Parameterized Complexity of Independence and Domination on Geometric Graphs

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Abstract. We investigate the parameterized complexity of Maximum Independent Set and Dominating Set restricted to certain geometric graphs. We show that Dominating Set is W[1]-hard for the intersection graphs of unit squares, unit disks, and line segments. For Maximum Independent Set, we show that the problem is W[1]-complete for unit segments, but fixed-parameter tractable if the segments are axis-parallel.

1 Introduction

For a set V of geometric objects, the *intersection graph* of V is a graph with vertex set V where two vertices are connected if and only if the corresponding two objects have non-empty intersection. Intersection graphs of disks, rectangles, line segments, and other objects arise in applications such as facility location [5], frequency assignment [3], and map labeling [1].

In this paper we investigate the parameterized complexity of Maximum Independent Set and Dominating Set restricted to certain geometric graphs. Both of these problems are W[1]-hard on general graphs, but fixed-parameter tractable when restricted to planar graphs. Geometric intersection graphs are in some sense intermediate between these two classes: they still have lot of geometric structure that might be used in algorithms, but we lose some of the simplicity of planar graphs. Therefore, it is an interesting question to investigate the complexity of these problems on different types of geometric graphs.

This line of research was pursued in [4], where Maximum Independent Set was proved to be W[1]-complete for unit disk and unit square graphs. Here we extend the results by considering the intersection graphs of line segments and the Dominating Set problem.

In Section 2, we introduce a general framework that can be used to prove W[1]-hardness for geometric problems. We give a semi-formal definition of what properties the gadgets of the reduction have to satisfy; in later sections the only thing we have to do for each W[1]-hardness proof is to define the problem-specific gadgets and verify the required properties.

In Section 3, we show that Dominating Set is W[1]-hard for unit disk graphs and unit square graphs. In general, Dominating Set is W[2]-complete, but it

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turns out that Dominating Set is in W[1] (hence W[1]-complete) for unit square graphs. As far as we know, this is the first example when Dominating Set restricted to some class of graphs is not W[2]-complete, but not fixed-parameter tractable either. Section 4 shows that Dominating Set is W[1]-complete also for the intersection graphs of axis-parallel line segments.

Section 5 considers the Maximum Independent Set problem for the intersection graphs of line segments. If the segments are axis-parallel (or more generally, if they belong to at most d different directions), then the problem is fixed-parameter tractable. However, if there is no restriction on the number of different directions, then the problem becomes W[1]-complete, even if every segment has the same length.

2 General Framework

All the W[1]-hardness proofs in the paper follow the same general framework. In this section we present a general reduction technique that can be used to prove hardness of a geometric or planar problem. The reduction creates an instance that consists of some number of gadgets, and connections between gadgets. The exact details of the gadgets and the connections are problem specific, and will be given in later sections separately for each problem. However, we show here that if for a particular problem there is a gadget satisfying certain properties, then the problem is W[1]-hard.

The W[1]-hardness proof is by parameterized reduction from Maximum Clique. Given a graph G and an integer k, it has to be decided if G has a clique of size k. For convenience, we assume that G has n vertices and n edges. The set of vertices and the set of edges are identified with the set $\{1, 2, \ldots, n\}$.

The constructed instance contains k^2 copies of the gadget, arranged in k rows and k columns. The gadget in row i and column j will be denoted by $G_{i,j}$. Adjacent gadgets in the same row are connected by a *horizontal connection* and adjacent gadgets in the same column are connected by a *vertical connection*.

Let $\iota : \{1, \ldots, n^2\} \to \{1, \ldots, n\} \times \{1, \ldots, n\}$ be an arbitrary one-to-one mapping, and let $\iota(s) = (\iota_1(s), \iota_2(s))$ for every s. For technical reasons, in this paper we always use the mapping defined by $s = (\iota_1(s) - 1)n + \iota_2(s)$. The crucial property of the gadget is that in every optimum solution it represents an integer number between $1 \leq s \leq n^2$, which can be also interpreted as the pair $\iota(s)$. The role of the horizontal connections is to ensure that if the values of the two gadgets are s and s', then $\iota_1(s) = \iota_1(s')$, i.e., they agree in the first component. Therefore, in an optimum solution the same value v_i will be represented by the first component of every gadget in row i. Similarly, the vertical connections ensure that if s and s' are the values of two adjacent gadgets in a column, then $\iota_2(s) = \iota_2(s')$. Thus the second component has the same value v'_j in column j.

