structure which is statistically isotropic, so that pressure gradients applied in different directions produce the same flux, we may write

$$\nabla \bar{p} = -\mu \bar{\mathbf{u}}/k, \tag{4.8.21}$$

where k is a constant, called the *permeability*, which depends on the size and shape of the interstices (being proportional to the square of their linear dimensions, for a given shape). The relation (4.8.21) is known as Darcy's law (Darcy 1856), and has a long history of use in soil mechanics for a wide variety of porous media. Its justification rests partly on the above theoretical argument, and partly on its agreement with measurements of the flow produced by an applied pressure gradient in homogeneous media like sand.

The relation (4.8.21) implies that, when the porous medium is statistically uniform and k is independent of position, the velocity $\bar{\mathbf{u}}$ is irrotational, with the velocity potential ϕ proportional to the pressure as in the case of the Hele Shaw cell. The equation of mass conservation, averaged in the above way, leads to $\nabla . \bar{\mathbf{u}} = 0$, and so

$$\nabla^2 \phi = 0. \tag{4.8.22}$$

This equation is to be solved subject to the conditions of zero normal derivative of ϕ at an impermeable surface and a given constant value of ϕ (or $\bar{p}$) at a free surface of the fluid. (In the case of water percolating through soil, a free surface for the water in the form of an air–water boundary, or 'water table', may occur in the interior of the soil.) In this way many practical problems concerned with seepage from dams, variations in the level of the water table near a well, movement of ground water near a coast due to tidal variations of pressure, etc., have been solved.

Two-dimensional flow in a corner

Suppose that one rigid plane is sliding steadily over another, with constant inclination θ_0, as sketched in figure 4.8.2 for the case $\theta_0 = \frac{1}{2}\pi$. Fluid in the region between the planes is set in motion, as might happen in a cylinder with a moving piston, or at the edge of a blade used to scrape up liquid on a table. Near the intersection O, the gradients of velocity become very large, because the velocity has different values at the two rigid boundaries, and it is a reasonable presumption that viscous forces are dominant. The velocity distribution in the neighbourhood of O will be determined on this assumption, which will then be checked *a posteriori*.

The problem can be made one of steady motion by choosing an origin of co-ordinates at (and moving with) O. In cases of two-dimensional motion with negligible inertia forces, it is convenient to introduce the stream function ψ (§2.2) so that the equation of mass conservation is satisfied identically and the one non-zero component of vorticity becomes $-\nabla^2\psi$. The first of equations (4.8.3) is then

$$\nabla^2(\nabla^2\psi) = 0, \tag{4.8.23}$$

and the boundary conditions, in terms of polar co-ordinates (r, θ), are

$$\frac{\partial \psi}{\partial r} = 0, \quad \frac{1}{r}\frac{\partial \psi}{\partial \theta} = -U, \quad \text{at } \theta = 0,$$

$$\frac{\partial \psi}{\partial r} = 0, \quad \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0, \quad \text{at } \theta = \theta_0.$$

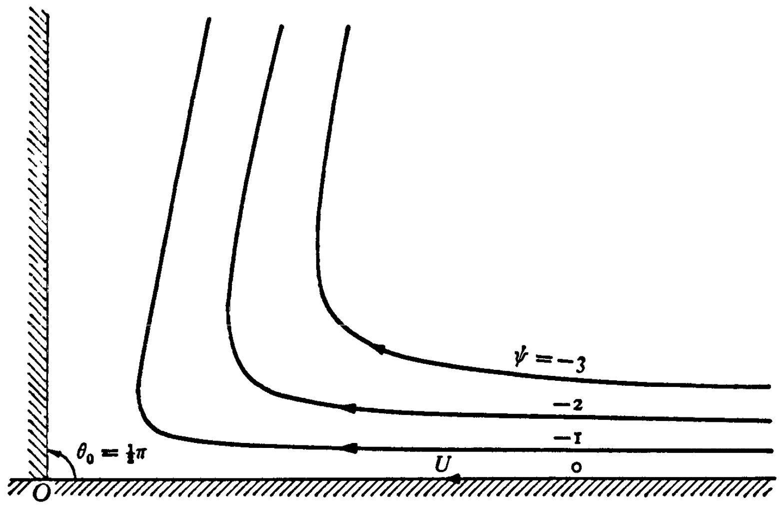

Figure 4.8.2. Two-dimensional flow in a corner due to one rigid plane sliding on another (arbitrary units for ψ).

