# Maschinelles Lernen 1 - Assignment 2
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## 1 Gauging the Risk

$$
\begin{aligned}
f &= \text{a house will be flooded} & \neg f &= \text{a house won't be flooded} \\
x &= \text{house stands in a high risk area} & \neg x &= \text{house stands in a low risk area} \\
P(f) &= 0.0005 & P(\neg f) &= 0.9995 \\
P(x) &= 0.04 & P(\neg x) &= 0.96 \\
P(x|f) &= 0.8 & P(\neg x|f) &= 0.2 \\
\alpha_1 &= \text{buying insurance} & \alpha_2 &= \text{not buying insurance} \\
\lambda(\alpha_1|f) &= 1100€ & \lambda(\alpha_1|\neg f) &= 1100€ \\
\lambda(\alpha_2|f) &= 100000€ & \lambda(\alpha_2|\neg f) &= 0€
\end{aligned}
$$

### a)

Bayes' formula is

$$
P(A|B) = \frac{P(B|A)P(A)}{P(B)}
$$

We want to determine the probability that a house in a high risk area will be flooded, i. e., we want to calculate $P(f|x)$:

$$
P(f|x) = \frac{P(x|f)P(f)}{P(x)} = \frac{0.8 \cdot 0.0005}{0.04} = 0.01
$$

There is a 1% probability that our house will be flooded next year.

**b)**

We want to compute how much loss we can expect depending on whether we decide to buy insurance or not. Therefore we compute $R(\alpha_1|x)$ and $R(\alpha_2|x)$:

$$P(\neg f|x) = 1 - P(f|x) = 0.99$$
$$R(\alpha_1|x) = \lambda(\alpha_1|f)P(f|x) + \lambda(\alpha_1|\neg f)P(\neg f|x) = 1100\euro \cdot 0.01 + 1100\euro \cdot 0.99 = 1100\euro$$
$$R(\alpha_2|x) = \lambda(\alpha_2|f)P(f|x) + \lambda(\alpha_2|\neg f)P(\neg f|x) = 100000\euro \cdot 0.01 + 0\euro \cdot 0.99 = 1000\euro$$

Since $R(\alpha_2|x) < R(\alpha_1|x)$ it would be cheaper not to buy an insurance.

**c)**

We feel that the existing calculation matches this group's common sense.

## 2 Bounds on the Error

**a)**

*Proof.* We have

$$\min[P(\omega_1|x), P(\omega_2)|x)] \overset{!}{\leq} 2P(\omega_1|x)P(\omega_2|x)$$

WLOG let $P(\omega_1|x) \leq P(\omega_2|x)$ so that $0 \leq P(\omega_1|x) \leq 0.5$. Then

$$P(\omega_1|x) \leq 2P(\omega_1|x)P(\omega_2|x).$$

It holds that $P(\omega_2|x) = 1 - P(\omega_1|x)$ so

$$P(\omega_1|x) \leq 2P(\omega_1|x)(1 - P(\omega_1|x))$$
$$\leq 2P(\omega_1|x) - 2P(\omega_1|x)^2 \Leftrightarrow$$
$$2P(\omega_1|x)^2 \leq P(\omega_1|x)$$

For $P(\omega_1|x) = 0$, the inequality holds. Otherwise we can divide by that term and get

$$P(\omega_1|x) \leq 0.5$$

which is fulfilled because of our assumptions. $\qquad\square$

**b)**

Let $P(\omega_1|x) = P(\omega_2|x) = 0.5$. Then

$$P(\text{error}|x) = \min[P(\omega_1|x, P(\omega_2|x)] \overset{!}{\leq} \alpha P(\omega_1|x)P(\omega_2|x)$$
$$0.5 \leq 0.25\alpha \Leftrightarrow$$
$$\alpha \geq 2$$

which violates the assumption $\alpha < 2$.

# 3 Gaussian Densities

## a)

Because $P(\text{error}|x) = P(\omega_2|x)$ and the definition of $P(\text{error}|x)$ we know that $P(\omega_2|x) \leq P(\omega_1|x)$. Using Bayes' rule, it holds that

$$\frac{P(x|\omega_2)P(\omega_2)}{\cancel{P(x)}} \leq \frac{P(x|\omega_1)P(\omega_1)}{\cancel{P(x)}}$$

We reformulate this to

$$\frac{P(x|\omega_2)}{P(x|\omega_1)} \leq \frac{P(\omega_1)}{P(\omega_2)} \tag{1}$$

and using the definitions we get

$$\frac{\exp(-(x+\mu)^2/(2\sigma^2))}{\exp(-(x-\mu)^2/(2\sigma^2))} \leq \frac{P(\omega_1)}{P(\omega_2)}$$

Applying the logarithm gives

$$\frac{1}{2\sigma^2}\left((x-\mu)^2 - (x+\mu)^2\right) = \frac{1}{2\sigma^2}(-4x\mu) \leq \ln\frac{P(\omega_1)}{P(\omega_2)}$$

If we want the relationship between $P(\omega_2|x)$ and the conditional error to hold, then

$$\ln P(\omega_2|x) \leq \ln P(\omega_1|x) + \frac{2x\mu}{\sigma^2} \Leftrightarrow$$

$$P(\omega_2|x) \leq e^{2x\mu/\sigma^2} P(\omega_1|x)$$

## b)

We assume that $P(x|\omega_1) = ce^{-|x-\mu_1|/\sigma^2}, \mu_1 \in \mathbb{R}$ and $P(x|\omega_2) = ce^{-|x-\mu_2|/\sigma^2}, \mu_2\mathbb{R}$ where $c \in \mathbb{R}$. To simplify the discussion we assume that $\mu_1 \leq \mu_2$. If we insert this into inequality 1, then

$$\frac{P(x|\omega_2)}{P(x|\omega_1)} = \frac{\exp(-|x-\mu_2|/\sigma^2)}{\exp(-|x-\mu_1|/\sigma^2)} = \exp\left(\frac{|x-\mu_1| - |x-\mu_2|}{\sigma^2}\right)$$

For different $x$ we get

$$\ln\left(\frac{P(x|\omega_2)}{P(x|\omega_1)}\right) = \begin{cases} \frac{\mu_1+\mu_2}{\sigma^2}, & \text{if } x \leq \mu_1 \\ \frac{\mu_1+\mu_2-2x}{\sigma^2}, & \text{if } \mu_1 < x < \mu_2 \\ \frac{\mu_1-\mu_2}{\sigma^2}, & \text{if } x \geq \mu_2 \end{cases}$$

So if $x$ is outside of the open interval $(\mu_1, \mu_2)$ the conditional error does not depend on the exact value of $x$. If we assume additionally $\mu_2 = \mu_1$ for the Laplacian distribution, then $P(x|\omega_1) = P(x|\omega_2) > 0$ and using Bayes' rule gives

$$P(x|\omega_2)P(\omega_2) \leq P(x|\omega_1)P(\omega_1) \Leftrightarrow P(\omega_2) \leq P(\omega_1)$$