Graph Coloring

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- Swiss mathematician and physician
- Led ot of works about infinitesimal calculus and complex numbers
- Studied moon movement
- Laid the foundation of Graph Theory with the *Koenigsberg bridges problem*.

Plan

1. History
   - Euler
   - Koenigsberg bridges problem
2. Theory
   - Basic Notions
   - Graph coloring problem
   - Resolution method
3. Examples
   - Exams Affectation Problem
   - Antennas Frequencies Problem
4. Conclusion

Problem Presentation

FIGURE: Koenigsberg

Problem Presentation

- City in Prussia (nowadays Kaliningrad in Russia) based on the both sides of *Pregolya River*

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- Cut into 4 parts
- 7 bridges connecting the four sides together

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- Cut into 4 parts
- 7 bridges connecting the four sides together
- Problem : Find a walk through he city that passes through the seven bridges once and only once
- Solved by Euler in 1735

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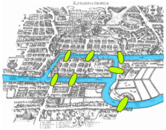
Graph Representation



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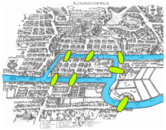

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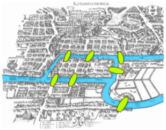

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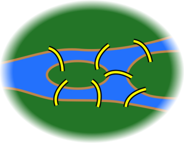

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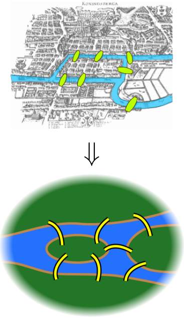

- Each bridge is represented by an edge
- Each part of the town is represented by a node

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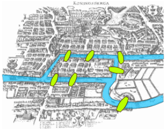

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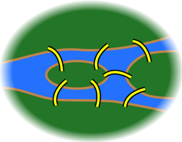

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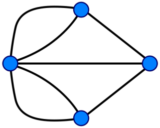

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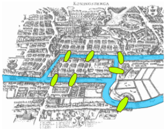

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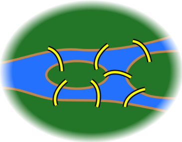

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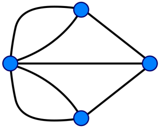

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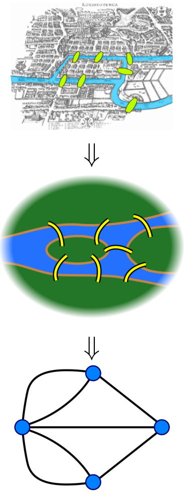

- Each bridge is represented by an edge
- Each part of the town is represented by a node
- The result is a graph
- Every node reached by the walk is left
- The number of edges on each node is even
- That's why it's impossible to find a solution

Graph Theory Celebrities

- **Alexandre-Théophile Vandermonde** (1735 - 1796) : led reasearch on matrix and binary trees

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- **Francis Guthrie** (1831 - 1899) : expounded the *Four colour postulate*
- **Hugo Hedwiger** (1908 - 1981) : laid down the mathematical basics of the presented algorithm

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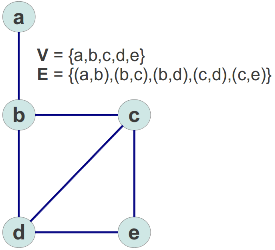
What's a graph ?

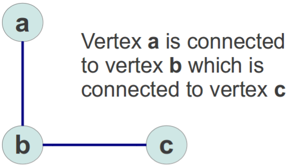


- A graph G is composed of :
  - a set V of n vertices ;
  - a set E of m edges.

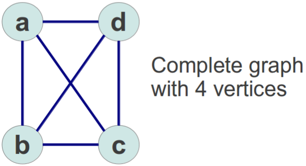
- An edge e = (u,v) is a pair of vertices $u, v \in V$. It means there is a way between u and v.

Other Notions

- Two vertices u and v are connected (or adjacent) if and only if $(u, v) \in E$.

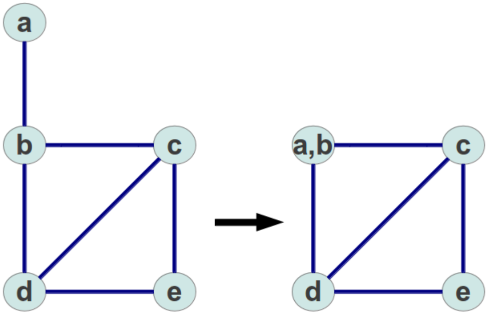


- A graph G is said 'complete' if and only if for each $u, v \in V$ there is an edge $(u, v) \in E$.



Edge contracting example

- To contract an edge e = (u,v), both vertices are unified as a single one.

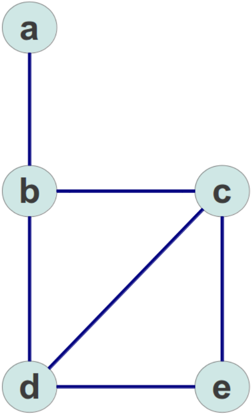


- Same graph with edge (a,b) contracted.

Plan

Graph coloring problem

- The problem of graph coloring consists of setting a color for each vertex of the graph as there is no two connected vertices with same color.



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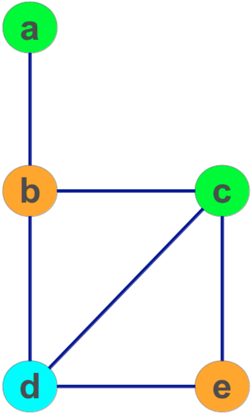
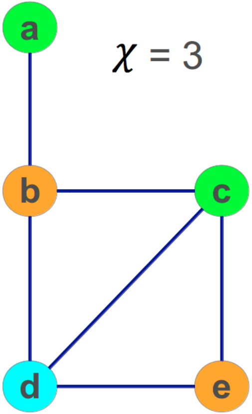


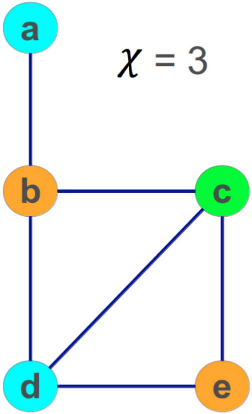
Graph coloring notions

- The chromatic number χ of a graph is the minimal number of colors needed to match the graph coloring problem constraints.



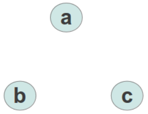
Graph coloring notions

- The chromatic polynom P of a graph is the number of possibilities to match the graph coloring problem constraints with a specific number of colors k.

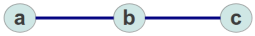


Some specific graphs

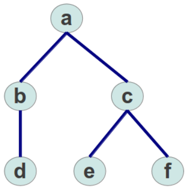
- Stable : there is no edge ($\#E = 0$).



- Way : edges make a way between all vertices : $E = (v_1, v_2), (v_2, v_3), (v_3, v_4), ..., (v_{n-1}, v_n)$;

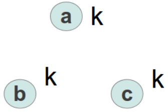


- Tree : there is no circuit in the graph.

