

# Maschinelles Lernen 1 - Assignment 2

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## 1 Gauging the Risk

$f$  = a house will be flooded

$x$  = house stands in a high risk area

$$P(f) = 0.0005$$

$$P(x) = 0.04$$

$$P(x|f) = 0.8$$

$\alpha_1$  = buying insurance

$$\lambda(\alpha_1|f) = 1100\text{€}$$

$$\lambda(\alpha_2|f) = 100000\text{€}$$

$\neg f$  = a house won't be flooded

$\neg x$  = house stands in a low risk area

$$P(\neg f) = 0.9995$$

$$P(\neg x) = 0.96$$

$$P(\neg x|f) = 0.2$$

$\alpha_2$  = not buying insurance

$$\lambda(\alpha_1|\neg f) = 1100\text{€}$$

$$\lambda(\alpha_2|\neg f) = 0\text{€}$$

**a)**

Bayes' formula is

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

We want to determine the probability that a house in a high risk area will be flooded, i. e., we want to calculate  $P(f|x)$ :

$$P(f|x) = \frac{P(x|f)P(f)}{P(x)} = \frac{0.8 \cdot 0.0005}{0.04} = 0.01$$

There is a 1% probability that our house will be flooded next year.

**b)**

We want to compute how much loss we can expect depending on whether we decide to buy insurance or not. Therefore we compute  $R(\alpha_1|x)$  and  $R(\alpha_2|x)$ :

$$P(\neg f|x) = 1 - P(f|x) = 0.99$$

$$R(\alpha_1|x) = \lambda(\alpha_1|f)P(f|x) + \lambda(\alpha_1|\neg f)P(\neg f|x) = 1100\text{€} \cdot 0.01 + 1100\text{€} \cdot 0.99 = 1100\text{€}$$

$$R(\alpha_2|x) = \lambda(\alpha_2|f)P(f|x) + \lambda(\alpha_2|\neg f)P(\neg f|x) = 100000\text{€} \cdot 0.01 + 0\text{€} \cdot 0.99 = 1000\text{€}$$

Since  $R(\alpha_2|x) < R(\alpha_1|x)$  it would be cheaper not to buy an insurance.

**c)**

We feel that the existing calculation matches this group's common sense.

## 2 Bounds on the Error

**a)**

*Proof.* We have

$$\min[P(\omega_1|x), P(\omega_2|x)] \stackrel{!}{\leq} 2P(\omega_1|x)P(\omega_2|x)$$

WLOG let  $P(\omega_1|x) \leq P(\omega_2|x)$  so that  $0 \leq P(\omega_1|x) \leq 0.5$ . Then

$$P(\omega_1|x) \leq 2P(\omega_1|x)P(\omega_2|x).$$

It holds that  $P(\omega_2|x) = 1 - P(\omega_1|x)$  so

$$\begin{aligned} P(\omega_1|x) &\leq 2P(\omega_1|x)(1 - P(\omega_1|x)) \\ &\leq 2P(\omega_1|x) - 2P(\omega_1|x)^2 \Leftrightarrow \\ 2P(\omega_1|x)^2 &\leq P(\omega_1|x) \end{aligned}$$

For  $P(\omega_1|x) = 0$ , the inequality holds. Otherwise we can divide by that term and get

$$P(\omega_1|x) \leq 0.5$$

which is fulfilled because of our assumptions. □

**b)**

Let  $P(\omega_1|x) = P(\omega_2|x) = 0.5$ . Then

$$\begin{aligned} P(\text{error}|x) = \min[P(\omega_1|x), P(\omega_2|x)] &\stackrel{!}{\leq} \alpha P(\omega_1|x)P(\omega_2|x) \\ 0.5 &\leq 0.25\alpha \Leftrightarrow \\ \alpha &\geq 2 \end{aligned}$$

which violates the assumption  $\alpha < 2$ .

### 3 Gaussian Densities

**a)**

Because  $P(\text{error}|x) = P(\omega_2|x)$  and the definition of  $P(\text{error}|x)$  we know that  $P(\omega_2|x) \leq P(\omega_1|x)$ . Using Bayes' rule, it holds that

$$\frac{P(x|\omega_2)P(\omega_2)}{\cancel{P(x)}} \leq \frac{P(x|\omega_1)P(\omega_1)}{\cancel{P(x)}}$$

We reformulate this to

$$\frac{P(x|\omega_2)}{P(x|\omega_1)} \leq \frac{P(\omega_1)}{P(\omega_2)}$$

and using the definitions we get

$$\frac{\exp(-(x+\mu)^2/(2\sigma^2))}{\exp(-(x-\mu)^2/(2\sigma^2))} \leq \frac{P(\omega_1)}{P(\omega_2)}$$

Applying the logarithm gives

$$\frac{1}{2\sigma^2} ((x-y)^2 - (x+y)^2) = \frac{1}{2\sigma^2} (-4x\mu) \leq \ln \frac{P(\omega_1)}{P(\omega_2)}$$

If we want the relationship between  $P(\omega_2|x)$  and the conditional error to hold, then

$$\begin{aligned} \ln P(\omega_2|x) &\leq \ln P(\omega_1|x) + \frac{2x\mu}{\sigma^2} \Leftrightarrow \\ P(\omega_2|x) &\leq e^{2x\mu/\sigma^2} P(\omega_1|x) \end{aligned}$$

**b)**

If we furthermore also assume  $\mu_2 = -\mu_1$  for a Laplacian distribution, we need to make a differentiation.

- for  $x \geq \mu$ , condition for  $P(\omega_2)$  does not depend on  $x$  any more.
- for  $x < \mu$ , condition for  $P(\omega_2)$  does not depend on  $\mu$  any more.