Now we encode the graph G into the instance by restricting certain gadgets. Restricting a gadget to the subset $S \subseteq \{1, 2, \ldots, n^2\}$ means that the gadget is modified such that it can represent values only from S. For every $1 \leq i \leq k$, we restrict the gadget $G_{i,i}$ to the set $\{s : \iota_1(s) = \iota_2(s)\}$. This ensures that the first component in row i is the same as the second component in column i, i.e., $v_i = v'_i$ for every $1 \leq i \leq k$. To encode the structure of the graph, we restrict $G_{i,j}$ (for every $i \neq j$) to the set $\{s : \iota_1(s)$ and $\iota_2(s)$ are adjacent vertices$\}$. It is clear that if every gadget has a value that respects these restrictions, then $v_1, v_2, \ldots, v_k$ are all distinct and they form a clique of size k: if v_i and v_j are not adjacent, then the value $(v_i, v_j) = (v_i, v'_j)$ does not respect the restriction on gadget $G_{i,j}$. On the other hand, if $v_1, v_2, \ldots, v_k$ is a clique of size k, then we can assign value $\iota^{-1}((v_i, v_j))$ to gadget $G_{i,j}$. This assignment respects the restrictions on the gadgets and the connections.

In summary, the gadgets have to satisfy the following requirements:

Definition 1 (Matrix Gadget). *A gadget satisfies the following properties:*

1. **(The gadget)** *In every solution of the constructed instance, each gadget represents a number between* 1 *and* n^2*.*
2. **(Restriction)** *The gadget can be restricted to a set* $\emptyset \neq S \subseteq \{1, \ldots, n^2\}$ *such that in every solution the gadget represents a number in* S*.*
3. **(Horizontal connection)** *If two gadgets are connected by a horizontal connection, then the values they represent agree in the first component.*
4. **(Vertical connection)** *If two gadgets are connected by a vertical connection, then the values they represent agree in the second component.*
5. **(Constructing a solution)** *If it is possible to assign values to the gadgets such that this assignment respects the restrictions and respects the connections, then the constructed instance has a solution.*

The first four requirements ensure that if the instance described above has a solution, then G has a clique of size k. The other direction of the reduction follows from the last requirement: if $v_1, \ldots, v_k$ is a clique of size k, then giving the value (v_i, v_j) to gadget $G_{i,j}$ respects the restrictions and the connections, thus there is a solution.

3 Dominating Set for Squares and Disks

The first problem we consider is Dominating Set: given a graph G, the task is to find a set S of k vertices such that each vertex of the graph is either in S or is a neighbor of a member of S. In this section we prove hardness results for the problem in the case of unit disk graphs and unit square graphs.

Theorem 1. Dominating Set *is* W[1]*-hard for axis-parallel unit squares.*

Proof. The proof uses the framework of Section 2. Let $\epsilon < 1/3n^2$. In this proof it does not matter if the squares are open or closed. In the constructed instance of Dominating Set the lower left corner of each square is of the form $(i+\alpha\epsilon, j+\beta\epsilon)$, where i and j are integers, and $-n \leq \alpha, \beta \leq n$. If two squares have the same i, j values, then they belong to the same *block;* the blocks form a partition of the squares. If the lower left corner of a square S is $(i + \alpha\epsilon, j + \beta\epsilon)$ then α (resp., β) is the *horizontal* (resp., *vertical*) offset of S, and we set offset$(S) = (\alpha, \beta)$.

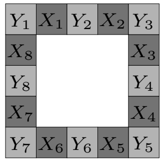


Fig. 1. The gadget used in the proof of Theorem 1

The gadget. The gadget used in the reduction is shown in Figure 1. It consists of 16 blocks $X_1, \ldots, X_8, Y_1, \ldots, Y_8$. Each block X_i contains n^2 squares $X_{i,1}, \ldots, X_{i,n^2}$, while each block Y_i contains n^2+1 squares $Y_{i,0}, \ldots, Y_{i,n^2}$. The offsets of the squares are defined as follows:

$$\begin{aligned}
&\text{offset}(X_{1,j}) = (j, -\iota_2(j)) &&\text{offset}(Y_{1,j}) = (j+0.5, j+0.5)\\
&\text{offset}(X_{2,j}) = (j, \iota_2(j)) &&\text{offset}(Y_{2,j}) = (j+0.5, -n)\\
&\text{offset}(X_{3,j}) = (-\iota_1(j), -j) &&\text{offset}(Y_{3,j}) = (j+0.5, -j-0.5)\\
&\text{offset}(X_{4,j}) = (\iota_1(j), -j) &&\text{offset}(Y_{4,j}) = (-n, -j-0.5)\\
&\text{offset}(X_{5,j}) = (-j, \iota_2(j)) &&\text{offset}(Y_{5,j}) = (-j-0.5, -j-0.5)\\
&\text{offset}(X_{6,j}) = (-j, -\iota_2(j)) &&\text{offset}(Y_{6,j}) = (-j-0.5, n)\\
&\text{offset}(X_{7,j}) = (\iota_1(j), j) &&\text{offset}(Y_{7,j}) = (-j-0.5, j+0.5)\\
&\text{offset}(X_{8,j}) = (-\iota_1(j), j) &&\text{offset}(Y_{8,j}) = (n, j+0.5)
\end{aligned}$$

