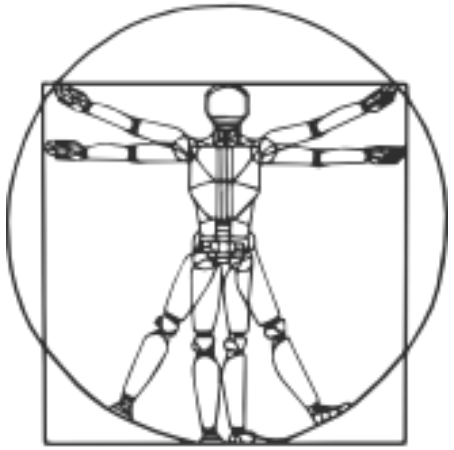
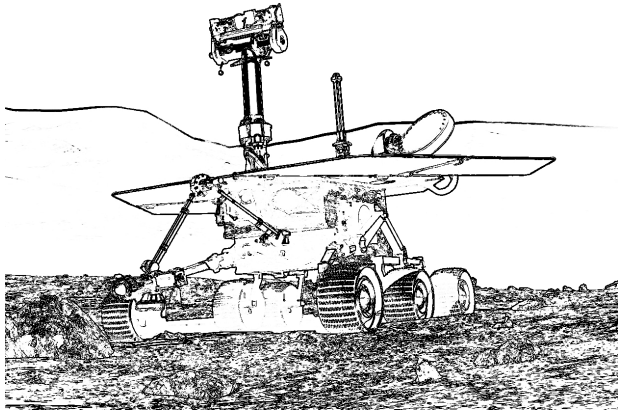




COMP417
Introduction to Robotics and Intelligent Systems

Lecture 17: Extended Kalman Filter

Adapted from an earlier version by Florian Shkurti @ U of T

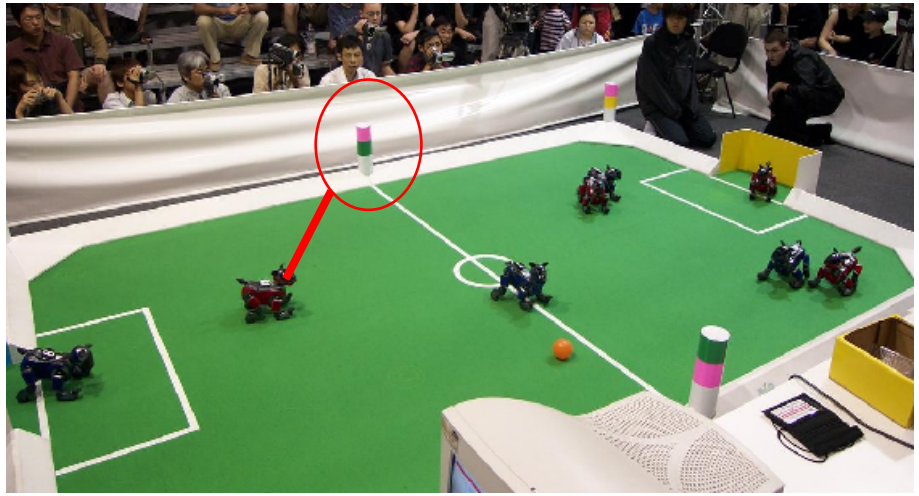
McGill

MRL Mobile Robotics Lab at McGill University

Beyond the KF Assumptions

- We used the linear forms of motion and measurement (e.g., z=Hx+noise) at several stages to make the equations easier
- But, this was not a good thing in general. Real robots almost all have ***nonlinear motions and measurements!***
  - E.g., the robots below sense the distance and angle to the landmarks (l) at the edge of the field
  - To form this equation, need trigonometry and square root of squares
  - These are not linear, so are we stuck?

$$z_t^{(i)} = h_i(x_t) = \begin{bmatrix} \sqrt{(p_x(t) - l_x^{(i)})^2 + (p_y(t) - l_y^{(i)})^2} \\ \text{atan2}(p_y(t) - l_y^{(i)}, \quad p_x(t) - l_x^{(i)}) - \theta(t) \end{bmatrix} + n_t$$



Plan for today

- Happily, we are not stuck!
- Let's extend the KF ideas to nonlinear models:
  - First look at what goes wrong exactly
  - Then the math to fix it
- We'll end with an algorithm called the Extended Kalman filter, which is the one that is actually used everywhere
  - Main math ideas come from KF, but extended to practical robots

Recommended reading

- Chapter 3.3 in Probabilistic Robotics for EKF
- Chapter 7.4 in Probabilistic Robotics for EKF-Localization
- Chapter 10.2 in Probabilistic Robotics for EKF-SLAM

Kalman Filter: an instance of Bayes' Filter

Assumptions guarantee that if the prior belief before the prediction step is Gaussian

$$
\begin{aligned}
bel(x_t) &= p(x_t|u_{0:t-1}, z_{0:t}) \\
&= \eta\, p(z_t|x_t) \int p(x_t|u_{t-1}, x_{t-1})\, bel(x_{t-1})\, dx_{t-1}
\end{aligned}
$$

then the prior belief after the prediction step will be Gaussian

and the posterior belief (after the update step) will be Gaussian.

Linear observations with Gaussian noise

$$z_t = Hx_t + n_t \quad \text{with noise } n_t \sim \mathcal{N}(0, R)$$

Linear dynamics with Gaussian noise

$$x_t = Ax_{t-1} + Bu_{t-1} + Gw_{t-1} \quad \text{with noise } w_{t-1} \sim \mathcal{N}(0, Q)$$

Initial belief is Gaussian

$$bel(x_0) \sim \mathcal{N}(\mu_0, \Sigma_0)$$

Kalman Filter: an instance of Bayes' Filter

Suppose you replace the linear models with nonlinear models.

Does the posterior $bel(x_t)$ remain Gaussian?

Assumptions guarantee that if the prior belief before the prediction step is Gaussian

$$bel(x_t) = p(x_t|u_{0:t-1}, z_{0:t})$$
$$= \eta\, p(z_t|x_t) \int p(x_t|u_{t-1}, x_{t-1})\, bel(x_{t-1})\, dx_{t-1}$$

then the prior belief after the prediction step will be Gaussian

and the posterior belief (after the update step) will be Gaussian.

Linear observations with Gaussian noise

$$z_t = h(x_t) + n_t$$
with noise $n_t \sim \mathcal{N}(0, R)$

Linear dynamics with Gaussian noise

$$x_t = f(x_{t-1}, u_{t-1}) + Gw_{t-1}$$
with noise $w_{t-1} \sim \mathcal{N}(0, Q)$

Initial belief is Gaussian

$$bel(x_0) \sim \mathcal{N}(\mu_0, \Sigma_0)$$

Kalman Filter: an instance of Bayes' Filter

Suppose you replace the linear models with nonlinear models.

Does the posterior $bel(x_t)$ remain Gaussian? NO

Assumptions guarantee that if the prior belief before the prediction step is Gaussian

$$bel(x_t) = p(x_t|u_{0:t-1}, z_{0:t})$$
$$= \eta\, p(z_t|x_t) \int p(x_t|u_{t-1}, x_{t-1})\, bel(x_{t-1})\, dx_{t-1}$$

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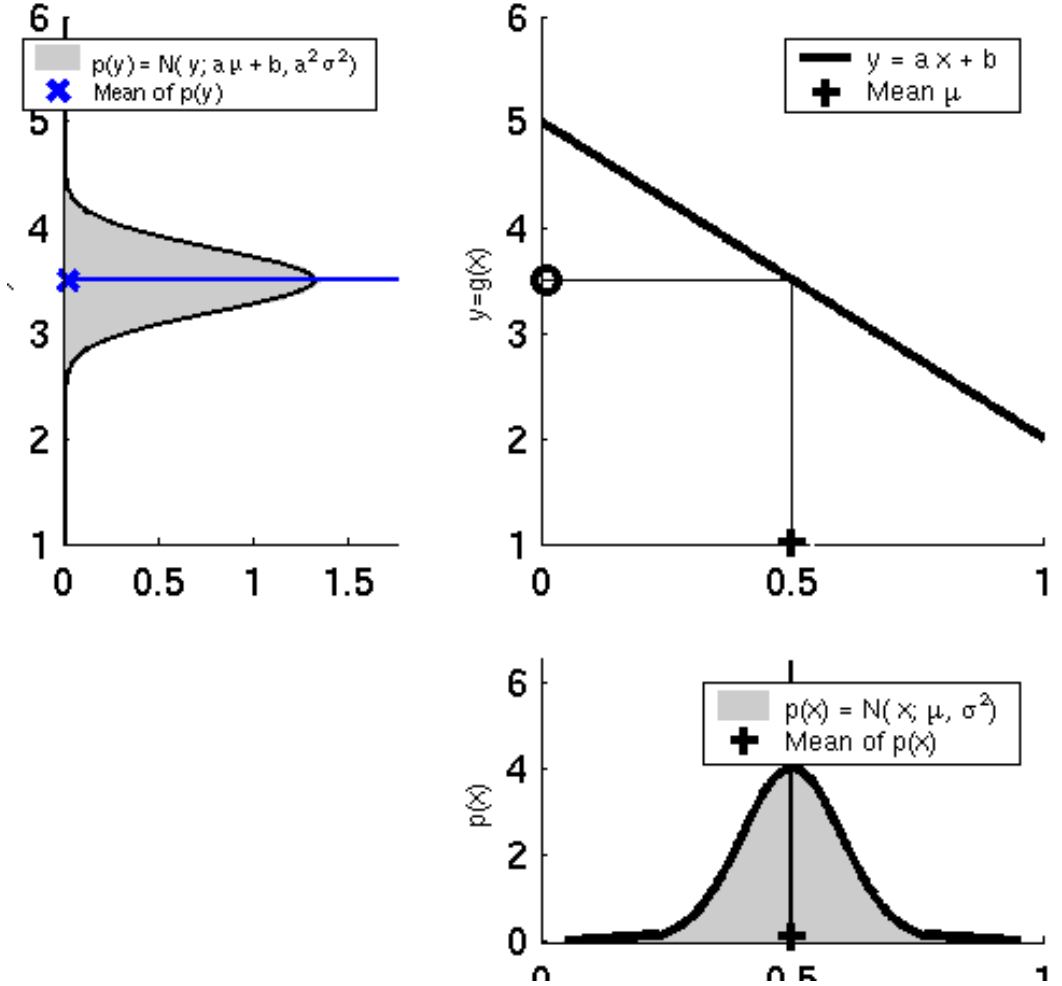
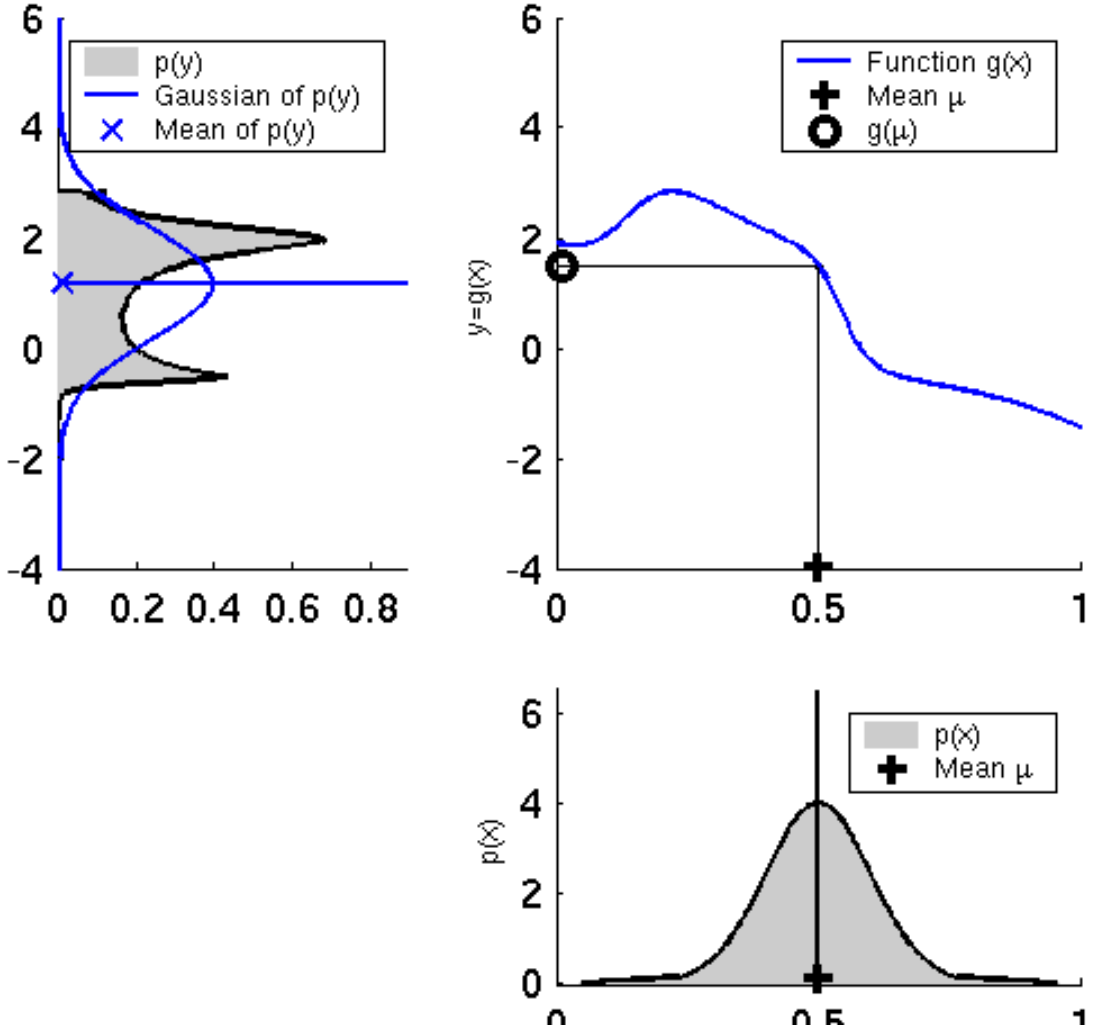
Linearity Assumption Revisited