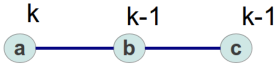


Associated chromatic polynom

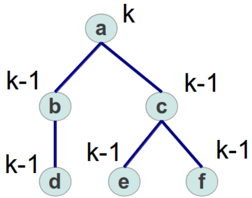
- Chromatic polynom of a stable is equals to $k^{\#V}$;



- Chromatic polynom of a way is equals to $k(k-1)^{\#V-1}$;



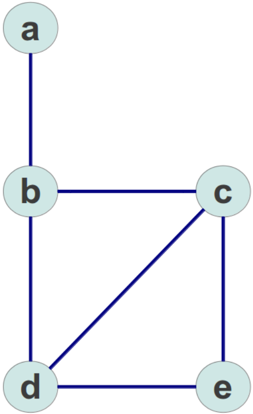
- Chromatic polynom of a tree is equals to $k(k-1)^{\#V-1}$.



Plan

Simple application

- Objective : how much colors will be needed to color this graph ?
- Resolution :
  - **find the chromatic polynom of the graph ;**
  - try to find a solution to this polynom with the smallest k.



Hadwiger magic formula

- The chromatic polynom of a graph G is equals to the chromatic polynom of this graph modified by suppressing an edge minus the chromatic polynom of this graph modified by contracting this edge.

- $P(G)_k = P(G - \{e\})_k - P(G \smile \{e\})_k$

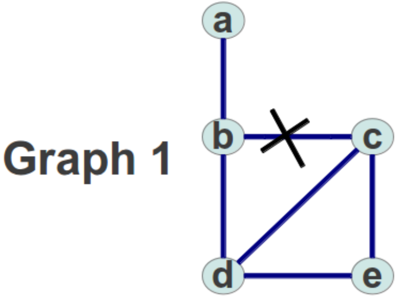
Algorithm 1 ChromaticPolynom

Require: graph **G** : **V** (set of **n** vertices), **E** (set of **m** vertices)
Ensure: P_k^G : chromatic polynom of G

```
if isStable(G) then
    return k^n
else
    if isWay(G) or isTree(G) then
        return k(k − 1)^n
    else
        Edge e ← getEdge(G)
        Graph G1 ← G
        Graph G2 ← G
        removeEdge(G1,e)
        contractEdge(G2,e)
        return (ChromaticPolynom(G1) - ChromaticPolynom(G2))
    end if
end if
```

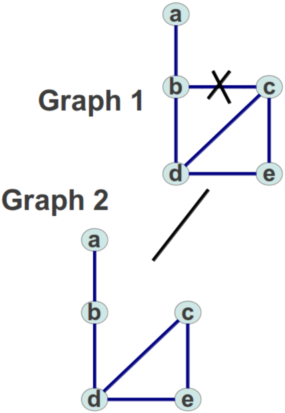
Simple application

- Choose an edge ;



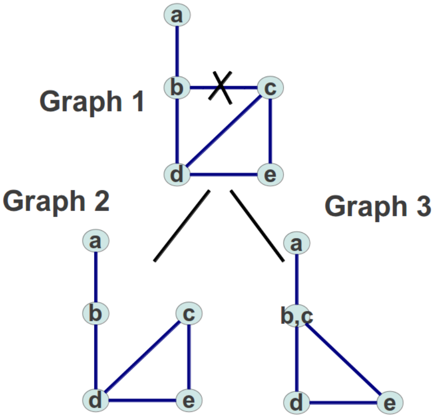
Simple application

- Choose an edge ;
- Draw the graph 2 with this edge suppressed ;



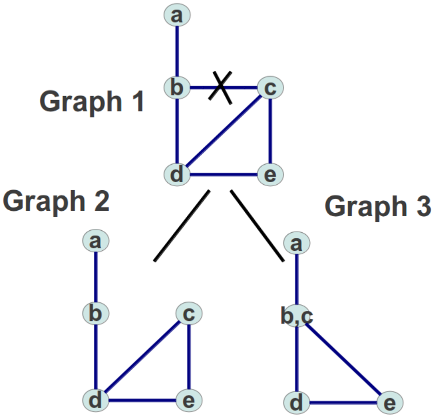
Simple application

- Choose an edge ;
- Draw the graph 2 with this edge suppressed ;
- Draw the graph 3 with this edge contracted.



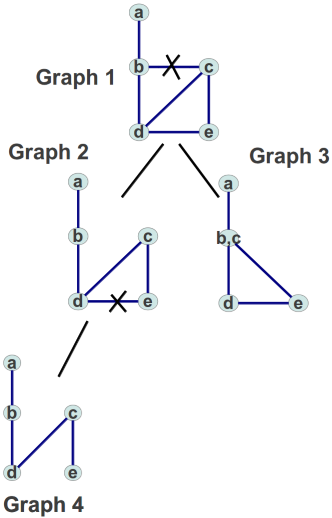
Simple application

- Choose an edge ;
- Draw the graph 2 with this edge suppressed ;
- Draw the graph 3 with this edge contracted.
- These graphs are not simple so apply the formula again.



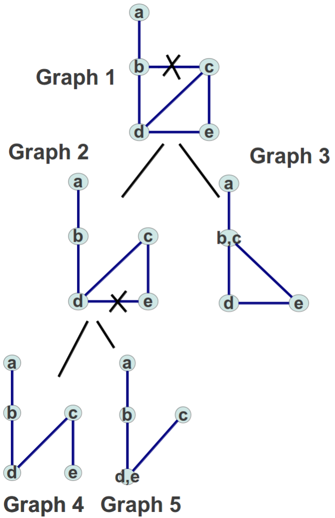
Simple application

- Choose an edge from graph 2 ;
- Draw the graph 4 with this edge suppressed ;



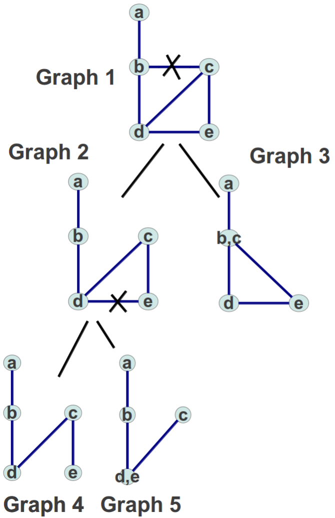
Simple application

- Choose an edge from graph 2 ;
- Draw the graph 4 with this edge suppressed ;
- Draw the graph 5 with this edge contracted.



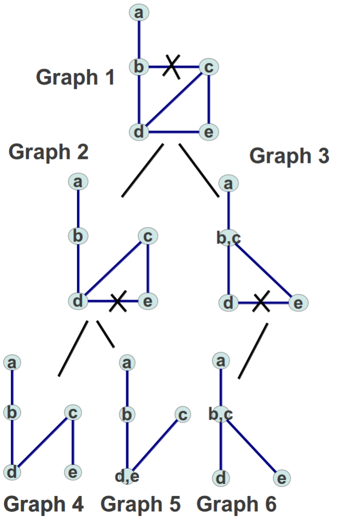
Simple application

- Choose an edge from graph 2 ;
- Draw the graph 4 with this edge suppressed ;
- Draw the graph 5 with this edge contracted.
- These graphs are 'ways' it is easy to determine their chromatic polynom.