Observe that two squares can intersect only if they belong to the same or adjacent blocks. For example, the squares in block X_2 have positive vertical offsets and the squares in X_3 have negative vertical offset, hence they do not intersect. The crucial property of the construction is that two squares X_{i,j_1}, X_{i+1,j_2} dominate every square of Y_{i+1} if and only if $j_1 \geq j_2$. This follows from the fact that X_{i,j_1} dominates exactly $Y_{i+1,0}, \ldots, Y_{i+1,j_1-1}$ from block Y_{i+1} and X_{i+1,j_2} dominates exactly $Y_{i+1,j_2}, \ldots, Y_{i+1,n^2}$ from block Y_{i+1}.

Lemma 1. *Assume that a gadget is part of an instance such that none of the blocks Y_i are intersected by squares outside the gadget. If there is a dominating set D of the instance that contains exactly 8 squares from the gadget, then there is a dominating set D' with $|D'| \leq |D|$, and there is an integer $1 \leq j \leq n^2$ such that D' contains exactly the squares $X_{1,j}, \ldots, X_{8,j}$ from the gadget.*

Proof. If D contains no square from any X_i, then it has to contain at least one square from each Y_i. Remove these squares, and add the squares $X_{1,1}, \ldots, X_{8,1}$ to D instead. This does not increase the size of D, and every square of the gadget will be dominated. Furthermore, as a square from Y_i cannot dominate anything outside the gadget, the modified set is also a dominating set, and we are done.

We show that D' can be chosen such that it contains exactly one square from each X_i, and consequently, it contains no squares from the blocks Y_i. Observe that the squares in Y_i cannot be all dominated by squares only from X_{i-1} or by squares only from X_i (the indices of the blocks are modulo 8). This implies that if D contains no square from Y_i, then D contains at least one square from X_{i+1}

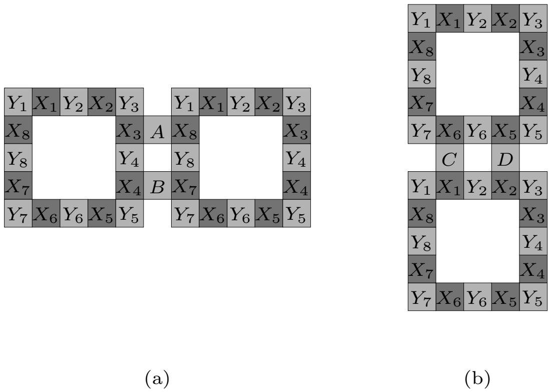


Fig. 2. The horizontal (a) and vertical (b) connections used in the proof of Theorem 1

and at least one square from X_{i-1}. Assume that $D \cap X_i = \emptyset$ for some i, but $D \cap (X_1 \cup \cdots \cup X_8)$ is maximal. Since D contains a square from some X_i, there are integers a, b such that $D \cap X_a \neq \emptyset$, $D \cap X_b \neq \emptyset$, and $D \cap X_i = \emptyset$ for every $a < i < b$. Therefore, $D \cap Y_i \neq \emptyset$ for every $a < i \leq b$. Let $X_{a,j}$ be a member of $X_a \cap D$. Set $D' := (D \setminus (Y_{a+1} \cup \cdots \cup Y_b) \cup X_{a+1,j} \cup \cdots \cup X_{b,j}$. Clearly, $|D'| \leq |D|$, and D is also a dominating set: the squares in Y_i are dominated by $X_{i-1,j}$ and $X_{i,j}$ for every $a < i \leq b$. This contradicts the maximality of $D \cap (X_1 \cup \cdots \cup X_8)$.

Assume that D contains squares $X_{1,j_1}, \ldots, X_{8,j_8}$, this means that D contains no other square from the gadget. As we have observed above, the squares in Y_i are dominated only if $j_{i-1} \geq j_i$. This gives the chain of inequalities $j_1 \geq j_2 \geq \cdots \geq j_8 \geq j_1$, thus all these values are the same integer j. Thus D contains exactly the squares $X_{1,j}, \ldots, X_{8,j}$ from the gadget. □

The constructed instance contains k^2 copies of the gadget, and it will be true that gadgets are connected to the rest of the instance only via the X_i blocks. The new parameter (the size of the dominating set to be found) is $k' = 8k^2$. At least 8 squares are required to dominate the Y_i blocks of a gadget, thus every solution has to contain exactly 8 squares from each gadget. In this case, Lemma 1 defines a number j for each gadget, which will be called the *value* of the gadget. Therefore, Property 1 of Definition 1 is satisfied.