If $y = ax + b$
and $x \sim \mathcal{N}(\mu, \sigma^2)$

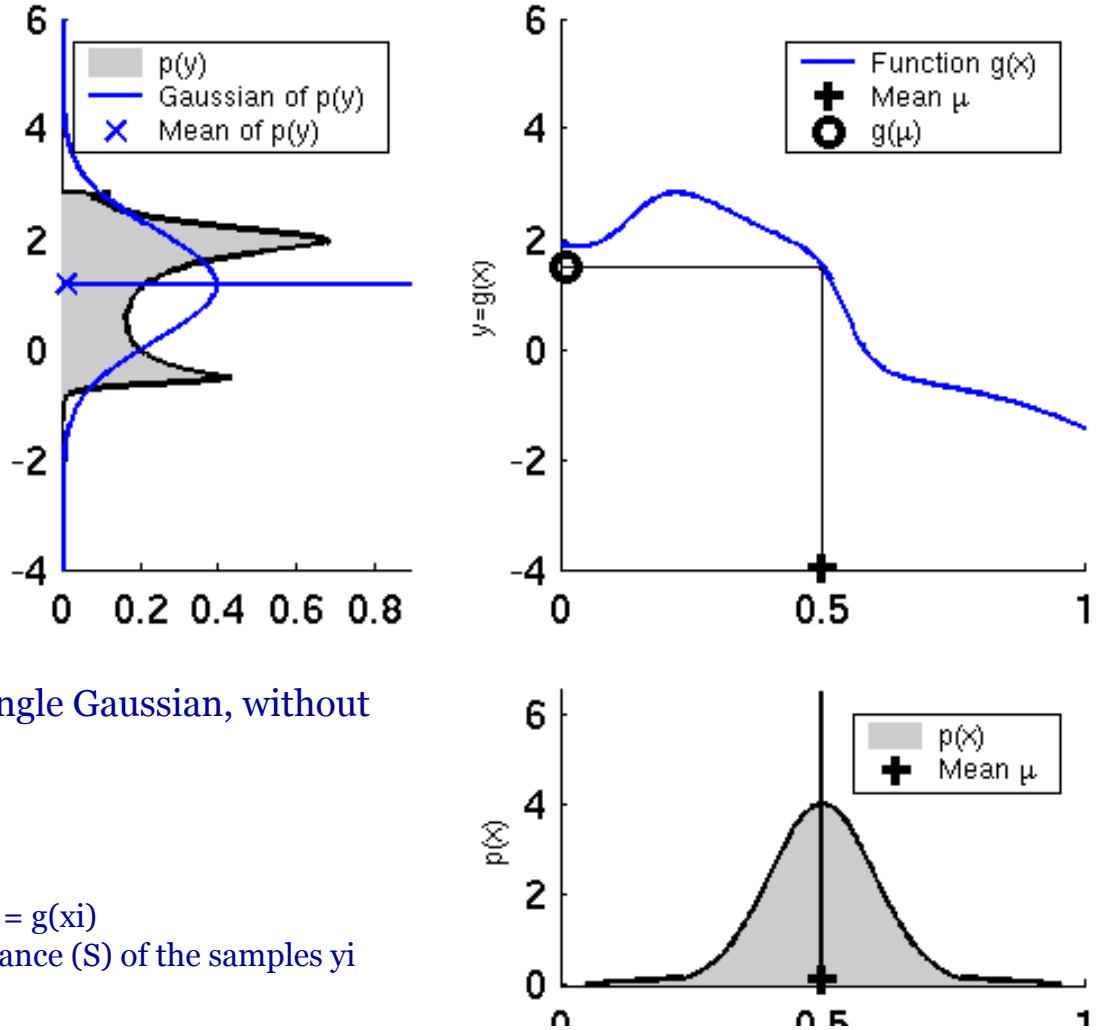
then $y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

Nonlinear Function



If $y = g(x)$
and $x \sim \mathcal{N}(\mu, \sigma^2)$
then y is not necessarily distributed as a Gaussian.

Nonlinear Function



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then y is not necessarily distributed as a Gaussian.

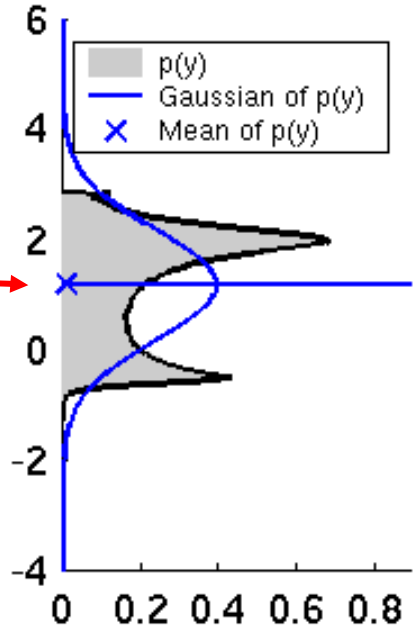
How can we approximate p(y) using a single Gaussian, without having a formula for p(y)?

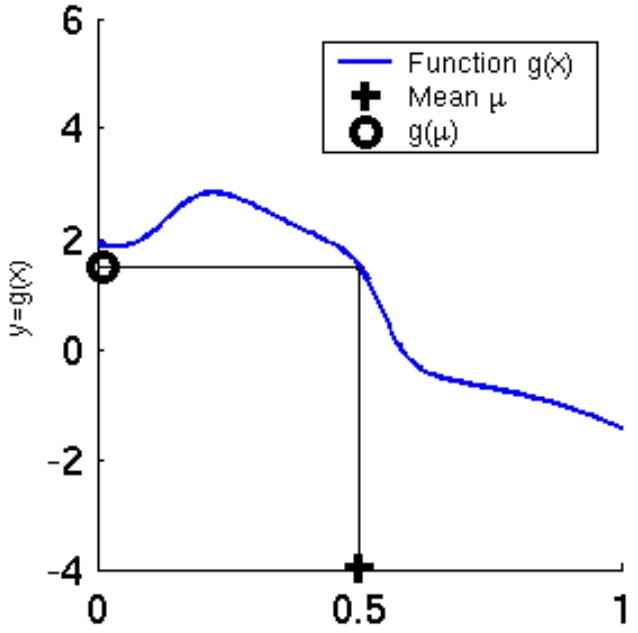
IDEA #1: MONTE CARLO SAMPLING
- Draw many (e.g. 10^6) samples xi ~ p(x)
- Pass them through the nonlinear function yi = g(xi)
- Compute the empirical mean (m) and covariance (S) of the samples yi
- Return Normal(m, S)

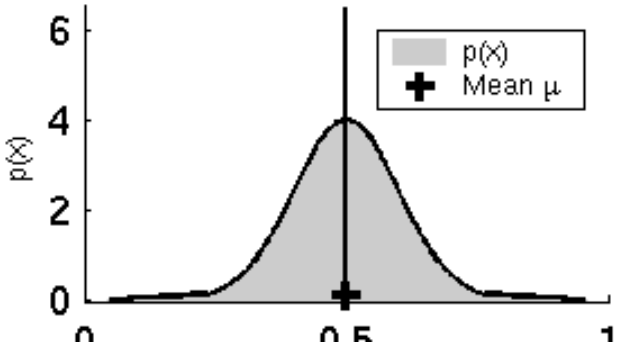
IDEA #2: LINEARIZE THE NONLINEAR FUNCTIONS f, h
- Then we are in the case y = Gx +c, so p(y) is a Gaussian

Nonlinear Function

This is how we computed the Gaussian approximation of p(y).







