Computerized Simulation
Exercise No. 3

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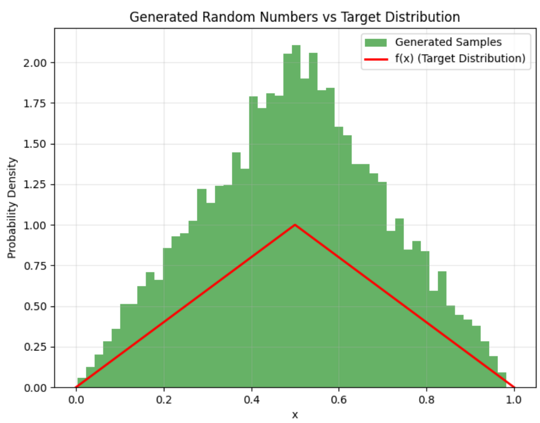
Contents

1 Exercise 1

Write a program using the **accept-reject method** to generate random numbers following a custom probability distribution defined by the piecewise function as follows:

$$f(x) = \begin{cases} 2x & 0 \leq x \leq 0.5 \\ 2(1-x) & 0.5 < x \leq 1. \end{cases}$$

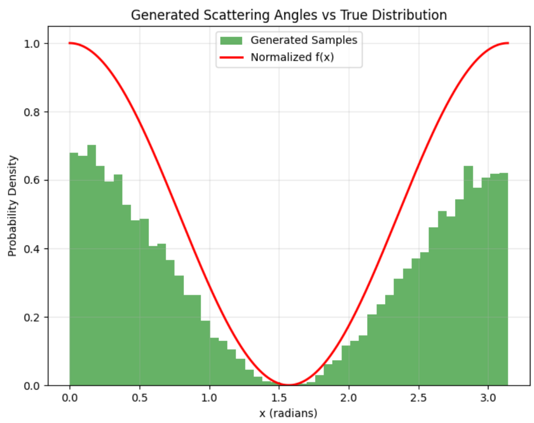
```python
import numpy as np
import matplotlib.pyplot as plt

def piecewise_function(x):
    return np.where((0 <= x) & (x <= 0.5), 2 * x,
           np.where((0.5 < x) & (x <= 1), 2 * (1 - x), 0))

def accept_reject_sampling(target_func, domain, n_samples, max_val):
    samples = []
    x_min, x_max = domain
    while len(samples) < n_samples:
        x = np.random.uniform(x_min, x_max)
        u = np.random.uniform(0, 1)
        if u <= target_func(x) / max_val:
            samples.append(x)
    return np.array(samples)

def plot_results(samples, target_func, domain, bins=50):
    x_vals = np.linspace(domain[0], domain[1], 1000)
    f_vals = target_func(x_vals)
    plt.figure(figsize=(8, 6))
    plt.hist(samples, bins=bins, density=True, alpha=0.6, color='g', label='Generated Samples')
    plt.plot(x_vals, f_vals, 'r-', linewidth=2, label='f(x) (Target Distribution)')
    plt.xlabel('x')
    plt.ylabel('Probability Density')
    plt.title('Generated Random Numbers vs Target Distribution')
    plt.legend()
    plt.grid(True, alpha=0.3)
    plt.show()

n_samples = 10000
domain = (0, 1)
max_val = 1
np.random.seed(0)
random_numbers = accept_reject_sampling(piecewise_function, domain, n_samples, max_val)
plot_results(random_numbers, piecewise_function, domain)
```



2 Exercise 2

Simulate the distribution of angles at which particles scatter when the probability distribution of scattering angles is proportional to $\cos^2(x)$. Use the **accept-reject method** to generate the angles.

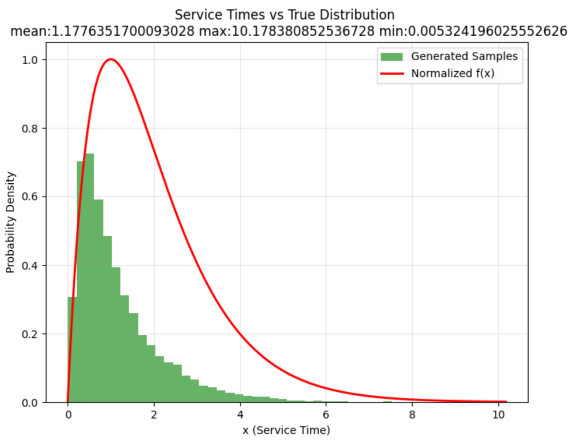
```python
import numpy as np
import matplotlib.pyplot as plt

def scattering_distribution(x):
    return np.cos(x)**2

def accept_reject_sampling(target_func, domain, n_samples, max_val):
    samples = []
    x_min, x_max = domain
    while len(samples) < n_samples:
        x = np.random.uniform(x_min, x_max)
        u = np.random.uniform(0, 1)
        if u <= target_func(x) / max_val:
            samples.append(x)
    return np.array(samples)

def plot_results(samples, target_func, domain, bins=50):
    x_vals = np.linspace(domain[0], domain[1], 1000)
    f_vals = target_func(x_vals)
    plt.figure(figsize=(8, 6))
    plt.hist(samples, bins=bins, density=True, alpha=0.6, color='g', label='Generated Samples')
    plt.plot(x_vals, f_vals / np.max(f_vals), 'r-', linewidth=2, label='Normalized f(x)')
    plt.xlabel('x (radians)')
    plt.ylabel('Probability Density')
    plt.title('Generated Scattering Angles vs True Distribution')
    plt.legend()
    plt.grid(True, alpha=0.3)
    plt.show()

n_samples = 10000
domain = (0, np.pi)
max_val = 1
np.random.seed(0)
scattering_angles = accept_reject_sampling(scattering_distribution, domain, n_samples, max_val)
plot_results(scattering_angles, scattering_distribution, domain)
```