The form of these boundary conditions is such that ψ could be proportional to r everywhere, and it is worthwhile to enquire if the differential equation allows such a possibility. We therefore write

$$\psi(r, \theta) = rf(\theta), \tag{4.8.24}$$

and substitute in (4.8.23) to find

$$\nabla^2\left\{\frac{1}{r}(f+f'')\right\} = \frac{1}{r^3}(f+2f''+f^{\text{iv}}) = 0.$$

The solution of this equation for f is

$$f(\theta) = A\sin\theta + B\cos\theta + C\theta\sin\theta + D\theta\cos\theta, \tag{4.8.25}$$

and we need to choose values of A, B, C, D such that

$$f(0) = 0, \quad f'(0) = -U, \quad f(\theta_0) = 0, \quad f'(\theta_0) = 0.$$

The required values are

$$A, B, C, D = (-\theta_0^2, 0, \theta_0 - \sin\theta_0\cos\theta_0, \sin^2\theta_0) \times \frac{U}{\theta_0^2 - \sin^2\theta_0}. \tag{4.8.26}$$

Thus we have a solution which satisfies the boundary conditions and the equations of motion with inertia forces neglected. The components of acceleration of the fluid at any point, evaluated according to this solution, are proportional to U^2/r, with a constant of proportionality which varies with θ and is of order unity. The viscous forces, also evaluated according to this solution, are of order $\mu U/r^2$, so that the assumption of negligible inertia forces is self-consistent if $\rho r U/\mu \ll 1$; that is, the solution obtained is valid within a neighbourhood of the intersection O defined by

$$r \ll \mu/\rho U.$$

For lubricating oil at normal temperatures, and $U = 10$ cm/sec, this condition is $r \ll 0{\cdot}4$ cm.

For any flow of the form (4.8.24), the velocity components are independent of r and streamlines with equal increments in ψ cut any radial line at equidistant points. The motion is reversible, since the governing equations and boundary conditions are linear and homogeneous. In the particular case $\theta_0 = \frac{1}{2}\pi$, the solution becomes

$$\psi = \frac{rU}{\frac{1}{4}\pi^2 - 1}\left(-\tfrac{1}{4}\pi^2 \sin\theta + \tfrac{1}{2}\pi\theta\sin\theta + \theta\cos\theta\right), \tag{4.8.27}$$

and the streamlines (of the motion relative to O) for this case are sketched in figure 4.8.2.

It will be noticed that both the normal and tangential components of stress in the fluid vary as r^{-1}, so that the total force exerted on the planes $\theta = 0$ and $\theta = \theta_0$ by the fluid is logarithmically infinite. In practice two plane rigid boundaries do not make perfect geometrical contact, and the maximum stress is finite; what we learn from the above solution is that the total force on the plane boundaries depends on the precise shape of the two boundaries very close to their intersection and that the force increases as the clearance is diminished.

An analogous solution of the equation (4.8.23) of the form $\psi = r^2 f(\theta)$ can be used to describe the two-dimensional flow in the neighbourhood of the point of intersection of two straight rigid boundaries which are in relative rotation about that point. Yet another, of the form $\psi = r^3 f(\theta)$, can be used to describe the flow in the neighbourhood of a point of zero tangential stress at a straight rigid boundary; and since this latter flow has some relevance to the considerations of § 5.10 the solution will be described briefly.

