Solution

Homework 1; Mathematical Foundations of Machine Learning

Release Date: September 5th, 2024

DUE: **September 19th, 2024, 11:59 PM CAT**

1. Systems of Linear equations (70 points)

1.1 A system of linear equations I (20 points)

Consider the system of linear equations below:

$$
\begin{aligned}
3x + y &= 6 \\
-2x + 2y + 8z &= -8 \\
4x + 4y + 8z &= 4
\end{aligned}
$$

1. Express the system in a compact matrix form, $\mathbf{Ax} = \mathbf{b}$

$$
\begin{bmatrix} 3 & 1 & 0 \\ -2 & 2 & 8 \\ 4 & 4 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 4 \end{bmatrix}
$$

2. Use Gaussian elimination to determine if the system has no solution, one unique solution or infinitely many solutions and justify your answer

$$
\begin{bmatrix} 3 & 1 & 0 & | & 6 \\ -2 & 2 & 8 & | & -8 \\ 4 & 4 & 8 & | & 4 \end{bmatrix}
$$

$$
\begin{array}{r} \\ \frac{1}{2}\text{Row2} \\ \frac{1}{4}\text{Row3} \end{array}
\begin{bmatrix} 3 & 1 & 0 & | & 6 \\ -1 & 1 & 4 & | & -4 \\ 1 & 1 & 2 & | & 1 \end{bmatrix}
$$

$$
\begin{array}{r} \\ \text{Row1} + 3\text{Row2} \\ \text{Row1} - 3\text{Row3} \end{array}
\begin{bmatrix} 3 & 1 & 0 & | & 6 \\ 0 & 4 & 12 & | & -6 \\ 0 & -2 & -6 & | & 3 \end{bmatrix}
$$

$$
\begin{array}{r} \\ \\ \text{Row2} + 2\text{Row3} \end{array}
\begin{bmatrix} 3 & 1 & 0 & | & 6 \\ 0 & 4 & 12 & | & -6 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}
$$

Since there are 2 pivots, 3 and 4, the matrix has a rank of 2. The system does not have a unique solution since the matrix is not full ranked (i.e rank(A) != 3). The system is not inconsistent. Hence there are infinitely many solutions.

3. If the system has a solution, what is the solution?

We proceed with Gaussian elimination till we obtain the reduced row-echelon form

$$\text{Row1} - \tfrac{1}{4}\text{Row2} \begin{bmatrix} 3 & 0 & -3 & | & \frac{15}{2} \\ 0 & 4 & 12 & | & -6 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{array}{r} \frac{1}{3}\text{Row1} \\ \frac{1}{4}\text{Row2} \\ \end{array} \begin{bmatrix} 1 & 0 & -1 & | & \frac{5}{2} \\ 0 & 1 & 3 & | & -\frac{3}{2} \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\text{Particular solution: } \mathbf{x} = \begin{bmatrix} \frac{5}{2} \\ -\frac{3}{2} \\ 0 \end{bmatrix}$$

Solution to Ax' = 0:

$$\begin{bmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & 3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\text{x'} = \lambda \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}, \text{for all } \lambda \in \mathbb{R}$$

$$\textbf{General solution: } \mathbf{x} = \begin{bmatrix} \frac{5}{2} \\ -\frac{3}{2} \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}, \text{for all } \lambda \in \mathbb{R}$$

1.1.2 Determine whether the following systems of equations (or matrix equations) described below have no solution, one unique solution, or infinitely many solutions, and justify your answer.

(a)

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

where a, b, c, d are scalars satisfying $\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$.

Performing Gaussian elimination to row-echelon form

$$\begin{bmatrix} a & b & | & c \\ d & e & | & f \end{bmatrix}$$

$$\text{Row2} - \tfrac{d}{a}\text{Row1} \begin{bmatrix} a & b & | & c \\ 0 & e - \frac{db}{a} & | & f - \frac{dc}{a} \end{bmatrix}$$

$$\text{Since } \frac{a}{b} = \frac{b}{e}; e = \frac{db}{a}$$

$$\text{Since } \frac{a}{d} = \frac{c}{f}; f = \frac{dc}{a}$$

Augmented matrix becomes:

$$\begin{bmatrix} a & b & | & c \\ 0 & 0 & | & 0 \end{bmatrix}$$

There are infinitely many solutions because A is not full ranked as shown from Gaussian elimination. And the system is consistent.

(b) $A\mathbf{x} = \mathbf{0}$, where A is a non-singular matrix.

