On Multiway Cut parameterized above lower bounds

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Abstract

In this paper we consider two *above lower bound* parameterizations of the NODE MULTIWAY CUT problem — above the maximum separating cut and above a natural LP-relaxation — and prove them to be fixed-parameter tractable. Our results imply $O^*(4^k)$ algorithms for VERTEX COVER ABOVE MAXIMUM MATCHING and ALMOST 2-SAT as well as an $O^*(2^k)$ algorithm for NODE MULTIWAY CUT with a standard parameterization by the solution size, improving previous bounds for these problems.

1 Introduction

The study of cuts and flows is one of the most active fields in combinatorial optimization. However, while the simplest case, where we seek a cut separating two given vertices of a graph, is algorithmically tractable, the problem becomes hard as soon as one starts to deal with multiple terminals. For instance, given three vertices in a graph it is NP-hard to decide what is the smallest size of a cut that separates every pair of them (see [4]). The generalization of this problem — the well-studied NODE MULTIWAY CUT problem — asks for the size of the smallest set separating a given set of terminals. The formal definition is as follows:

NODE MULTIWAY CUT
Input: A graph $G = (V, E)$, a set $T \subseteq V$ of *terminals* and an integer k.
Question: Does there exist a set $X \subseteq V \setminus T$ of size at most k such that any path between two different terminals intersects X?

For various approaches to this problem we refer the reader for instance to [8, 2, 4, 13].

Before describing our results, let us discuss the methodology we will be working with. We will be studying NODE MULTIWAY CUT (and several other problems) from the parameterized complexity point of view. Note that since the solution to our problem is a set of k vertices and it is easy to verify whether a solution is correct, we can solve the problem by enumerating and verifying all the $O(|V|^k)$ sets of size k. Therefore, for every fixed value of k, our problem can be solved in polynomial time. This approach, however, is not feasible even for, say, $k = 10$. The idea of parameterized complexity is to try to split the (usually exponential) dependency on k from the (hopefully uniformly polynomial) dependency on $|V|$ — so we look for an algorithm where the degree of the polynomial does not depend on k, e.g., an $O(C^k |V|^{O(1)})$ algorithm.

Formally, a parameterized problem Q is a subset of $\Sigma^* \times \mathbb{N}$ for some finite alphabet Σ, where the integer is the *parameter*. We say that the problem is *fixed parameter tractable* (*FPT*) if there exists an algorithm solving any instance (x, k) in time $f(k)\text{poly}(|x|)$ for some (usually exponential) computable function f. It is known that a problem is FPT iff it is kernelizable: a kernelization algorithm for a problem Q takes an instance (x, k) and in time polynomial in $|x| + k$ produces an equivalent instance (x', k') (i.e., $(x, k) \in Q$ iff $(x', k') \in Q$) such that $|x'| + k' \leq g(k)$ for some computable function g. The function g is the *size of the kernel*, and if it is polynomial, we say that Q admits a polynomial kernel. The reader is invited to refer to now classical books by Downey and Fellows [5], Flum and Grohe [7] and Niedermeier [15].

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The typical parameterization takes the solution size as the parameter. For instance, Chen et al. [2] have shown an algorithm solving NODE MULTIWAY CUT in time $O(4^k n^{O(1)})$, improving upon the previous result of Daniel Marx [13]. However, in many cases it turns out we have a natural lower bound on the solution size — for instance, in the case of the VERTEX COVER problem the cardinality of the maximal matching is such a lower bound. It can happen that this lower bound is large — rendering algorithms parameterized by the solution size impractical. For some problems, better answers have been obtained by introducing the so called *parameterization above guaranteed value*, i.e. taking as the parameter the difference between the expected solution size and the lower bound. The idea was first proposed in [12]. An overview of this currently active research area can be found in the introduction to [11].

We will consider two natural lower bounds for NODE MULTIWAY CUT — the separating cut and the LP-relaxation solution. Let $I = (G, T, k)$ be a NODE MULTIWAY CUT instance and let $s = |T|$. By a *minimum solution* to I we mean a set $X \subseteq V \setminus T$ of minimum cardinality that disconnects the terminals, even if $|X| > k$.