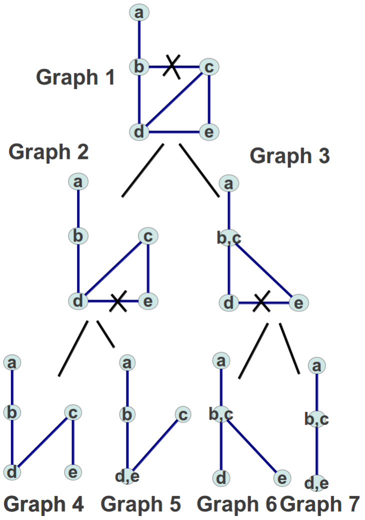
Simple application

- Choose an edge from graph 3 ;
- Draw the graph 6 with this edge suppressed ;



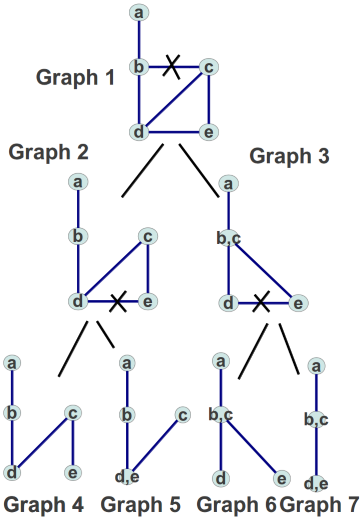
Simple application

- Choose an edge from graph 3 ;
- Draw the graph 6 with this edge suppressed ;
- Draw the graph 7 with this edge contracted.



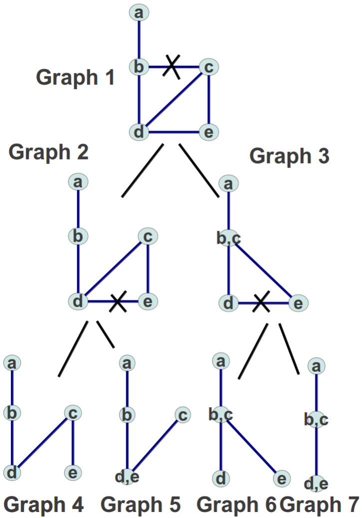
Simple application

- Choose an edge from graph 3 ;
- Draw the graph 6 with this edge suppressed ;
- Draw the graph 7 with this edge contracted.
- Graph 6 is a 'tree' and graph 7 a 'way' it is easy to determine their chromatic polynom.



Simple application

- $P(G4) = k(k-1)^4$;
- $P(G5) = k(k-1)^3$;
- $P(G6) = k(k-1)^3$;
- $P(G7) = k(k-1)^2$.



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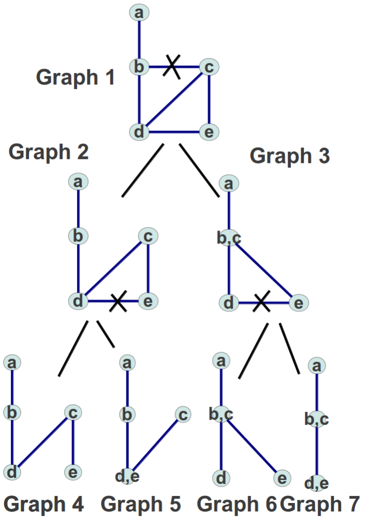
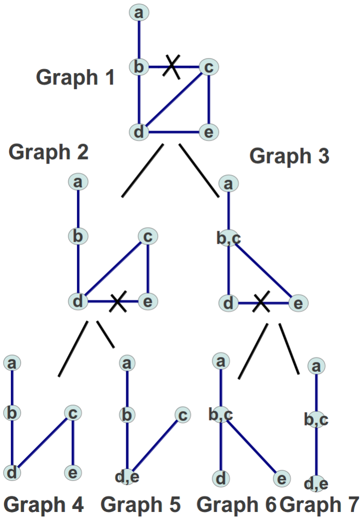
- $P(G7) = k(k-1)^2$.

- $P(G2) = P(G4)$ - $P(G5)$
  $= k(k-1)^4 - k(k-1)^3$;
- $P(G3) = P(G6)$ - $P(G7)$
  $= k(k-1)^3 - k(k-1)^2$;



Simple application

- $P(G2) = k(k-1)^4 - k(k-1)^3$;
- $P(G3) = k(k-1)^3 - k(k-1)^2$;
- $P(G1) = P(G2) - P(G3) = k((k-1)^4 - 2*(k-1)^3 + (k-1)^2)$



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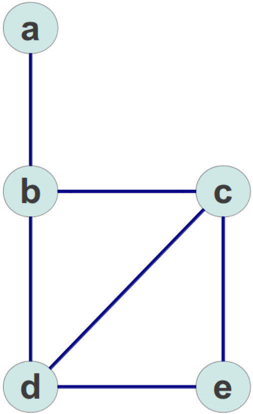
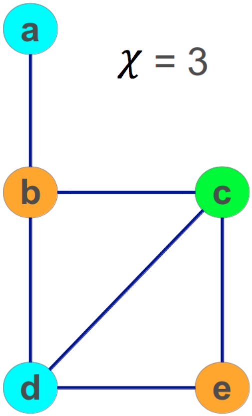


Simple application

- $P(G1) = k((k-1)^4 - 2*(k-1)^3 + (k-1)^2)$.
- Find the smallest k :
  - $P(G1)_{k=1} = 0$ possibility to color the graph with only 1 color.
  - $P(G1)_{k=2} = 0$ possibility to color the graph with 2 colors.
  - $P(G1)_{k=3} = 12$ possibilities to color the graph with 3 colors.
- $\chi(G1) = 3$ colors are needed to color this graph.

Simple application



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Exams Affectation Problem

Problem presentation :

- Some students have to pass their exams

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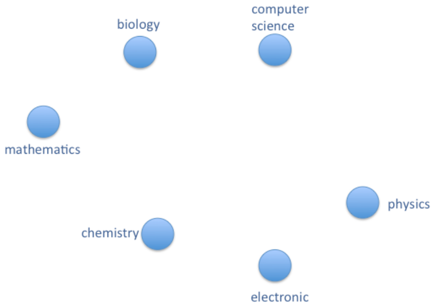
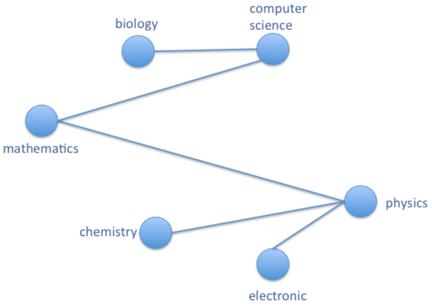


Problem presentation :

- Some students have to pass their exams
- They have 2 courses from 6

Exams Affectation Problem



Problem presentation :

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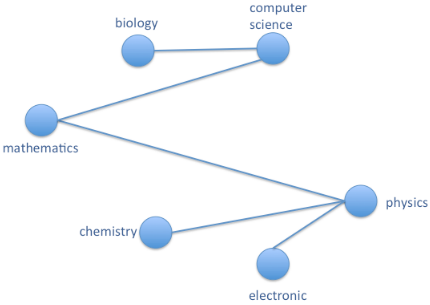
- They have 2 courses from 6
- There are 5 different careers

Exams Affectation Problem

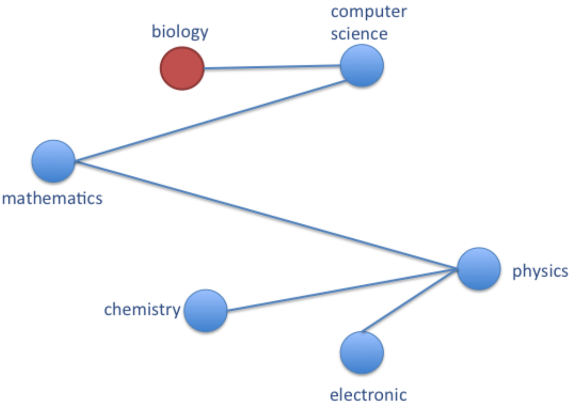


Problem presentation :

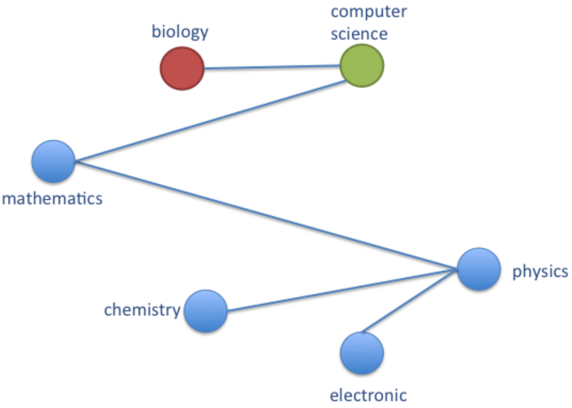
- Some students have to pass their exams
- They have 2 courses from 6
- There are 5 different careers
- How many slots does the administrator need to organize these exams ?