Since A is non-singular, it has full rank and is invertible. Also, the zero vector is in the column space of A. Therefore there is a unique solution

(c) A homogeneous system of 3 equations in 4 unknowns.

This is of the form $Ax = b$ where A is an 3×4 matrix. Even if A has full rank, the 4th column will always be linearly dependent on the first 3 columns. As such, there would be infinitely many solutions.

(d) $A\mathbf{x} = \mathbf{b}$, where the row-reduced echelon form of the augmented matrix $[A \mid \mathbf{b}]$ looks as follows:

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

No solution. The system is inconsistent.

1.2 Pseudo-inverse (20 points)

Given the matrix A $= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$

1. Compute $\mathbf{A}^\top \mathbf{A}$ and show that it is symmetric

$$\mathbf{A}^\top = \begin{bmatrix} 0 & 1 & -2 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \mathbf{A}^\top \mathbf{A} &= \begin{bmatrix} 0 & 1 & -2 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 6 & -3 & -2 \\ -3 & 3 & 2 \\ -2 & 2 & 3 \end{bmatrix} \end{aligned}$$

This matrix is symmetric because for any $i, j \in [1, 2, 3]$, $(\mathbf{A}^\top \mathbf{A})_{ij} = (\mathbf{A}^\top \mathbf{A})_{ji}$

2. Compute $(\mathbf{A}^\top \mathbf{A})^{-1}$ using Gaussian Elimination

Augmented matrix is as follows:

$$\left[\begin{array}{ccc|ccc} 6 & -3 & -2 & 1 & 0 & 0 \\ -3 & 3 & 2 & 0 & 1 & 0 \\ -2 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

Performing Gaussian elimination till reduced row-echelon form:

$$\begin{array}{r} \\ 2\text{Row2} + \text{Row1} \\ 3\text{Row3} + \text{Row1} \end{array} \left[\begin{array}{ccc|ccc} 6 & -3 & -2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 2 & 0 \\ 0 & 3 & 7 & 1 & 0 & 3 \end{array}\right]$$

$$\begin{array}{r} \\ \\ \text{Row3} - \text{Row2} \end{array} \left[\begin{array}{ccc|ccc} 6 & -3 & -2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 2 & 0 \\ 0 & 0 & 5 & 0 & -2 & 3 \end{array}\right]$$

$$\begin{array}{r} \text{Row1} + \text{Row2} \\ \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 6 & 0 & 0 & 2 & 2 & 0 \\ 0 & 3 & 2 & 1 & 2 & 0 \\ 0 & 0 & 5 & 0 & -2 & 3 \end{array}\right]$$

$$\begin{array}{r} \\ \text{Row2} - \frac{2}{5}\text{Row3} \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 6 & 0 & 0 & 2 & 2 & 0 \\ 0 & 3 & 0 & 1 & \frac{14}{5} & -\frac{6}{5} \\ 0 & 0 & 5 & 0 & -2 & 3 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{6}\text{Row1} \\ \frac{1}{3}\text{Row2} \\ \frac{1}{5}\text{Row3} \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & 0 & \frac{1}{3} & \frac{14}{15} & -\frac{6}{15} \\ 0 & 0 & 1 & 0 & -\frac{2}{5} & \frac{3}{5} \end{array}\right]$$

Therefore, $(\mathbf{A}^\top\mathbf{A})^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{14}{15} & -\frac{6}{15} \\ 0 & -\frac{2}{5} & \frac{3}{5} \end{bmatrix}$

3. Let $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, if $\mathbf{Ax} = \mathbf{b}$ show that $\mathbf{x} = (\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top\mathbf{b}$ and obtain the value of $\mathbf{x}$. [Note: $\mathbf{x} = (\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top\mathbf{b}$ is called the Normal equation]

$$\begin{aligned} \mathbf{x} &= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{14}{15} & -\frac{6}{15} \\ 0 & -\frac{2}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 0 & 1 & -2 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{14}{15} & -\frac{6}{15} \\ 0 & -\frac{2}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ \frac{4}{5} \\ -\frac{1}{5} \end{bmatrix} \end{aligned}$$

1.3 A system of linear equations II (30 points)

CMU-Africa is trying to understand the pricing strategy used by Delight Canteen. The Canteen sells various types of food, each with a different price, but only charges a total price for all food types on a student's plate. As a machine learning engineer, you want to help students determine the price of their food based on the quantity (measured in Grams) of each foot type they add to their plate. In other to achieve this, you collect data from 6 of your friends on the quantity of each food type they served and the total price. The observations from your friends are recorded below.