For a terminal $t \in T$ a set $S \subseteq V \setminus T$ is a *separating cut* of t if t is disconnected from $T \setminus \{t\}$ in $G[V \setminus S]$ (the subgraph induced by $V \setminus S$). Let $m(I, t)$ be the size of a minimum isolating cut of t. Notice that for any t the value $m(I, t)$ can be found in polynomial time using standard max-flow techniques. Moreover, this value is a lower bound for the size of the minimum solution to I — any solution X has, in particular, to separate t from all the other terminals.

Now we consider a different approach to the problem, stemming from linear programming. Let $\mathcal{P}(I)$ denote the set of all simple paths connecting two different terminals in G. Garg et al. [8] gave a 2-approximation algorithm using the following natural LP-relaxation:

$$\text{minimize} \quad \sum_{v \in V \setminus T} d_v \tag{1}$$
$$\text{subject to} \quad \sum_{v \in P \cap (V \setminus T)} d_v \geq 1 \qquad \qquad \forall P \in \mathcal{P}(I)$$
$$d_v \geq 0 \qquad \qquad \forall v \in V \setminus T$$

In other words, the LP-relaxation asks to assign for each vertex $v \in V \setminus T$ a non-negative weight d_v, such that the distance between pair of terminals, with respect to the weights d_v, is at least one. This is indeed a relaxation of the original problem — if we restrict the values d_v to be integers, we obtain the original NODE MULTIWAY CUT.

The above LP-relaxation has exponential number of constraints, as $\mathcal{P}(I)$ can be exponentially big in the input size. However, the optimal solution for this LP-relaxation can be found in polynomial time either using separation oracle and ellipsoid method or by solving an equivalent linear program of polynomial size (see [8] for details). By $LP(I)$ we denote the cost of the optimal solution of the LP-relaxation (1). As the LP-relaxation is less restrictive than the original NODE MULTIWAY CUT problem, $LP(I)$ is indeed a lower bound on the size of the minimum solution.

We can now define two *above lower bound* parameters: $L(I) = k - LP(I)$ and $C(I) = k - \max_{t \in T} m(I, t)$, and denote by NMWC-A-LP (NODE MULTIWAY CUT ABOVE LP-RELAXATION) and NMWC-A-CUT (NODE MULTIWAY CUT ABOVE MAXIMUM SEPARATING CUT) the NODE MULTIWAY CUT problem parameterized by $L(I)$ and $C(I)$, respectively.

We say that a parameterized problem Q is in XP, if there exists an algorithm solving any instance (x, k) in time $|x|^{f(k)}$ for some computable function f, i.e., polynomial for any constant value of k. The NMWC-A-CUT problem was defined and shown to be in XP by Razgon in [16].

Our results In Section 2, using the ideas of Xiao [19] and building upon analysis of the LP relaxation by Guillemot [9], we prove a NODE MULTIWAY CUT instance I can be solved in $O^*(4^{L(I)})$ time[1], which easily yields an $O^*(2^{C(I)})$-time algorithm. Both algorithms run in polynomial space. Consequently we prove both NMWC-A-LP and NMWC-A-CUT problems to be FPT, solving an open problem of Razgon [16]. Observe

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[1] $O^*()$ is the $O()$ notation with suppressed factors polynomial in the size of the input.

that if $C(I) > k$ the answer is trivially negative, hence as a by-product we obtain an $O^*(2^k)$ time algorithm for the NODE MULTIWAY CUT problem, improving the previously best known $O^*(4^k)$ time algorithm by Chen et al. [2].

By considering a line graph of the input graph, it is easy to see that an edge-deletion variant of MULTIWAY CUT is easier than the node-deletion one, and our results hold also for the edge-deletion variant. We note that the edge-deletion variant, parameterized above maximum separating cut, was implicitly proven to be FPT by Xiao [19].

Furthermore we observe that VERTEX COVER ABOVE MAXIMUM MATCHING is a special case of NMWC-A-LP, while it is known that VERTEX COVER ABOVE MAXIMUM MATCHING is equivalent to ALMOST 2-SAT from the point of view of parameterized complexity [10, 18]. The question of an FPT algorithm for those two problems was a long-standing open problem until Razgon and O'Sullivan gave an $O^*(15^k)$-time algorithm in 2008. Our results improve this bound to $O^*(4^k)$ for both VERTEX COVER ABOVE MAXIMUM MATCHING and ALMOST 2-SAT. The details are gathered in Section 3.