Restriction. Let $S \subseteq \{1, 2, \ldots, n^2\}$ be an arbitrary set. We restrict the gadget by removing every square $X_{i,j}$ for $1 \leq i \leq 8$ and $j \notin S$. It can be checked that Lemma 1 remains true for gadgets modified this way. Obviously, if $X_{1,j}$ is removed, then the gadget cannot represent value j, thus the value represented by the gadget will be a member of S.

Horizontal connections. The horizontal connections required by Property 3 are shown in Figure 2a. We add a block A that is adjacent to block X_3 of the first gadget and block X_8 of the second, and we add a block B adjacent to X_4 of the first gadget and X_7 of the second. Blocks A and B contain $n + 1$ squares each:

square A_j has offset $(-j-0.5, -n^2-1)$ and square B_j has offset $(j+0.5, n^2+1)$ $(0 \leq j \leq n)$. These blocks do not intersect the Y_i blocks.

Assume that a dominating set does not contain any of the squares from A and B, it contains exactly the squares $X_{1,j}$, …, $X_{8,j}$ from the first gadget, and it contains exactly the squares $X_{1,j'}$, …, $X_{8,j'}$ from the second gadget. We claim that $\iota_1(j) = \iota_1(j')$. If $\iota_1(j) > \iota_1(j')$, then $X_{3,j}$ of the first gadget dominates the squares $A_{\iota_1(j)}$, …, A_{n^2} and $X_{8,j}$ of the second gadget dominates squares A_0, …, $A_{\iota_1(j')-1}$, thus $A_{\iota_1(j')}$ is not dominated. If $\iota_1(j) < \iota_1(j')$, then no square dominates $B_{\iota_1(j)}$ of block B. Thus $\iota_1(j) = \iota_1(j')$, and the values of the two gadgets agree in the first component.

Vertical connections. Vertical connections are defined analogously (see Figure 2b). Square C_j of block C has offsets $(n^2+1, -\iota_2(j))$ and square D_j has offsets $(-n^2-1, \iota_2(j))$ $(0 \leq j \leq n)$.

Constructing a solution. It is straightforward to see that if every gadget has a correct value, then a dominating set of size $8k^2$ can be found: if the value of a gadget is j, then select the 8 squares $X_{1,j}$, …, $X_{8,j}$ from the gadget. □

The same reduction shows hardness for unit disks: it can be shown that if each square in the constructed instance is replaced by a disk and ϵ is sufficiently small, then the intersection structure does not change. Details omitted.

Theorem 2. *Maximum independent set is* W[1]*-hard for the intersection graphs of unit disks in the plane.* □

For general graphs Dominating Set set is W[2]-complete, therefore Theorem 1 leaves open the question whether the problem is W[1]-complete or W[2]-complete when restricted to these graph classes. For unit squares (and more generally, for axis-parallel rectangles) we show that dominating set is in W[1]. This is the first example when a restriction of dominating set is easier than the general problem, but it is not fixed-parameter tractable.

Theorem 3. Dominating Set *is in* W[1] *for the intersection graphs of axis-parallel rectangles.*

Proof. We prove membership in W[1] by reducing Dominating Set to Short Turing Machine Computation. We construct a Turing machine (with unbounded nondeterminism) that accepts the empty string in k' steps if and only if there is a dominating set of size k. Henceforth $L(S)$ (resp., $R(S)$) denotes the x-coordinate of the left (resp., right) edge of open rectangle S, and $T(S)$ (resp., $B(S)$) denotes the y-coordinate of the top (resp., bottom) edge.

The tape alphabet of the Turing machine consists of one symbol for each rectangle in the instance plus two special symbols 0 and 1. In the fist k steps the machine nondeterministically writes k symbols $x_1, \ldots, x_k$ on the tape, which is a guess at a size k dominating set. Next $4k^2$ symbols $h_{1,1}, \ldots, h_{k,k}, h'_{1,1}, \ldots, h'_{k,k}$, $v_{1,1}, \ldots, v_{k,k}, v'_{1,1}, \ldots, v'_{k,k}$ are written, each of these symbols is either 0 or 1. The intended meaning of $h_{i,j}$ is the following: it is 1 if and only if $R(x_i) \leq L(x_j)$. Similarly, we will interpret $h'_{i,j} = 1$ as $R(x_i) \leq R(x_j)$. The symbols $v_{i,j}$ and $v'_{i,j}$ have similar meaning, but with B and T instead of L and R.

The rest of the computation is deterministic. First we check the consistency of the symbols $h_{i,j}$ with the symbols x_i, x_j. For each $1 \leq i, j \leq k$, we make a full scan of the tape, and store in the internal state of the machine the symbols $h_{i,j}$, x_i, x_j. If these symbols are not consistent (e.g., $R(x_i) > L(x_j)$ but $h_{i,j} = 1$) then the machine rejects. The length of the tape is $k + 4k^2$, and we repeat the check for k^2 pairs i, j, thus the checks take a constant number of steps.