Transaction	Food a (g)	Food b (g)	Food c (g)	Food d (g)	Total Cost (RWF)
1	100	50	150	200	2500
2	50	50	100	300	2300
3	100	150	200	100	3000
4	50	200	300	50	2900
5	200	50	250	50	3100
6	300	50	50	200	4300

From the observations in the table above, you are expected to obtain the price per quantity of each food type (mearsured in RWF/g).

1. Describe the above using a system of linear equations

$$
\begin{aligned}
100a + 50b + 150c + 200d &= 2500 \\
50a + 50b + 100c + 300d &= 2300 \\
100a + 150b + 200c + 100d &= 3000 \\
50a + 200b + 300c + 50d &= 2900 \\
200a + 50b + 250c + 50d &= 3100 \\
300a + 50b + 50c + 200d &= 4300
\end{aligned}
$$

2. Write the system of linear equations in a compact matrix form, $\mathbf{A}\mathbf{x} = \mathbf{b}$

$$
\begin{bmatrix} 100 & 50 & 150 & 200 \\ 50 & 50 & 100 & 300 \\ 100 & 150 & 200 & 100 \\ 50 & 200 & 300 & 50 \\ 200 & 50 & 250 & 50 \\ 300 & 50 & 50 & 200 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2500 \\ 2300 \\ 3000 \\ 2900 \\ 3100 \\ 4300 \end{bmatrix}
$$

3. Use Gaussian elimination to obtain a solution for the system.

$$\begin{bmatrix} 100 & 50 & 150 & 200 & | & 2500 \\ 50 & 50 & 100 & 300 & | & 2300 \\ 100 & 150 & 200 & 100 & | & 3000 \\ 50 & 200 & 300 & 50 & | & 2900 \\ 200 & 50 & 250 & 50 & | & 3100 \\ 300 & 50 & 50 & 200 & | & 4300 \end{bmatrix}$$

$$\begin{array}{r} \\ \text{Row2} - \frac{1}{2}\text{Row1} \\ \text{Row3} - \text{Row1} \\ \text{Row4} - \frac{1}{2}\text{Row1} \\ \text{Row5} - 2\text{Row1} \\ \text{Row6} - 3\text{Row1} \end{array} \begin{bmatrix} 100 & 50 & 150 & 200 & | & 2500 \\ 0 & 25 & 25 & 200 & | & 1050 \\ 0 & 100 & 50 & -100 & | & 500 \\ 0 & 175 & 225 & -50 & | & 1650 \\ 0 & -50 & -50 & -350 & | & -1900 \\ 0 & -100 & -400 & -400 & | & -3200 \end{bmatrix}$$

$$\begin{array}{r} \\ \\ \text{Row3} - 4\text{Row2} \\ \text{Row4} - 7\text{Row2} \\ \text{Row5} + 2\text{Row2} \\ \text{Row6} + 4\text{Row2} \end{array} \begin{bmatrix} 100 & 50 & 150 & 200 & | & 2500 \\ 0 & 25 & 25 & 200 & | & 1050 \\ 0 & 0 & -50 & -900 & | & -3700 \\ 0 & 0 & 50 & -1450 & | & -5700 \\ 0 & 0 & 0 & 50 & | & 200 \\ 0 & 0 & 300 & 400 & | & 1000 \end{bmatrix}$$

$$\begin{array}{r} \\ \\ \\ \text{Row4} + \text{Row3} \\ \\ \text{Row6} - 6\text{Row3} \end{array} \begin{bmatrix} 100 & 50 & 150 & 200 & | & 2500 \\ 0 & 25 & 25 & 200 & | & 1050 \\ 0 & 0 & -50 & -900 & | & -3700 \\ 0 & 0 & 0 & -2350 & | & -9400 \\ 0 & 0 & 0 & 50 & | & 200 \\ 0 & 0 & 0 & 5800 & | & 23200 \end{bmatrix}$$

$$\begin{array}{r} \\ \\ \\ \\ \text{Row5} + \frac{1}{47}\text{Row4} \\ \text{Row6} + \frac{5800}{2350}\text{Row4} \end{array} \begin{bmatrix} 100 & 50 & 150 & 200 & | & 2500 \\ 0 & 25 & 25 & 200 & | & 1050 \\ 0 & 0 & -50 & -900 & | & -3700 \\ 0 & 0 & 0 & -2350 & | & -9400 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