One of the major open problems in kernelization is the question of a polynomial kernel for NODE MULTIWAY CUT , parameterized by the solution size. Our results show that the number of terminals can be reduced to $2k$ in polynomial-time, improving a quadratic bound due to Razgon [17]. Moreover, our algorithm includes a number of polynomial-time reduction rules, that may be of some interest from the point of view of kernelization.

Finally, we consider the NODE MULTICUT problem, a generalization of NODE MULTIWAY CUT , which was recently proven to be FPT when parameterized by the solution size [14, 1]. In Section 4 we show that NODE MULTICUT, when parameterized above a natural LP-relaxation, is significantly more difficult and even not in XP.

Notation Let us introduce some notation. All considered graphs are undirected and simple. Let $G = (V, E)$ be a graph. For $v \in V$ by $N(v)$ we denote the set of neighbours of v, $N(v) = \{u \in V : uv \in E\}$, and by $N[v]$ the closed neighbourhood of v, $N[v] = N(v) \cup \{v\}$. We extend this notation to subsets of vertices $S \subseteq V$, $N[S] = \bigcup_{v \in S} N[v]$, $N(S) = N[S] \setminus S$. By *removing* a vertex v we mean transforming G to $(V \setminus v, E \setminus \{uv, vu : u \in V\})$. The resulting graph is denoted by $G \setminus v$. By *contracting an edge uv* we mean the following operation: we remove vertices u and v, introduce a new vertex x_{uv} and connect it to all vertices previously connected to u or v. The resulting graph is denoted by G/uv. If $u \in T$ and $v \notin T$, we somewhat abuse the notation and identify the new vertex x_{uv} with u, so that the terminal set remains unchanged. In this paper we do not contract any edge that connects two terminals.

2   Algorithms for MULTIWAY CUT

Let $I = (G, T, k)$, where $G = (V, E)$, be a NODE MULTIWAY CUT instance. First, let us recall the two known facts about the LP-relaxation (1).

Definition 2.1 ([8, 9])**.** *Let $(d_v)_{v \in V \setminus T}$ be a feasible solution to the LP-relaxation (1) of I. For a terminal t, the* zero area *of t, denoted by U_t, is the set of vertices within distance zero from t with respect to weights d_v.*

Lemma 2.2 ([8])**.** *Given an optimal solution $(d_v^*)_{v \in V \setminus T}$ to the LP-relaxation (1), let us construct an assignment $(d_v)_{v \in V \setminus T}$ as follows. First, for each terminal t compute its zero area U_t with respect to weights $(d_v^*)_{v \in V \setminus T}$. Second, for $v \in V \setminus T$ we take $d_v = 1$ if $v \in N(U_t)$ for at least two terminals t, $d_v = 1/2$ if $v \in N(U_t)$ for exactly one terminal t, and $d_v = 0$ otherwise. Then $(d_v)_{v \in V \setminus T}$ is also an optimal solution to the LP-relaxation (1).*

Lemma 2.3 ([9], Lemma 3)**.** *Let $(d_v^*)_{v \in V \setminus T}$ be any optimal solution to the LP-relaxation (1) of I. Then there is a minimum solution to I that is disjoint with $\bigcup_{t \in T} U_t$.*

Our algorithm consists of two parts. The first part is a set of several polynomial-time reduction rules. At any moment, we apply the lowest-numbered applicable rule. We shall prove that the original instance I is a

YES–instance if and only if the reduced instance is a YES–instance, we will say this means the reduction is *sound*. We prove that no reduction rule increases the parameter $L(I)$ or the graph size. If no reduction rule can be applied, we proceed to the branching rule. The branching rule outputs two subcases, each with the parameter $L(I)$ decreased by at least $1/2$ and a smaller graph. If the answer to any of the two subcases is YES, we return YES from the original instance, otherwise we return NO. As the parameter $L(I)$ decreases by at least $1/2$ with each branching, and we can trivially return NO if $L(I)$ is negative, we obtain the claimed $O^*(4^{L(I)})$ running time.

Reduction 1. *If two terminals are connected by an edge or $L(I) < 0$, return NO.*

The first part of the above rule is obviously sound, as we only remove vertices, not edges. The second part is sound as the optimal cost of the LP-relaxation (1) is a lower bound for the size of the minimum solution to the instance I.