For technical reasons we add four dummy rectangles D_L, D_R, D_T, D_B. The rectangle D_L is to the left of the other rectangles, i.e., $R(D_L) \leq L(S)$ for every other rectangle S. Similarly, the rectangles D_R, D_T, D_B are to the right, top, bottom of the other rectangles, respectively. Instead of testing whether the k rectangles $x_1, \ldots, x_k$ form a dominating set, we will test whether $x_1, \ldots, x_k$, D_L, D_R, D_T, D_B are dominating. Clearly, the answer is the same.

We say that the $k + 4$ selected rectangles contain an *invalid window* if there are four selected rectangles S_L, S_R, S_T, S_B with the following properties.

- $R(S_L) \leq L(S_R)$ and $T(S_B) \leq B(S_T)$. Let A be the rectangle with left edge $R(S_L)$, right edge $L(S_R)$, bottom edge $T(S_B)$, top edge $B(S_T)$.
- There is no selected rectangle that intersects A.
- There is a rectangle S that is completely contained in A.

If the selected rectangles contain an invalid window, then they are not dominating since rectangle S is not dominated. On the other hand, if there is a rectangle S which is not dominated, then the selected squares contain an invalid window: by extending S into the four directions until we reach the edge of some selected rectangles, we obtain the window A. The four rectangles that stopped us from further extending A can be used as S_L, S_R, S_T, S_B.

In the rest of the computation, the Turing machine checks whether the selected rectangles contain an invalid window. For each quadruple i_L, i_R, i_T, i_B it has to be checked whether the rectangles x_{i_L}, x_{i_R}, x_{i_T}, x_{i_B} form an invalid window. Using the symbols $h_{i,j}$ etc. on the tape, it can be tested in a constant number of steps whether a selected rectangle intersects the window determined by these four rectangles. If not, then the machine reads into its internal state the four values x_{i_L}, x_{i_R}, x_{i_T}, x_{i_B}, and rejects if there is a rectangle in the window determined by these squares. There are $(k + 4)^4$ possible quadruples and each check can be done in a constant number of steps; therefore, the whole computation takes a constant number k' of steps. □

4 Dominating Set for Line Segments

In this section we use the framework of Section 2 to prove that Dominating Set is W[1]-complete also for axis-parallel line segments.

Theorem 4. Dominating Set *is* W[1]*-complete for axis-parallel segments.*

Proof. Membership in W[1] follows from Theorem 3. Therefore, only W[1]-hardness has to be proven here. In the constructed instance of Dominating

Set, there are k^2 gadgets, and the new parameter is $k' = 12k^2$. Every dominating set has to contain at least 12 segments from each gadget, hence every solution contains exactly 12 segments from each gadget.

The gadget. The gadget satisfying the requirements of Definition 1 is shown in Figure 3. Unless stated otherwise, the segments are open in this proof. The line segments in the gadget can be dominated only by at least 12 segments, since a segment can dominate at most one of a', b', ..., ℓ'. Furthermore, we claim that there are exactly n^2 dominating sets of size 12: they are of the form $a_s, b_s, \ldots, \ell_s$ for $1 \le s \le n^2$. First, it is easy to see that a size 12 dominating set has to contain exactly one segment from $a_1, \ldots, a_{n^2}$, exactly one segment from $b_1, \ldots, b_{n^2}$, etc. For example, if none of $a_1, \ldots, a_{n^2}$ is selected, then we have to select both a' and a'', which makes the size of the dominating set greater than 12. Assume that $a_{s_a}, b_{s_b}, \ldots, \ell_{s_\ell}$ is a dominating set. Segment a_{s_a} dominates $b_1, b_2, \ldots, b_{s_a-1}$ (see Fig. 3) and c_{s_c} dominates $b_{s_c+1}, \ldots, b_{n^2}$. Therefore, if $s_c > s_a$ then neither b_{s_a} nor b_{s_c} are dominated. At most one of these two segments can be selected, thus there would be a segment that is neither selected nor dominated. We can conclude that $s_c \le s_a$. Moreover, it is also true that if $s_c = s_a$, then neither a_{s_a} nor c_{s_c} dominates $b_{s_a} = b_{s_c}$, hence $s_b = s_a = s_c$ follows. By a similar argument, $s_e \le s_c$ with equality only if $s_c = s_d = s_e$. Continuing further we obtain $s_a \ge s_c \ge s_e \ge s_g \ge s_i \ge s_k \ge s_a$, thus there are equalities throughout, implying $s_a = s_b = s_c = \cdots = s_\ell$, as required. This means that the gadget represents a value between 1 and n^2 in every solution.

Restriction. Restricting a gadget to set S is implemented by removing the segments $a_s, b_s, \ldots, \ell_s$ from the gadget for every $s \notin S$.