The velocity gradient $\partial u/\partial y$ (the y-axis being normal to the boundary) at the wall changes sign at a point of zero friction O, and there must exist a streamline OP which divides flow coming towards O from the right and from the left, as sketched in figure 4.8.3. If the inclination of this dividing streamline to the boundary is taken as θ_0, it is readily found that a solution of (4.8.23) which satisfies the no-slip condition at the wall and satisfies $\psi = 0$ at $\theta = \theta_0$ is

$$\psi(r, \theta) = Ar^3 \sin^2\theta \sin(\theta_0 - \theta), \tag{4.8.28}$$

where A is an arbitrary constant. The streamlines corresponding to this solution are shown in the figure for $\theta_0 = \frac{1}{3}\pi$ and $A < 0$, and again the velocity may equally be taken to be everywhere in the opposite direction. The region near O in which the solution (4.8.28) is self-consistent is given by $r^3 \ll \mu/\rho\,|A|$. The values of A and θ_0 in this solution are evidently determined by circumstances outside the region in which the solution is valid. However, it can be deduced readily that, whatever the signs of A and $\cos\theta_0$, the directions of the pressure gradient (which, according to the solution (4.8.28), is uniform) and the velocity at $\theta = \theta_0$ lie in the same quadrant.

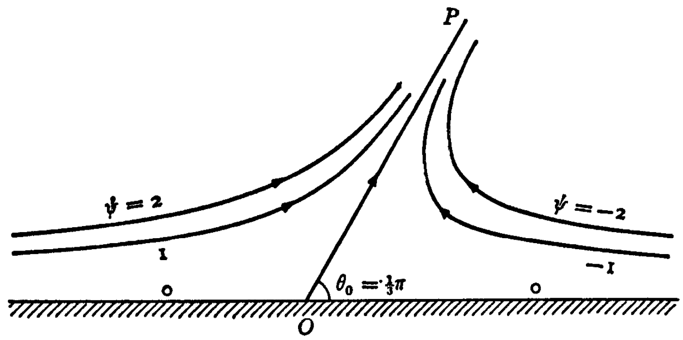

Figure 4.8.3. Two-dimensional flow near a point of zero friction at a plane rigid boundary, according to (4.8.28), with $\theta_0 = \frac{1}{3}\pi$ and $A < 0$ (arbitrary units for ψ).

Uniqueness and minimum dissipation theorems

We shall show that there cannot be more than one solution for the velocity distribution for flow in a given region with negligible inertia forces and consistent with prescribed values of the velocity vector at the boundary of the region (including a hypothetical boundary at infinity when the fluid is of infinite extent). The argument is very similar in form to that used in § 2.7 to establish uniqueness of the solution for the potential ϕ of a solenoidal irrotational velocity field with certain prescribed boundary conditions. An interesting related result is that flow with negligible inertia forces has a smaller total rate of dissipation than any other incompressible flow in the same region with the same values of the velocity vector everywhere on the boundary of the region. (Both results were given first by Helmholtz (1868*a*).)

First let us suppose that u_i, p, e_{ij} and u_i^*, p^*, e_{ij}^* are two sets of distributions of the velocity, pressure and rate-of-strain tensor in a certain region of volume V, both of which satisfy the equations (4.8.1) and (4.8.2); and suppose further that $u_i = u_i^*$ at all points on the boundary of the region (A). Then

$$\begin{aligned}\int (e_{ij}^* - e_{ij})(e_{ij}^* - e_{ij})\, dV &= \int \frac{\partial(u_i^* - u_i)}{\partial x_j}(e_{ij}^* - e_{ij})\, dV \\ &= \int (u_i^* - u_i)(e_{ij}^* - e_{ij})\, n_j\, dA \\ &\quad -\tfrac{1}{2}\int (u_i^* - u_i)\, \nabla^2 (u_i^* - u_i)\, dV \\ &= -\frac{1}{2\mu}\int (u_i^* - u_i)\frac{\partial(p^* - p)}{\partial x_i}\, dV \\ &= -\frac{1}{2\mu}\int (p^* - p)(u_i^* - u_i)\, n_i\, dA = 0,\end{aligned}$$

showing that the rates of strain e_{ij}^* and e_{ij} must be identical everywhere. The