1. $T(v_1) = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$, $T(v_2) = \begin{bmatrix} 0 \\ 8 \\ 10 \end{bmatrix}$

$$A \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 4 & 8 \\ 6 & 10 \end{bmatrix}$$

$$A \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 4 & 8 \\ 6 & 10 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 0 \\ 4 & 8 \\ 6 & 10 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

finding the inverse

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 2 & 0 & 1 & 1 \\ 0 & -2 & -1 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1/2 & 1/2 \\ 0 & -1 & -1/2 & 1/2 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 1/2 & -1/2 \end{array}\right]$$

$$A = \begin{bmatrix} 2 & 0 \\ 4 & 8 \\ 6 & 10 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 6 & -2 \\ 8 & -2 \end{bmatrix}$$

$$T(x) = Ax$$

if $x = \begin{bmatrix} x \\ y \end{bmatrix}$

$$T(x) = \begin{bmatrix} 1 & 1 \\ 6 & -2 \\ 8 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ 6x - 2y \\ 8x - 2y \end{bmatrix}$$

4) $T(b_1) = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}, \quad T(b_2) = \begin{bmatrix} 3 \\ 7 \\ 0 \end{bmatrix}$

$$T(b_1) = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$T(b_2) = \begin{bmatrix} 3 \\ 7 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

$$\begin{bmatrix} 5 & 3 \\ 1 & 7 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$

$$\begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 5 & 3 \\ 1 & 7 \\ 4 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right]$$

$$\begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 1 & 7 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & 5 \\ 0 & 2 \end{bmatrix}$$

Hence, the matrix $\begin{bmatrix} 4 & -2 \\ 1 & 5 \\ 0 & 2 \end{bmatrix}$ is the transformation matrix

that transforms coordinates in V wrt $\mathcal{B}$ into

coordinates in W wrt C

Verify: b_1

$T(b_1)$:

coordinates for $b_1 \rightarrow 1\, b_1 + 0\, b_2$

$\rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 4 & -2 \\ 1 & 5 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \text{Coordinates in } W \text{ wrt } C$$

Check: $4C_1 + 1C_2 + 0C_3 = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$

i.e $4\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$ //

Hence, the transformation matrix of T is

$$A_T = \begin{bmatrix} 4 & -2 \\ 1 & 5 \\ 0 & 2 \end{bmatrix}$$

Note that A_T transforms coordinates wrt B into coordinates wrt C

It's advisable to always work with coordinates whenever the basis vectors are known

If you prefer to work with the original vectors, then

$T(b_1)$:

$$\begin{bmatrix} 4 & -2 \\ 1 & 5 \\ 0 & 2 \end{bmatrix} \underbrace{\begin{bmatrix} | & | \\ b_1 & b_2 \\ | & | \end{bmatrix}^{-1} \begin{bmatrix} | \\ b_1 \\ | \end{bmatrix}}_{\text{equivalent to the coordinate of } b_1 \text{ wrt } B}$$

the result of this is a coordinate wrt C

and $\underbrace{\begin{bmatrix} | & | & | \\ c_1 & c_2 & c_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 1 & 5 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} | & | \\ b_1 & b_2 \\ | & | \end{bmatrix}^{-1}}_{\text{Transformation matrix that transforms vectors wrt } B \text{ to vectors wrt } C.} \begin{bmatrix} | \\ b_1 \\ | \end{bmatrix}$ gives the vector wrt C

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 1 & 5 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 4 \\ -3 & 4 \\ 2 & 2 \end{bmatrix}$$

Hence $\begin{bmatrix} 1 & 4 \\ -3 & 4 \\ 2 & 2 \end{bmatrix}$ is the transformation matrix that transforms vectors wrt B onto vectors wrt C

Verify: b_1

$T(b_1)$:

$$\begin{bmatrix} 1 & 4 \\ -3 & 4 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$$

So, we can have a transformation matrix that operates on the coordinates wrt the basis or directly on the vectors.

$$A_T = \begin{bmatrix} 4 & -2 \\ 1 & 5 \\ 0 & 2 \end{bmatrix}$$ on coordinates

or $$\begin{bmatrix} 1 & 4 \\ -3 & 4 \\ 2 & 2 \end{bmatrix}$$ on vectors

You should work with the coordinates.

Change of basis is only possible when working with the coordinates.

Now, change of Basis

new basis, $\tilde{B} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$, $\tilde{C} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$$\tilde{B} \longrightarrow B$$

$$f(\tilde{b}_1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$f(\tilde{b}_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = c \begin{bmatrix} 1 \\ 1 \end{bmatrix} + d \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & 1/2 \\ -1 & -1/2 \end{bmatrix}$$

Hence, the transformation matrix that transforms coordinates wrt $\tilde{B}$ to coordinates wrt B is

$$\begin{bmatrix} 0 & 1/2 \\ -1 & -1/2 \end{bmatrix}$$

Let $S = \begin{bmatrix} 0 & 1/2 \\ -1 & -1/2 \end{bmatrix}$

$$\tilde{C} \longrightarrow C$$

$$f(\tilde{c}_1) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$f(\tilde{c}_2) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = d\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + e\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + f\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$- \qquad = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

$$f(\tilde{C}_3) = g \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + h \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} g \\ h \\ i \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

$$\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right]$$

$$\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3/2 & 1 & -1 \\ -1/2 & 0 & 1 \\ -1/2 & 0 & 0 \end{bmatrix}$$

$$T = \begin{bmatrix} 3/2 & 1 & -1 \\ -1/2 & 0 & 1 \\ -1/2 & 0 & 0 \end{bmatrix}$$

$$\tilde{A}_T = T^{-1} A_T S$$

must be ~~for~~ for coordinates

T^{-1}

$$\left[\begin{array}{ccc|ccc} 3/2 & 1 & -1 & 1 & 0 & 0 \\ -1/2 & 0 & 1 & 0 & 1 & 0 \\ -1/2 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{ccc|ccc} 3 & 2 & -2 & 2 & 0 & 0 \\ 0 & 2 & 4 & 2 & 6 & 0 \\ 0 & 2 & -2 & 2 & 0 & 6 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 3 & 2 & -2 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 & 0 & 2 \end{array}\right] \quad \left[\begin{array}{ccc|ccc} 3 & 2 & -2 & 2 & 0 & 0 \\ 0 & 2 & 4 & 2 & 6 & 0 \\ 0 & 0 & -6 & 0 & -6 & 6 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 3 & 0 & -6 & 0 & -6 & 0 \\ 0 & 2 & 4 & 2 & 6 & 0 \\ 0 & 0 & -6 & 0 & -6 & 6 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 0 & 0 & -6 \\ 0 & 2 & 4 & 2 & 6 & 0 \\ 0 & 0 & -6 & 0 & -6 & 6 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 0 & 0 & -6 \\ 0 & 6 & 12 & 6 & 18 & 0 \\ 0 & 0 & -12 & 0 & -12 & 12 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 0 & 0 & -6 \\ 0 & 6 & 0 & 6 & 6 & 12 \\ 0 & 0 & -12 & 0 & -12 & 12 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right]$$

$$\therefore \tilde{A}_T = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 1 & 5 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1/2 \\ -1 & -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -4 \\ 5 & 7 \\ 1 & 7 \end{bmatrix} \begin{bmatrix} 0 & 1/2 \\ -1 & -1/2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -7 & -1 \\ -7 & -3 \end{bmatrix}$$