This is now in Row Echelon form. We can proceed to reduced row-echelon form

$$\begin{array}{r} \text{Row1} - 2\text{Row2} \\ \\ \\ \\ \\ \\ \end{array} \begin{bmatrix} 100 & 0 & 100 & -200 & | & 400 \\ 0 & 25 & 25 & 200 & | & 1050 \\ 0 & 0 & -50 & -900 & | & -3700 \\ 0 & 0 & 0 & -2350 & | & -9400 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{array}{r} \text{Row1} + 3\text{Row3} \\ \text{Row2} + \frac{1}{2}\text{Row3} \\ \\ \\ \\ \\ \end{array} \begin{bmatrix} 100 & 0 & 0 & -2000 & | & -7000 \\ 0 & 25 & 0 & -250 & | & -800 \\ 0 & 0 & -50 & -900 & | & -3700 \\ 0 & 0 & 0 & -2350 & | & -9400 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{array}{r} \text{Row1} - \frac{2000}{2350}\text{Row4} \\ \text{Row2} - \frac{250}{2350}\text{Row4} \\ \text{Row3} - \frac{900}{2350}\text{Row4} \\ \\ \\ \\ \end{array} \begin{bmatrix} 100 & 0 & 0 & 0 & | & 1000 \\ 0 & 25 & 0 & 0 & | & 200 \\ 0 & 0 & -50 & 0 & | & 100 \\ 0 & 0 & 0 & -2350 & | & -9400 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{array}{r} \frac{1}{100}\text{Row1} \\ \frac{1}{25}\text{Row2} \\ -\frac{1}{50}\text{Row3} \\ -\frac{1}{2350}\text{Row4} \\ \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, $a = 10, b = 8, c = 2,$ and $d = 4$

4. What is the rank of the matrix $\mathbf{A}$ of the compact matrix form

   4

5. What is rank of the augmented matrix $[\mathbf{A}|\mathbf{b}]$

   4

6. Using python and numpy show that the solution obtained using Gaussian Elimination is the same as that of the normal equation.
   Note that the normal equation is expressed as $\mathbf{x} = (\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top\mathbf{b}$. Use the function numpy.linalg.inv to compute the inverse of a matrix.

2. Linear Independence, Basis and Rank (30 points)

2.1 Which of the following sets are subspaces of $\mathbb{R}^3$?

1. $\mathbf{U}_1 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_1 \leq 0 \right\}$

   Not a subspace.

   Fails closure under scalar multiplication:
   for any $\lambda \in \mathbb{R}$,
   if $\lambda < 0$, then $\lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \notin \mathbf{U}_1$ since $\lambda x_1 \geq 0$

2. $\mathbf{U}_2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_1 + 2x_2 + x_3 = 0 \right\}$

   It is a subspace.

   Satisfies closure under scalar multiplication: for any $\lambda \in \mathbb{R}$, $\lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbf{U}_1$

   because; $\lambda x_1 + 2\lambda x_2 + \lambda x_3 = \lambda(x_1 + 2x_2 + x_3) = \lambda(0) = 0$

   Satisfies closure under vector addition: for any $u, v \in \mathbf{U}_2$ if $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

   $u + v = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix} \in \mathbf{U_2}$

because; $(u_1+v1)+2(u_2+v_2)+(u_3+v_3) = (u_1+2u_2+u_3)+(v_1+2v_2+v_3) = 0+0 = 0$

Contains the zero vector; $\mathbf{0} \in \mathbf{U}_2$
because if $x_1 = x_2 = x_3 = 0$ then $x_1 + 2x_2 + x_3 = 0 + 2(0) + 0 = 0$

3. $\mathbf{U}_3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_3 = 1, x_2 = 2x_1 \right\}$

   Not a subspace.

   Does not contain the zero vector; $\mathbf{0} \notin \mathbf{U}_3$
   This is because x_3 can never be 0

4. $\mathbf{U}_4 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_3 = 0 \right\}$

   It is a subspace.