$\Rightarrow$ #slot = #colour

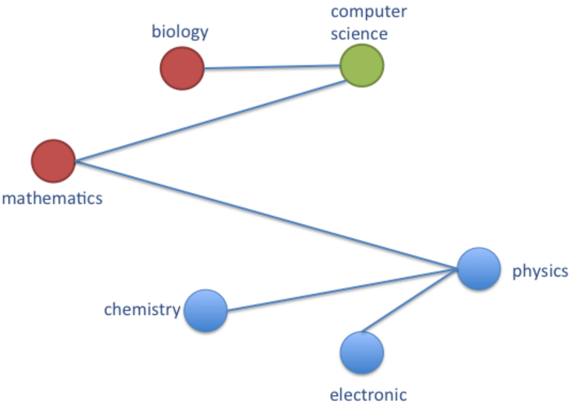
Problem Resolution



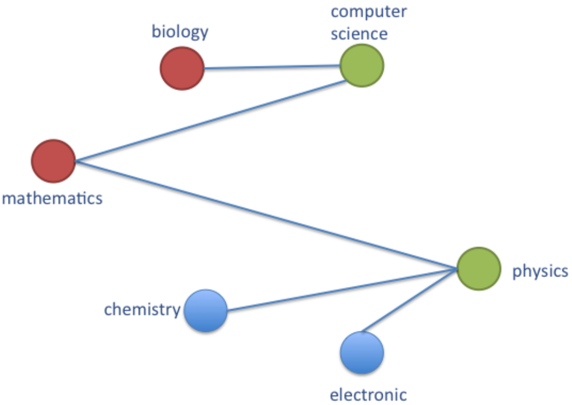
Problem Resolution



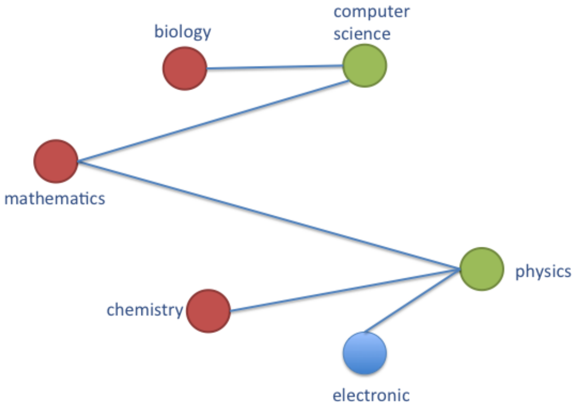
Problem Resolution



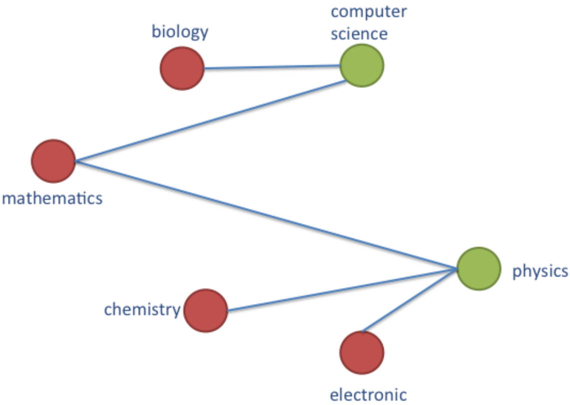
Problem Resolution



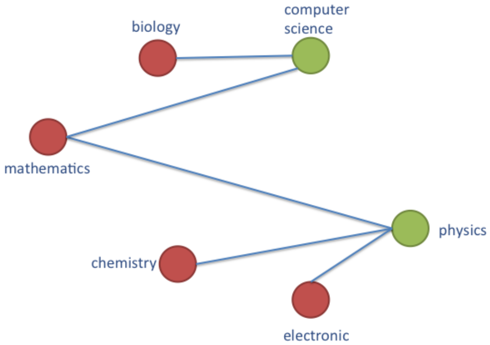
Problem Resolution



Problem Resolution

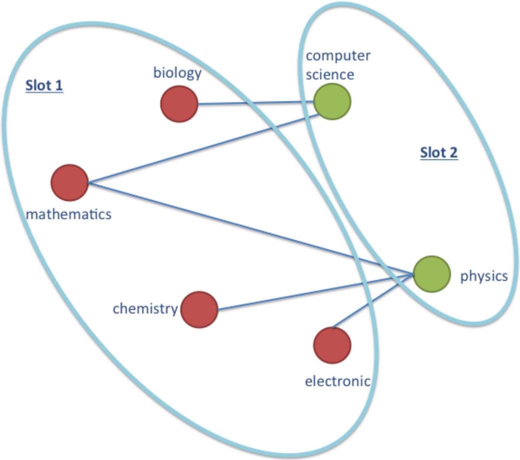


Problem Resolution



The administrator needs 2 slots to organize exams.

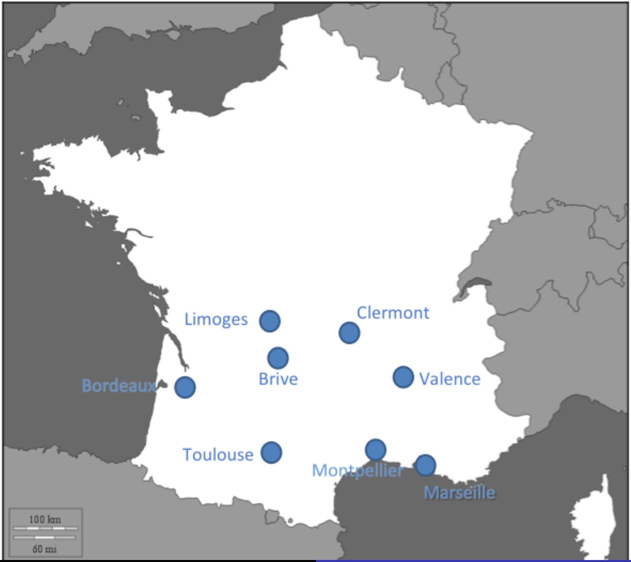
Problem Resolution



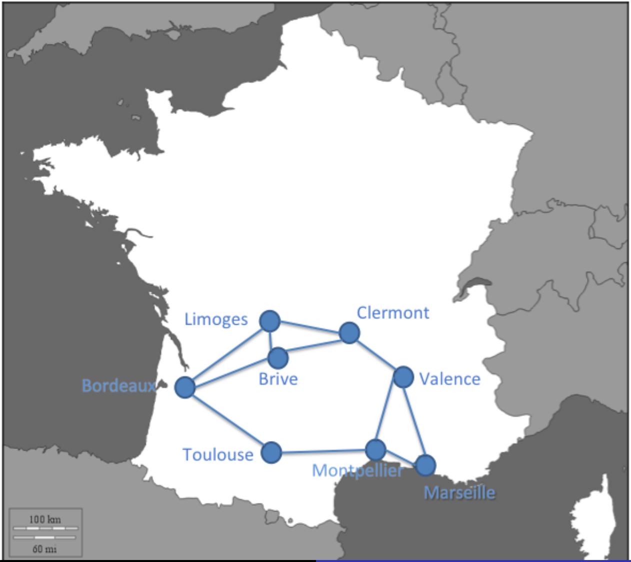
The administrator needs 2 slots to organize exams.

