Maschinelles Lernen 1 - Assignment 2

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1 Gauging the Risk

$$\begin{aligned}
f &= \text{a house will be flooded} & \neg f &= \text{a house won't be flooded}\\
x &= \text{house stands in a high risk area} & \neg x &= \text{house stands in a low risk area}\\
P(f) &= 0.0005 & P(\neg f) &= 0.9995\\
P(x) &= 0.04 & P(\neg x) &= 0.96\\
P(x|f) &= 0.8 & P(\neg x|f) &= 0.2\\
\alpha_1 &= \text{buying insurance} & \alpha_2 &= \text{not buying insurance}\\
\lambda(\alpha_1|f) &= 1100€ & \lambda(\alpha_1|\neg f) &= 1100€\\
\lambda(\alpha_2|f) &= 100000€ & \lambda(\alpha_1|\neg f) &= 0€
\end{aligned}$$

(a) Bayes Rule:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Compute $P(f|x)$:

$$P(f|x) = \frac{P(x|f)P(f)}{P(x)} = \frac{0.8 \cdot 0.0005}{0.04} = 0.01$$

(b) Compute $R(\alpha_1|x)$ and $R(\alpha_2|x)$:

$$\begin{aligned}
P(\neg f|x) &= 1 - P(f|x) = 0.99\\
R(\alpha_1|x) &= \lambda(\alpha_1|f)P(f|x) + \lambda(\alpha_1|\neg f)P(\neg f|x) = 1100€ * 0.01 + 1100€ * 0.99 = 1100€\\
R(\alpha_2|x) &= \lambda(\alpha_2|f)P(f|x) + \lambda(\alpha_2|\neg f)P(\neg f|x) = 100000€ * 0.01 + 0€ * 0.99 = 1000€
\end{aligned}$$

Since $R(\alpha_2|x) < R(\alpha_1|x)$ it would be more viable not to buy an insurance.

(c) insert text

2 Bounds on the Error

a)

Proof. We have

$$\min[P(\omega_1|x), P(\omega_2)|x)] \overset{!}{\leq} 2P(\omega_1|x)P(\omega_2|x)$$

WLOG let $P(\omega_1|x) \leq P(\omega_2|x)$ so that $0 \leq P(\omega_1|x) \leq 0.5$. Then

$$P(\omega_1|x) \leq 2P(\omega_1|x)P(\omega_2|x).$$

It holds that $P(\omega_2|x) = 1 - P(\omega_1|x)$ so

$$\begin{aligned} P(\omega_1|x) &\leq 2P(\omega_1|x)(1 - P(\omega_1|x)) \\ &\leq 2P(\omega_1|x) - 2P(\omega_1|x)^2 \Leftrightarrow \\ 2P(\omega_1|x)^2 &\leq P(\omega_1|x) \end{aligned}$$

For $P(\omega_1|x) = 0$, the inequality holds. Otherwise we can divide by that term and get

$$P(\omega_1|x) \leq 0.5$$

which is fulfilled because of our assumptions. □

b)

Let $P(\omega_1|x) = P(\omega_2|x) = 0.5$. Then

$$\begin{aligned} P(\text{error}|x) = \min[P(\omega_1|x, P(\omega_2|x)] &\overset{!}{\leq} \alpha P(\omega_1|x)P(\omega_2|x) \\ 0.5 &\leq 0.25\alpha \Leftrightarrow \\ \alpha &\geq 2 \end{aligned}$$

which violates the assumption $\alpha < 2$.

3 Gaussian Densities

(a)

$$
\begin{aligned}
& P(\omega_2|x) \leq P(\omega_1|x) \\
\Leftrightarrow \quad & \frac{P(x|\omega_2)P(\omega_2)}{P(x)} \leq \frac{P(x|\omega_1)P(\omega_1)}{P(x)} \\
\Leftrightarrow \quad & P(x|\omega_2)P(\omega_2) \leq P(x|\omega_1)P(\omega_1) \\
\Leftrightarrow \quad & \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{(x+\mu)^2}{2\sigma^2}} P(\omega_2) \leq \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{(x-\mu)^2}{2\sigma^2}} P(\omega_1) \\
\Leftrightarrow \quad & e^{\frac{(x+\mu)^2}{2\sigma^2}} P(\omega_2) \leq e^{\frac{(x-\mu)^2}{2\sigma^2}} P(\omega_1) \\
\Leftrightarrow \quad & \frac{(x+\mu)^2}{2\sigma^2} + \ln(P(\omega_2)) \leq \frac{(x-\mu)^2}{2\sigma^2} + \ln(P(\omega_1)) \\
\Leftrightarrow \quad & -(x+\mu)^2 + 2\sigma^2 \ln(P(\omega_2)) \leq -(x-\mu)^2 + 2\sigma^2 \ln(P(\omega_1)) \\
\Leftrightarrow \quad & -x^2 - x\mu - \mu^2 + 2\sigma^2 \ln(P(\omega_2)) \leq -x^2 + x\mu - \mu^2 + 2\sigma^2 \ln(P(\omega_1)) \\
\Leftrightarrow \quad & -x^2 - x\mu - \mu^2 + 2\sigma^2 \ln(P(\omega_2)) \leq -x^2 + x\mu - \mu^2 + 2\sigma^2 \ln(P(\omega_1)) \\
\Leftrightarrow \quad & \ln(P(\omega_2)) \leq \frac{2x\mu}{2\sigma^2} + \ln(P(\omega_1)) \\
\Leftrightarrow \quad & P(\omega_2) \leq e^{\frac{x\mu}{\sigma^2}} P(\omega_1)
\end{aligned}
$$

Therefore:

$$\int_{x=-\infty}^{\infty} P(error|x)p(x)dx = \int_{x=-\infty}^{\infty} min[P(\omega_1|x); P(\omega_2|x)]dx = \int_{x=-\infty}^{\infty} P(\omega_2|x)p(x)dx$$

if $P(\omega_2) \leq e^{\frac{x\mu}{\sigma^2}} P(\omega_1)$.

(b) If we furthermore also assume $\mu_2 = -\mu_1$ for a Laplacian distribution, we need to make a differentiation.

- for $x \geq \mu$, condition for $P(\omega_2)$ does not depend on x any more.
- for $x < \mu$, condition for $P(\omega_2)$ does not depend on μ any more.