Reduction 2. *If there exists a vertex $w \in V \setminus T$ that is adjacent to two terminals $t_1, t_2 \in T$, remove w from G and decrease k by one.*

The above rule is sound, as such a vertex w has to be included in any solution to I. Let us now analyse how the parameter $L(I)$ is influenced by this rule. Let $I' = (G \setminus w, T, k-1)$ be the output instance. Notice that any feasible solution $(d_v)_{v \in V \setminus (T \cup \{w\})}$ to I' can be extended to a feasible solution of I by putting $d_w = 1$. Thus $LP(I) \le LP(I') + 1$, and we infer $L(I) \ge L(I')$.

Reduction 3. *Let $w \in V \setminus T$ be a neighbour of a terminal $t \in T$. Let $(d_v^\circ)_{v \in V \setminus T}$ be a solution to the LP-relaxation (1) with an additional constraint $d_w = 0$. If the cost of the solution $(d_v^\circ)_{v \in V \setminus T}$ is equal to $LP(I)$, contract the edge tw.*

As $(d_v^\circ)_{v \in V \setminus T}$ is a feasible solution to the LP-relaxation (1), its cost is at least $LP(I)$. If the rule is applicable, $(d_v^\circ)_{v \in V \setminus T}$ is an optimal solution to the LP-relaxation (1) and $w \in U_t$. The soundness of Reduction 3 follows from Lemma 2.3. Moreover, note that if I' is the output instance of Reduction 3, we have $LP(I) = LP(I')$, as $(d_v^\circ)_{v \in V \setminus (T \cup \{w\})}$ is a feasible solution to the LP-relaxation (1) for the instance I'. We infer that $L(I) = L(I')$.

The following lemma summarizes properties of an instance, assuming none of the above reduction rules is applicable.

Lemma 2.4. *If Reductions 1, 2 and 3 are not applicable, then:*

1. *An assignment $(d_v)_{v \in V \setminus T}$ that assigns $d_v = 1/2$ if $v \in N(T)$ and $d_v = 0$ otherwise is an optimal solution to the LP-relaxation (1).*

2. *For each terminal $t \in T$, the set $N(t)$ is the unique minimum separating cut of t.*

Proof. Let $(d_v^*)_{v \in V \setminus T}$ be any optimal solution to the LP-relaxation (1). As Reduction 3 is not applicable, $d_w^* > 0$ for any $w \in N(T)$. As Reduction 2 is not applicable, if we invoke Lemma 2.2 on the assignment $(d_v^*)_{v \in V \setminus T}$, we obtain the assignment $(d_v)_{v \in V \setminus T}$. Thus the first part of the lemma is proven.

For the second part, obviously $N(t)$ is a separating cut of t. Let $C \subseteq V \setminus T$ be any other separating cut of t and assume $|C| \le |N(t)|$. Let $d_v' = d_v + 1/2$ if $v \in C \setminus N(t)$, $d_v' = d_v - 1/2$ if $v \in N(t) \setminus C$ and $d_v' = d_v$ otherwise. It is easy to see that d_v' is a feasible solution to the LP-relaxation (1). As $|C| \le |N(t)|$, $\sum_{v \in V \setminus T} d_v' \le \sum_{v \in V \setminus T} d_v$ and we infer that $(d_v')_{v \in V \setminus T}$ is an optimal solution to the LP-relaxation (1). However, $d_v' = 0$ for $v \in N(t) \setminus C$, and Reduction 3 would be applicable. $\square$

Branching rule. *Let $w \in V \setminus T$ be a neighbour of a terminal $t \in T$. Branch into two subcases, either w is included in a solution to the* Node Multiway Cut *instance I or not. In the first branch, we remove w from the graph and decrease k by one. In the second one, we contract the edge tw.*

The soundness of the branching rule is straightforward. We now prove that in both subcases the parameter $L(I)$ drops by at least $1/2$. Let $I_1 = (G \setminus w, T, k-1)$ and $I_2 = (G/tw, T, k)$ be the output instances in the first and second cases, respectively.

