provided the number of hidden nodes is not limited. This is similar to how unlimited bond dimensions render MPSs fully expressible.

<sup>4</sup> As explained in [17], Deep and Unrestricted Boltzmann machines are essentially equivalent.

course, this comes at the expense of some representational power[17]. Nevertheless, as mentioned before, RBM-NNSs can still represent many complex and highly entangled quantum states[13, 15–17]. Thus, building on the study of quantum operations, we propose a method to evolve an initial RBM-NNS in such a way that the final state is also an easy to sample RBM-NNS. The central idea is that, whenever a quantum operation transform the input state to a UBM-NNS, this output state is *projected* back to the family of RBM-NNS. Two projection procedures are presented and discussed. Finally, this ideas are tested showing that it is possible to optimize RBM-NNSs in order to approximate the ground state of the transverse field Ising model in one dimension without employing stochastic methods like VMC. To the best of our knowledge, this is the first example of a method in which RBM-NNS are optimized in a deterministic manner, providing an alternative to stochastic methods.

This article is organized as follows: in section II we review the definition of RBM-NNS and show how the action of simple one-body unitaries takes them to UBM-NNS. In section IIB we define general UBM-NNS and show how the network must be modified to take into account the action of K-body operations. In section III we compare the sampling of UBM-NNS to usual QMC methods. In section IV we propose a method to continuously project the evolved state back to the family of RBM-NNS. Finally, in Section V we apply this ideas and show how to approximate the ground state of an Ising chain.

## II. RBM-NNSs AND ONE-BODY OPERATIONS

Boltzmann machines are described by a set of nodes (neurons), representing stochastic variables, with bidirectional weighted connections between them forming a neural network. These nodes are usually split in two groups: visible nodes and hidden nodes. The restriction in Restricted Boltzmann Machines (RBMs) is that no connections are allowed between nodes of the same group, as depicted in Fig. 1.

In classical RBMs an energy function is assigned to the network, which is typically a quadratic function of the node values:

$$E_{\text{RBM}} = -a^t v - b^t h - h^t W v,$$

where  $v = (v_1, \dots, v_N)^t$  and  $h = (h_1, \dots, h_M)^t$  are column vectors with the values of the  $N$  visible nodes and the  $M$  hidden nodes, respectively. The vectors  $a = (a_1, \dots, a_N)^t$  and  $b = (b_1, \dots, b_M)^t$ , along with the  $M \times N$  matrix  $W$ , are the parameters of the network. The constants  $a_k$  and  $b_k$  are known as offsets, and the components of the matrix  $W$  indicate the weights of the connections. In classical applications the parameters  $a$ ,  $b$ , and  $W$  are real, and it is assumed that the probability distribution for the stochastic variables in  $h$  and  $v$ ,

$P(v, h|a, b, W)$ , is of the Boltzmann form:

$$P(v, h|a, b, W) = \frac{1}{Z} e^{-E_{\text{RBM}}}$$

where  $Z = \sum_{v,h} e^{-E_{\text{RBM}}}$  is the partition function. Now, this network can be used to learn a target probability distribution over the visible nodes. Given a training set of configurations for the visible nodes, with a distribution  $P_T(v)$ , training the network means to adjust the network parameters  $a$ ,  $b$ , and  $W$  in order to minimize some measure of distance<sup>5</sup> between the marginal distribution over the visible nodes

$$P(v) = \sum_{h_1, \dots, h_M} P(v, h|a, b, W) \quad (1)$$

and  $P_T(v)$ . This is an optimization problem that can be attacked with different iterative methods, such as so-called contrastive divergence[23]. The fact that hidden nodes are not connected between them in RBMs (i.e, the energy is only linear in  $h$ ), allows to explicitly perform the sum to find  $P(v)$ . If it is assumed for simplicity that the hidden variables  $h_k$  are binomial, taking the values  $\{-1, 1\}$ , then from Eq. (1) we obtain:

$$P(v) \propto e^{a^t v} \prod_{m=1}^M \cosh(b_m + (Wv)_m),$$

apart from a normalization constant.

As proposed in [14] it is possible to extend this approach to learn, or model, a quantum state or wavefunction instead of a probability distribution. For this, let us consider a many body system with  $N$  subsystems, each with  $R$  levels  $\{|s\rangle\}_{s=1, \dots, R}$ . A computational basis for the whole system can be given by the product states  $|s_1, s_2, \dots, s_N\rangle = \otimes_{k=1}^N |s_k\rangle$ . Any many-body state  $|\Psi\rangle$  can then be considered as the mapping  $\Psi(s_1, \dots, s_N) \equiv \langle s_1, \dots, s_N | \Psi \rangle$  of each of these product states to a complex probability amplitude. Thus, by associating each variable  $s_k$  with a visible node of a RBM, we can model this mapping as:

$$\Psi(s) \propto \sum_{h_1, \dots, h_M} e^{a^t s + b^t h + h^t W s} \quad (2)$$

where  $s = (s_1, \dots, s_N)^t$ , and the parameters  $a$ ,  $b$ , and  $W$  are allowed to be complex. As in the previous equation, in the rest of this work we will describe quantum states up to an unspecified normalization constant. It is not required to know this constant since in order to estimate the expectation value of physical quantities for a given state we employ stochastic methods which only need to evaluate the ratios  $\Psi(s)/\Psi(s')$  for different configurations  $s$  and  $s'$ , as explained in detail in [14].

<sup>5</sup> For example, the Kullback-Leibler divergence[22].

In the following we will focus in the case in which each subsystem is a two-level system (i.e., a spin-1/2 or a qubit), so that the visible nodes are also binomial variables and each  $s_k$  can only take the values  $\{-1, 1\}$  (i.e.,  $R = 2$ ). We will refer to quantum states written as in Eq. (2) as a RBM-NNS.

### A. Action of one-body operations

Let us consider a linear operation  $U^{(j)}$  acting on the Hilbert space of subsystem  $j$  (a single spin-1/2). How does this operation act on a given RBM-NNS? First, we note that if  $U_{s,s'}^{(j)} = \langle s | U^{(j)} | s' \rangle$  are the matrix elements of  $U^{(j)}$ , and  $\Psi(s)$  is an arbitrary wavefunction, then the wavefunction corresponding to the state  $|\Psi'\rangle = U^{(j)} |\Psi\rangle$  is:

$$\Psi'(s) = \sum_{s'_j} U_{s,s'_j}^{(j)} \Psi(s_1, \dots, s_{j-1}, s'_j, s_{j+1}, \dots, s_N).$$

We will assume for the moment that the matrix elements of the operation  $U^{(j)}$  can be expressed as:

$$U_{s,s'}^{(j)} = A e^{\alpha s + \beta s' + \omega s s'}, \quad (3)$$

with complex parameters  $\alpha$ ,  $\beta$  and  $\omega$ . If the operation  $U^{(j)}$  is required to be unitary, then the value of the constant  $A$  should be such that  $\det(U^{(j)}) = 1$ . As we explain later, up to a global phase, any spin-1/2 unitary can be described in this way. However, this parametrization also allows for non-unitary operations. Then, if the initial state  $|\Psi\rangle$  is a RBM-NNS with  $M$  hidden variables and parameters  $a$ ,  $b$  and  $W$ , we have:

$$\begin{aligned} \Psi'(s) = & \sum_{h_1, \dots, h_M, s'_j} \exp \left( \sum_{n \neq j} a_n s_n + \sum_i b_i h_i + \sum_i \sum_{n \neq j} h_i W_{i,n} s_n \right) \\ & \times A \exp \left( a_j s'_j + \sum_i h_i W_{i,j} s'_j + \alpha s_j + \beta s'_j + \omega s_j s'_j \right). \end{aligned}$$

This last expression is already written in a form that suggest us to consider the sum index  $s'_j$  as a new hidden node. Indeed,  $\Psi'(s)$  can be expressed as:

$$\Psi'(s) = A \sum_{h_1, \dots, h_{M+1}} e^{\tilde{a}^t s + \tilde{b}^t \tilde{h} + \tilde{h}^t \tilde{W} s + \tilde{h}^t \tilde{X} \tilde{h} / 2}, \quad (4)$$

which is similar to Eq. (2) but with an additional term  $\tilde{h}^t \tilde{X} \tilde{h} / 2$  describing interactions between hidden nodes. In the previous expression, the updates to the original vectors  $a$ ,  $b$  and  $h$ , denoted  $\tilde{a}$ ,  $\tilde{b}$  and  $\tilde{h}$ , are

$$\begin{aligned} \tilde{a} &= (a_1, \dots, a_{j-1}, \alpha, a_{j+1}, \dots, a_N)^t, \\ \tilde{b} &= (b_1, \dots, b_M, \beta + a_j)^t, \\ \tilde{h} &= (h_1, \dots, h_M, h_{M+1})^t, \end{aligned} \quad (5)$$

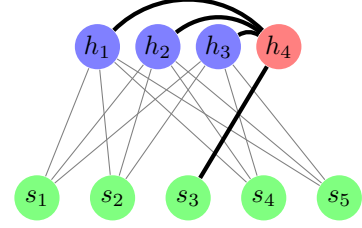


FIG. 2: Unrestricted Boltzmann machine with 5 visible units and 4 hidden units, resulting from applying a one-body operation to the spin represented by the visible node 3 of Fig. 1. The new hidden node is colored in red and the new connections are marked by thick lines.

and the new matrices  $\tilde{W}$  and  $\tilde{X}$  are given by

$$\begin{aligned} \tilde{W} &= \begin{pmatrix} | & & | & 0 & | & & | \\ W_{i,1} & \cdots & W_{i,j-1} & \vdots & W_{i,j+1} & \cdots & W_{i,N} \\ | & & | & 0 & | & & | \\ 0 & \cdots & 0 & \omega & 0 & \cdots & 0 \end{pmatrix}, \\ \tilde{X} &= \begin{pmatrix} 0 & \cdots & 0 & W_{1,j} \\ \vdots & \ddots & \vdots & W_{2,j} \\ & & & \vdots \\ 0 & \cdots & 0 & W_{M,j} \\ W_{1,j} & W_{2,j} & \cdots & W_{M,j} & 0 \end{pmatrix}. \end{aligned} \quad (6)$$

Therefore, the state  $|\Psi'\rangle = U^{(j)} |\Psi\rangle$  can be written as a NNS with one more hidden variable with respect to  $|\Psi\rangle$  and, more importantly, in terms of an *Unrestricted Boltzmann machine* (UBM) where the interaction between hidden variables is described by the matrix  $\tilde{X}$ . Figure 2 illustrates the new network giving rise to  $|\Psi'\rangle$ , Eq. (4), as if a one-body operation was applied to the subsystem 3 in Fig. 1. We see that the new hidden node only connects to the visible node corresponding to the subsystem where the operation was applied, and also, in principle, to all the preexistent hidden nodes.

