Some Beautiful Problems in Astrophysics

Shubhayan Ghosal
Dept.of.Math , Jadavpur University

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1 Problem

Consider a spherical star of mass M and radius R that is rotating uniformly with angular velocity ω. Assume that the star is made of an ideal gas with adiabatic index γ. Find the equation of state of the star, i.e., the relation between the pressure P, the density ρ, and the distance r from the center of the star.

1.1 Solution

The equation of state of the rotating star can be derived by balancing the gravitational force, the centrifugal force, and the pressure gradient force. Using the spherical coordinates (r, θ, ϕ), we have

$$\frac{dP}{dr} = -\rho \frac{GM_r}{r^2} + \rho r \omega^2 \sin^2 \theta,$$

where M_r is the mass enclosed within radius r. Integrating this equation from the center to the surface of the star, we get

$$P(R) - P(0) = -\int_0^R \rho \frac{GM_r}{r^2} dr + \int_0^R \rho r \omega^2 \sin^2 \theta dr.$$

Using the ideal gas law, $P = \frac{\rho k_B T}{\mu m_H}$, where k_B is the Boltzmann constant, T is the temperature, μ is the mean molecular weight, and m_H is the mass of hydrogen, we can write

$$\frac{\rho(R) k_B T(R)}{\mu m_H} - \frac{\rho(0) k_B T(0)}{\mu m_H} = -\int_0^R \rho \frac{GM_r}{r^2} dr + \int_0^R \rho r \omega^2 \sin^2 \theta dr.$$

Assuming that the temperature is constant throughout the star, we can simplify this equation as

$$\rho(R) - \rho(0) = -\frac{\mu m_H}{k_B T} \left(\int_0^R \rho \frac{GM_r}{r^2} dr - \int_0^R \rho r \omega^2 \sin^2 \theta dr \right).$$

Using the adiabatic relation, $P \propto \rho^\gamma$, we can write

$$\rho(R)^\gamma - \rho(0)^\gamma = -\gamma \frac{\mu m_H}{k_B T} \left(\int_0^R \rho \frac{GM_r}{r^2} dr - \int_0^R \rho r \omega^2 \sin^2 \theta dr \right).$$

This is the equation of state of the rotating star, relating the pressure, the density, and the distance from the center of the star.

2 Problem

A binary system consists of two stars of masses m_1 and m_2 orbiting each other in a circular orbit of radius a. The stars are emitting gravitational waves, which carry away energy and angular momentum from the system. Find the rate of change of the orbital radius a due to the gravitational wave emission.

2.1 Solution

The rate of change of the orbital radius a due to the gravitational wave emission can be found by using the conservation of energy and angular momentum. The total energy of the binary system is given by

$$E = -\frac{Gm_1m_2}{2a},$$

where G is the gravitational constant. The total angular momentum of the binary system is given by

$$L = m_1m_2\sqrt{\frac{Ga}{m_1+m_2}}.$$

The rate of change of the energy and the angular momentum due to the gravitational wave emission are given by

$$\frac{dE}{dt} = -\frac{32}{5}\frac{G^4m_1^2m_2^2(m_1+m_2)}{c^5a^5},$$

and

$$\frac{dL}{dt} = -\frac{32}{5}\frac{G^{7/2}m_1^2m_2^2(m_1+m_2)^{1/2}}{c^5a^{7/2}},$$

where c is the speed of light. Using the chain rule, we can write

$$\frac{dE}{dt} = \frac{dE}{da}\frac{da}{dt},$$

and

$$\frac{dL}{dt} = \frac{dL}{da}\frac{da}{dt}.$$

Solving for $\frac{da}{dt}$, we get

$$\frac{da}{dt} = -\frac{64}{5}\frac{G^3m_1m_2(m_1+m_2)}{c^5a^3}\left(1+\frac{29}{12}\frac{m_1m_2}{(m_1+m_2)^2}\right).$$

This is the rate of change of the orbital radius a due to the gravitational wave emission.

3 Problem

A black hole of mass M is surrounded by an accretion disk of gas with inner radius R_{in} and outer radius R_{out}. The gas in the disk is heated by the viscous dissipation of the orbital energy and radiates as a blackbody. Find the luminosity of the accretion disk as a function of the radius R.