In the first subcase, it is sufficient to prove that $LP(I_1) \geq LP(I) - 1/2$, i.e., that the cost of the optimal solution to the LP-relaxation (1) drops by at most half. Assume the contrary, that $LP(I_1) < LP(I) - 1/2$. Let $(d_v)_{v \in V \setminus T \setminus \{w\}}$ be a half-integral optimal solution to the LP-relaxation (1) for I_1, as asserted by Lemma 2.2. Note that if we put $d_w = 1$, then $(d_v)_{v \in V \setminus T}$ is a feasible solution to the LP-relaxation (1) for I, and $LP(I_1) \geq LP(I) - 1$. By half-integrality, $LP(I_1) = LP(I) - 1$ and $(d_v)_{v \in V \setminus T}$ is an optimal half-integral solution to the LP-relaxation (1) for I. As Reduction 3 is not applicable, $d_v > 0$ for all $v \in N(T)$. As $(d_v)_{v \in V \setminus T}$ is half-integral, $d_v \geq 1/2$ for all $v \in N(T)$. However, the assignment given by Lemma 2.4 has strictly smaller cost than $(d_v)_{v \in V \setminus T}$ (as $d_w = 1$), a contradiction to the fact that $(d_v)_{v \in V \setminus T}$ is an optimal solution to the LP-relaxation (1) for I. Thus $LP(I_1) \geq LP(I) - 1/2$ and $L(I_1) \leq L(I) - 1/2$.

In the second subcase note that, as Reduction 3 is not applicable, $LP(I_2) > LP(I)$. As the LP-relaxation (1) has half-integral solutions, we have $LP(I_2) \geq LP(I) + 1/2$. This implies that $L(I_2) \leq L(I) - 1/2$.

Since Reduction 1 stops when the parameter $L(I)$ becomes negative, we obtain the following theorem.

Theorem 2.5. *There exists an algorithm that solves a* NODE MULTIWAY CUT *instance I in $O^*(4^{L(I)})$ time.*

To solve NODE MULTIWAY CUT parameterized by $C(I)$, we introduce one more reduction rule. Recall s denotes the number of terminals.

Reduction 4. *If $C(I) \geq \frac{s-2}{s-1} \cdot k$ or $C(I) \leq 2L(I)$, return YES.*

Now we show that Reduction 4 is sound. Let $t_0 \in T$ be the terminal with the largest separating cut, i.e., $m(I, t_0) = \max_{t \in T} m(I, t)$. Let $X = N(T \setminus \{t_0\})$. Obviously no two terminals are in the same connected component of $G[V \setminus X]$. We claim that $|X| \leq k$.

If $C(I) \geq \frac{s-2}{s-1} \cdot k$, $|N(t)| = m(I, t)$ by Lemma 2.4, and:

$$|X| = \sum_{t \in T \setminus \{t_0\}} m(I, t) \leq (s-1)m(I, t_0) = (s-1)(k - C(I)) \leq k.$$

In the second case, the condition $C(I) \leq 2L(I)$ is equivalent to $2LP(I) - m(I, t_0) \leq k$. From the structure of the optimum half-integral solution given by Lemma 2.4, we infer that $2LP(I) \geq |N(T)|$. By Lemma 2.4, $|N(t_0)| = m(I, t_0)$. Since Reduction 2 is not applicable, $N(t_0) \cap N(t) = \emptyset$ for $t \in T \setminus \{t_0\}$. We infer that $|X| = 2LP(I) - m(I, t_0) \leq k$, and Reduction 4 is sound.

Corollary 2.6. *There exists an algorithm that solves a* NODE MULTIWAY CUT *instance I in $O^*(2^{\min(C(I), \frac{s-2}{s-1} \cdot k)})$ time. In the case of three terminals, this yields a $O^*(2^{k/2})$-time algorithm.*

Finally, we would like to note that all our reduction rules are polynomial-time and could be used in a hypothetical algorithm to find a polynomial kernel for NODE MULTIWAY CUT . Let us supply them with one additional clean-up rule.

Reduction 5. *If there exists a connected component of G with at most one terminal, remove it.*

The following lemma shows that our reductions improve the quadratic bound on the number of terminals due to Razgon [17].

Lemma 2.7. *If Reductions 1, 2, 3 and 5 are not applicable, then $|T| \leq 2k$.*

Proof. As noted before, the optimal half-integral solution given by Lemma 2.4 implies that $|N(T)| = 2LP(I)$. However, if Reduction 5 is not applicable, $N(t) \neq \emptyset$ for any $t \in T$, and $|T| \leq |N(T)|$ by Reduction 2. We infer that $2LP(I) \geq |T|$. If $|T| > 2k$, then $L(I) < 0$ and Reduction 1 would return NO. $\square$

3 From Node Multiway Cut to Almost 2-SAT

We start with problem definitions. For a graph G by $\mu(G)$ we denote the size of a maximum matching in G.