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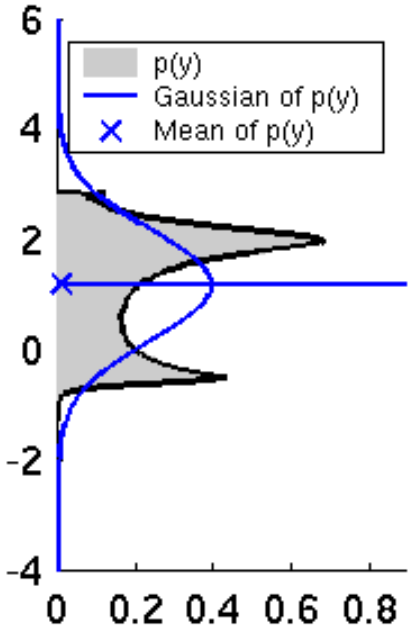
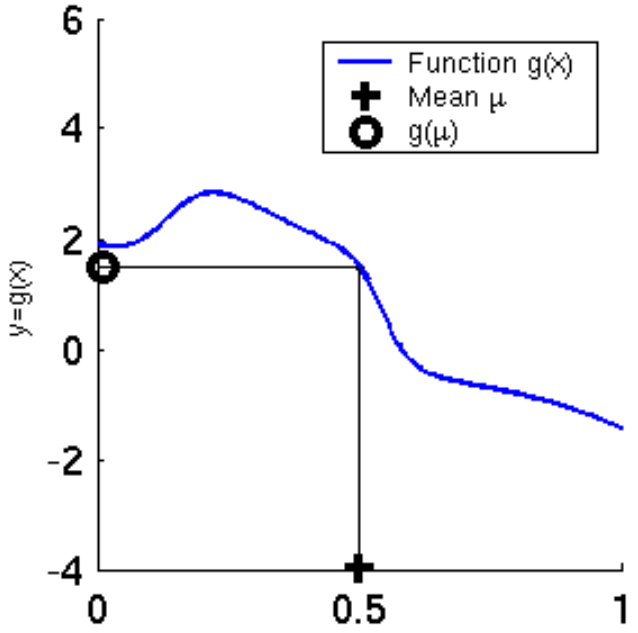
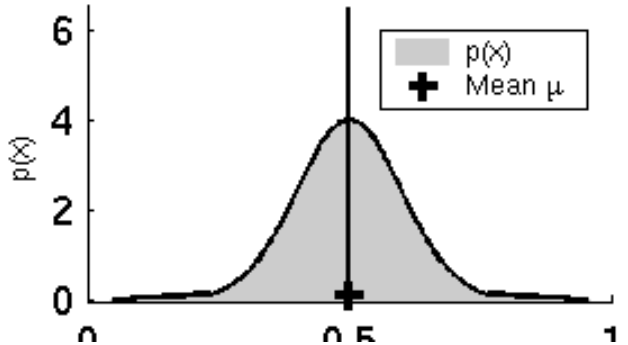
Nonlinear Function







If $y = g(x)$
and $x \sim \mathcal{N}(\mu, \sigma^2)$
then y is not necessarily distributed as a Gaussian.

Bad idea to use in Kalman Filter because we want to approximate p(y) efficiently.

How can we approximate p(y) using a single Gaussian, without having a formula for p(y)?

IDEA #1: MONTE CARLO SAMPLING
- Draw many (e.g. 10^6) samples xi ~ p(x)
- Pass them through the nonlinear function yi = g(xi)
- Compute the empirical mean (m) and covariance (S) of the samples yi
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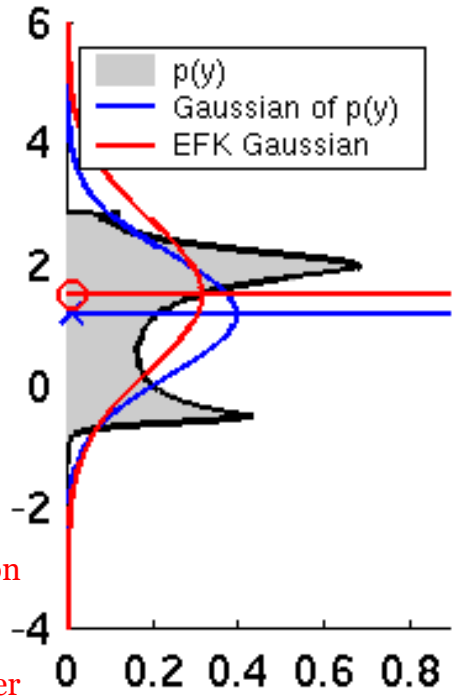
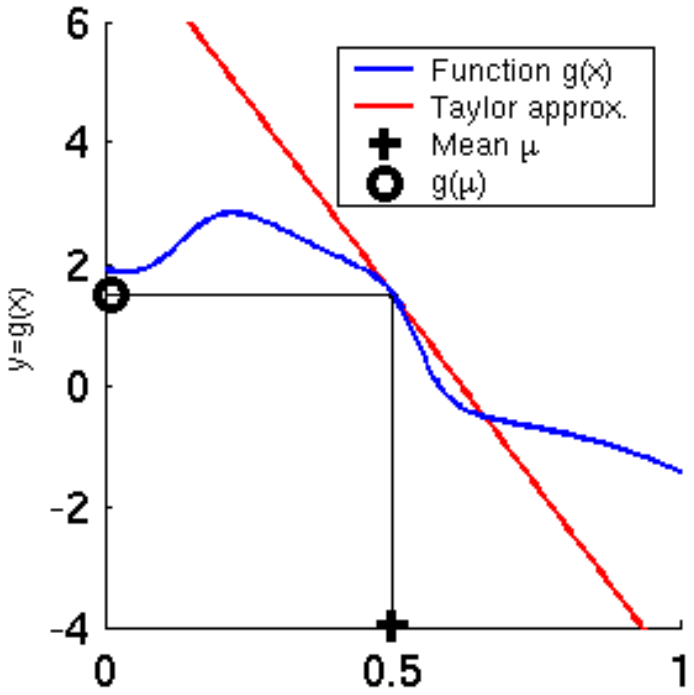
IDEA #2: LINEARIZE THE NONLINEAR FUNCTIONS f, h
- Then we are in the case y = Gx +c, so p(y) is a Gaussian

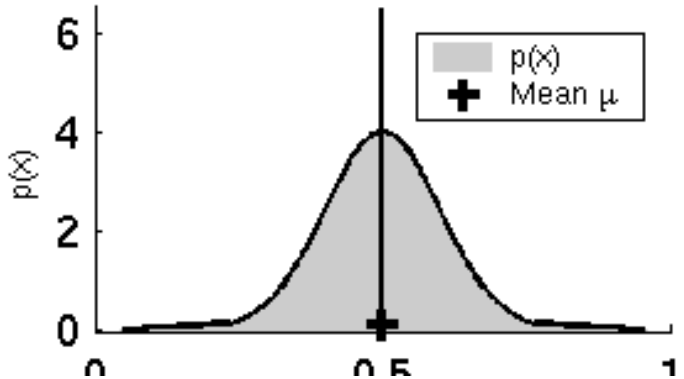
Linearization

Notice how the Linearization approximation differs from the Monte Carlo approximation (which is better, provided sufficiently many samples).

That said, the Linearization approximation can be computed efficiently, and can be integrated into the Kalman Filter estimator → Extended Kalman Filter





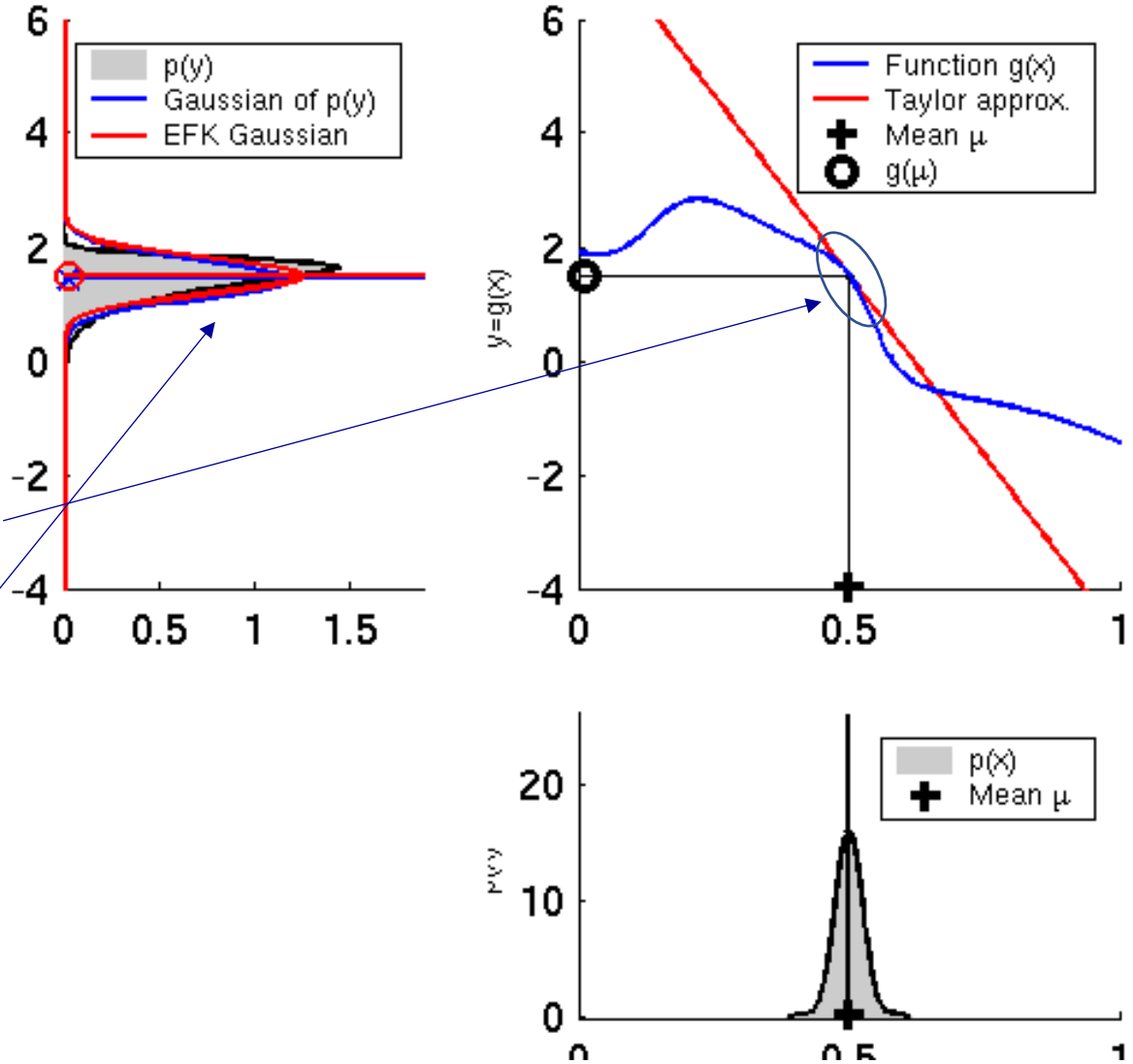


Linearization with low approximation error

The quality of the linearization approximation depends on the uncertainty of p(x) but also on the shape of the nonlinearity g(x).

In this example p(x) has small variance so most points will be concentrated around 0.5 and will pass through a very small region of g(x), where g(x) is close to linear.

In this case p(y) is nearly Gaussian, and the Linearization approximation matches the Monte Carlo approximation.