We now turn to analyze the family of linear operations given by Eq. 3. First, we note that for  $U^{(j)}$  to be unitary  $\alpha$  and  $\beta$  must be imaginary, and  $\text{Im}(w) = \pi(n + 1/2)/2$  for any integer  $n$  (we take  $n = 0$  in what follows). Thus, we can rewrite Eq. (3) as:

$$U^{(j)} = \frac{e^{i\pi/4}}{\sqrt{2 \cosh(2\omega')}} \begin{pmatrix} e^{i(\alpha' + \beta')} e^{\omega'} & -i e^{i(\alpha' - \beta')} e^{-\omega'} \\ -i e^{-i(\alpha' - \beta')} e^{-\omega'} & e^{-i(\alpha' + \beta')} e^{\omega'} \end{pmatrix}. \quad (7)$$

In this form, the new parameters  $\alpha'$ ,  $\beta'$  and  $\omega'$  are real numbers. Up to a global phase, Eq. 7 is equivalent to any spin-1/2 unitary operation. It is particularly interesting to analyze the case of operations that are diagonal in the computational basis (rotations around  $\hat{z}$ ). We see that such operations are recovered in the limit  $\omega' \rightarrow +\infty$ . However, in that case the new hidden node  $h_{M+1}$  can be identified with  $s_j$  and eliminated (only the terms in which  $h_{M+1} = s_j$  survive when the sum in Eq. 4 is performed).

Thus, rotations around the  $\hat{z}$  axis of subsystem  $j$  can be implemented without adding new hidden nodes, and just updating the value of  $a_j$  according to the rule  $a_j \rightarrow a_j - i\theta/2$ , where  $\theta$  is the rotation angle. We also note that the representation of infinitesimal operations requires large values of  $\omega'$ . As an example, for the infinitesimal rotation  $U^{(j)} = \mathbb{1} - (i\theta/2)\sigma_j^x$  we have  $\omega' = (-1/2)\log(\theta/2)$ . This will be relevant for the analysis of the projection method presented in Section IV.

### B. UBM-NNSs and K-body operations

The results from the previous section motivate us to define an extended family of NNSs, in which the wavefunction is represented in terms of an Unrestricted Boltzmann Machine (UBM). In this case, internal connections in the groups of hidden and visible nodes are allowed. Then, we consider wavefunctions that can be written as:

$$\Psi(s) = \sum_{h_1, \dots, h_M} e^{a^t s + b^t h + h^t W s + h^t X h / 2 + s^t Y s / 2}, \quad (8)$$

where the vectors  $a$ ,  $s$ ,  $b$  and  $h$ , and the matrix  $W$  are defined as before, while the symmetric matrices  $X$  and  $Y$  contain the weights of the connections within the hidden and visible layer, respectively. They also have null diagonals (any non zero diagonal element on  $X$  or  $Y$  will not have any effect if all the nodes can only take the values  $\pm 1$ ). We will refer to states written in this way as UBM-NNS.

We are interested in finding a family of linear operations that can be efficiently applied to the previous states. We will first consider two-body operations, acting on subsystems  $j$  and  $k$ , such that their matrix elements  $U_{rs, r's'}^{(j,k)} = \langle r, s | U^{(j,k)} | r', s' \rangle$  can be expressed as:

$$U_{rs, r's'}^{(j,k)} = A \exp \left( \alpha^t q + \beta^t q' + \frac{1}{2} (q^t \ q'^t) \begin{pmatrix} 0 & \lambda & \Omega \\ \lambda & 0 & 0 \\ \Omega^t & 0 & \gamma \end{pmatrix} \begin{pmatrix} q \\ q' \end{pmatrix} \right) \quad (9)$$

where  $q = \begin{pmatrix} r \\ s \end{pmatrix}$ ,  $q' = \begin{pmatrix} r' \\ s' \end{pmatrix}$ ,  $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ ,  $\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ ,  $\lambda$  and  $\gamma$  are constants, and  $\Omega$  is a  $2 \times 2$  matrix. The parameters  $\alpha$ ,  $\beta$ ,  $\lambda$ ,  $\gamma$  and  $\Omega$  can in principle be complex valued. In the next section we explain that, at variance with the single qubit case, **not any unitary over two qubits can be written in this way.**

As in the previous section, it can be seen that if the wave function of the state  $|\Psi\rangle$  is given by Eq. (8), then the wavefunction of  $|\Psi'\rangle = U^{(j,k)} |\Psi\rangle$  can also be expressed as a UBM-NNS with new vectors:

$$\begin{aligned} \tilde{a} &= (a_1, \dots, a_{j-1}, \alpha_1, a_{j+1}, \dots, a_{k-1}, \alpha_2, a_{k+1}, \dots, a_N)^t \\ \tilde{b} &= (b_1, \dots, b_M, \beta_1 + a_j, \beta_2 + a_k)^t \\ \tilde{h} &= (h_1, \dots, h_M, h_{M+1}, h_{M+2})^t, \end{aligned} \quad (10)$$

and matrices:

$$\begin{aligned} \tilde{W} &= \begin{pmatrix} | & 0 & | & 0 & | & | \\ W_{i,1} & \dots & \vdots & W_{i,j+1} & \dots & \vdots & W_{i,k+1} & \dots & W_{i,N} \\ | & 0 & | & 0 & | & | \\ Y_{j,1} & \dots & \Omega_{11} & Y_{j,j+1} & \dots & \Omega_{12} & Y_{j,k+1} & \dots & Y_{j,N} \\ Y_{k,1} & & \Omega_{21} & Y_{k,j+1} & & \Omega_{22} & Y_{k,k+1} & & Y_{k,N} \end{pmatrix}, \\ \tilde{X} &= \begin{pmatrix} & & & W_{1,j} & W_{1,k} \\ & & & W_{2,j} & W_{2,k} \\ & & X & \vdots & \vdots \\ & & & W_{M,j} & W_{M,k} \\ W_{1,j} & W_{2,j} & \dots & W_{M,j} & 0 & \gamma + Y_{j,k} \\ W_{1,k} & W_{2,k} & \dots & W_{M,k} & \gamma + Y_{j,k} & 0 \end{pmatrix}, \\ \tilde{Y}_{n,n'} &= \begin{cases} Y_{n,n'} & n, n' \neq j, k \\ \lambda \delta_{n',k} & n = j \\ \lambda \delta_{n',j} & n = k \end{cases}. \end{aligned} \quad (11)$$

Thus, we see that under the action of the two-body operation given by Eq. 9, the resulting NNS can also be described by a UBM but with two additional hidden nodes.

We have focused on two-body operations but these results can be directly extended to the case of  $K$ -body operations. Thus, we can consider operations  $U$  acting on  $K$  two-level subsystems whose matrix elements  $U_{q_1 \dots q_K, q'_1 \dots q'_K} = \langle q_1, \dots, q_K | U | q'_1 \dots q'_K \rangle$  can be written as:

$$U_{q,q'} = A \exp \left( \alpha^t q + \beta^t q' + \frac{1}{2} (q^t \ q'^t) \begin{pmatrix} \Lambda & \Omega \\ \Omega^t & \Gamma \end{pmatrix} \begin{pmatrix} q \\ q' \end{pmatrix} \right) \quad (12)$$

where  $q = (q_1, \dots, q_K)^t$ ,  $q' = (q'_1, \dots, q'_K)^t$ ,  $\alpha$  and  $\beta$  are column vectors with  $K$  components, and  $\Lambda$ ,  $\Gamma$  and  $\Omega$  are  $K \times K$  matrices.  $\Lambda$  and  $\Gamma$  are symmetric with null diagonals. Operations in this family can be applied to any UBM-NNS by adding  $K$  hidden nodes and modifying the offsets and connections weights in a way that is a direct extension of Eqs. 10 and 11 for the case of two-body operations. We refer to operations that can be written as in Eq. 12 as Neural Network Operations (NNOs), since they can also be represented by a network of nodes or neurons with associated complex offsets and arbitrary connections between them (with complex weights), as is depicted in Figure 3. Note that in this case there are no hidden nodes, although composition of two or more NNOs leads to networks with hidden nodes.