3.1 Solution

The luminosity of the accretion disk as a function of the radius R can be found by using the conservation of mass and angular momentum. The mass accretion rate $\dot{M}$ is constant throughout the disk, i.e.,

$$\dot{M} = 2\pi R\Sigma v_R,$$

where Σ is the surface density of the disk and v_R is the radial velocity of the gas. The specific angular momentum l of the gas is given by

$$l = \sqrt{GMR},$$

where G is the gravitational constant and M is the mass of the black hole. The rate of change of the specific angular momentum due to the viscous torque T is given by

$$\frac{dl}{dt} = \frac{T}{\dot{M}} = \frac{1}{2\pi R\Sigma v_R}\frac{d}{dR}\left(2\pi R^3\Sigma\nu\frac{dl}{dR}\right),$$

where ν is the kinematic viscosity of the gas. The viscous dissipation per unit area of the disk is given by

$$D = \frac{1}{2}\Sigma\nu\left(\frac{dl}{dR}\right)^2.$$

The luminosity per unit area of the disk is given by

$$L = \sigma T^4,$$

where σ is the Stefan-Boltzmann constant and T is the temperature of the gas. Assuming that the disk is in thermal equilibrium, i.e., $D = L$, we can write

$$\sigma T^4 = \frac{1}{2}\Sigma\nu\left(\frac{dl}{dR}\right)^2.$$

Substituting the expression for l and simplifying, we get

$$T = \left(\frac{3GM\dot{M}}{8\pi\sigma R^3}\right)^{1/4}\left(1 - \sqrt{\frac{R_{in}}{R}}\right)^{1/4}.$$

The luminosity of the disk as a function of the radius R is then given by

$$L(R) = 2\pi RL = 4\pi\sigma RT^4 = \frac{3GM\dot{M}}{2R^2}\left(1 - \sqrt{\frac{R_{in}}{R}}\right).$$

This is the luminosity of the accretion disk as a function of the radius R.

4 Problem

A neutron star of mass M and radius R is spinning with angular frequency Ω. The neutron star has a magnetic dipole moment $\vec{\mu}$ that is misaligned with its spin axis by an angle α. The neutron star emits electromagnetic radiation from its magnetic poles, which sweep across the sky as the star rotates. Find the period and the duty cycle of the pulsar signal observed by a distant observer.

4.1 Solution

The period of the pulsar signal is simply the time it takes for the neutron star to complete one rotation. This is given by the reciprocal of the angular frequency, Ω. So, the period T is:

$$T = \frac{2\pi}{\Omega}$$

The duty cycle of the pulsar signal is the fraction of the period during which the signal is observed. This depends on the geometry of the situation, specifically the angle α between the magnetic axis and the spin axis. If we assume that the emission is confined within a cone of half-angle α around the magnetic axis, then the fraction of the sky covered by this cone as the star rotates is approximately α/π. Therefore, the duty cycle D is:

$$D = \frac{\alpha}{\pi}$$

5 Problem

Consider a system of three stars with masses m_1, m_2, and m_3, and positions $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$, respectively. Assume that the stars are point masses and that their mutual gravitational attraction is the only force acting on them. Derive the equations of motion for the system using Lagrangian mechanics, and show that the total energy and angular momentum of the system are conserved.

5.1 Solution

The potential energy of a pair of stars is given by $U = -G\frac{m_i m_j}{|\vec{r}_i - \vec{r}_j|}$, where G is the gravitational constant. For three bodies, the total potential energy is the sum of the potential energies of each pair:

$$U = -G\left(\frac{m_1 m_2}{|\vec{r}_1 - \vec{r}_2|} + \frac{m_1 m_3}{|\vec{r}_1 - \vec{r}_3|} + \frac{m_2 m_3}{|\vec{r}_2 - \vec{r}_3|}\right)$$

The kinetic energy of each star is given by $T = \frac{1}{2}m\vec{v}^2$, and the total kinetic energy is the sum of the kinetic energies of each star:

$$T = \frac{1}{2} m_1 \vec{v}_1^2 + \frac{1}{2} m_2 \vec{v}_2^2 + \frac{1}{2} m_3 \vec{v}_3^2$$

The Lagrangian of the system is then $L = T - U$. We can use the Euler-Lagrange equation to find the equations of motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}_i}\right) - \frac{\partial L}{\partial \vec{r}_i} = 0$$

This will give us three second-order differential equations, one for each star.