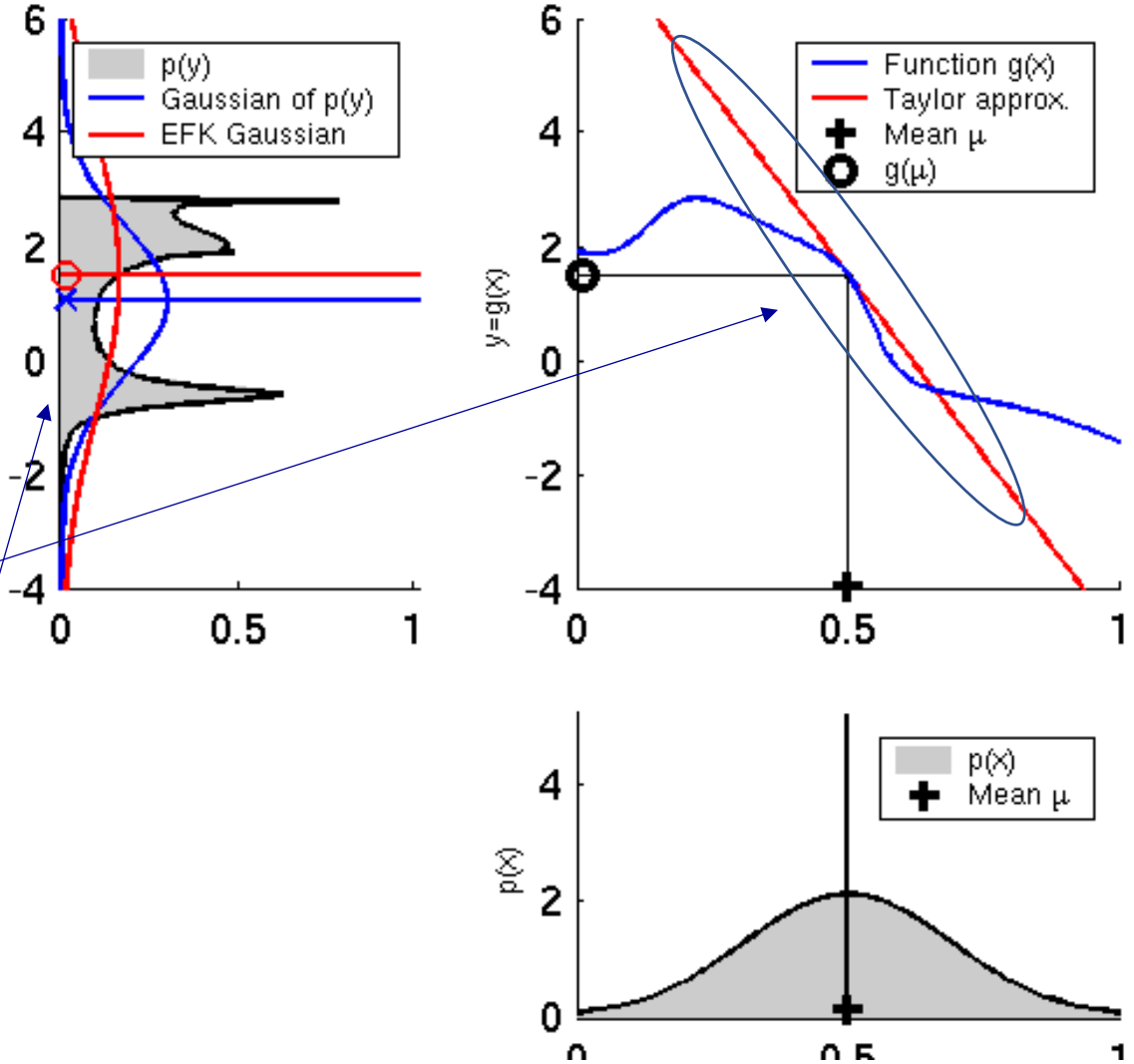


Linearization with high approximation error

The quality of the linearization approximation depends on the uncertainty of p(x) but also on the shape of the nonlinearity g(x).

In this example p(x) has high variance so points g(x) will be spread out around g(0.5), where g(x) is not close to linear.

In this case p(y) is multimodal, and the Linearization approximation matches the Monte Carlo approximation are both suboptimal approximations. Again Monte Carlo is better, provided sufficient samples.



How do we linearize?

- Using the first order Taylor expansion around the mean of the previous update step's state estimate:

$$
\begin{aligned}
x_{t+1} &= f(x_t, u_t) + w_t \\
&\approx f(\mu_{t|t}, u_t) + \frac{\partial f}{\partial x}(\mu_{t|t}, u_t)(x_t - \mu_{t|t}) + w_t \\
&= f(\mu_{t|t}, u_t) + F_t(x_t - \mu_{t|t}) + w_t \\
&= F_t x_t + \boxed{f(\mu_{t|t}, u_t) - F_t \mu_{t|t}} + w_t \\
&= F_t x_t + \bar{u}_t + w_t
\end{aligned}
$$

Constant term with respect to the state

Recall how to compute the Jacobian matrix. For example, if

$$f(x_1, x_2, u) = [x_1 + x_2^2,\ x_2 + 3u,\ x_1^4 - u^2] \in \mathbb{R}^3$$

then the Jacobian of f with respect to (x_1, x_2) at (μ_1, μ_2, u) is

$$
\begin{aligned}
\frac{\partial f}{\partial x_{1:2}}(\mu_1, \mu_2, u_1) &= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} \end{bmatrix} (\mu_1, \mu_2, u_1) \\
&= \begin{bmatrix} 1 & 2\mu_2 \\ 0 & 1 \\ 4\mu_1^3 & 0 \end{bmatrix}
\end{aligned}
$$

How do we linearize?

- Using the first-order Taylor expansion around the mean of the previous prediction step's state estimate:

$$
\begin{aligned}
z_{t+1} &= h(x_{t+1}) + n_{t+1} \\
&\approx h(\mu_{t+1|t}) + \frac{\partial h}{\partial x}(\mu_{t+1|t})(x_t - \mu_{t+1|t}) + n_{t+1} \\
&= h(\mu_{t+1|t}) + H_{t+1}(x_{t+1} - \mu_{t+1|t}) + n_{t+1} \\
&= H_{t+1}x_{t+1} + \boxed{h(\mu_{t+1|t}) - H_{t+1}\mu_{t+1|t}} + n_{t+1} \\
&= H_{t+1}x_{t+1} + \bar{c}_{t+1} + n_{t+1}
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\frac{\partial h}{\partial x_{1:2}}(\mu_1, \mu_2) &= \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} \\ \frac{\partial h_3}{\partial x_1} & \frac{\partial h_3}{\partial x_2} \end{bmatrix} (\mu_1, \mu_2) \\
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Extended Kalman Filter: an instance of Bayes' Filter

Assumptions guarantee that if the prior belief before the prediction step is Gaussian

$$
\begin{aligned}
bel(x_t) &= p(x_t|u_{0:t-1}, z_{0:t}) \\
&= \eta\, p(z_t|x_t) \int p(x_t|u_{t-1}, x_{t-1})\, bel(x_{t-1})\, dx_{t-1}
\end{aligned}
$$

then the prior belief after the prediction step will be Gaussian

and the posterior belief (after the update step) will be Gaussian.

Linear observations with Gaussian noise

$$
\begin{aligned}
z_t &= H_t x_t + \bar{c}_t + n_t \\
&\text{with noise } n_t \sim \mathcal{N}(0, R)
\end{aligned}
$$

Linear dynamics with Gaussian noise

$$
\begin{aligned}
x_t &= F_{t-1} x_{t-1} + \bar{u}_{t-1} + G w_{t-1} \\
&\text{with noise } w_{t-1} \sim \mathcal{N}(0, Q)
\end{aligned}
$$

Initial belief is Gaussian

$$bel(x_0) \sim \mathcal{N}(\mu_0, \Sigma_0)$$

Dynamics

$$x_{t+1} = f(x_t, u_t) + Gw_t$$
$$w_t \sim \mathcal{N}(0, Q)$$

Measurements

$$z_t = h(x_t) + n_t$$
$$n_t \sim \mathcal{N}(0, R)$$

EKF in N dimensions

Init

$$bel(x_0) \sim \mathcal{N}(\mu_{0|0}, \Sigma_{0|0})$$

Prediction Step

$$\mu_{t+1|t} = f(\mu_{t|t}, u_t)$$
$$\Sigma_{t+1|t} = F_t \Sigma_{t|t} F_t^T + GQG^T$$

Update Step

Received measurement $\bar{z}_{t+1}$ but expected to receive $\mu_{z_{t+1}} = h(\mu_{t+1|t})$

Prediction residual is a Gaussian random variable $\delta z \sim \mathcal{N}(\bar{z}_{t+1} - \mu_{z_{t+1}}, S_{t+1})$
where the covariance of the residual is $S_{t+1} = H_{t+1} \Sigma_{t+1|t} H_{t+1}^T + R$

Kalman Gain (optimal correction factor): $K_{t+1} = \Sigma_{t+1|t} H_{t+1}^T S_{t+1}^{-1}$

$$\mu_{t+1|t+1} = \mu_{t+1|t} + K_{t+1}(\bar{z}_{t+1} - \mu_{z_{t+1}})$$
$$\Sigma_{t+1|t+1} = \Sigma_{t+1|t} - K_{t+1} H_{t+1} \Sigma_{t+1|t}$$

EKF Summary

- Efficient: Polynomial in measurement dimensionality k and state dimensionality n:
  $O(k^{2.376} + n^2)$

As in KF, inverting the covariance of the residual is O(k^2.376)

- Not optimal (unlike the Kalman Filter for linear systems)
- Can diverge if nonlinearities are large
- Works surprisingly well even when all assumptions are violated

Example #1: EKF-Localization

"Using sensory information to locate the robot in its environment is the most fundamental problem to providing a mobile robot with autonomous capabilities." [Cox '91]

- **Given**
  - Map of the environment.
  - Sequence of sensor measurements.
- **Wanted**
  - Estimate of the robot's position.
- **Problem classes**
  - Position tracking
  - Global localization
  - Kidnapped robot problem (recovery)