In Appendix A it is shown that an operation  $U$  given by Eq. 12 will be unitary if and only if the followings conditions hold: (i) The components of  $\alpha, \beta, \Gamma$ , and  $\Lambda$  are purely imaginary, (ii) the matrix  $\Omega$  should have only one element different from zero in each row, and (iii) the imaginary part  $x$  of each of these elements should be such that  $\cos(2x) = 0$ . Thus, the number of independent real parameters is  $n_K = K^2 + 2K$ . This should be compared with the number  $m_K = 2^{2K} - 1$  of independent real parameters for arbitrary  $K$ -body unitaries (apart from global phases). For  $K = 1$  we have  $n_1 = m_1 = 3$ , as we



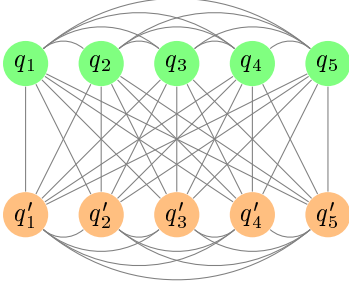


FIG. 3: Neural network representing an arbitrary five-body NNO. For given values of  $q_1, \dots, q_N$  and  $q'_1, \dots, q'_N$ , evaluation of the network gives the complex matrix element  $\langle q_1, \dots, q_N | U | q'_1, \dots, q'_N \rangle$ .

expect since we saw that any one-body unitary is a NNO. On the other hand, already for  $K = 2$  we have  $n_2 = 8$  and  $m_2 = 15$ . Thus, **only a restricted set of two-body unitaries can be cast as NNO**. However, **this restricted set includes entangling operations**. As a simple example, we note that for  $\alpha = \beta = 0$ ,  $\Gamma = 0$ ,  $\Lambda = i\lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ , and  $\Omega = i(2n+1)\pi/4 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ , we obtain the following two-body unitary:

$$U = \frac{1}{2} \begin{pmatrix} ie^{i\lambda} & e^{i\lambda} & e^{i\lambda} & -ie^{i\lambda} \\ e^{-i\lambda} & ie^{-i\lambda} & -ie^{-i\lambda} & e^{-i\lambda} \\ e^{-i\lambda} & -ie^{-i\lambda} & ie^{-i\lambda} & e^{-i\lambda} \\ -ie^{i\lambda} & e^{i\lambda} & e^{i\lambda} & ie^{i\lambda} \end{pmatrix}, \quad (13)$$

which takes product states into maximally entangled states for  $\lambda = \pi/4$ . **This operation and all the one-qubit operations form a universal set of quantum gates that can be expressed as NNOs**. From this, it follows that the resulting state of any quantum circuit with  $G$  one-qubit and two-qubit gates can be cast as a UBM-NNS with a number of hidden nodes that is linear in  $G$ , provided that the initial state is also a UBM-NNS. However, we note that the **direct application of a  $K$ -body operation could be more efficient, in terms of the number of hidden nodes added to the network, than decomposing it in terms of a set of one-body and two-body primitives**. Thus, it is interesting to investigate what kind of  $K$ -body operations can be expressed as NNOs.

As explained in section II, the action of one-body unitaries that are diagonal in the computational basis (rotations around  $\hat{z}$ ) can be implemented without adding new **hidden** nodes. The same happens for diagonal two-body unitaries. Indeed, the controlled rotation  $\exp(-i(\theta/2)\sigma_z^{(j)}\sigma_z^{(k)})$  can be obtained as a NNO in the limit  $\text{Re}(\Omega) \rightarrow +\infty$ , and can be implemented without adding hidden nodes and just updating the matrix  $Y$  according to  $Y_{j,k} \rightarrow Y_{j,k} - i\theta/2$ .

### III. SAMPLING OF UBM-NNSs

The previous results are interesting and promising, but are not directly useful. In fact, it is not clear how to extract information out of the representation given by Unrestricted Boltzmann Machines. In contrast to RBMs, it is not possible to analytically perform the sum over the hidden nodes of a UBM, since these nodes interact with each other. Therefore, for general UBM-NNSs,  $\Psi(s)$  cannot be evaluated in an efficient and exact way. Approximate solutions are in principle possible, but, as we will see, they suffer from the well known ‘sign problem’ of standard quantum Monte Carlo (QMC) techniques[21, 24].

In Eq. (8) the task is to evaluate the sum over the variables  $h_1, \dots, h_M$ . Without restricting in some way the matrix  $X$ , this is at least as hard as computing the partition function of an arbitrary classical Ising system (which is intractable[25]). An approximate numerical solution might be to employ a Metropolis-like sampling strategy, and only consider the terms of the sum with larger contributions. However, due to the ‘sign problem’, this approach can only be applied in a restricted family of problems. To explain this we will focus on a particular example: the determination of the ground state of the one dimensional transverse field Ising model (TFI-1D) via a Susuki-Trotter evolution in imaginary time. The elementary interactions of this model give rise to operations that can be easily represented as one-body and two-body NNOs. In this way, we will be able to explicitly construct a UBM-NNS that approximates the ground state of the model. The Hamiltonian of this model is:

$$H = -J \left( \sum_{j=1}^{N-1} \sigma_j^z \sigma_{j+1}^z + h \sum_{j=1}^N \sigma_j^x \right) \quad (14)$$

where  $J > 0$ , and we consider open boundary conditions. We want to prepare the ground state of this Hamiltonian via imaginary time evolution of the following initial state:  $|\Psi(0)\rangle = \otimes_{k=1}^N [(|-1\rangle + |1\rangle)/\sqrt{2}]$ , which can be considered as a RBM-NNS with  $N$  visible nodes,  $M = 0$  hidden nodes,  $a = 0$  and  $Y = 0$ . This state is ‘evolved’ in imaginary time  $t$  with the operator  $V(t) = e^{-tH}$ . If  $|\Psi(0)\rangle$  has a non-vanishing projection in the ground state subspace, then  $|\Psi(t)\rangle = V(t)|\Psi(0)\rangle$  will belong to that subspace for  $t \rightarrow +\infty$ . We can approximate the operator  $V(t)$  as a periodic circuit using the first order Susuki-Trotter approximation:

$$V(t) = e^{-tH} = \left( e^{-tH/S} \right)^S \simeq \left( \prod_{k=1}^{N-1} g_2(k) \prod_{k=1}^N g_1(k) \right)^S \quad (15)$$

where  $S$  is the total amount of steps and

$$\begin{aligned} g_1(k) &= e^{\tau J h \sigma_k^x} \\ g_2(k) &= e^{\tau J \sigma_k^z \sigma_{k+1}^z} \end{aligned} \quad (16)$$

and are one-body and two-body elementary operations, respectively ( $\tau = t/S$ ). A circuit representing this decomposition of  $V(t)$  is shown in Figure 4 for  $N = 4$  and  $S = 2$ .

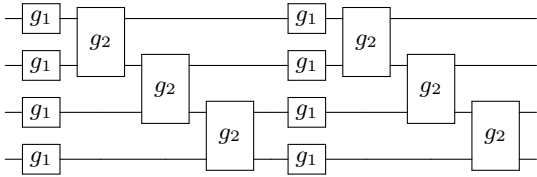


FIG. 4: Approximation of  $V(t)$  as a circuit for  $N = 4$  spins and  $S = 2$  Trotter steps.

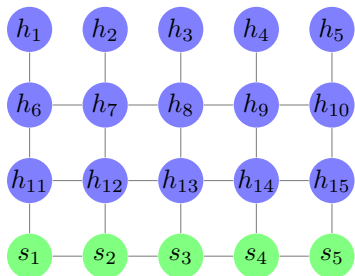


FIG. 5: UBM representing  $|\Psi(t)\rangle = V(t)|\Psi(0)\rangle$  for  $N = 5$  spins and  $S = 3$  Trotter steps. The application of each Trotter step adds a layer of hidden nodes to the network.

Figure 5 illustrates the UBM representing the final state after the application of the decomposition of Eq. (15) to  $|\Psi(0)\rangle$  for  $N = 5$  and  $S = 3$ . Since  $g_2$  is diagonal in the computational basis, it can be implemented without adding hidden nodes, as explained in the previous section. However, as shown in section II, each application of  $g_1$  adds one hidden node to the network. The new hidden nodes organize themselves in a two-dimensional structure with interactions between first neighbors. The weights of the vertical and horizontal edges are  $w_v = \log(\coth(\tau J h))/2$  and  $w_h = \tau J$ , respectively. Of course, we are just recovering the well known correspondence, or duality, between the TFI-1D model and the 2D classical anisotropic Ising model [20]. We see that the hidden nodes of the UBM representation of  $\Psi(t)$  act as the classical spins of the corresponding classical model. This can be generalized to more complex models in higher dimensions.

Now, in order to evaluate the probability amplitude  $\Psi(s, t)$  according to Eq. (8), we could implement an importance sampling strategy to numerically approximate the sum. As in usual QMC methods, that are also based on a quantum-classical correspondence, this will only work reliably, in principle, if the parameters  $b$ ,  $W$  and  $X$  in Eq. (8) are real, such that the factor  $\exp(b^t h + h^t W s + h^t X h/2)$  is always real and positive

for all  $h$  and  $s$ <sup>6</sup>. Otherwise the numerical ‘sign problem’ will hamper the accurate estimation of  $\Psi(s, t)$  [21, 24].

Thus, so far the situation is completely analogous to the one faced by standard QMC techniques: a dual classical system is constructed from a quantum Hamiltonian, and properties of the quantum model are obtained by stochastic sampling of the classical model, whenever it is free from the sign problem. However, the representation of quantum states via RBM suggest a way around this, as we explain in the next section.

#### IV. PROJECTING UBM-NNSs ONTO RBM-NNSs

As explained in section II, the sum over hidden nodes of a RBM can be performed analytically. Therefore, the amplitudes  $\Psi(s)$  corresponding to any RBM-NNS can be efficiently and exactly computed even for complex  $b$  and  $W$  (see Eq. (2)). RBM-NNSs are then free from the sign problem, since no importance sampling is necessary to evaluate  $\Psi(s)$ . This is true also if the RBM-NNS is allowed to have interactions between its visible nodes (i.e., if  $Y \neq 0$ ). Thus, in this section we consider an extended definition of RBM-NNS that allows for those interactions: a RBM-NNS is just a UBM-NNS with  $X = 0$ . It should be noted also that the interactions in the visible layer can be alternatively represented as mediated by additional hidden nodes (at most  $N(N-1)/2$ , one for each possible interaction in the visible layer) [16, 17].