Plan

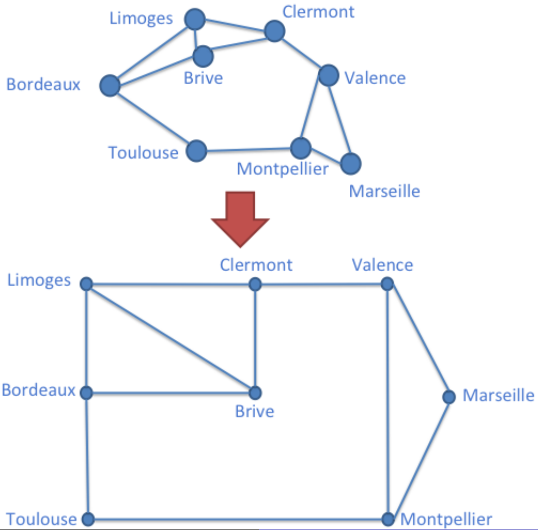
Antennas Frequencies Problem



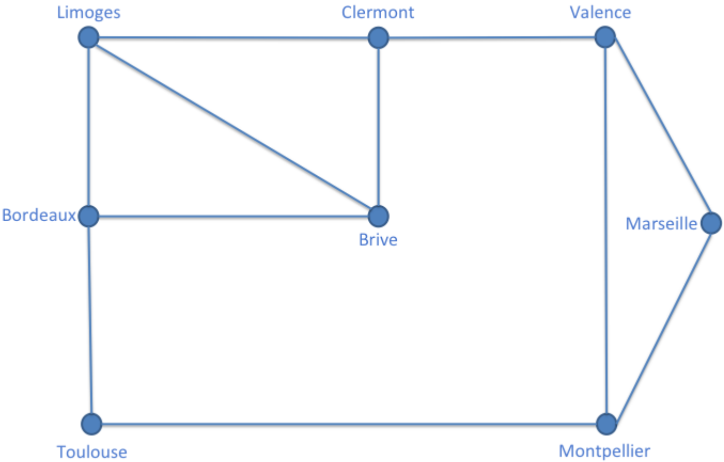
Antennas Frequencies Problem



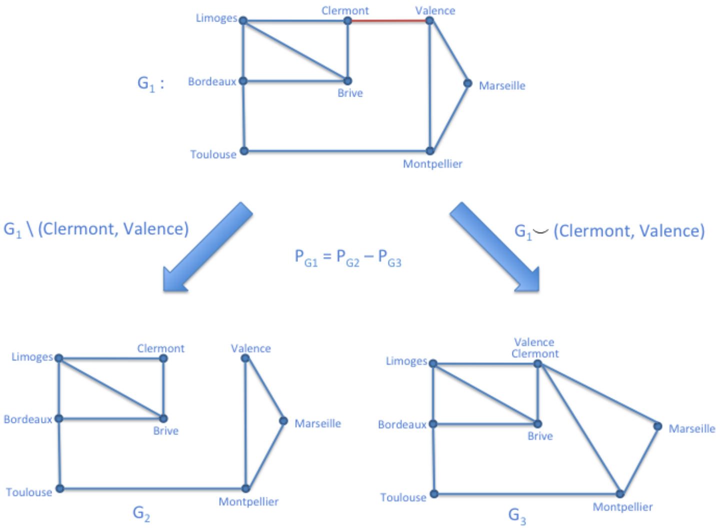
Antennas Frequencies Problem



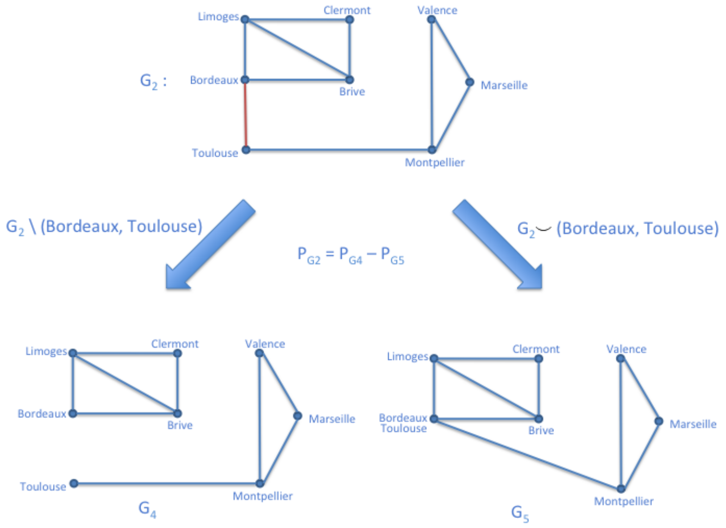
Antennas Frequencies Problem



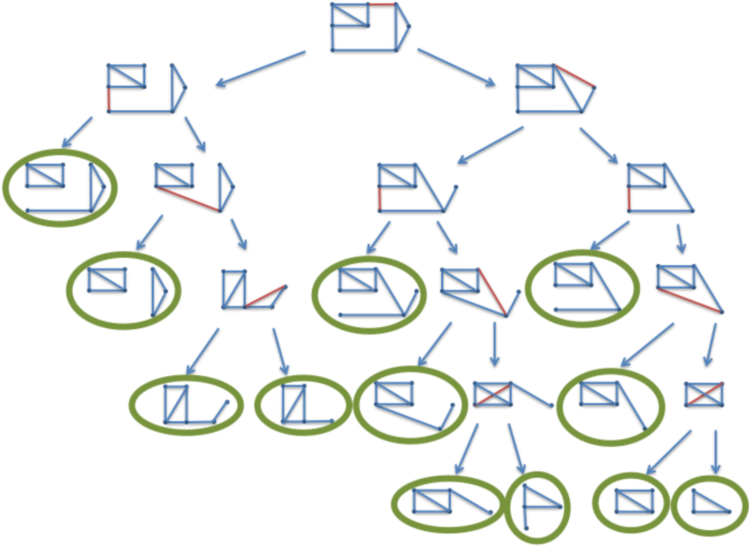
Resolution


G_1 :
Limoges
Clermont
Valence
Bordeaux
Brive
Marseille
Toulouse
Montpellier
$G_1 \setminus$ (Clermont, Valence)
$P_{G1} = P_{G2} - P_{G3}$
$G_1 \smile$ (Clermont, Valence)
Limoges
Clermont
Valence
Bordeaux
Brive
Marseille
Toulouse
Montpellier
G_2
Limoges
Valence
Clermont
Bordeaux
Brive
Marseille
Toulouse
Montpellier
G_3