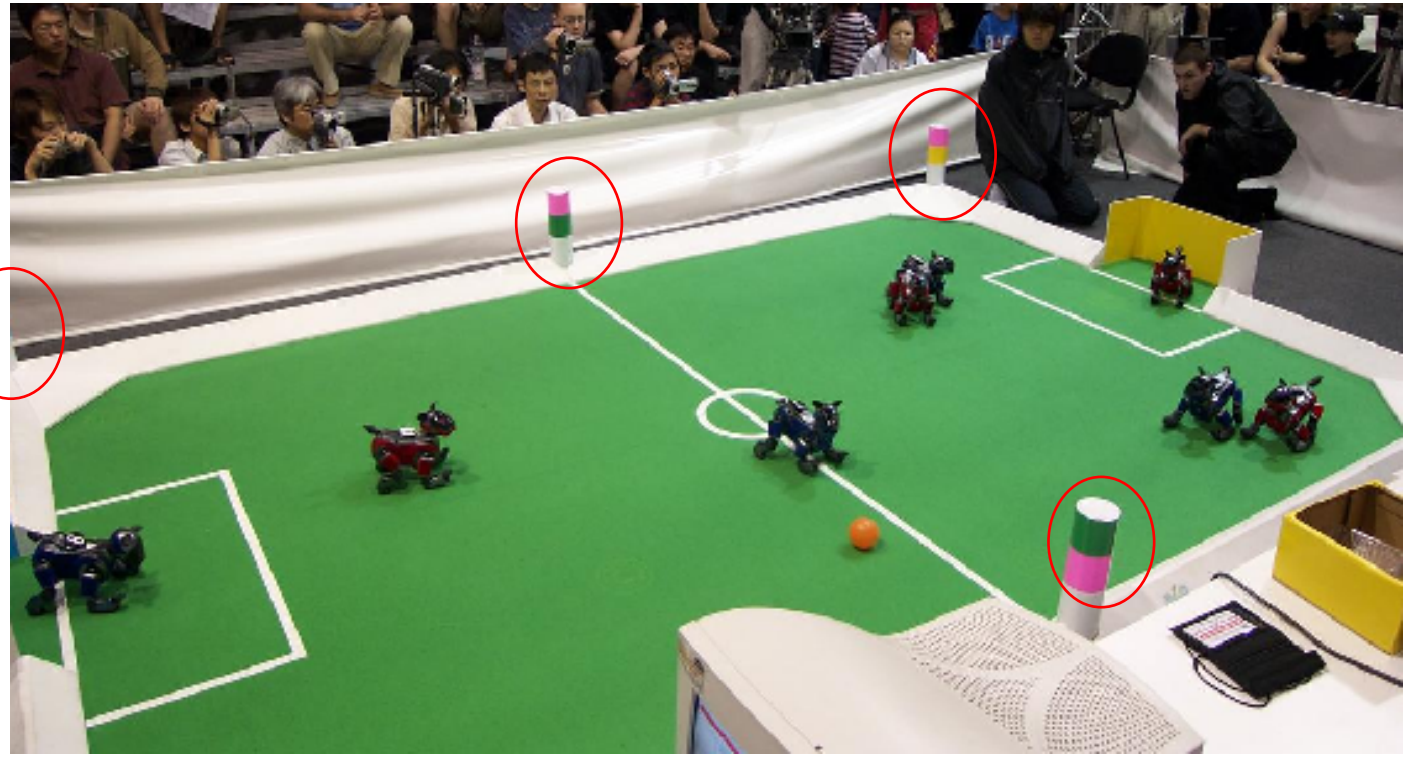
Landmark-based Localization

Landmarks, whose position $(l_x^{(i)}, l_y^{(i)})$ in the world is known.

Each robot measures its range and bearing from each landmark to localize itself.

State of a robot:

$$x_t = \begin{bmatrix} p_x(t) \\ p_y(t) \\ \theta(t) \end{bmatrix}$$



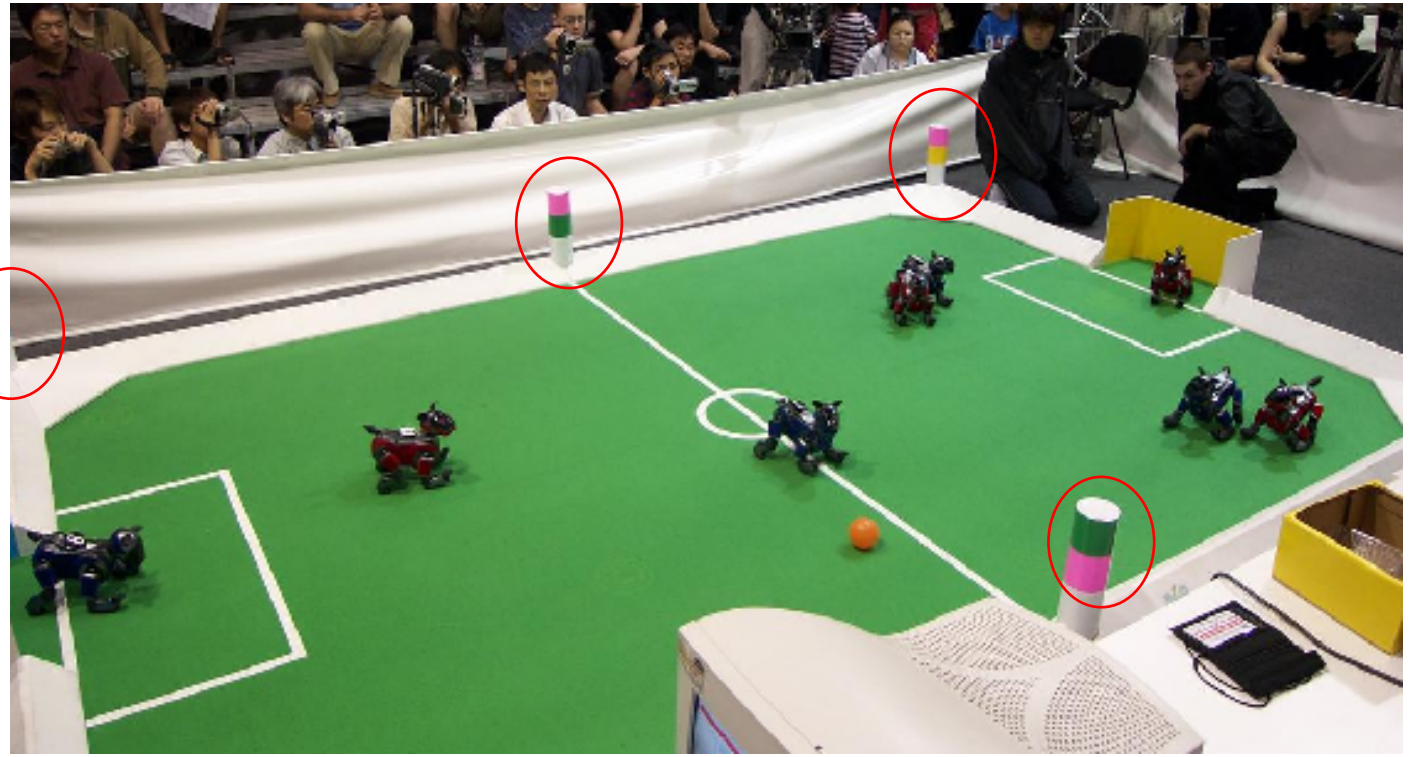
Landmark-based Localization

Measurement at time t,

$$z_t = \begin{bmatrix} \dots \\ z_t^{(i)} \\ \dots \end{bmatrix}$$

is a variable-sized vector, depending on the landmarks that are visible at time t.

Each measurement is a 2D vector, containing range and bearing from the robot to a landmark.



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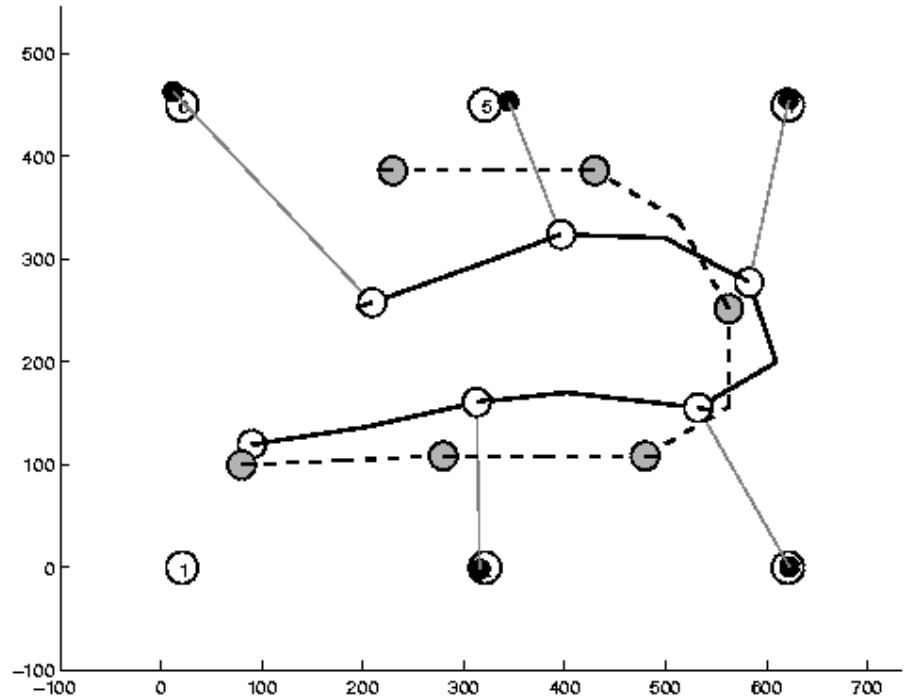
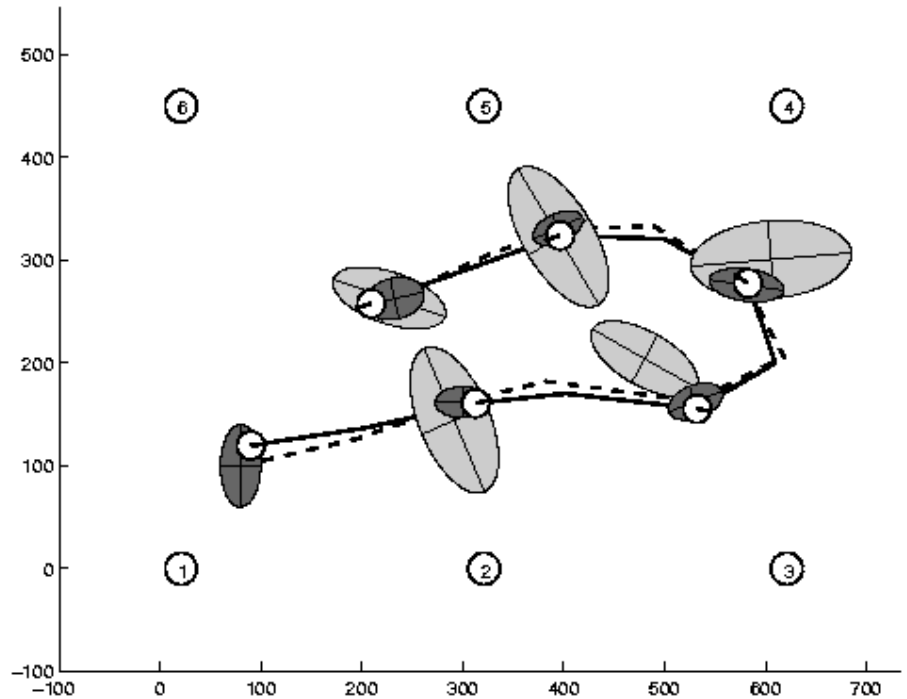
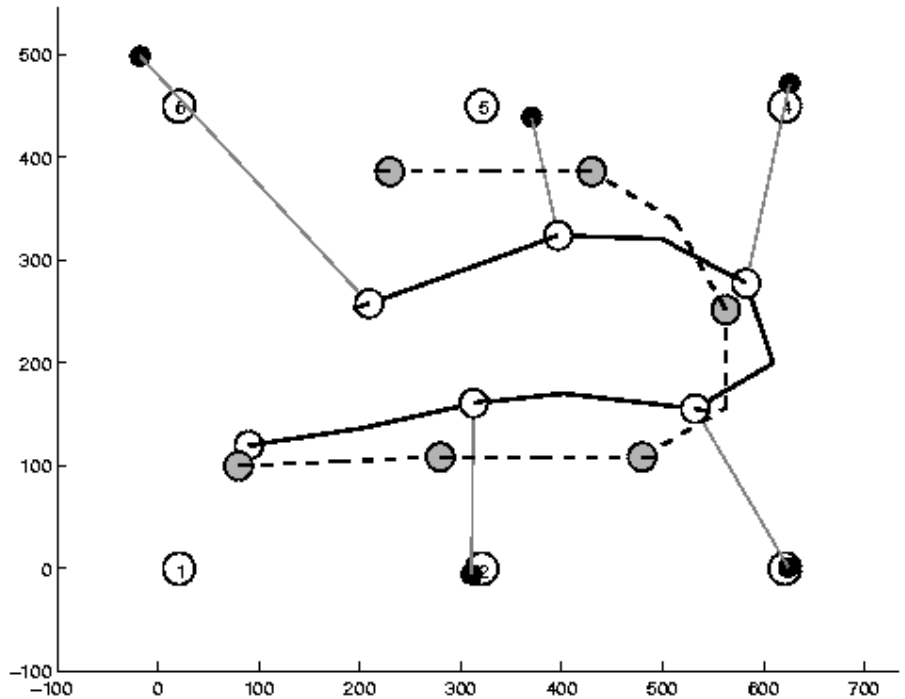
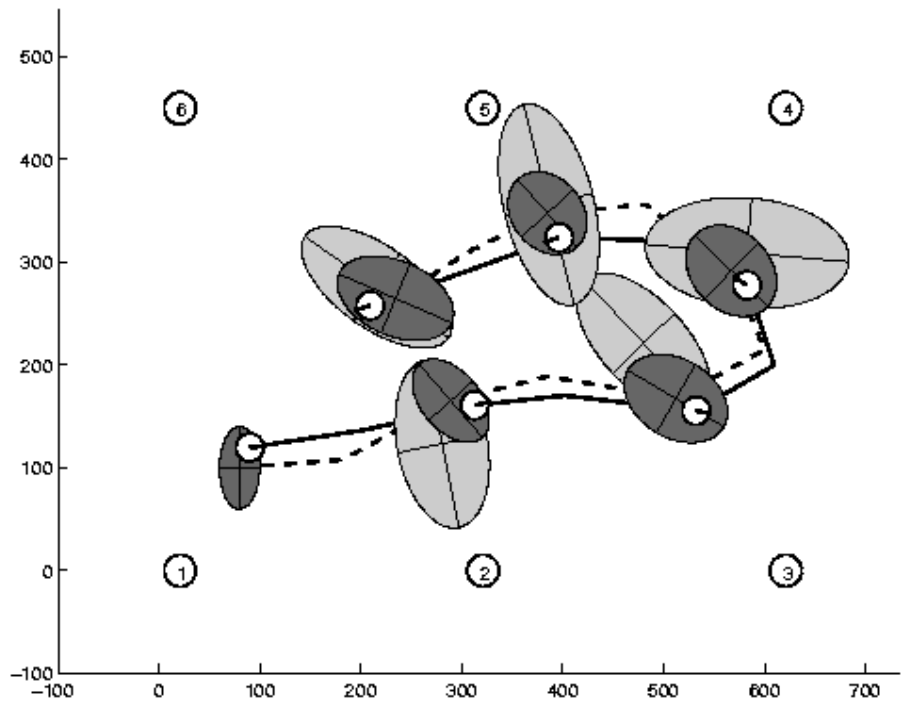
Estimation Sequence (1)