   Satisfies closure under scalar multiplication; for any $\lambda \in \mathbb{R}$, $\lambda \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} \in \mathbf{U}_4$

   Satisfies closure under vector addition
   for any $\begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix} \in \mathbf{U}_4$, $\begin{bmatrix} u_1 + v_1 \\ u_2 + u_2 \\ 0 \end{bmatrix} \in \mathbf{U}_4$

   Contains the zero vector; $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in \mathbf{U}_4$

2.2 Consider the following vectors

$$\mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix} \in \mathbb{R}^3$$

and

$$\alpha, \beta \in \mathbb{R}$$

1. Express $\mathbf{w}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$, of the form,

$$\mathbf{w} = \alpha \mathbf{u} + \beta \mathbf{v}$$

   Are $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ linearly independent ? What is the rank of the matrix whose columns are $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$?

$$\begin{bmatrix} 3 & 5 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix}$$

Performing Gaussian elimination on the augmented matrix:

$$\left[\begin{array}{cc|c} 3 & 5 & 3 \\ 2 & 3 & 3 \\ 3 & 4 & 6 \end{array}\right]$$

$$\begin{array}{r} 2\text{Row1} \\ 3\text{Row2} \\ 2\text{Row3} \end{array} \left[\begin{array}{cc|c} 6 & 10 & 6 \\ 6 & 9 & 9 \\ 6 & 8 & 12 \end{array}\right]$$

$$\begin{array}{r} \\ \text{Row2} - \text{Row1} \\ \text{Row3} - \text{Row1} \end{array} \left[\begin{array}{cc|c} 6 & 10 & 6 \\ 0 & -1 & 3 \\ 0 & -2 & 6 \end{array}\right]$$

$$\begin{array}{r} \\ \\ \frac{1}{2}\text{Row3} \end{array} \left[\begin{array}{cc|c} 6 & 10 & 6 \\ 0 & -1 & 3 \\ 0 & -1 & 3 \end{array}\right]$$

$$\begin{array}{r} \\ \\ \text{Row3} - \text{Row2} \end{array} \left[\begin{array}{cc|c} 6 & 10 & 6 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{r} \text{Row1} + 10\text{Row2} \\ \\ \\ \end{array} \left[\begin{array}{cc|c} 6 & 0 & 36 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{6}\text{Row1} \\ -1\text{Row2} \\ \\ \end{array} \left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore, $\alpha = 6, \beta = -3$

$\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ are not linearly independent

The rank of $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is 2 (There only 2 pivots in the augmented matrix above)

2. Let $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \in \mathbb{R}^3$. Show that the set of vectors $\{\mathbf{u}, \mathbf{v}, \mathbf{x}\}$ are linearly independent. What is the rank of the matrix whose columns are $\mathbf{u}, \mathbf{v}$, and $\mathbf{x}$?

$$\begin{bmatrix} 3 & 5 & 3 \\ 2 & 3 & 1 \\ 3 & 4 & 2 \end{bmatrix}$$

Determinant = 3(6-4) -5(4-3) + 3(8-9) = 6 -5 -3 = -2
Since the determinant is not zero, then the matrix above has full rank and the columns are linearly independent. Hence $\mathbf{u}, \mathbf{v}$, and $\mathbf{x}$ are linearly independent.

The rank of the matrix is 3

3. Find $a, b, c \in \mathbb{R}$ such that

$$a\mathbf{u} + b\mathbf{v} + c\mathbf{x} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$$

Performing Gaussian elimination on the augmented matrix:

$$\begin{bmatrix} 3 & 5 & 3 & | & 5 \\ 2 & 3 & 1 & | & 2 \\ 3 & 4 & 2 & | & 1 \end{bmatrix}$$

$$\begin{array}{r} 2\text{Row1} \\ 3\text{Row2} \\ 2\text{Row3} \end{array} \begin{bmatrix} 6 & 10 & 6 & | & 10 \\ 6 & 9 & 3 & | & 6 \\ 6 & 8 & 4 & | & 2 \end{bmatrix}$$

$$\begin{array}{r} \\ \text{Row2} - \text{Row1} \\ \text{Row3} - \text{Row1} \end{array} \begin{bmatrix} 6 & 10 & 6 & | & 10 \\ 0 & -1 & -3 & | & -4 \\ 0 & -1 & -1 & | & -4 \end{bmatrix}$$

$$\begin{array}{r} \\ \\ \text{Row3} - \text{Row2} \end{array} \begin{bmatrix} 6 & 10 & 6 & | & 10 \\ 0 & -1 & -3 & | & -4 \\ 0 & 0 & 2 & | & 0 \end{bmatrix}$$

$$\begin{array}{r} \text{Row1} + 10\text{Row2} \\ \\ \\ \end{array} \begin{bmatrix} 6 & 0 & -24 & | & -30 \\ 0 & -1 & -3 & | & -4 \\ 0 & 0 & 2 & | & 0 \end{bmatrix}$$

$$\begin{array}{r} \text{Row1} + 12\text{Row3} \\ \text{Row2} + \frac{3}{2}\text{Row3} \\ \\ \end{array} \begin{bmatrix} 6 & 0 & 0 & | & -30 \\ 0 & -1 & 0 & | & -4 \\ 0 & 0 & 2 & | & 0 \end{bmatrix}$$

$$\begin{array}{r} \frac{1}{6}\text{Row1} \\ -\text{Row2} \\ \frac{1}{2}\text{Row3} \end{array} \begin{bmatrix} 1 & 0 & 0 & | & -5 \\ 0 & 1 & 0 & | & 4 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