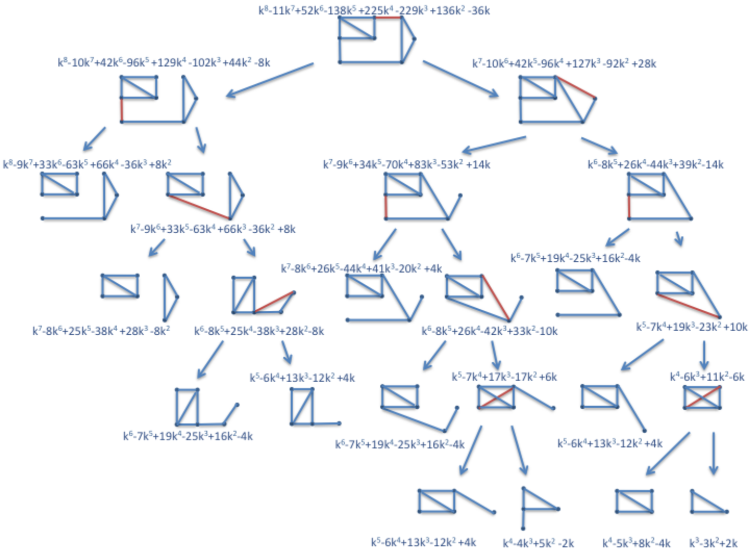
Resolution


G2 :
Limoges
Clermont
Valence
Bordeaux
Brive
Marseille
Toulouse
Montpellier
G2 \ (Bordeaux, Toulouse)
PG2 = PG4 – PG5
G2 (Bordeaux, Toulouse)
Limoges
Clermont
Valence
Bordeaux
Brive
Marseille
Toulouse
Montpellier
G4
Limoges
Clermont
Valence
Bordeaux
Toulouse
Brive
Marseille
Montpellier
G5

Resolution



Resolution


$k^8-11k^7+52k^6-138k^5+225k^4-229k^3+136k^2-36k$
$k^8-10k^7+42k^6-96k^5+129k^4-102k^3+44k^2-8k$
$k^7-10k^6+42k^5-96k^4+127k^3-92k^2+28k$
$k^8-9k^7+33k^6-63k^5+66k^4-36k^3+8k^2$
$k^7-9k^6+34k^5-70k^4+83k^3-53k^2+14k$
$k^6-8k^5+26k^4-44k^3+39k^2-14k$
$k^7-9k^6+33k^5-63k^4+66k^3-36k^2+8k$
$k^6-7k^5+19k^4-25k^3+16k^2-4k$
$k^7-8k^6+26k^5-44k^4+41k^3-20k^2+4k$
$k^7-8k^6+25k^5-38k^4+28k^3-8k^2$
$k^6-8k^5+25k^4-38k^3+28k^2-8k$
$k^6-8k^5+26k^4-42k^3+33k^2-10k$
$k^5-7k^4+19k^3-23k^2+10k$
$k^5-6k^4+13k^3-12k^2+4k$
$k^5-7k^4+17k^3-17k^2+6k$
$k^4-6k^3+11k^2-6k$
$k^6-7k^5+19k^4-25k^3+16k^2-4k$
$k^6-7k^5+19k^4-25k^3+16k^2-4k$
$k^5-6k^4+13k^3-12k^2+4k$
$k^5-6k^4+13k^3-12k^2+4k$
$k^4-4k^3+5k^2-2k$
$k^4-5k^3+8k^2-4k$
k^3-3k^2+2k

Solutions

We're searching the greatest value of k such as the polynom isn't null :

$$P(k) = k^8 - 11k^7 + 52k^6 - 138k^5 + 225k^4 - 229k^3 + 136k^2 - 36k$$

- $k = 1$:

$$(1)^8 - 11(1)^7 + 52(1)^6 - 138(1)^5 + 225(1)^4 - 229(1)^3 + 136(1)^2 - 36 * 1$$

$$= 1 - 11 + 52 - 138 + 225 - 229 + 136 - 36 = 0$$

- $k = 2$:

$$(2)^8 - 11(2)^7 + 52(2)^6 - 138(2)^5 + 225(2)^4 - 229(2)^3 + 136(2)^2 - 36 * 2$$

$$= 256 - 1408 + 3328 - 4416 + 3600 - 1832 + 544 - 72 = 0$$

Solutions

- $k = 3$:

$$(3)^8 - 11(3)^7 + 52(3)^6 - 138(3)^5 + 225(3)^4 - 229(3)^3 + 136(3)^2 - 36 * 3$$

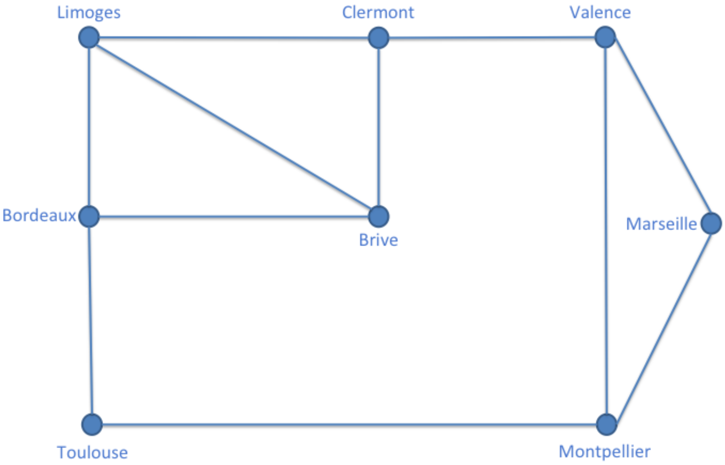
$$= 6561 - 24057 + 37908 - 33534 + 18225 - 6183 + 1224 - 108$$

$$= 36$$

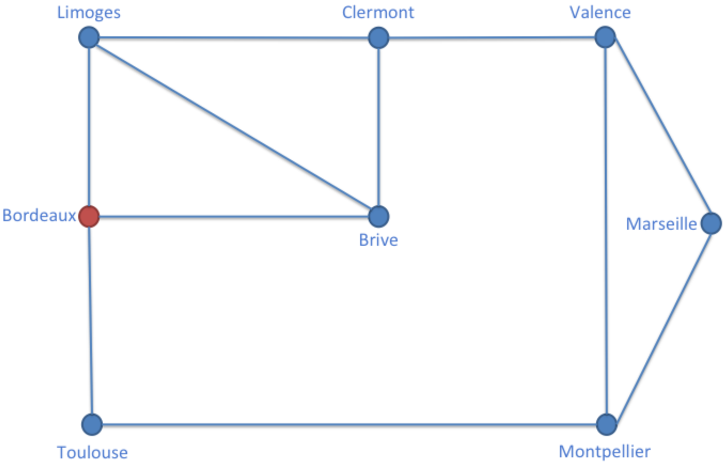
There are 36 possibilities to color this graph with three colours.

Let see the way to color the graph...