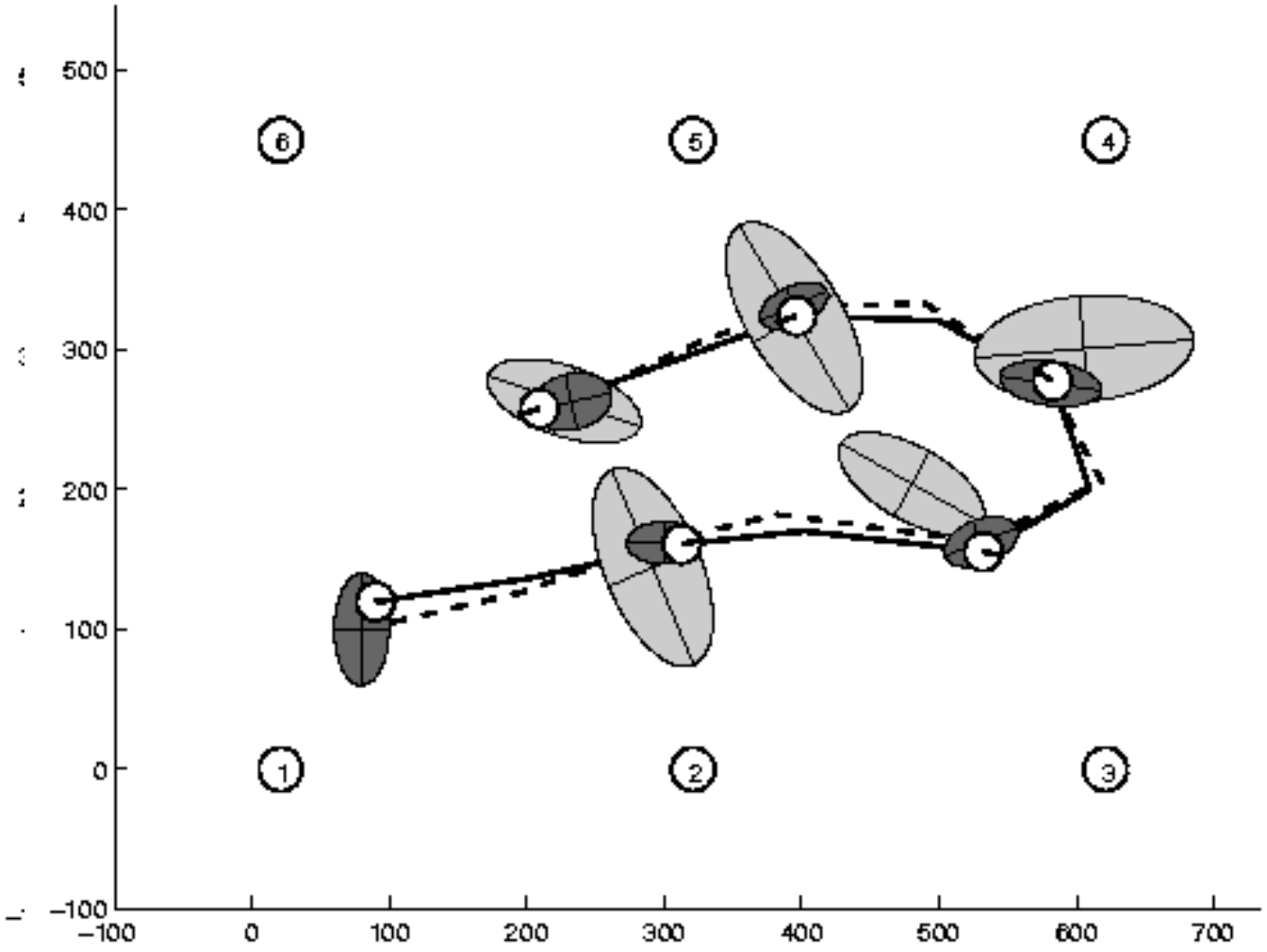


Estimation Sequence (2)





Comparison to true trajectory



Example #2: EKF-SLAM

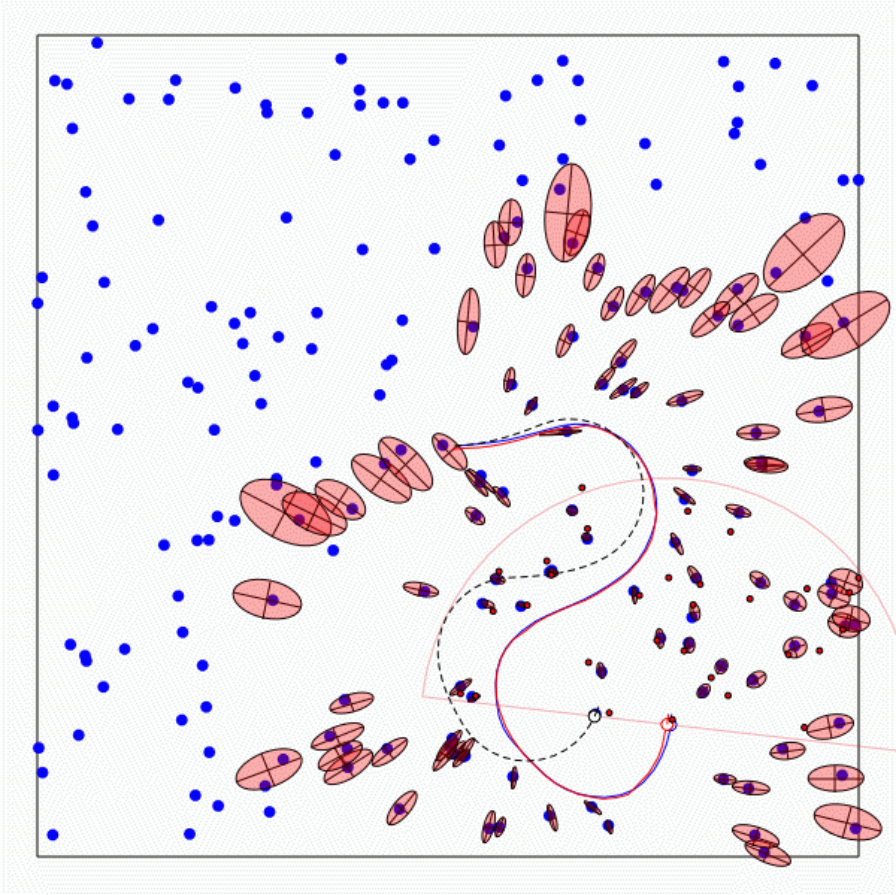
A robot is exploring an unknown, static environment.