Horizontal connections. Figure 4a shows how to connect two adjacent gadgets by a horizontal connection. We add $2n$ new segments $x_1, \ldots, x_n, y_1, \ldots, y_n$. The right end point of h_i (resp., j_i) in the first gadget is modified to be its intersection with $x_{\iota_1(i)}$ (resp., $y_{\iota_1(i)}$), and this end point is set to be a closed end point. The left end point of d_i and b_i are similarly modified, but these end points are set to be open. Assume that there is a dominating set that contains 12 segments from each of the gadgets and contains none of the segments $x_1, \ldots, x_n, y_1, \ldots, y_n$. Furthermore, assume that the pair $i = (\iota_1(i), \iota_2(i))$ is the value of the first gadget and the $i' = (\iota_1(i'), \iota_2(i'))$ is the value of the second gadget. In particular, this means that h_i, j_i are selected in the first gadget, and $b_{i'}$, $d_{i'}$ are selected in the second. Now if $\iota_1(i) < \iota_1(i')$, then $x_{\iota_1(i')}$ is not dominated, and if $\iota_1(i) > \iota_1(i')$, then $y_{\iota_1(i')}$ is not dominated, thus $\iota_1(i) = \iota_1(i')$ follows.

Vertical connections. Done analogously, see Figure 4b. □

5 Maximum Independent Set for Line Segments

In this section we turn our attention to the Maximum Independent Set problem. The problem is fixed-parameter tractable for axis-parallel line segments, or more generally, if the lines have only a fixed number of different directions:

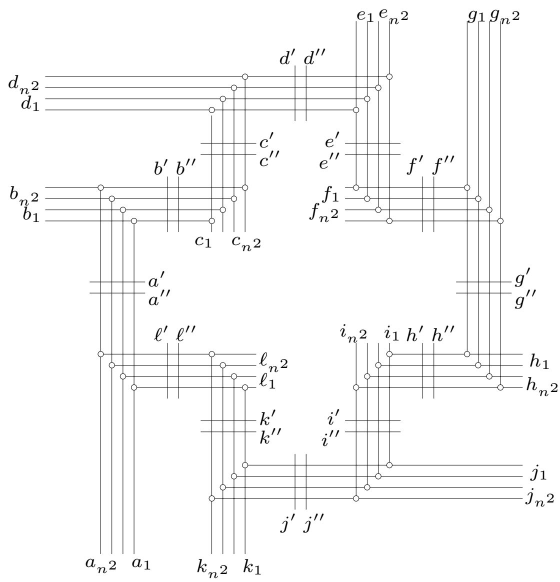


Fig. 3. The gadget used in the proof of Theorem 4

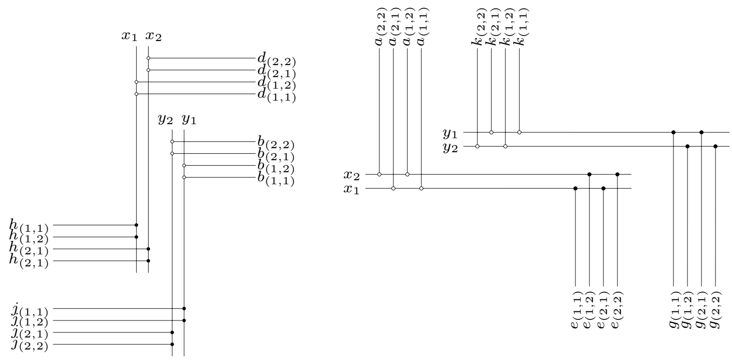
Theorem 5. MAXIMUM INDEPENDENT SET *for the intersection graphs of line segments in the plane can be solved in* $2^{O(k^2 d^2 \log d)} n \log n$ *time if the lines are allowed to have at most* d *different directions.*

Proof. Let $L_1, L_2, \ldots, L_d$ be the partition of the line segments according to their directions. The segments in L_i lie on n_i parallel lines $\ell_{i,1}, \ldots, \ell_{i,n_i}$. If $n_i \geq k$, then we can select k parallel segments from L_i that are on different lines, hence we have an independent set of size k. Thus it can be assumed that $n_i < k$ for every i. Therefore, the $n_1 + n_2 + \cdots + n_d$ lines have at most $\binom{d}{2}(k-1)^2$ intersection points, which will be called the *special points*. Apart from the special points, every point in the plane is covered by segments of at most one direction only. In a solution a special point is either not covered, or covered by a segment in one of $L_1, L_2, \ldots, L_d$. We try all $d^{\binom{d}{2}(k-1)^2}$ possibilities: each special point is assigned to one of the d directions. After deleting the segments that cross a special point from the wrong direction, we get d independent problems: segments with different directions do not cross each other. Furthermore, problem L_i consists of n_i independent problems: the parallel lines do not intersect. Therefore, the solution for this case can be obtained by selecting from each line as many independent segments as possible. It is well-known that this can be done in $O(n \log n)$ time. □

A similar result was independently obtained by Kára and Kratochvíl (see [2] elsewhere in this volume). Their algorithm is somewhat faster and works even if only the intersection graph is given (not the segments themselves).