The total energy of the system, $E = T + U$, is conserved because the Lagrangian does not explicitly depend on time. The total angular momentum, $\vec{L} = \sum_i m_i \vec{r}_i \times \vec{v}_i$, is conserved because the Lagrangian is invariant under rotations (the potential energy depends only on the distance between the stars, not their absolute positions).

This is a general result in mechanics: if the Lagrangian of a system has a certain symmetry, then there is a corresponding quantity that is conserved. This is known as Noether's theorem. In this case, time-invariance leads to energy conservation, and rotational invariance leads to angular momentum conservation.

6 Problem

Suppose that a planet with mass m_p and radius R_p orbits a star with mass m_s and radius R_s in a circular orbit with radius a. A moon with mass m_m and radius R_m orbits the planet in a circular orbit with radius b. Assume that the orbits are coplanar and that the tidal forces are negligible. Find the conditions for the stability of the system, i.e., the ranges of values for a, b, m_p, m_m, R_p, and R_m that prevent the moon from escaping the planet's gravity or colliding with the planet or the star.

6.1 Solution

The stability of the system depends on several factors. Here are the conditions that need to be satisfied:

1. The moon must not escape the planet's gravity: The gravitational force between the moon and the planet must be greater than the centrifugal force on the moon due to its orbit around the planet. This gives us the inequality:

$$G\frac{m_p m_m}{b^2} \geq \frac{m_m v_m^2}{b}$$

where v_m is the speed of the moon in its orbit around the planet. This simplifies to:

$$b \leq G\frac{m_p}{v_m^2}$$

2. The moon must not collide with the planet: The distance between the moon and the planet must be greater than the sum of their radii. This gives us the inequality:

$$b \geq R_p + R_m$$

3. The moon must not collide with the star: The distance between the moon and the star must be greater than the sum of their radii. Since the moon is in the same plane as the planet and the star, the minimum distance between the moon and the star is $a - b$. This gives us the inequality:

$$a - b \geq R_s + R_m$$

4. The planet must not escape the star's gravity: Similarly to condition 1, the gravitational force between the planet and the star must be greater than the centrifugal force on the planet due to its orbit around the star. This gives us the inequality:

$$G\frac{m_s m_p}{a^2} \geq \frac{m_p v_p^2}{a}$$

where v_p is the speed of the planet in its orbit around the star. This simplifies to:

$$a \leq G\frac{m_s}{v_p^2}$$

5. The planet must not collide with the star: The distance between the planet and the star must be greater than the sum of their radii. This gives us the inequality:

$$a \geq R_s + R_p$$

These conditions provide a set of inequalities that a, b, m_p, m_m, R_p, and R_m must satisfy for the system to be stable. Note that the speeds v_m and v_p are not independent variables, but are determined by the gravitational forces and the radii of the orbits. They can be found from the conditions for circular motion:

$$v_m = \sqrt{G\frac{m_p}{b}}, \quad v_p = \sqrt{G\frac{m_s}{a}}$$

Substituting these into the inequalities gives a set of conditions involving only the physical parameters of the system. These conditions can be solved to find the ranges of values for a, b, m_p, m_m, R_p, and R_m that result in a stable system. Note that these are necessary, but not sufficient, conditions for stability. A full analysis would also need to consider perturbations and the long-term evolution of the system.

7 Few Books

1. *Mathematical Methods for Physics and Engineering* by Riley, Hobson, and Bence
2. *Astrophysics for Physicists* by Arnab Rai Choudhuri
3. *An Introduction to Modern Astrophysics* by Bradley W. Carroll and Dale A. Ostlie
4. *Mathematical Foundations of Quantum Mechanics* by John von Neumann