Vertex Cover above Maximum Matching   **Parameter:** k
Input: A graph $G = (V, E)$ and an integer k.
Question: Does there exist a vertex cover in G of size at most $\mu(G) + k$?

Almost 2-SAT   **Parameter:** k
Input: A 2-SAT formula Φ and an integer k.
Question: Does there exist a set X of at most k clauses of Φ, whose deletion makes Φ satisfiable?

Now we prove that Vertex Cover above Maximum Matching is a special case of NMWC-a-LP.

Theorem 3.1. *There exists an algorithm that solves* Vertex Cover above Maximum Matching *in* $O^*(4^k)$ *time.*

Proof. Let $I = (G = (V, E), k)$ be a Vertex Cover above Maximum Matching instance. We construct a Node Multiway Cut instance $I' = (G', T, k')$ as follows. For each $v \in V$ we create a terminal t_v and connect it to v, thus $T = \{t_v : v \in V\}$ and each terminal in G' is of degree one. Moreover we take $k' = \mu(G) + k$.

We claim that $X \subseteq V$ is a vertex cover in G if and only if each connected component of $G'[(V \setminus X) \cup T]$ contains at most one terminal. If $X \subseteq V$ is a vertex cover in G, $G[V \setminus X]$ is an independent set, thus every edge in $G'[(V \setminus X) \cup T]$ is of type $t_v v$. In the other direction, note that if $uv \in E$ and $u, v \notin X$, then t_u and t_v are connected in $G'[(V \setminus X) \cup T]$.

We now show that $LP(I') \geq \mu(G)$. Let M be a maximum matching in G and let $(d_v)_{v \in V}$ be an optimal solution to the LP-relaxation (1) for I'. For each $uv \in M$, the path consisting of vertices t_u, u, v and t_v is in $\mathcal{P}(I')$, thus $d_u + d_v \geq 1$. As M is a matching, we infer that $\sum_{v \in v} d_v \geq |M| = \mu(G)$.

Since $LP(I') \geq \mu(G)$ and $k' = k + \mu(G)$, we have $L(I') \leq k$. We apply algorithm from Theorem 2.5 to the instance I' and the time bound follows. $\square$

We now reproduce the reduction from Almost 2-SAT to Vertex Cover above Maximum Matching to prove the following theorem.

Theorem 3.2. *There exists an algorithm that solves* Almost 2-SAT *in* $O^*(4^k)$ *time.*

Proof. Let $I = (\Phi, k)$ be an Almost 2-SAT instance. First, we replace each clause $C \in \Phi$ that consists of a single literal l with a clause $(l \vee l)$. From now we assume that each clause of Φ consists of two, possibly equal, literals.

Let x be a variable of Φ. By $n(x)$ we denote the number of occurrences of the variable x in the formula Φ (if $l = x$ or $l = \neg x$, a clause $(l \vee l)$ counts as two occurrences). Let us arbitrarily number those occurrences and for any $1 \leq i \leq n(x)$, by $C(x, i)$ we denote the clause where x occurs the i-th time.

We now construct a Vertex Cover above Maximum Matching instance $I' = (G, k)$. For each variable x and for each $1 \leq i \leq n(x)$ we create two vertices $v(x, i)$ and $v(\neg x, i)$. For $l \in \{x, \neg x\}$ we denote $V(l) = \{v(l, i) : 1 \leq i \leq n(x)\}$. For each variable x and for each $1 \leq i, j \leq n(x)$ we connect $v(x, i)$ and $v(\neg x, j)$ by an edge, i.e., we make a full bipartite subgraph with sides $V(x)$ and $V(\neg x)$.

Furthermore, if $C(x, i) = C(y, j)$ for some variables x, y and indices $1 \leq i \leq n(x)$, $1 \leq j \leq n(y)$ (possibly $x = y$, but $(x, i) \neq (y, j)$), we introduce an edge $v(l_x, i)v(l_y, j)$, where $C(x, i) = C(y, j) = (l_x \vee l_y)$, l_x is the i-th occurrence of x and l_y is the j-th occurrence of y. Such an edge is called a *clause edge*. Note that we introduce exactly one clause edge for each clause of Φ and no two clause edges share an endpoint in G.