However, the problem is W[1]-hard with arbitrary directions:

Theorem 6. MAXIMUM INDEPENDENT SET *is* W[1]-*complete for intersection graphs of unit line segments.*



(a) (b)

Fig. 4. Connecting two gadgets in the same row (a) or column (b)

Proof. The proof uses the framework of Section 2. The new parameter $k' := 4k^2 + 2k(k-1)$ is 4 times the number of gadgets plus 2 times the number of connections. It is not possible to select more than 4 (resp., 2) independent segments from a gadget (resp., connection), hence every solution has to contain exactly that many segments from every gadget and connection.

The gadget. Henceforth we assume that the line segments are open. Each gadget consists of $4n^2$ line segments. For the gadget *centered at point* (x, y) the segments a_1, b_1, c_1, d_1 are arranged as shown in Figure 5. Set $\theta = 1/2n^6$. For $2 \le i \le n^2$, the lines a_i, b_i, c_i, d_i are obtained by rotating counterclockwise the four lines in Figure 5 around (x, y) by $(i-1)\theta$ radians. As discussed above, the parameter of the Maximum Independent Set problem is set in such a way that every solution contains 4 independent segments of the gadget. We say that the gadget represents value i in a solution if these four segments are a_i, b_i, c_i, d_i. The following lemma shows that every gadget represents a value in a solution:

Lemma 2. *At most 4 segments can be selected from each gadget. If S is an independent set of size 4 in a gadget, then $S = \{a_i, b_i, c_i, d_i\}$ for some $1 \le i \le n^2$.*

Proof. Since a_i and $a_{i'}$ intersect each other, at most one segment can be selected from $\{a_i : 1 \le i \le n^2\}$. Similarly, we can select at most one segment from the b_i's, c_i's, and d_i's, hence an independent set cannot have size more than 4.

Assume now that a_{i_a}, b_{i_b}, c_{i_c}, d_{i_d} is an independent set in the gadget. First we show that $i_a \le i_b$. It is sufficient to show that every a_j with $j > 1$ intersects b_1, since a_{i_a} and b_{i_b} has the same relation as $a_{i_a - i_b + 1}$ and b_1. The upper end point of a_j has y-coordinate greater than $y + 0.5$, while the y-coordinate of the other end point is smaller than y, thus it is easy to see that it intersects b_1. Similar arguments show that $i_a \le i_b \le i_c \le i_d \le i_a$, hence $i_a = i_b = i_c = i_d$ $\square$

Restriction. To restrict the gadget to a set S, we remove a_i, b_i, c_i, d_i from the gadget for every $i \notin S$.

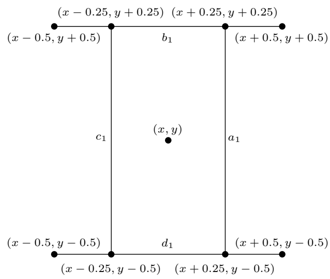


Fig. 5. The four segments of the gadget

Horizontal connections. If two gadgets are connected by a horizontal connection, then their distance is $1+\delta$ (where the constant $\delta > 0$ is to be determined later), i.e., they are centered at (x_0, y_0) and $(x'_0, y'_0) = (x_0 + 1 + \delta, y_0)$. Let A_i be the intersection of the line $y = y_0 + 0.1$ and segment a_i of the first gadget. Let C_i be the intersection of the same line and segment c_i of the second gadget. We want to add n segments in such a way that segment e_j $(1 \le j \le n)$ intersects only segments $a_1, \ldots, a_{(j-1)n}$ of the first gadget and segments $c_{jn+1}, \ldots, c_{n^2}$ of the second gadget. This can be achieved if (open) segment e_j $(1 \le j \le n)$ connects $A_{(j-1)n+1}$ and C_{jn}. The segments e_j have different lengths, but it is possible to modify the x-coordinates of the end points and set δ such that every e_j has unit length (details omitted).

The e_j's ensure that if a_i, b_i, c_i, d_i are selected from the first gadget, $a_{i'}$, $b_{i'}$, $c_{i'}$, $d_{i'}$ are selected from the second gadget, and a segment e_j is also selected, then $\iota_1(i) \ge \iota_1(i')$. Recall that $i = (\iota_1(i) - 1)n + \iota_2(i)$ and $i' = (\iota_1(i') - 1)n + \iota_2(i')$. As e_j intersects $a_1, \ldots, a_{(j-1)n}$, it follows that $\iota_1(i) \ge j$, otherwise e_j would intersect a_i. Segment e_j intersects segments $c_{jn+1}, \ldots, c_{n^2}$ of the second gadget, hence $i' \le (j-1)n + n$ and $\iota_1(i') \le j \le \iota_1(i)$ follows.