3 Exercise 3

Simulate service times for customers in a queue. Assume service times follow a specific distribution (e.g., $f(x) = xe^{-x}$), and use the **accept-reject method** to generate the times. Analyze the average waiting time.

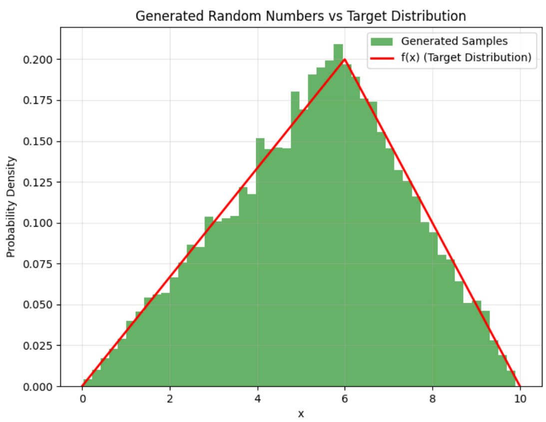
```python
import numpy as np
import matplotlib.pyplot as plt

def service_time_distribution(x):
    return x * np.exp(-x) if x > 0 else 0

def exponential_distribution(x):
    return np.exp(-x) if x > 0 else 0

def accept_reject_sampling(target_func, proposal_func, domain, n_samples, max_val):
    samples = []
    while len(samples) < n_samples:
        x = np.random.exponential(1)
        u = np.random.uniform(0, 1)
        if u <= target_func(x) / (max_val * proposal_func(x)):
            samples.append(x)
    return np.array(samples)

def plot_results(samples, target_func, bins=50):
    x_vals = np.linspace(0, max(samples), 1000)
    f_vals = np.array([target_func(x) for x in x_vals])
    plt.figure(figsize=(8, 6))
    plt.hist(samples, bins=bins, density=True, alpha=0.6, color='g', label='Generated Samples')
    plt.plot(x_vals, f_vals / f_vals.max(), 'r-', linewidth=2, label='Normalized f(x)')
    plt.xlabel('x (Service Time)')
    plt.ylabel('Probability Density')
    plt.title(f'Service Times vs True Distribution \n mean:{np.mean(samples)} max:{np.max(samples)}
min:{np.min(samples)}')
    plt.legend()
    plt.grid(True, alpha=0.3)
    plt.show()

n_samples = 10000
domain = (0, np.inf)
max_val = 1 / np.exp(1)
np.random.seed(0)
service_times = accept_reject_sampling(service_time_distribution, exponential_distribution, domain,
n_samples, max_val)
plot_results(service_times, service_time_distribution)
```



4 Exercise 4

Generate random numbers for a triangular distribution defined on $[a, b]$ with mode c.

```python
import numpy as np
import matplotlib.pyplot as plt

def triangular_distribution_function(x, a, b, c):
    return np.where((a <= x) & (x < c), 2 * (x - a) / ((b - a) * (c - a)),
           np.where((c <= x) & (x <= b), 2 * (b - x) / ((b - a) * (b - c)), 0))

def accept_reject_sampling(target_func, domain, n_samples, max_val, *params):
    samples = []
    x_min, x_max = domain
    while len(samples) < n_samples:
        x = np.random.uniform(x_min, x_max)
        u = np.random.uniform(0, 1)
        if u <= target_func(x, *params) / max_val:
            samples.append(x)
    return np.array(samples)

def plot_results(samples, target_func, domain, params, bins=50):
    x_vals = np.linspace(domain[0], domain[1], 1000)
    f_vals = target_func(x_vals, *params)
    plt.figure(figsize=(8, 6))
    plt.hist(samples, bins=bins, density=True, alpha=0.6, color='g', label='Generated Samples')
    plt.plot(x_vals, f_vals, 'r-', linewidth=2, label='f(x) (Target Distribution)')
    plt.xlabel('x')
    plt.ylabel('Probability Density')
    plt.title('Generated Random Numbers vs Target Distribution')
    plt.legend()
    plt.grid(True, alpha=0.3)
    plt.show()

a, b, c = 0, 10, 6
n_samples = 10000
max_val = 2 / (b - a)
domain = (a, b)
np.random.seed(0)
random_numbers = accept_reject_sampling(triangular_distribution_function, domain, n_samples, max_val,
a, b, c)
plot_results(random_numbers, triangular_distribution_function, domain, (a, b, c))
```