We claim that I is an Almost 2-SAT YES-instance if and only if I' is a Vertex Cover above Maximum Matching YES-instance. First note that G has a perfect matching consisting of all edges of the type $v(x, i)v(\neg x, i)$.

Assume I is a YES-instance. Let $X \subseteq \Phi$ be a set of clauses, such that there exists a truth assignment ϕ of all variables of Φ that satisfies all clauses of $\Phi \setminus X$. We now construct a vertex cover Y of G. For each

variable x and for each index $1 \leq i \leq n(x)$, we take into Y the vertex $v(x, i)$ if x is true in the assignment ϕ, and $v(\neg x, i)$ otherwise. Moreover, for each clause $C \in X$ we take into Y any endpoint of the clause edge for C.

Clearly $|Y| \leq \mu(G) + |X|$. Each non-clause edge $v(x, i)v(\neg x, j)$ is covered by Y, as $v(x, i) \in Y$ if x is true in ϕ, and $v(\neg x, j) \in Y$ otherwise. Let $e_C = v(l_x, i)v(l_y, j)$ be a clause edge for clause $C = (l_x \vee l_y)$. If $C \in X$, then one of the endpoints of e_C is chosen into Y. Otherwise, l_x or l_y is true in ϕ and the corresponding vertex is chosen into Y.

In the other direction, let us assume that I' is a YES-instance and let Y be a vertex cover of G. We construct a truth assignment ϕ as follows. Let x be a variable of Φ. Recall that G has a complete bipartite subgraph with sides $V(x)$ and $V(\neg x)$. Thus $V(l) \subseteq Y$ for some $l \in \{x, \neg x\}$, and we take l to be true in ϕ (if $V(x) \cup V(\neg x) \subseteq Y$, we choose whether x is true or false arbitrarily). Let X be the set of clauses of Φ that are not satisfied by ϕ. We claim that $|X| \leq |Y| - \mu(G)$.

Let Y_1 be the union of all sets $V(l)$ for which l is true under ϕ. Obviously $Y_1 \subseteq Y$ and $|Y_1| = \mu(G)$. Let $Y_2 = Y \setminus Y_1$. Take any $C \in X$. As C is not satisfied by ϕ, the clause edge e_C corresponding to C does not have an endpoint in Y_1. Since Y is a vertex cover in G, e_C has an endpoint in Y_2. Finally, recall that no two clause edges share an endpoint. This implies that $|Y_2| \geq |X|$ and $|X| \leq |Y| - \mu(G)$.

We infer that the instances I and I' are equivalent. As the above construction can be done in polynomial time, the running time follows from Theorem 3.1. $\qquad\square$

4 Hardness of Node Multicut parameterized above LP-relaxation

Recall the definition of Node Multicut, which is a natural generalization of Node Multiway Cut .

Node Multicut
Input: A graph $G = (V, E)$, a set $\mathcal{T}$ of pairs of terminals, and an integer k.
Question: Does there exist a set X of at most k non-terminal vertices, whose removal disconnects all pairs of terminals in $\mathcal{T}$?

The LP-relaxation (1) for Node Multiway Cut naturally generalizes to Node Multicut as follows. Let T be the set of all terminals in the given Node Multicut instance $I = (G, \mathcal{T}, k)$. In the LP-relaxation we ask for an assignment of non-negative weights $(d_v)_{v \in V \setminus T}$, such that for each pair $(s, t) \in \mathcal{T}$ the distance between s and t with respect to the weights $(d_v)_{v \in V \setminus T}$ is at least one. Clearly, if X is a solution to I, an assignment that takes $d_v = 1$ if $v \in X$ and $d_v = 0$ otherwise, is a feasible solution to the LP-relaxation. Let $LP(I)$ be the cost of an optimal solution to this LP-relaxation. We denote by NMC-A-LP the Node Multicut problem parameterized by $L(I) = k - LP(I)$, i.e., parameterized above LP lower bound.

In this section we prove that NMC-A-LP does not even belong to XP, by the following lemma.