When a RBM-NNS is subjected to a non trivial evolution, interactions between hidden nodes will appear and the resulting state will be described by a UBM-NNS for which, in general, no efficient and accurate way of computing  $\Psi(s)$  is available. One possible approach to avoid this problem is to continuously project the quantum state back to the family of RBM-NNSs during its evolution. In this section we explore a possible way to perform this projection.

In first place, we choose the set of all the one-body unitaries plus the controlled rotations  $\exp(-i(\theta/2)\sigma_z^{(j)}\sigma_z^{(k)})$  as a universal set of gates in terms of which we will decompose any global unitary operation. As we mentioned before, the controlled rotations can be implemented without adding hidden nodes and *without inducing interactions between the preexisting hidden nodes*. Therefore, when applying a given evolution (decomposed as a quantum circuit) to a NNS, hidden nodes will be added to the network only for one-body operations, and only then will interactions between the new and preexisting hidden nodes be induced (see Figures 1 and 2). However, if a one-body NNO is applied to a RBM-NNS (that, by def-

<sup>6</sup> This is in fact the case for the given example, but the imaginary time evolution in more complex models, or even the real time evolution in the TFI-1D model, leads to UBMs with complex  $X$ .

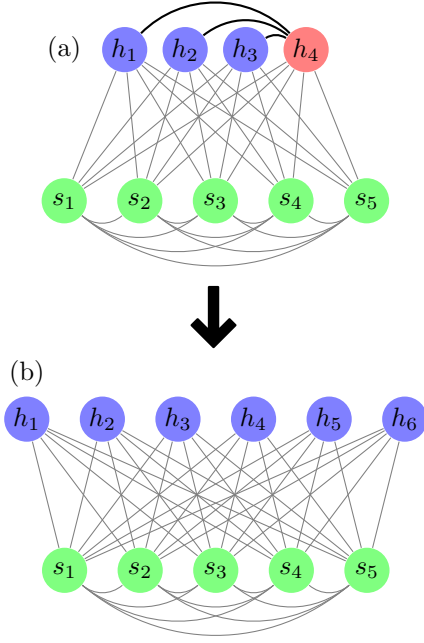


FIG. 6: Reduction of UBM-NNS (a) to a RBM-NNS (b). Note that the number of hidden nodes in (b) can be in principle larger than in (a), and the fidelity of the projection is expected to improve with more hidden nodes.

initiation, has no interactions between hidden nodes), the resulting state will be a UBM-NNS with a very special interaction structure among its hidden nodes: in principle all hidden nodes will interact with the newly added hidden node, but not between them. We consider the problem of projecting this new state back to the RBM-NNS family, as depicted in Figure 6. Given a procedure to perform this projection, then by applying it every time an one-body NNO is applied during the execution of a quantum circuit, the final state after the full evolution will be an easy to sample RBM-NNS. Of course, any projection procedure is expected to induce errors, i.e., the fidelity between the original and projected states will be less than unity. These errors will accumulate during the execution of a circuit, and this will severely limit the accuracy of the results.

In fact, it is important to note that a general and efficient solution to the proposed problem is not expected to exist. Previous works[17] have used complexity theoretical arguments showing that for many important classes of quantum states, e.g. those which are efficiently generated by quantum circuits, those which are representable by PEPS, and those which are ground states of  $k$ -local Hamiltonians, there exist instances which cannot be efficiently represented by RBMs (exactly nor to high precision). The existence of a RBM representation of such states would cause the collapse of the polynomial hierarchy ( $PH$ ) to the third level. In our approach we, seemingly, attempt to do more: we aim to efficiently find these representations, given quantum circuits as input.

The possibility of generically solving such a task has even more dramatic complexity-theoretic consequences, *e.g.*, if an algorithm converting between a given circuit, and the RBM representing the output state to exponential precision which is runnable in polynomial time were to exist, then  $\#P$  problems could be solved in polynomial time as well. This would imply a complete collapse of the polynomial hierarchy, *i.e.*  $P \supseteq PH^7$ , and, in particular  $P = NP = BQP$ . However, these arguments do not imply that no useful states have efficient RBM representations, or algorithms which construct them. Consequently, any heuristic method which attempts this, may be to a larger or smaller extent applicable to a given setting, which is one of the motivations of this work.

Since the structure of the hidden interactions in the states to be reduced is very simple (see Fig. 6-(a)), the sum over hidden nodes in Eq. 8 can still be performed analytically. Indeed, if there are  $M$  hidden nodes and it is considered, without loss of generality, that the last of them is the one that interacts with all the others, then:

$$\Psi(s) = e^{a^t s + s^t Y s/2} \times \left[ e^{b_M + w_M s} \prod_{k=1}^{M-1} 2 \cosh(w_k s + b_k + X_{k,M}) + e^{-b_M - w_M s} \prod_{k=1}^{M-1} 2 \cosh(w_k s + b_k - X_{k,M}) \right], \quad (17)$$

where  $w_k$  is the  $k$ -th row of  $W$ , and it was used that only the elements  $X_{k,M} = X_{M,k}$  of  $X$  are different from zero. Our goal is to approximate  $\Psi(s)$  with a RBM-NNS  $\Psi'(s)$ , for which the sum over the hidden nodes evaluates to:

$$\Psi'(s) = e^{a'^t s + s^t Y' s/2} \prod_{k=1}^{M'} 2 \cosh((W' s + b')_k), \quad (18)$$

for a number  $M'$  of hidden nodes and new parameters  $a'$ ,  $b'$ ,  $W'$ , and  $Y'$ . In principle it could be  $M' \geq M$ , since the quality of the approximation is expected to improve with larger  $M'$ . In fact, from the previous considerations about computational complexity, the number  $M'$  of new hidden nodes is expected to increase exponentially with  $N$  and  $M$  in the general case (for an exact mapping). Here, we propose two simple methods to perform the approximation of  $\Psi(s)$  by  $\Psi'(s)$ , and provide numerical evidence in favor of the feasibility of these approaches. We begin by rearranging the expression in Eq. 17 in the following way:

$$\Psi(s) = e^{a^t s + s^t Y s/2} 2 \cosh(\log(\chi(s))) \prod_{k=1}^{M-1} 2 f_k(s) \quad (19)$$

<sup>7</sup> This holds as  $P$  is self-low *i.e.*  $P^P \subseteq P$ . If  $\#P$  was solvable in poly-time, then  $P \supseteq PP$ , and since  $P^{PP} \supseteq PH$  (Toda's Theorem), we have that  $P$  contains  $PH$ .



where  $\chi(s)$  is given by:

$$\chi(s) = e^{b_M + w_M s} \prod_{k=1}^{M-1} \sqrt{\frac{\cosh(w_k s + b_k + X_{k,M})}{\cosh(w_k s + b_k - X_{k,M})}} \quad (20)$$

and

$$f_k(s) = \sqrt{\cosh(w_k s + b_k + X_{k,M}) \cosh(w_k s + b_k - X_{k,M})} \quad (21)$$

Equation 19 has a factored structure similar to a RBM-NNS wavefunction, as in Eq. 18. So far, no approximation has been made. The first method, explained in the following, consists in giving appropriate approximations for the functions  $\log(\chi(s))$  and  $f_k(s)$ .

### A. First method

Our goal is to take Eq. (19) to a form comparable to Eq. (18). A simple way to do that is to propose a linear approximation for the function  $\log(\chi(s))$  and to approximate each factor  $f_k(s)$  as the one corresponding to a single hidden node of a RBM-NNS. Explicitly, we propose to find new vectors  $w'_k$  and new constants  $b'_k$  such that:

$$f_k(s) \simeq c_k \cosh(w'_k s + b'_k) \quad (22)$$

for each  $k = 1, \dots, M-1$ , where  $c_k$  is an unimportant proportionality factor. Also, we approximate  $\log(\chi(s))$  as:

$$\log(\chi(s)) \simeq w'_M s + b'_M \quad (23)$$

for some vector  $w'_M$  and offset  $b'_M$ . In this way, the original state  $\Psi(s)$  given by Eq. (17) is approximated by a RBM-NNS with parameters  $a' = a$ ,  $Y' = Y$ ,  $b' = (b'_1, \dots, b'_M)^t$ , and a matrix  $W'$  with rows  $w'_1, \dots, w'_M$ . In Appendix B we give details about the numerical implementation of this method, which is based on the requirement that the proposed approximations in Eqs. 22 and 23 hold exactly when  $w_k s = \pm \|w_k\|_1$  and  $w_k s = 0$  ( $\|\cdot\|_1$  is the  $\ell_1$  vector norm). Here we discuss two limiting cases in which the proposed approximations can be explicitly found and are exact.

In first place we consider the limit of strong hidden connections in which  $|X_{k,M}| \gg |w_k s + b_k|$  for  $k = 1, \dots, M-1$ . To first order in this limit (that is, to first order in  $|X_{k,M}|^{-1}$ ) equation (23) holds exactly with parameters  $w'_M$  and  $b'_M$  given by:

$$w'_M = w_M + \sum_{k=1}^{M-1} \tanh(X_{k,M}) w_k, \quad (24)$$

and

$$b'_M = b_M + \sum_{k=1}^{M-1} \tanh(X_{k,M}) b_k, \quad (25)$$

while the parameters  $b'_k$  and  $w'_k$  for  $k = 1, \dots, M-1$  vanish. Thus, in this limit all the hidden nodes in the original state are condensed into a single one. This is natural, since for large  $|X_{k,M}|$  hidden nodes  $M$  and  $k$  are highly correlated.