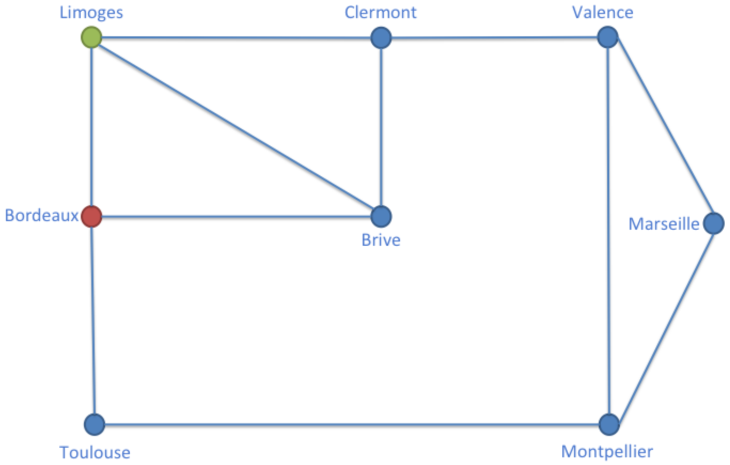
Solutions


Limoges
Clermont
Valence
Bordeaux
Brive
Marseille
Toulouse
Montpellier

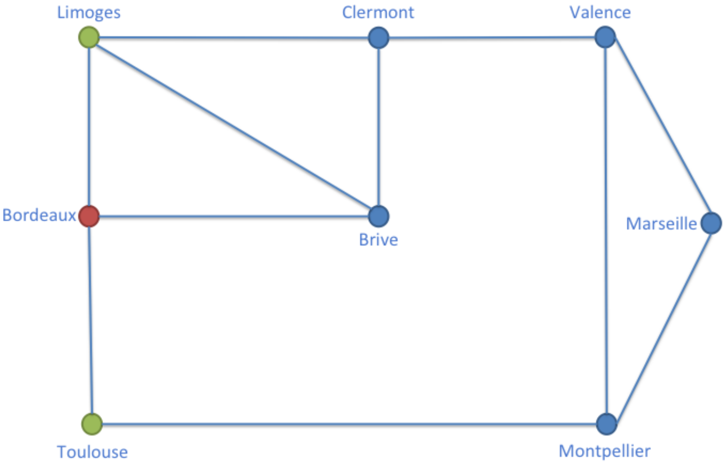
Solutions


Limoges
Clermont
Valence
Bordeaux
Brive
Marseille
Toulouse
Montpellier

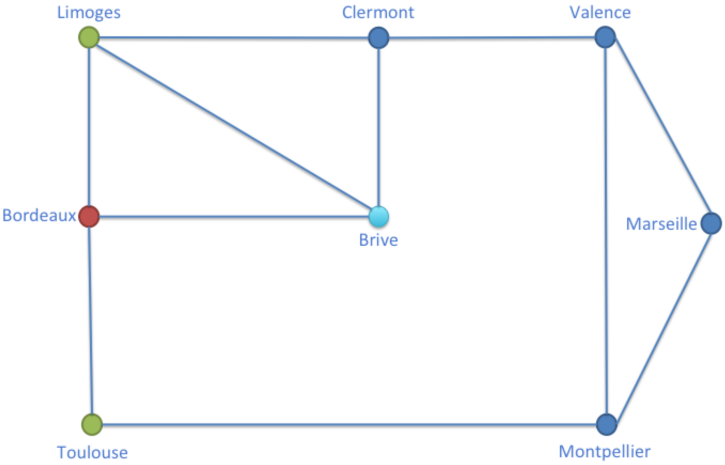
Solutions


Limoges
Clermont
Valence
Bordeaux
Brive
Marseille
Toulouse
Montpellier

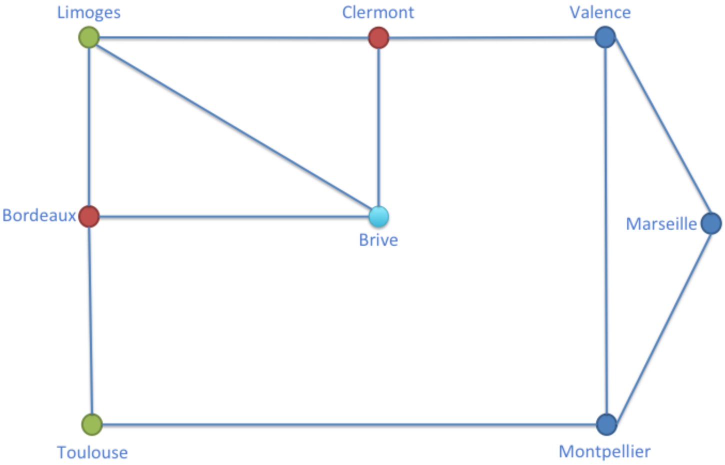
Solutions


Limoges
Clermont
Valence
Bordeaux
Brive
Marseille
Toulouse
Montpellier

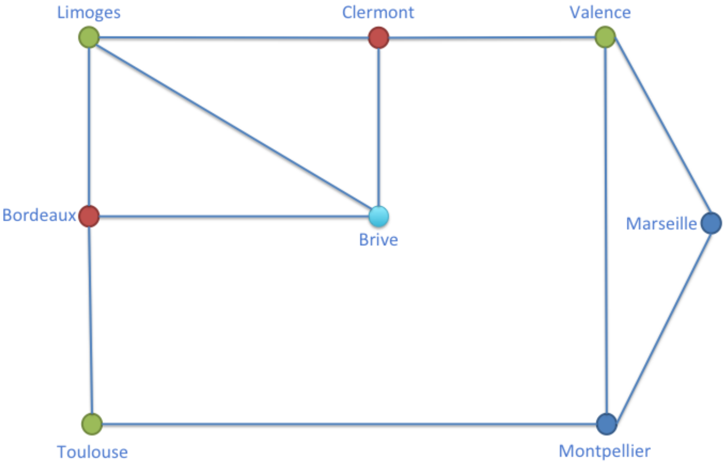
Solutions


Limoges
Clermont
Valence
Bordeaux
Brive
Marseille
Toulouse
Montpellier

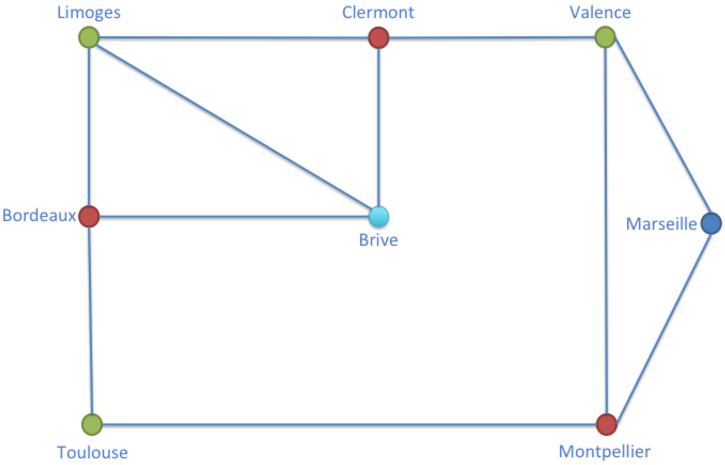
Solutions


Limoges
Clermont
Valence
Bordeaux
Brive
Marseille
Toulouse
Montpellier

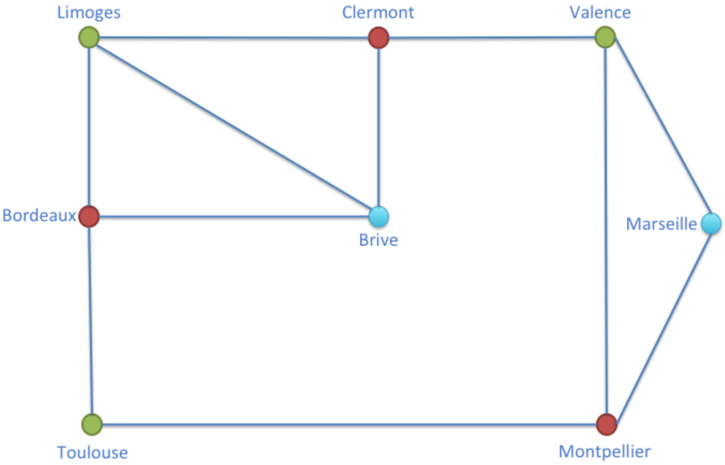
Solutions


Limoges
Clermont
Valence
Bordeaux
Brive
Marseille
Toulouse
Montpellier

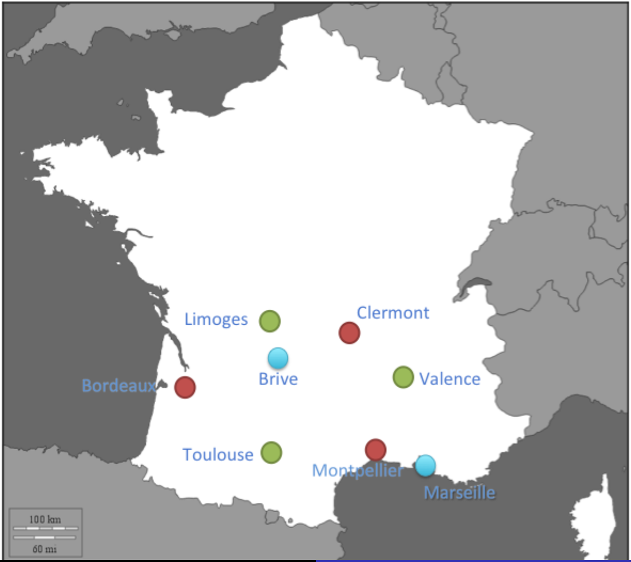
Solutions


Limoges
Clermont
Valence
Bordeaux
Brive
Marseille
Toulouse
Montpellier

Solutions


Limoges
Clermont
Valence
Bordeaux
Brive
Marseille
Toulouse
Montpellier

Solutions


Limoges
Clermont
Brive
Valence
Bordeaux
Toulouse
Montpellier
Marseille
100 km
60 mi

Research Opening

Graph coloring is a recent problem which is used in many research fields such as :

Research Opening

Graph coloring is a recent problem which is used in many research fields such as :

- Algorithms Improvement

Research Opening

Graph coloring is a recent problem which is used in many research fields such as :

- Algorithms Improvement
- Network Clustering

Research Opening

Graph coloring is a recent problem which is used in many research fields such as :

- Algorithms Improvement
- Network Clustering
- Electronic Memory Management

Research Opening

Graph coloring is a recent problem which is used in many research fields such as :

- Algorithms Improvement
- Network Clustering
- Electronic Memory Management
- Musicology
- ...

Network Clustering

- Cluster organising and hierarchy

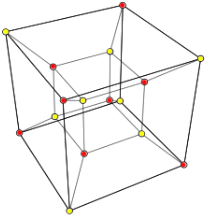
Link : *http ://liesp.insa-lyon.fr/v2/ ?q=fr/node/2246*

Network Clustering

- Cluster organising and hierarchy
- Each colour represents a part of the data

Link : *http ://liesp.insa-lyon.fr/v2/ ?q=fr/node/2246*

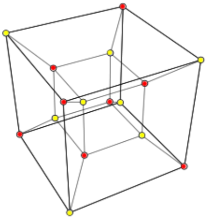
Network Clustering



- Cluster organising and hierarchy
- Each colour represents a part of the data
- Research of new architecture of computers layout

Link : *http ://liesp.insa-lyon.fr/v2/ ?q=fr/node/2246*

Network Clustering



- Cluster organising and hierarchy
- Each colour represents a part of the data
- Research of new architecture of computers layout
- Optimisation of data distribution in backcones based on hypercube

Link : *http ://liesp.insa-lyon.fr/v2/ ?q=fr/node/2246*

Electronic Memory Management

Depending on the data structure to store we have to :