Given:

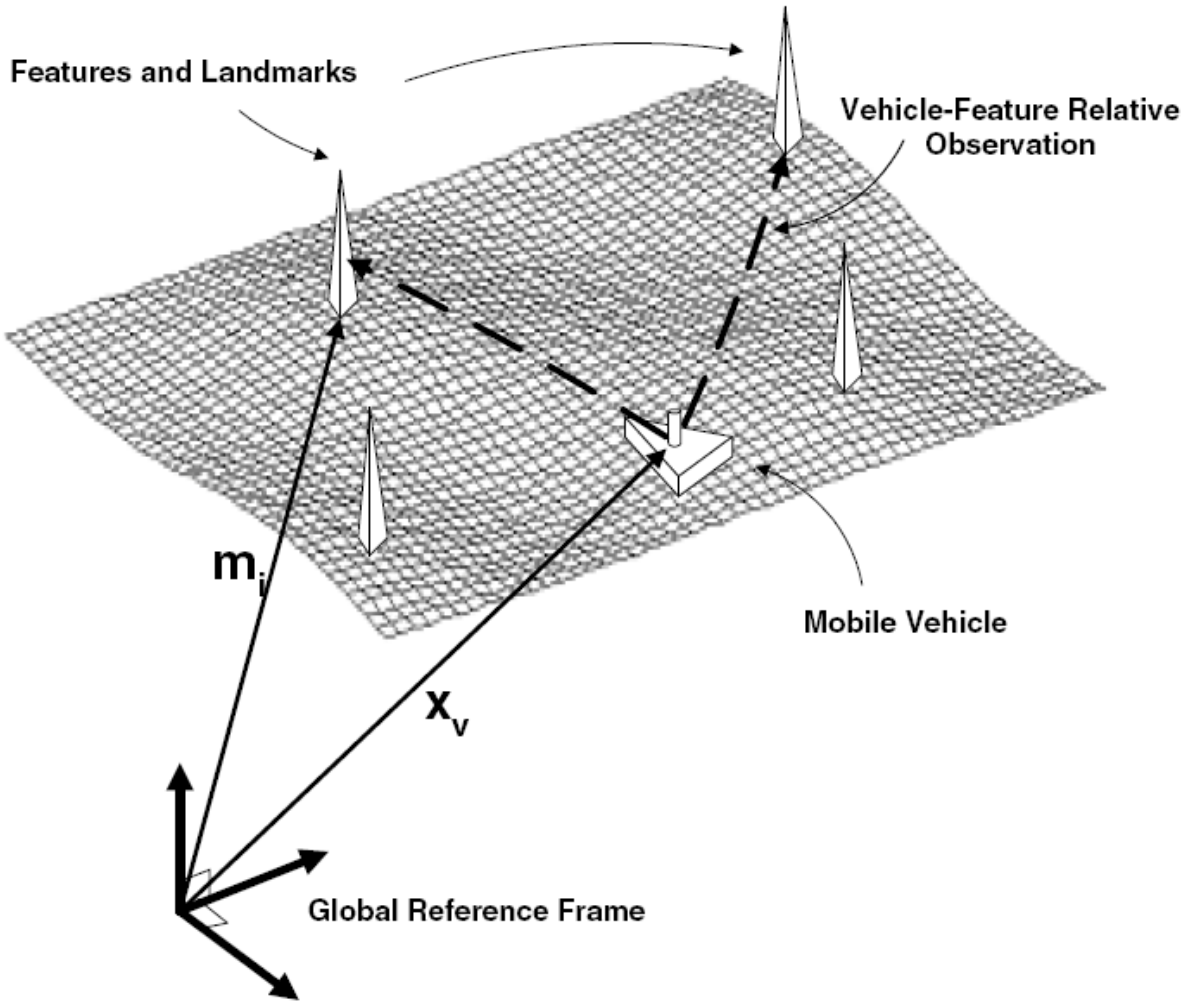
- The robot's controls
- Observations of nearby features

Estimate:

- Map of features
- Path of the robot

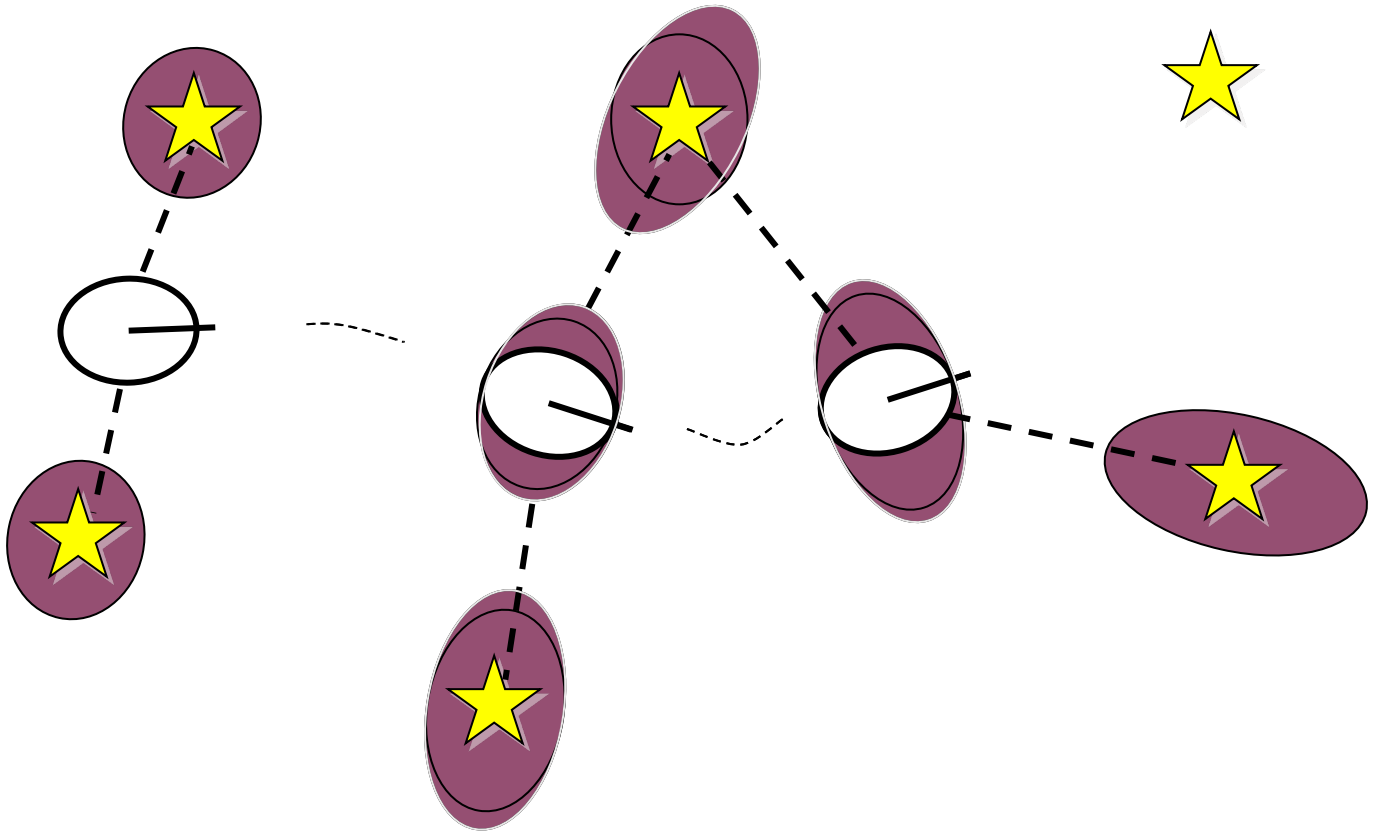


Structure of Landmark-based SLAM



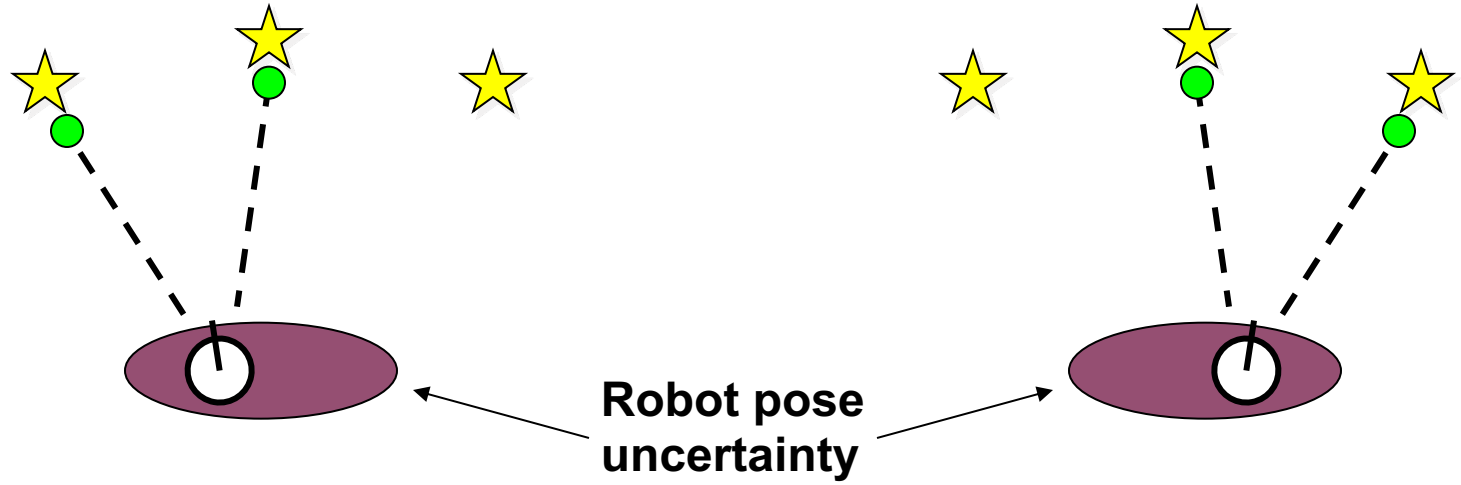
Why is SLAM a hard problem?

SLAM: robot path and map are both **unknown**



Robot path error correlates errors in the map

Why is SLAM a hard problem?



- In the real world, the mapping between observations and landmarks is unknown
- Picking wrong data associations can have catastrophic consequences
- Pose error correlates data associations

EKF-SLAM

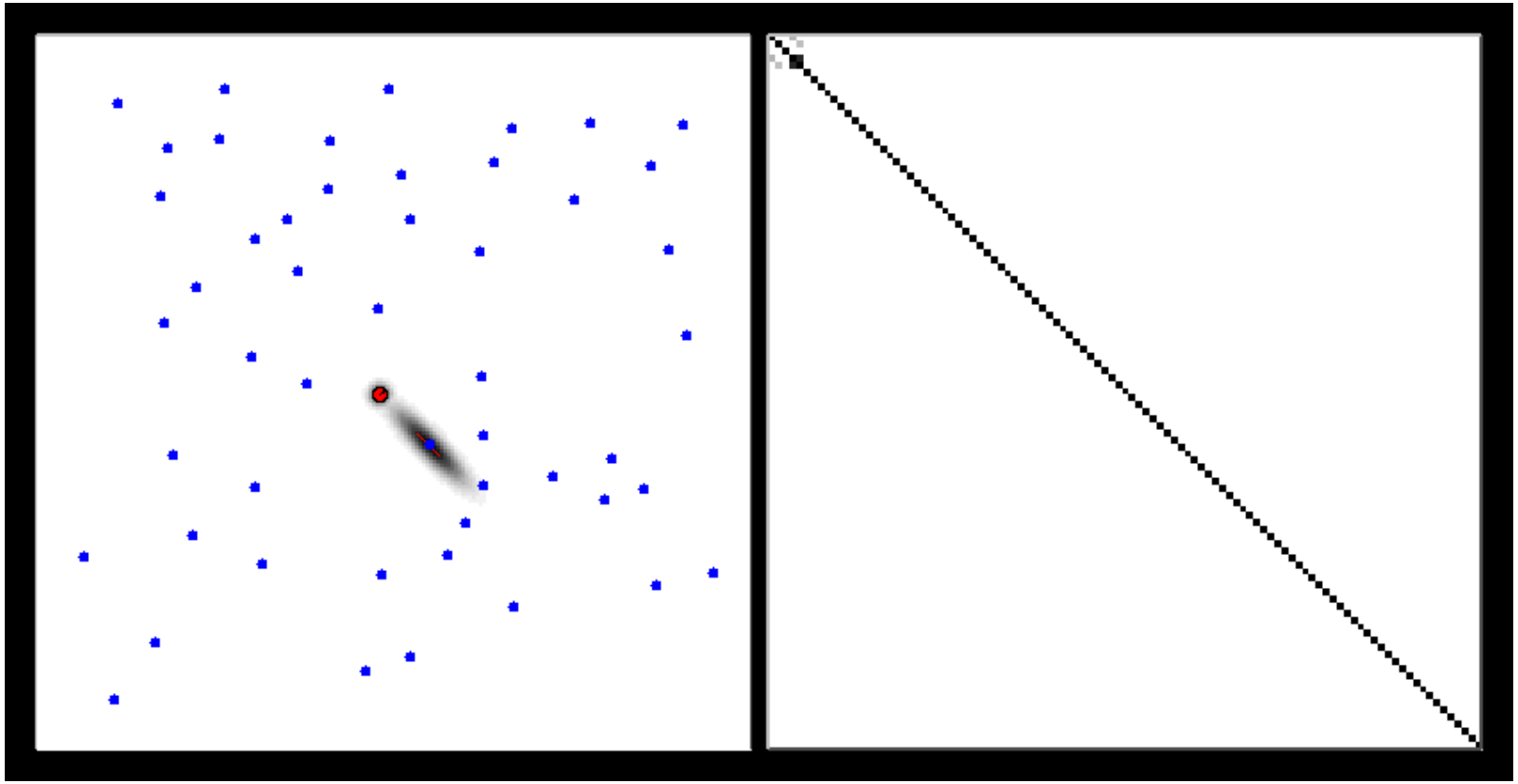
- Map with N landmarks: (3+2N)-dimensional Gaussian