Now we analyze the opposite limit of weak hidden connections, i.e.,  $|X_{k,M}| \ll |w_k s + b_k|$  for  $k = 1, \dots, M-1$ . This time, to first order in  $|X_{k,M}|$  we have  $f_k(s) = \cosh(w_k s + b_k)$ . Therefore, Eq. (22) holds with  $b'_k = b_k$  and  $w'_k = w_k$  for  $k = 1, \dots, M-1$ . Thus, the first  $M-1$  hidden nodes retain their original parameters. However, also to first order in  $|X_{k,M}|$ , the function  $\log(\chi(s))$  is given by:

$$\log(\chi(s)) \simeq w'_M s + b'_M + \sum_{k=1}^{M-1} X_{k,M} \tanh(w_k s + b_k) \quad (26)$$

Thus, we see that the condition of weak hidden connections is not enough to assure that a linear approximation for the function  $\log(\chi(s))$  holds. In principle, it is also necessary to assume that the components of  $w_k$  are small, so that a linear approximation to each function  $\tanh(w_k s + b_k)$  in the sum of the last equation can be given. In particular, to first order in  $w_k$  and  $b_k$ , we obtain the following expressions for the parameters of the hidden node  $M$ .

$$w'_M = w_M + \sum_{k=1}^{M-1} X_{k,M} w_k, \quad (27)$$

and

$$b'_M = b_M + \sum_{k=1}^{M-1} X_{k,M} b_k. \quad (28)$$

Then, under the conditions mentioned above, the parameters of the hidden node  $M$  in the obtained RBM-NNS are updated by small contributions of the other hidden nodes connected to it in the original UBM-NNS.

We test this projection method on randomly generated NNSs. For this, we consider NNSs with the same number  $N = M = 12$  of visible and hidden nodes. The components of the matrix  $W$  (or equivalently, of the vectors  $w_k$ ) are selected from a uniform distribution in the interval  $[-w, w]$ . The hidden connections  $X_{k,M}$  are selected from a uniform distribution in the interval  $[-x, x]$ . All the other parameters ( $a, b$ , and  $Y$ ) are zero. We fix  $w = 1/5$  and let  $x$  take values between 0 and 1. For each value of  $x$ , we generate 200 random NNSs and calculate the average fidelity  $\mathcal{F}$  between each generated state and the one obtained after the projection. This is done exactly so we limit to  $N = 12$ . Figure 7 shows the average infidelity  $\mathcal{I} = 1 - \mathcal{F}$  for three different versions of the method: i) the numerical implementation of the method explained in Appendix B ii) the limit of weak hidden connections, and iii) the limit of strong hidden connections. We see that in the first two cases the infidelity of the projection is indeed low for small  $x$ , but the numerical version of

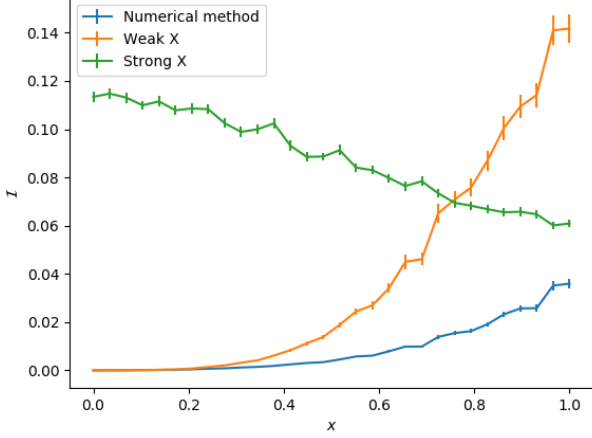


FIG. 7: Mean value of the infidelity  $\mathcal{I}$  as a function of the  $x$ , for  $w = 1/5$ ,  $N = 12$  and 200 randomly generated states for each point. The error bars indicate the uncertainty in the mean values. The blue line corresponds to the numerical implementation of the method, while the orange and green lines correspond to the projection rules obtained in the limit of weak and strong hidden connections, respectively.

the method is more robust for higher  $x$ . Also, the infidelity for the projection rules obtained for strong hidden connections decreases for increasing  $x$ , as expected.

As a final remark, we note that, as explained at the end of Section II, the action of infinitesimal one-body operations will add hidden nodes to the network, and that the weight of their connections to the visible layer will increase as the given operations approach the identity. This compromises the assumption that the components of the vectors  $w_k$  are small, that led to the last couple of equations. The second method that is presented below attempts to directly take into account the action of infinitesimal operations.

## B. Second method: infinitesimal operations

The previous method provides an approximated RBM representation of the UBM-NNS obtained by the application of a one-body operation to an initial RBM-NNS. We now present a different method, in which the action of the given operation is taken into account by just updating the parameters of the initial RBM-NNS. Consequently, the number of hidden nodes remains constant. This method only considers the action of infinitesimal one-body operations. If we apply such an operation to a given RBM-NNS, one might expect to be able to approximate the resulting state (with reasonable fidelity) by a new RBM-NNS with slightly different parameters. We will see that this is indeed the case, at least under some conditions that we discuss in the following.

Thus, we consider a RBM-NNS with parameters  $a, Y, b$

and  $W$ , and the corresponding wavefunction  $\Psi(s)$ . If the parameters are modified as  $x' = x + \delta x$  (where  $x = a, Y, b$  or  $W$ ), then, to first order in the variations  $\delta x$ , the new wavefunction  $\Psi'(s)$  is given by the following expression:

$$\begin{aligned} \frac{\Psi'(s)}{\Psi(s)} &\simeq 1 + \delta a^t s + s^t \delta Y s / 2 + \delta b^t T(s) + T(s)^t \delta W s \\ &\equiv 1 + \mathcal{P}(s), \end{aligned} \quad (29)$$

where the column vector  $T(s)$  has components

$$T_j(s) = \tanh(b_j + (W s)_j). \quad (30)$$

Now, we consider also a third state, that results from applying an infinitesimal one-body operation to the original state  $\Psi(s)$ . We assume for simplicity that the operation in question is the infinitesimal rotation  $U = e^{-i\delta\theta\sigma_x/2} \simeq \mathbb{1} + A\sigma_x$ , with  $A = -i\delta\theta/2$ . If this operation is applied to spin  $k$ , the resulting wavefunction  $\Psi''(s) = \sum_{s'} \langle s_k | U | s'_k \rangle \Psi(s')$  can be expressed as:

$$\frac{\Psi''(s)}{\Psi(s)} = 1 + A e^{-2a_k s_k - 2s_k (Y s)_k} C_k D_k(s) \equiv 1 + \mathcal{Q}(s), \quad (31)$$

where the factor  $C_k = \prod_{j=1}^M \cosh(2W_{j,k})$  is independent of  $s$  and the factor  $F_k(s)$  is given by:

$$D_k(s) = \prod_{j=1}^M (1 - T_j(s) \tanh(2W_{j,k} s_k)). \quad (32)$$

We now compute the fidelity between states  $\Psi'(s)$  and  $\Psi''(s)$ , given by  $\mathcal{F} = |\langle \Psi' | \Psi'' \rangle| / \sqrt{\langle \Psi' | \Psi' \rangle \langle \Psi'' | \Psi'' \rangle}$ . A rather long but straightforward calculation shows that to first non-trivial order in  $\mathcal{P}$  and  $\mathcal{Q}$  (defined in Eqs. (29) and (31)), the fidelity satisfies:

$$\mathcal{F}^2 = 1 - \text{Var}(\mathcal{P} - \mathcal{Q}), \quad (33)$$

where  $\text{Var}(\mathcal{X}) = \langle (\mathcal{X}^* - \langle \mathcal{X}^* \rangle)(\mathcal{X} - \langle \mathcal{X} \rangle) \rangle$  and the mean values are calculated according to the probability distribution given by the original wavefunction  $\Psi(s)$ :

$$\langle \mathcal{X} \rangle = \frac{1}{\sum_{s'} |\Psi(s')|^2} \sum_s |\Psi(s)|^2 \mathcal{X}(s). \quad (34)$$

Thus,  $\text{Var}(\mathcal{P} - \mathcal{Q})$  can be interpreted as the expected variance in the state  $\Psi(s)$  of an (in general non-hermitian) operator which is diagonal in the computational basis, with diagonal elements  $\mathcal{P}(s) - \mathcal{Q}(s)$ .