Lemma 4.1. NMC-A-LP, *restricted to instances where $L(I) = 0$, is NP-hard.*

Proof. We reduce from Multicoloured Independent Set which is NP-complete (see [6]). In this problem we are given a graph $G = (V, E)$ together with a partition of the vertex set into sets $V_1, V_2, \ldots, V_r$, such that $G[V_i]$ is a clique for $1 \leq i \leq r$, and we are to decide whether G contains an independent set of size r. Note that such an independent set needs to take exactly one vertex from each set V_i. W.l.o.g. we may assume that $|V_i| \geq 2$ for each $1 \leq i \leq r$. Let $|V| = n$ and let I be the given Multicoloured Independent Set instance.

We construct a Node Multicut instance $I' = (G', \mathcal{T}, n)$ as follows. We start with the graph G. Then, for each $v \in V$ we create a vertex v' and connect it to v. For each set V_i, we connect the vertices $\{v' : v \in V_i\}$ into a path P_i in an arbitrary order. We now add terminal pairs. Each terminal will be of degree one in the graph G'.

First, for each $v \in V$ we create a terminal t_v connected to v and we include in $\mathcal{T}$ all pairs (t_v, t_u) for $u, v \in V$, $u \neq v$. Second, for each $v \in V$ we create terminals s_v and s'_v, connected to v and v' respectively, and include (s_v, s'_v) in $\mathcal{T}$. Finally, for each set V_i, we create terminals a_i and b_i, connected to two endpoints of the path P_i, and include (a_i, b_i) in $\mathcal{T}$. This finishes the construction of the instance I'.

First note that for each $(s,t) \in \mathcal{T}$, we have $N(s) \cap N(t) = \emptyset$, due to the assumption $|V_i| \geq 2$ for each $1 \leq i \leq r$. Thus an assignment that takes $d_v = d_{v'} = 1/2$ for each $v \in V$ is a feasible solution to the LP-relaxation of cost n. Moreover, it is an optimal solution, as $d_v + d_{v'} \geq 1$ for each $v \in V$ due to the terminal pair (s_v, s'_v). Thus $LP(I') = n$.

Assume I is a YES-instance and let $X \subseteq V$ be an independent set of size r in G. Take $X' = \{v' : v \in X\}$ and $Y = (V \setminus X) \cup X'$. Clearly $|Y| = n$. To see that Y is a solution to the instance I' observe that $V \setminus X$ is a vertex cover of G.

In the other direction, let Y be a solution to the instance I'. Y needs to include v or v' for each $v \in V$, due to the terminal pair (s_v, s'_v). Thus $|Y| = n$ and Y includes exactly one vertex from the set $\{v, v'\}$ for each $v \in V$. Moreover, for each V_i, if $u', v' \in Y$, $u, v \in V_i$, then Y does not disconnect t_u from t_v. On the other hand, if $V_i \subseteq Y$, then Y does not intersect P_i and the pair (a_i, b_i) is not disconnected by Y. We infer that for each $1 \leq i \leq r$ there exists a vertex $v_i \in V_i$, such that $(V_i \setminus \{v_i\}) \cup \{v'_i\} \subseteq Y$. Moreover, if $v_i v_j \in E$ for some $1 \leq i < j \leq r$, then the pair (t_{v_i}, t_{v_j}) is not disconnected by Y. We infer that $\{v_i : 1 \leq i \leq r\}$ is an independent set in G, and the instances I and I' are equivalent. $\square$

5   Conclusions

In this paper, building upon work of Xiao [19] and Guillemot [9], we show that NODE MULTIWAY CUT is fixed-parameter tractable when parameterized above two lower bounds: largest isolating cut and the cost of the optimal solution of the LP-relaxation. We also believe that our results may be of some importance in resolving the question of an existence of a polynomial kernel for NODE MULTIWAY CUT.

One of the tools used in the parameterized complexity is the notion of important separators introduced by Marx in 2004 [13]. From that time important separators were used for proving several problems to be in FPT, including MULTIWAY CUT [13], DIRECTED FEEDBACK VERTEX SET [3], ALMOST 2-SAT [18] and MULTICUT [14, 1]. In this paper we show that in the NODE MULTIWAY CUT problem half-integral solutions of the natural LP-relaxation of the problem can be even more useful than important separators. Is it possible to use linear programming in other graph separation problems, for example to obtain a $O^*(c^k)$ algorithm for DIRECTED FEEDBACK VERTEX SET?

We have shown that NODE MULTICUT parameterized above LP-relaxation is not in XP. Is the edge-deletion variant similarly difficult?

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