In a similar way, we add segments $f_1, \ldots, f_n$, whose job is to ensure that $\iota_1(i') \ge \iota_1(i)$. We want to define the segments in such a way that f_j intersects $a_{jn+1}, \ldots, a_{n^2}$ of the first gadget and $c_1, \ldots, c_{(j-1)n}$ of the second gadget. This can be done analogously to the definition of the segments e_j, but this time we intersect the a_i's and c_i's with the line $y = y_0 - 0.1$. It can be shown, that if a_i of the first gadget, $c_{i'}$ of the second gadget, and segment f_j are independent, then $\iota_1(i') \ge \iota_1(i)$. Therefore, the horizontal connection effectively forces that $\iota_1(i) = \iota_2(i')$ if i and i' are the values represented by the two adjacent gadgets.

Vertical connections. The vertical connection consists of two sets of segments $g_1, \ldots, g_n$ and $h_1, \ldots, h_n$, where every g_i intersects every $g_{i'}$, and every h_j intersects every $h_{j'}$. These segments are defined in such a way that

- g_{j_1} intersects b_i of the lower gadget if and only if $\iota_2(i) > j_1$,
- g_{j_1} intersects $d_{i'}$ of the upper gadget if and only if $\iota_2(i') < j_1$,
- h_{j_2} intersects b_i of the lower gadget if and only if $\iota_2(i) < j_2$,
- h_{j_2} intersects $d_{i'}$ of the upper gadget if and only if $\iota_2(i') > j_2$.

It is easy to see that these segments do what is required from a vertical connection: if b_i of the first gadget, $d_{i'}$ of the second gadget, and g_{j_1}, h_{j_2} are independent segments, then $\iota_2(i) = \iota_2(i') = j_1 = j_2$. The only question is how to construct the segments such that they have the intersection structure defined above.

We modify the gadget centered at (x_0, y_0) as follows. Set $\gamma = 1/n^3$. For each segment b_i, consider the line ℓ_i containing this segment, and shift b_i along ℓ such that the x-coordinate of the right end point of b_i becomes $x_0+0.5+\iota_2(i)\gamma-\gamma/2$. The b_i's are "almost horizontal," thus it can be verified (details omitted) that

- the y-coordinate of the right end point of b_i is between $y_0+0.5$ and $y_0+0.5+\gamma$,
- the x-coordinate of the left end point of b_i is between $x_0-0.5+\iota_2(i)\gamma-0.6\gamma$ and $x_0 - 0.5 + \iota_2(i)\gamma - 0.4\gamma$,
- the y-coordinate of the left end point of b_i is between $y_0+0.5-\gamma$ and $y_0+0.5$.

In a symmetrical way, we can ensure that

- the x-coordinate of the left end point of d_i is $x_0 - 0.5 - \iota_2(i)\gamma + \gamma/2$.
- the y-coordinate of the left end point of d_i is between $y_0-0.5-\gamma$ and $y_0-0.5$,
- the x-coordinate of the right end point of d_i is between $x_0+0.5-\iota_2(i)\gamma+0.4\gamma$ and $x_0 + 0.5 - \iota_2(i)\gamma + 0.6\gamma$,
- the y-coord. of the right end point of d_i is between $y_0 - 0.5+\gamma$ and $y_0 - 0.5$.

In the vertical connection between the two gadgets centered at (x_0, y_0) and $(x_0, y_0 + 1.5)$, the segment g_j is a unit length segment that goes through the points $(x_0+0.5+j\gamma, y_0+0.5)$, $(x_0+0.5-(j-1)\gamma, y_0+1+\gamma)$, and the center point of g_j has $y_0 + 0.75$ as y-coordinate. As $\gamma < 1/n^2$, segment g_j is almost vertical; in particular, it reaches the line $y = y_0 + 0.5 + \gamma$ with an x-coordinate greater than $x_0 + 0.5 + j\gamma - \gamma/2$, and it reaches the line $y = y_0 + 1$ with x-coordinate less than $x_0 + 0.5 + (j - 1)\gamma + 0.4\gamma$. This means that g_j does not intersect a segment b_i if its right end point has x-coordinate at most $x_0 + j\gamma - \gamma/2$ (i.e., $\iota_2(i) > j$) and it does intersect a g_j if the x-coordinate of its right end point is greater than $x_0 + j\gamma$ (i.e., $\iota_2(i) \leq j$). Similarly, in the upper gadget, g_j intersects every d_i with $\iota_2(i) < j$, and does not intersect d_i if $\iota_2(i) > j$. □

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