In order to take into account the action of the operation  $U$  on spin  $k$  by updating the parameters  $a, Y, b$  and  $W$  of the original state, we should select the updates  $\delta a$ ,  $\delta Y$ ,  $\delta b$  and  $\delta W$  that minimize  $\text{Var}(\mathcal{P} - \mathcal{Q})$  (and thus maximize the fidelity  $\mathcal{F}$ ). This is again a complex and highly non-linear optimization problem. However, as we explain below, it can be easily solved to first order in the

parameters  $a, Y$  and  $W$ . In that regime, it is possible to select the updates  $\delta x$  in such a way that  $\mathcal{P}(s) - \mathcal{Q}(s)$  is a constant independent of  $s$ , and therefore has no variance (up to the considered order). This is done as follows. To first order in  $a$  and  $Y$  we can approximate the first non trivial factor appearing in Eq. (31) as:

$$e^{-2a_k s_k - 2s_k (Ys)_k} \simeq 1 - 2a_k s_k - 2s_k (Ys)_k + \dots \quad (35)$$

Also, to first order in the parameters  $W_{j,k}$ :

$$D_k(s) \simeq 1 - \sum_{j=1}^M T_j(s) \tanh(2W_{j,k}) s_k + \dots \quad (36)$$

Introducing these expansions back in Eq. (31) and comparing the result to Eq. (29) we can see how to select the updates  $\delta a$ ,  $\delta Y$ ,  $\delta b$  and  $\delta W$  in order for  $\mathcal{P}(s) - \mathcal{Q}(s)$  to be independent of  $s$ . The results are:

$$\begin{aligned} \delta a_j &= -2AC_k \delta_{j,k} a_k \\ \delta Y_{i,j} &= -2AC_k (\delta_{i,k} Y_{k,j} + \delta_{j,k} Y_{k,i}) \\ \delta b_j &= 2AC_k \tanh(2W_{j,k}) a_k \\ \delta W_{i,j} &= 2AC_k \tanh(2W_{i,k}) (Y_{k,j} - \delta_{k,j}/2) \end{aligned} \quad (37)$$

We stress that the above update rules are only expected to be useful in the regime where the parameters  $a, Y$  and  $W$  are sufficiently small. Also, the perturbative treatment is not consistent, since not all second order terms were considered but only the bilinear ones. Proper analytical consideration of higher order terms in Eqs. (35) and (36) might lead to better update rules valid on a wider regime, although in that case one also has to face the optimization problem of minimizing  $\text{Var}(\mathcal{P} - \mathcal{Q})$ . This will be explored in future works. In Appendix C the infidelity  $\mathcal{I}$  is explicitly evaluated to first non-trivial order in  $A$  and  $W$ , for the specific case in which  $a = 0$ ,  $b = 0$  and  $Y = 0$ .

### C. Remarks and comparisons between the two methods

The numerical implementation of the proposed approximations in the first method (method **I**) heavily rely on the assumption that the parameters  $\omega_k$  and  $X_{k,M}$  are real. The projection procedure can still be carried out for complex parameters, but it is not expected to offer a good approximation of the original state in that case. Consequently, this method is restricted to models already free from the sign problem. Be that as it may, this method can still be of practical relevance, since the RBM representation is more compact and easy to sample than a positive definite UBM-NNS.

In contrast, the update rules derived for infinitesimal operations (method **II**) are in principle valid for real as well for complex parameters. In the next section we show that they can be used to evolve states according to the TFI-1D Hamiltonian in real time, where complex

parameters for the instantaneous RBM-NNS are needed. They are however severely limited by the fact that they were derived only to first order in the original parameters. This, in turn, limits the total time to which states can be accurately evolved.

## V. OPTIMIZATION OF RBM-NNS

### A. Imaginary time evolution.

As a proof of concept, we apply the previously introduced ideas and methods to a simple problem: the approximation with a RBM-NNS of the ground state of the TFI-1D model. We will compare our results to those originally obtained in [14], where the same model was employed as a testbed, and its ground state was approximated with a RBM-NNS optimized via a Variational Monte Carlo algorithm.

We apply the Trotter evolution in imaginary time introduced in Section III, now with periodic boundary conditions. Thus, we repeatedly apply the following Trotter step:

$$S_\tau = \prod_{k=1}^N g_2(k) \prod_{k=1}^N g_1(k) \quad (38)$$

to the initial state  $|\Psi(0)\rangle = \otimes_{k=1}^N [(|-1\rangle + |1\rangle)/\sqrt{2}]$ . Again,  $g_1(k) = e^{\tau J h \sigma_k^x}$  and  $g_2(k) = e^{\tau J \sigma_k^z \sigma_{k+1}^z}$  (this time,  $N+1=1$  should be understood), and  $\tau$  is the time step (we take  $J=1$  in the following). We will first consider the projection method (**I**). Thus, after the application of each  $g_1$  operation, we project the resulting state back to a RBM-NNS according to the procedure explained in Appendix B.

After the application of a single Trotter step, the network representing the state gains  $N$  new hidden nodes. Thus, old hidden nodes could be deleted such that after each step the factor  $M/N$  does not exceeds some integer constant  $\alpha$  fixed before hand. This is the ‘hidden-variable density’ defined in [14]. However, the projection method **I** is such that for this model only the newest  $N$  hidden nodes remain connected to the visible layer, and therefore with this method the hidden node density  $\alpha$  is effectively always 1. With more sophisticated projection methods the value of  $\alpha$  could be selected at will.

We first consider an Ising chain with  $N = 40$  spins and a transverse field  $h = 0.5$  (which is sub-critical). In Figure 8 we show the mean value of energy per spin  $\epsilon = E/N$  for the RBM-NNS obtained after each Trotter step as a function of the total imaginary time, for different time steps. The mean value of the energy  $E$  was estimated stochastically in the same way as in [14]. We stress that this is only done to monitor the convergence of the method, but no information from this sampling is used to assist the optimization, and that the RBM-NNS after any number of steps can be obtained with no sampling at all. This is in fact the main advantage of

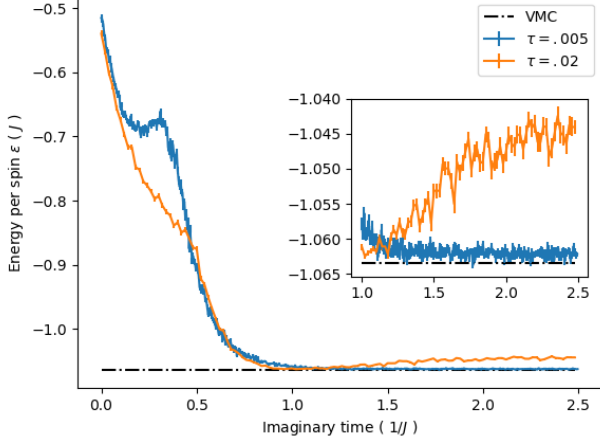


FIG. 8: Mean value of the energy per spin as a function of the total imaginary time, for two different values of the time step  $\tau$ . The parameters are  $N = 40$  and  $h = 0.5$ . The dashed black line correspond to the value of energy obtained via optimization of an RBM-NNS with VMC ( $\alpha = 1$ ) [26]. The inset shows in detail the last part of the optimization. (Projection method I).

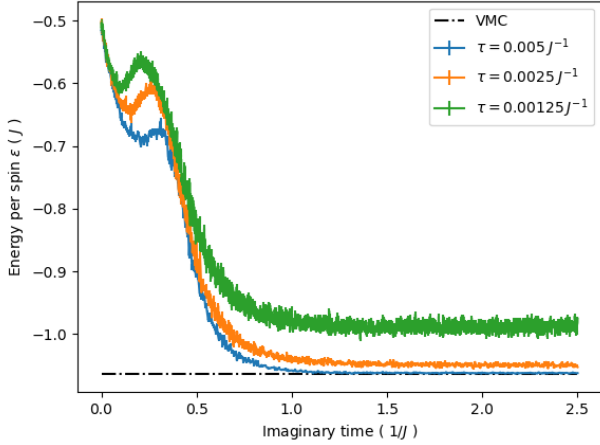


FIG. 9: Mean value of the energy per spin as a function of the total imaginary time, for decreasing values of the time step  $\tau$ . The parameters are the same that for Figure 8. (Projection method I).

the proposed method. We observe that in general the energy increases after reaching a minimum value (this is shown for  $\tau = 0.02J^{-1}$  but is also observed for higher values of  $\tau$ ). However, there are values of  $\tau$  for which this ‘rebound’ is not observed and the energy attains a minimum for large times (as shown for  $\tau \simeq 0.005J^{-1}$ ). In this last case, the mean value of the energy approaches that obtained with VMC [26], but does not improve it.

Figure 9 shows that not always a smaller value of the time step  $\tau$  produce better results. Thus, there seems

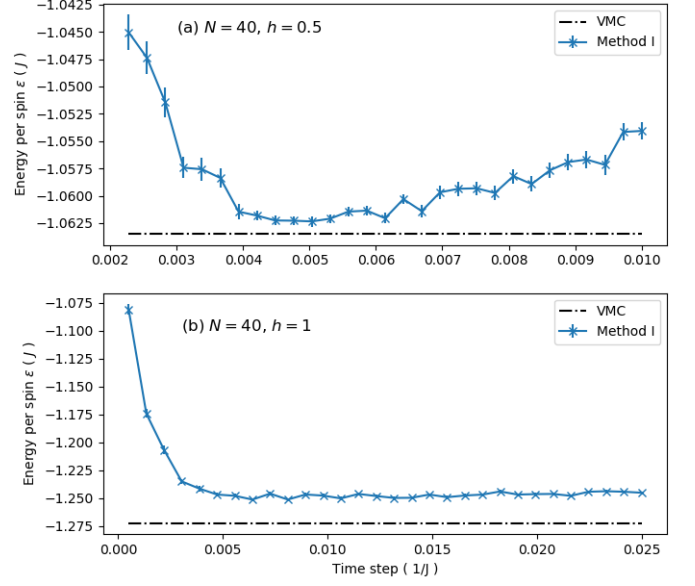


FIG. 10: Mean value of the energy per spin as a function of the time step, for a total imaginary time of  $t = 5$  (a), and  $t = 6$  (b). The dashed lines indicate the energy obtained via optimization of RBM-NNS with VMC in each case [26, 27]. (Projection method I).

to be an optimal value of  $\tau$ , for which the energy obtained for large times is minimum. This is also evident from Figure 10-(a), where the asymptotic value of the energy is plotted as a function of  $\tau$ . This behavior is due to the fact that the projection procedure introduces errors that are independent from the errors introduced by the discretization in the Suzuki-Trotter evolution. It can be qualitatively understood as follows. The errors introduced by the Suzuki-Trotter decomposition increase monotonically with the Trotter step  $\tau$ . Therefore, even if the projection method applied after each one-body operation were free from errors, the fidelity of the generated output states is expected to decrease as  $\tau$  increases. However, for low values of  $\tau$  the error of the projection method dominates over the error introduced by the Suzuki-Trotter discretization. This is so, as discussed in Section II, since the action of infinitesimal one-body operations requires strong connections between the hidden and visible layers. This, as it follows from the analysis of Section IV for weak hidden connections, is expected to reduce the fidelity of the projection method. Thus, there is a trade-off between these two sources of errors that determines an optimal value of the Trotter step  $\tau$ .