Therefore, $a = -5, b = 4,$ and $c = 0$

4. Let $\mathbf{y} = \begin{bmatrix} 4 \\ 2 \\ k \end{bmatrix} \in \mathbf{R}^3$, find k such that $\mathbf{u}, \mathbf{v}$ and $\mathbf{y}$ are linearly dependent.

$$\begin{bmatrix} 3 & 5 & 4 \\ 2 & 3 & 2 \\ 3 & 4 & k \end{bmatrix}$$

$$\text{Determinant} = 3(3k - 8) - 5(2k - 6) + 4(8 - 9) = 9k - 24 - 10k + 30 - 4 = -k + 2$$

If the columns are linearly dependent, then $-k + 2 = 0$

$$k = 2$$

5. Is $\{\mathbf{u}, \mathbf{v}, \mathbf{x}\}$ a basis of $\mathbb{R}^3$? why ?

Yes. $\mathbf{u}, \mathbf{v}$, and $\mathbf{x}$ are linearly independent as obtained in 2. above

6. If $\mathbf{B} = \{\mathbf{u}, \mathbf{v}, \mathbf{x}\}$, obtain $\begin{bmatrix} -1 \\ -6 \\ 6 \end{bmatrix}$ as a linear combination of $\mathbf{B}$

Performing Gaussian elimination on the augmented matrix:

$$
\begin{bmatrix} 3 & 5 & 3 & | & -1 \\ 2 & 3 & 1 & | & -6 \\ 3 & 4 & 2 & | & 6 \end{bmatrix}
$$

$$
\begin{array}{r} 2\text{Row1} \\ 3\text{Row2} \\ 2\text{Row2} \end{array}
\begin{bmatrix} 6 & 10 & 6 & | & -2 \\ 6 & 9 & 3 & | & -18 \\ 6 & 8 & 4 & | & 12 \end{bmatrix}
$$

$$
\begin{array}{r} \\ \text{Row2} - \text{Row1} \\ \text{Row3} - \text{Row1} \end{array}
\begin{bmatrix} 6 & 10 & 6 & | & -2 \\ 0 & -1 & -3 & | & -16 \\ 0 & -2 & -2 & | & 14 \end{bmatrix}
$$

$$
\begin{array}{r} \\ \\ \text{Row3} - 2\text{Row2} \end{array}
\begin{bmatrix} 6 & 10 & 6 & | & -2 \\ 0 & -1 & -3 & | & -16 \\ 0 & 0 & 4 & | & 46 \end{bmatrix}
$$

$$
\begin{array}{r} \text{Row1} + 10\text{Row2} \\ \\ \\ \end{array}
\begin{bmatrix} 6 & 0 & -24 & | & -162 \\ 0 & -1 & -3 & | & -16 \\ 0 & 0 & 4 & | & 46 \end{bmatrix}
$$

$$
\begin{array}{c} \text{Row1} - \text{Row3} \\ 4\text{Row2} \\ 3\text{Row3} \end{array}
\begin{bmatrix} 6 & 0 & 0 & | & 114 \\ 0 & -4 & -12 & | & -64 \\ 0 & 0 & 12 & | & 138 \end{bmatrix}
$$

$$
\begin{array}{r} \\ \text{Row2} + \text{Row3} \\ \\ \end{array}
\begin{bmatrix} 6 & 0 & 0 & | & 114 \\ 0 & -4 & 0 & | & 74 \\ 0 & 0 & 12 & | & 138 \end{bmatrix}
$$

$$
\begin{array}{c} \frac{1}{6}\text{Row1} \\ -\frac{1}{4}\text{Row2} \\ \frac{1}{12}\text{Row3} \end{array}
\begin{bmatrix} 1 & 0 & 0 & | & 19 \\ 0 & 1 & 0 & | & -18.5 \\ 0 & 0 & 1 & | & 11.5 \end{bmatrix}
$$

Hence, $\begin{bmatrix} -1 \\ -6 \\ 6 \end{bmatrix} = 19 \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} - 18.5 \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix} + 11.5 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$