$$Bel(x_t, m_t) = \left\langle \begin{pmatrix} x \\ y \\ \theta \\ l_1 \\ l_2 \\ \vdots \\ l_N \end{pmatrix}, \begin{pmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{x\theta} & \sigma_{xl_1} & \sigma_{xl_2} & \cdots & \sigma_{xl_N} \\ \sigma_{xy} & \sigma_y^2 & \sigma_{y\theta} & \sigma_{yl_1} & \sigma_{yl_2} & \cdots & \sigma_{yl_N} \\ \sigma_{x\theta} & \sigma_{y\theta} & \sigma_\theta^2 & \sigma_{\theta l_1} & \sigma_{\theta l_2} & \cdots & \sigma_{\theta l_N} \\ \sigma_{xl_1} & \sigma_{yl_1} & \sigma_{\theta l_1} & \sigma_{l_1}^2 & \sigma_{l_1 l_2} & \cdots & \sigma_{l_1 l_N} \\ \sigma_{xl_2} & \sigma_{yl_2} & \sigma_{\theta l_2} & \sigma_{l_1 l_2} & \sigma_{l_2}^2 & \cdots & \sigma_{l_2 l_N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{xl_N} & \sigma_{yl_N} & \sigma_{\theta l_N} & \sigma_{l_1 l_N} & \sigma_{l_2 l_N} & \cdots & \sigma_{l_N}^2 \end{pmatrix} \right\rangle$$

- Can handle hundreds of dimensions

Appendix

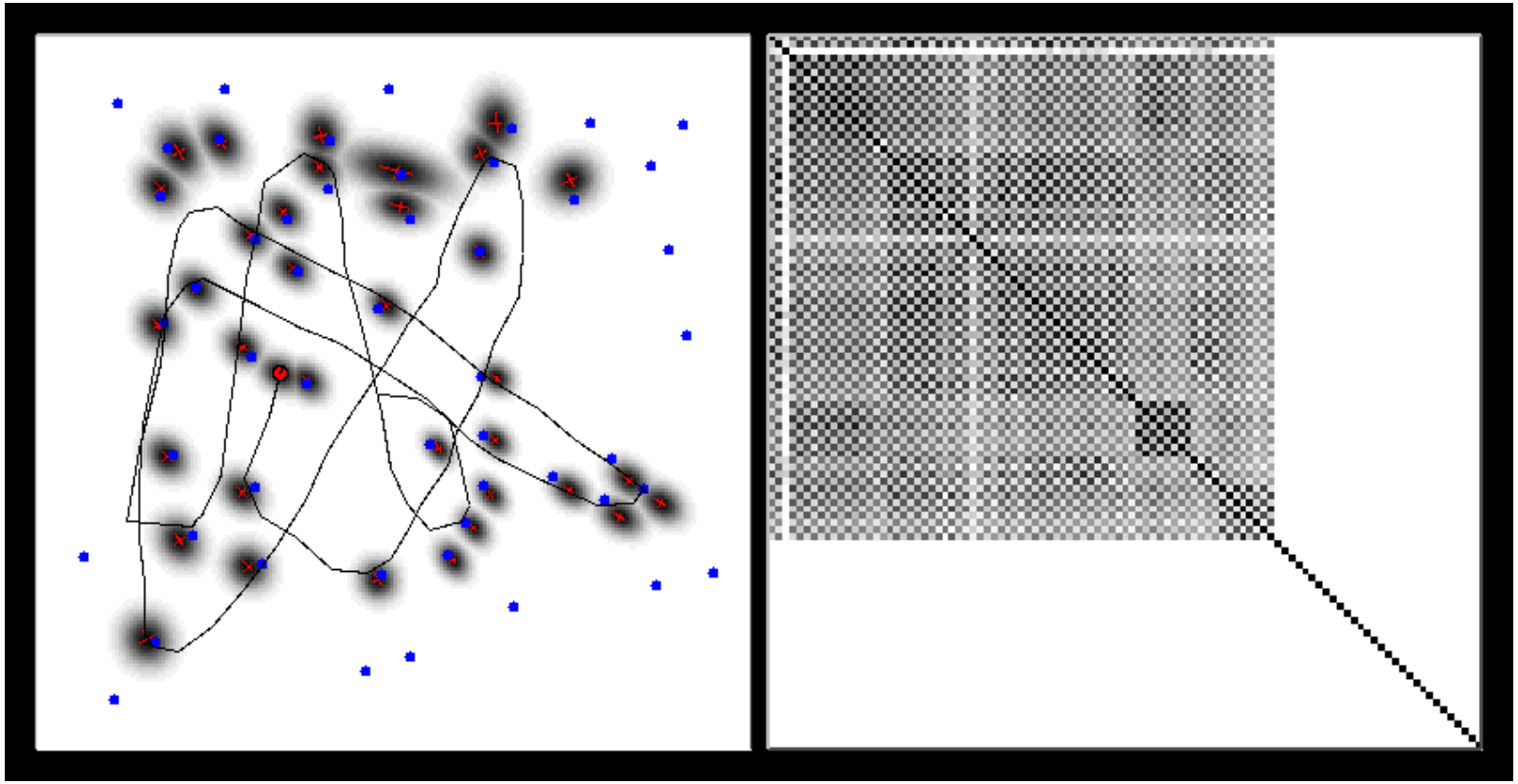
EKF-SLAM



Map

Covariance matrix

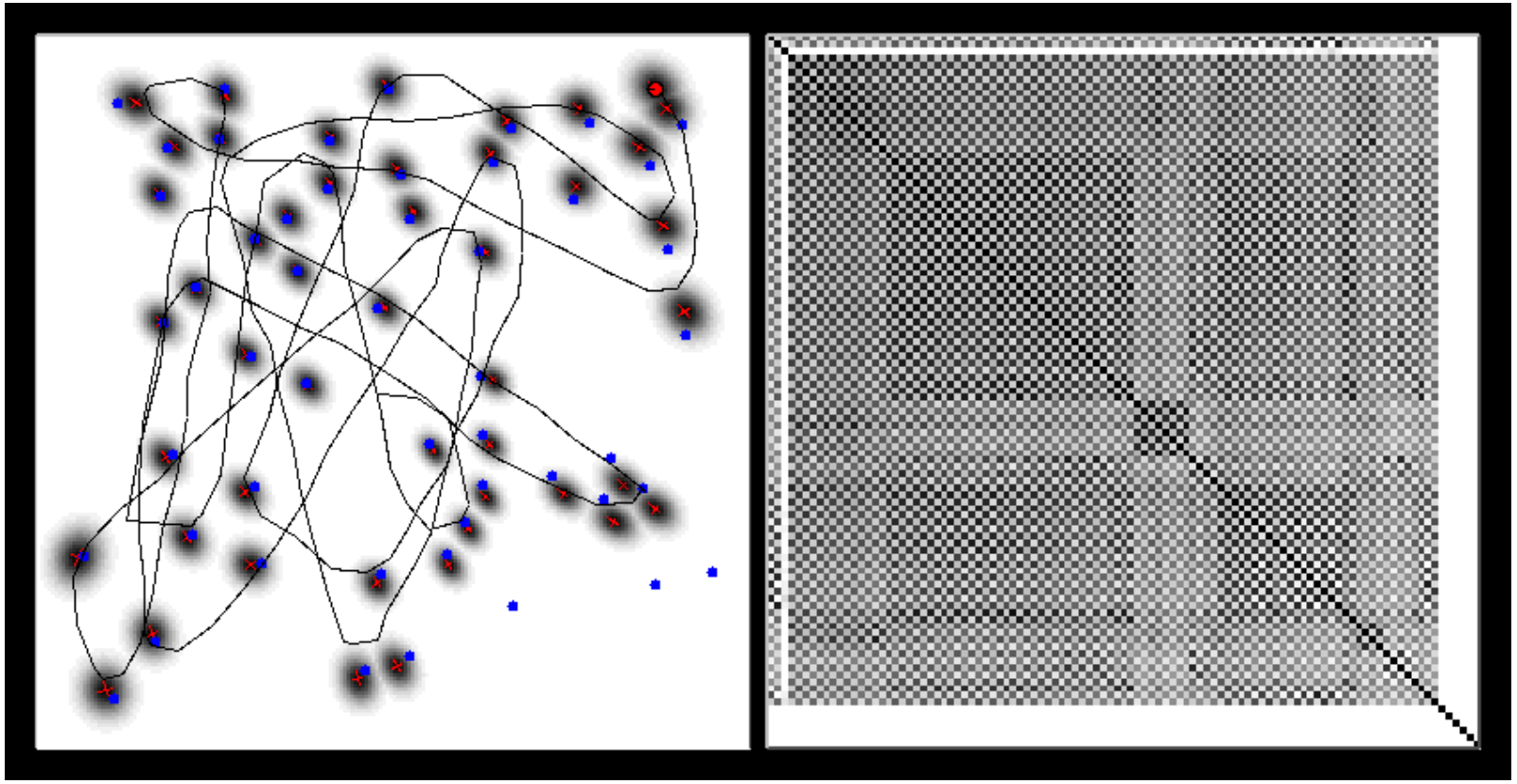
EKF-SLAM



Map

Covariance matrix

EKF-SLAM



Map

Covariance matrix

Properties of EKF-SLAM (Linear Case)

[Dissanayake et al., 2001]

Theorem:

The determinant of any sub-matrix of the map covariance matrix decreases monotonically as successive observations are made.

Theorem:

In the limit the landmark estimates become fully correlated