Figure 10-(b) shows that for the critical case in which  $h = 1$  the energy gap between our solution and that obtained with VMC increases. This fact might point out to some limitation of projection method I to deal with the long-range correlations present in the critical ground

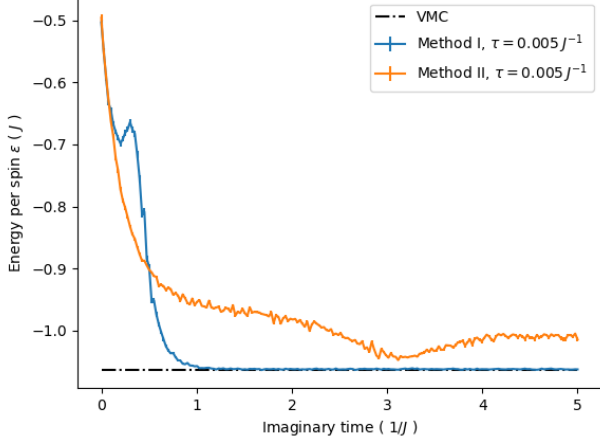


FIG. 11: Convergence of the energy optimization for the two projection methods **I** and **II**. The parameters are the same as in the Figure 8.

state, although this state has proven to be harder to approximate even with variational methods[14].

Now we turn to the consideration of the projection method **II** developed for infinitesimal one-body operations in Section IV B. In this case, we take into account the action of each one-body operation by just updating the parameters of the RBM-NNS according to Eqs. 37, without adding any hidden node. Thus, we need to provide an initial state with the desired number of hidden nodes. To motivate the choice of the initial state that is used in the following, we note that the two-body operations  $g_2(k) = e^{\tau J \sigma_k^z \sigma_{k+1}^z}$  can be exactly taken into account by adding a hidden node  $h$  which is connected to the visible nodes  $k$  and  $k+1$  with strength  $w = W_{h,k} = W_{h,k+1} = \text{arccosh}(e^{2\tau J})/2$ . Thus, we take as initial state a RBM-NNS with  $a = 0$ ,  $b = 0$ ,  $Y = 0$  and a matrix  $W$  with components  $W_{h,v} = w(\delta_{h,v} + \delta_{h+1,v})$ , for  $h = 1, \dots, N$  and  $N+1 \rightarrow 1$ . Then, the initial state has  $M = N$  hidden nodes ( $\alpha = 1$ ), each of which is connected with the same strength to two successive visible nodes, as if the operations  $g_2(k)$  ( $k = 1, \dots, N$ ) were already applied once to the state  $|\Psi(0)\rangle$  employed previously.

Figure 11 compares the convergence of method **I** and **II**. The two methods seems to follow the same curve for short times, although method **I** attains lower energies than method **II** for later times. In fact, for this last method the energy increases after reaching a minimum value.

We note that although the proposed methods are not able to improve the results obtained by Variational Monte Carlo, they can still be employed to efficiently obtain partially optimized states that can be afterward refined by stochastic methods. In this way, the total computational effort might be reduced, in comparison to a fully stochastic optimization.

## B. Real time evolution.

In all the above examples it was only necessary to deal with real parameters  $a, b, Y$  and  $W$  during the optimization of the RBM-NNSs. In fact, to study the ground state of the one-dimensional TFI model, we could have also sampled a ‘two-dimensional’ positive definite UBM-NNS wavefunction, as explained in Section III, since this model is free from the sign problem. To test if the projection method **II** is capable of dealing with RBM-NNSs with complex parameters, we study the real time evolution of the same model. In this case, complex parameters are necessary to track the time evolution.

Thus, we consider the Suzuki-Trotter decomposition of the unitary operator  $U(t) = e^{-itH}$ , which is the same as before (Eq. 38), this time in terms of elementary unitaries  $g_1(k) = e^{i\tau J h \sigma_k^x}$  and  $g_2(k) = e^{i\tau J \sigma_k^z \sigma_{k+1}^z}$ . As before we take an initial RBM-NNS with  $a = 0$ ,  $b = 0$ ,  $Y = 0$  and a matrix  $W$  with components  $W_{h,v} = w(\delta_{h,v} + \delta_{h+1,v})$ , for  $h = 1, \dots, N$  and  $N+1 \rightarrow 1$ . This time, however, we must take  $w = \text{arccosh}(e^{i2\tau J})/2$ . This initial state is (for small  $\tau$ ) a good approximation of the ground state of the considered model for  $h \rightarrow +\infty$ . We then evolve it in time with  $h = 2$  or  $h = 1/5$  and measure the expectation value  $\langle \sigma^x \rangle$  of the magnetization in the transverse direction as a function of time.

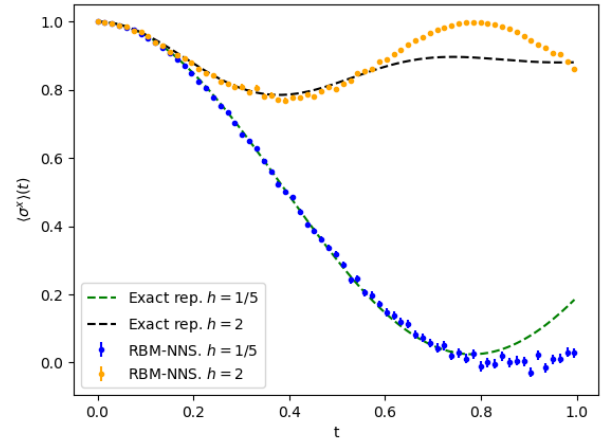


FIG. 12: Evolution of  $\langle \sigma^x \rangle$  for the two instantaneous quenches  $h = +\infty \rightarrow 2$  and  $h = +\infty \rightarrow 1/5$ . The number of spins is  $N = 24$  and the Trotter step is  $\tau = 0.005J^{-1}$ . Dashed lines correspond to the exact expectation value of  $\sigma^x$  obtained from the exact representation of the wavefunction, while dots indicate the stochastic estimates for the same quantity obtained by sampling the RBM-NNSs.

We compared the results with those obtained from the same Suzuki-Trotter evolution but using an exact representation of the wavefunction, to which the operations  $g_1(k)$  and  $g_2(k)$  are also applied exactly. For this reason we limit to  $N = 24$ . Figure 12 shows the evolution of  $\langle \sigma^x \rangle$  for the two quenches and the two representations



considered. For short times, the results obtained using RBM-NNSs and the update rules of method **II** qualitatively agree with the ones obtained from the exact representation.

## VI. DISCUSSION

In this work we considered the problem of evolving or optimizing Neural Network States based on Restricted Boltzmann Machines without relying on stochastic methods. Our aim was to identify simple and efficient update rules to take into account the action of elementary operations on quantum states, on the level of the neural network representation of those states. We showed that the application of very simple one-body unitaries to Neural Network States (NNSs) based on Restricted Boltzmann Machines (RBMs) motivates an extension of this class in order to include NNSs based on Unrestricted Boltzmann Machines (UBMs). We have parametrized a family of  $K$ -body operations that can be efficiently applied to states in this new class. We showed that there are universal quantum gates included in this family, and therefore that the action of any quantum circuit on a NNS can be efficiently represented by a new NNS with a number of new hidden nodes that grows linearly with the number of elementary operations in the circuit. This results are similar to the ones obtained recently in [17]. However, we give a more general parametrization of many body operations, which offer more freedom to choose a set of universal quantum gates in terms of which to decompose general quantum circuits. We also showed that the action of one-body rotations and two-body controlled rotations that are diagonal in the computational basis can be taken into account without leaving the family of RBM-NNSs.

As an application of our study of quantum operations, we investigated a procedure to optimize or evolve RBM-NNSs in such a way that the evolved state is still parametrized in this way. This procedure is based on the solution of a basic problem involving Boltzmann Ma-

chines: the reduction to a RBM of an UBM with only a single hidden node connected to the others. Two approximate methods to perform this reduction or projection were discussed. As a proof of concept, we applied these methods to the imaginary and real time evolution of the transverse field Ising model in one dimension. In the case of imaginary time evolution we compared our results with those obtained via Variational Monte Carlo (VMC) methods. Although the quality of the results offered by our methods is not comparable to the solutions obtained with VMC, we think that the proposed methods could be useful as a first stage in a global optimization. However, before applying these methods to more complex problems, it is necessary to perform a deeper study of their properties and limitations, in particular of the errors that they introduce. More elaborate and accurate projection methods might also be developed. In this work we limited ourselves to show that deterministic (as opposed to stochastic) optimization or evolution of RBM-NNSs is in principle possible.

While investigations of this kind are pragmatically motivated - we aim to find more efficient methods for computational physics - they could also establish connections between machine learning, computational complexity theory, and quantum physics. Further, they can serve as inspiration for novel quantum algorithms. For instance, we note that any UBM-NNS which can be reached starting from some simple initial state using a polynomial circuit is also a UBM whose distribution can be efficiently sampled on a quantum computer (but, likely, not classical). Therefore, this offers a route for identifying quantum algorithms for sampling Unrestricted of Deep Boltzmann Machines.