Link : *http ://www.roadef.org/forums/index.php ? action=vthread&forum=2&topic=572*

Electronic Memory Management

Depending on the data structure to store we have to :

- estimate the number of access which will be requested

Link : *http ://www.roadef.org/forums/index.php ? action=vthread&forum=2&topic=572*

Electronic Memory Management

Depending on the data structure to store we have to :

- estimate the number of access which will be requested
- spot an appropriate slot memory so that the location of highly-requested data structures is
- near enough to improve circuit efficiency

Link : *http ://www.roadef.org/forums/index.php ? action=vthread&forum=2&topic=572*

Electronic Memory Management

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- spot an appropriate slot memory so that the location of highly-requested data structures is
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- far enough to reduce the conflicts while concurrent access

Link : *http ://www.roadef.org/forums/index.php ? action=vthread&forum=2&topic=572*

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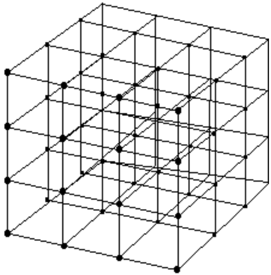
The memory is represented by a grid graph :



Link : *http ://www.roadef.org/forums/index.php ? action=vthread&forum=2&topic=572*

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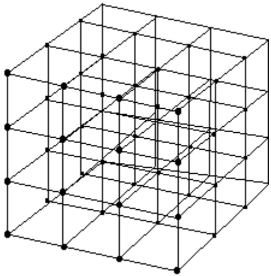


Each data structure is represented by a colour.

Link : *http ://www.roadef.org/forums/index.php ?*

Musicology

How does it work ?

Link : *http ://lla-creatis.univ-tlse2.fr/accueil/soutenances/soutenance-de-these-de-gilles-baroin-114385.kjsp*

Musicology

How does it work ?

- Scale representation as a Cartesian product of two graph :

Link : *http ://lla-creatis.univ-tlse2.fr/accueil/soutenances/soutenance-de-these-de-gilles-baroin-114385.kjsp*

Musicology

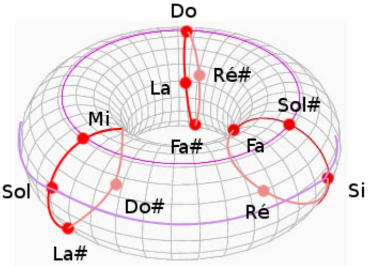
How does it work ?

- Scale representation as a Cartesian product of two graph :
- C_3 which represents notes spaced out by a minor third

Link : *http ://lla-creatis.univ-tlse2.fr/accueil/soutenances/soutenance-de-these-de-gilles-baroin-114385.kjsp*

Musicology

How does it work ?

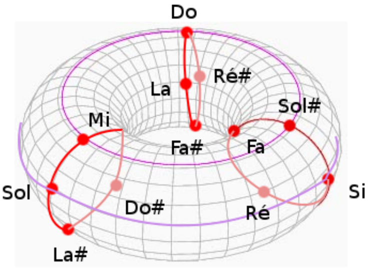


- Scale representation as a Cartesian product of two graph :
- C_3 which represents notes spaced out by a minor third
- C_4 which represents notes spaced out by a major third

Link : *http ://lla-creatis.univ-tlse2.fr/accueil/soutenances/soutenance-de-these-de-gilles-baroin-114385.kjsp*

Musicology

How does it work ?



- Scale representation as a Cartesian product of two graph :
- C_3 which represents notes spaced out by a minor third
- C_4 which represents notes spaced out by a major third
- use of graph coloration based on complex numbers

Link : *http ://lla-creatis.univ-tlse2.fr/accueil/soutenances/soutenance-de-these-de-gilles-baroin-114385.kjsp*

Thank you for your attention

Do you have some questions ?