## VII. ACKNOWLEDGEMENTS

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## Appendix A: Unitary $K$ -body NNOs

In this appendix we give sufficient and necessary conditions for a  $K$ -body Neural Network Operation to be unitary. As explained in the main text, these operations are defined with respect to a given computational basis  $\{|q_1, \dots, q_K\rangle\}$  as those whose matrix elements  $U_{q_1 \dots q_K, q'_1 \dots q'_K} = \langle q_1, \dots, q_K | U | q'_1 \dots q'_K \rangle$  can be written as:

$$U_{q,q'} = A \exp \left( \alpha^t q + \beta^t q' + \frac{1}{2} (q^t \ q'^t) \begin{pmatrix} \Lambda & \Omega \\ \Omega^t & \Gamma \end{pmatrix} \begin{pmatrix} q \\ q' \end{pmatrix} \right) \quad (\text{A1})$$

where  $q = (q_1, \dots, q_K)^t$ ,  $q' = (q'_1, \dots, q'_K)^t$ ,  $\alpha$  and  $\beta$  are column vectors with  $K$  components, and  $\Lambda$ ,  $\Gamma$  and  $\Omega$  are  $K \times K$  matrices.  $\Lambda$  and  $\Gamma$  are symmetric with null diagonals.

If the matrix  $U$  is unitary then it must be  $U^\dagger U = U U^\dagger = \mathbb{1}$ . From Eq. A1 the diagonal elements of  $U^\dagger U$  are:

$$(U^\dagger U)_{q,q} = |A|^2 \exp \left( (\beta + \beta^*)^t q + \frac{1}{2} q^t (\Gamma + \Gamma^*) q \right) \times \sum_r \exp \left( (\alpha + \alpha^*)^t r + r^t (\Omega + \Omega^*) q + \frac{1}{2} r^t (\Lambda + \Lambda^*) r \right) \quad (\text{A2})$$

where  $r = (r_1, \dots, r_K)^t$ . If  $U^\dagger U = \mathbb{1}$ , then the previous expression should be independent of  $q$ . For this to happen it is clear that it should be  $\text{Re}(\beta) = \text{Re}(\Gamma) = 0$ . Applying the same condition to  $U U^\dagger$  it follows that  $\text{Re}(\alpha) = \text{Re}(\Lambda) = 0$ . Therefore, the previous expression can be reduced to:

$$(U^\dagger U)_{q,q} = |A|^2 \sum_r \exp \left( r^t (\Omega + \Omega^*) q \right) = \prod_{i=1}^K 2 \cosh \left( \sum_j (\Omega_{i,j} + \Omega_{i,j}^*) q_j \right) \quad (\text{A3})$$

This last expression will be independent of  $q$  if and only if the matrix  $\text{Re}(\Omega)$  has at most one element per row different from zero (since each component of  $q$  is just  $\pm 1$  and  $\cosh(x)$  is an even function). Finally, the non-diagonal elements of  $U^\dagger U$  are:

$$(U^\dagger U)_{q,s} = |A|^2 \exp \left( \beta^t (s - q) + \frac{1}{2} (s^t \Lambda s - q^t \Lambda q) \right) \times \prod_{i=1}^K 2 \cosh \left( \frac{1}{2} \sum_j (\Omega_{i,j} + \Omega_{i,j}^*) (q_j + s_j) + (\Omega_{i,j} - \Omega_{i,j}^*) (s_j - q_j) \right). \quad (\text{A4})$$

Now, since  $\text{Re}(\Omega)$  has only one element different from zero in each row, and since we are assuming  $q \neq s$ , at least one of the factors in the last line of the last equation is equal to:

$$2 \cosh \left( \frac{1}{2} \sum_j (\Omega_{i,j} - \Omega_{i,j}^*) (s_j - q_j) \right). \quad (\text{A5})$$

This factor will always vanish if  $2 \times \text{Im}(\Omega)$  has also only one element different from zero in each row, in the same positions of the non-zero elements of  $\text{Re}(\Omega)$ , and each of them is such that it cosine vanishes.

### Appendix B: Implementation of the first projection method

In this Appendix we give details about the numerical implementation of the first projection method.

*First part.* Given  $w_k$ ,  $b_k$ , and  $X_{k,M}$  for  $1 \leq k \leq M-1$ , we need to find  $c_k$ ,  $w'_k$  and  $b'_k$  such that the factor  $f_k(s)$  defined in Eq. (21) is approximated by  $c_k \cosh(w'_k s + b'_k)$ , as in Eq. (22). We consider  $w'_k = \beta w_k$ , for some constant  $\beta$ , so we just need to determine the constants  $c_k$ ,  $\beta$  and  $b'_k$ . We choose them imposing that the right and left hand side of Eq. (22) coincide for  $ws = 0$ ,  $ws = \|w\|_1$ , and  $ws = -\|w\|_1$  ( $\|\cdot\|_1$  is the  $\ell_1$  vector norm). This is done numerically with an iterative method.

*Second part.* We need now to give a linear approximation of the function  $\log(\chi(s))$ . From Eq. 20, we have:

$$\log(\chi(s)) = b_M + w_M s + \frac{1}{2} \sum_{k=1}^{M-1} g_k(s) \quad (\text{B1})$$

where we define  $g_k(s) = \log(\cosh(w_k s + b_k + X_{k,M})) - \log(\cosh(w_k s + b_k - X_{k,M}))$  for each  $1 \leq k \leq M-1$ . We approximate each of these functions as  $g_k(s) \simeq \alpha_k w_k s + b_k$ , and the constants  $\alpha_k$  and  $\beta_k$  are obtained by requiring the approximation to be exact for  $w_k s = \|w\|_1$  and  $w_k s = -\|w\|_1$ . Thus, the new parameters for the hidden node  $M$  are  $b'_M = b_M + \sum_{k=1}^{M-1} \beta_k$  and  $w'_M = w_M + \sum_{k=1}^{M-1} \alpha_k w_k$ .

### Appendix C: Estimation of the error for infinitesimal operations

In this section we evaluate the expression in Eq. (33) for the projection fidelity of method **I** in a particular case. We consider that the original state has parameters  $a = 0$ ,  $b = 0$ , and  $Y = 0$ , and we will analyze how the infidelity  $\mathcal{I} = 1 - \mathcal{F}$  scales for small  $W$ . From Eq. (33) it is clear that to first order in  $\mathcal{P}$  and  $\mathcal{Q}$  (defined in Section IV B) the infidelity satisfies:

$$\mathcal{I} \simeq \text{Var}(\mathcal{P} - \mathcal{Q})/2, \quad (\text{C1})$$

where  $\text{Var}(\mathcal{X}) = \langle (\mathcal{X}^* - \langle \mathcal{X}^* \rangle)(\mathcal{X} - \langle \mathcal{X} \rangle) \rangle$  and the mean values are calculated according to the probability distribution given by the original wavefunction  $\Psi(s)$ :

$$\langle \mathcal{X} \rangle = \frac{1}{\sum_{s'} |\Psi(s')|^2} \sum_s |\Psi(s)|^2 \mathcal{X}(s) \quad (\text{C2})$$

For the particular case mentioned above ( $a = 0, b = 0$ ,  $Y = 0$  and a given  $W$ ), we obtain the following expression for  $\mathcal{X}(s) = \mathcal{P}(s) - \mathcal{Q}(s)$ :

$$\mathcal{X}(s) = AC_k \left[ 1 + \sum_{m,n} T_m(s) T_n(s) \tanh(2W_{m,k}) \tanh(2W_{n,k}) \right] \quad (\text{C3})$$

where, as defined in Eq. (30),  $T_n(s) = \tanh((Ws)_n)$  (recall that we are considering  $b = 0$ ). When the components of  $W$  are sufficiently small we can approximate  $T_n(s) \simeq (Ws)_n$  and therefore:

$$\mathcal{X}(s) = AC_k \left[ 1 + 4 \sum_{i,j} (W^t W)_{i,k} (W^t W)_{j,k} s_i s_j \right] \quad (\text{C4})$$

Now we note that for  $W \rightarrow 0$  the distribution  $|\Psi(s)|^2$  is completely flat, i.e, all configurations  $s$  are equiprobable, and as a consequence we have  $\langle s_i s_j \rangle \simeq \delta_{i,j}$ . Thus we can readily evaluate  $\langle \mathcal{X} \rangle$  for small  $W$ , obtaining:

$$\langle \mathcal{X}(s) \rangle = AC_k [1 + 4(W^t W W^t W)_{k,k}] \quad (\text{C5})$$

Also, in the same limit we have  $\langle s_i s_j s_k s_l \rangle = \delta_{j,k} \delta_{l,m} + \delta_{j,l} \delta_{k,m} + \delta_{j,m} \delta_{k,l} - 2\delta_{j,k,l,m}$ , and we can use this identity to evaluate  $\langle \mathcal{X}^* \mathcal{X} \rangle$ . In this way we arrive at the final result:

$$\begin{aligned} \mathcal{I} &= \text{Var}(\mathcal{X})/2 = (\langle \mathcal{X}^* \mathcal{X} \rangle - \langle \mathcal{X}^* \rangle \langle \mathcal{X} \rangle)/2 \\ &= 16|AC_k|^2 \sum_{i \neq j} (W^t W)_{i,k} (W^t W)_{i,k}^* (W^t W)_{j,k} (W^t W)_{j,k}^* \end{aligned} \quad (\text{C6})$$

From this expression we see that if  $\lambda$  is the typical scale of the components of the matrix  $W$ , the infidelity scales as  $\mathcal{I} \propto |A|^2 \lambda^8 N^2$ . This scaling with  $N$  corresponds actually to the worst case scenario in which there is no notion of locality in the matrix  $W$  (i.e, a given hidden neuron is in principle connected to all visible nodes, and not only to a group of them of restricted size).