5 Exercise 5

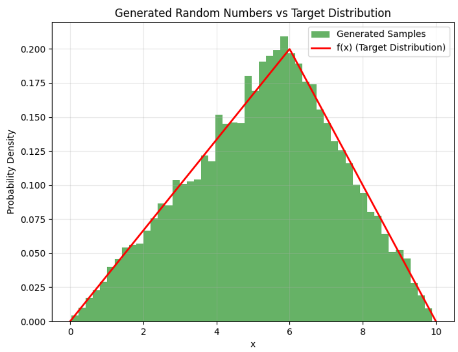
Implement a random number generator for the Rayleigh distribution with variance σ^2.

The Rayleigh distribution is a continuous probability distribution commonly used in signal processing, radar systems, and various statistical applications.

$$PDF : f(x;\sigma) = \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}, \quad x \geq 0,$$

$$CDF : F(x;\sigma) = 1 - e^{-\frac{x^2}{2\sigma^2}}, \quad x \geq 0.$$

```python
import numpy as np
import matplotlib.pyplot as plt

def triangular_distribution_function(x, a, b, c):
    return np.where((a <= x) & (x < c), 2 * (x - a) / ((b - a) * (c - a)),
           np.where((c <= x) & (x <= b), 2 * (b - x) / ((b - a) * (b - c)), 0))

def accept_reject_sampling(target_func, domain, n_samples, max_val, *params):
    samples = []
    x_min, x_max = domain
    while len(samples) < n_samples:
        x = np.random.uniform(x_min, x_max)
        u = np.random.uniform(0, 1)
        if u <= target_func(x, *params) / max_val:
            samples.append(x)
    return np.array(samples)

def plot_results(samples, target_func, domain, params, bins=50):
    x_vals = np.linspace(domain[0], domain[1], 1000)
    f_vals = target_func(x_vals, *params)
    plt.figure(figsize=(8, 6))
    plt.hist(samples, bins=bins, density=True, alpha=0.6, color='g', label='Generated Samples')
    plt.plot(x_vals, f_vals, 'r-', linewidth=2, label='f(x) (Target Distribution)')
    plt.xlabel('x')
    plt.ylabel('Probability Density')
    plt.title('Generated Random Numbers vs Target Distribution')
    plt.legend()
    plt.grid(True, alpha=0.3)
    plt.show()

a, b, c = 0, 10, 6
n_samples = 10000
max_val = 2 / (b - a)
domain = (a, b)
np.random.seed(0)
random_numbers = accept_reject_sampling(triangular_distribution_function, domain, n_samples, max_val,
a, b, c)
plot_results(random_numbers, triangular_distribution_function, domain, (a, b, c))
```



6 Exercise 6

Simulate a bank with multiple counters where:

- Customer arrival times follow an exponential distribution.
- Service times at counters also follow an exponential distribution.

```python
import numpy as np

def sample_exponential(rate):
    return np.random.exponential(1 / rate)

def simulate_bank(num_counters, arrival_rate, service_rate, simulation_time):
    current_time = 0
    next_arrival = sample_exponential(arrival_rate)
    counters_free_at = [0] * num_counters
    waiting_queue = []
    total_waiting_times = []
    total_system_times = []
    counter_usage = [0] * num_counters

    while current_time <= simulation_time:
        next_departure = min([t for t in counters_free_at if t > current_time], default=float('inf'))

        if next_arrival < next_departure:
            current_time = next_arrival
            # Check for free counters
            free_counters = [i for i, t in enumerate(counters_free_at) if t <= current_time]
            if free_counters:
                # Assign to a free counter
                assigned_counter = free_counters[0]
                service_time = sample_exponential(service_rate)
                counters_free_at[assigned_counter] = current_time + service_time
                counter_usage[assigned_counter] += service_time
                total_system_times.append(service_time)
            else:
                waiting_queue.append(current_time)
            next_arrival = current_time + sample_exponential(arrival_rate)
        elif next_departure != float('inf'):
            current_time = next_departure
            finished_counter = counters_free_at.index(next_departure)
            if waiting_queue:
                waiting_time_start = waiting_queue.pop(0)
                waiting_time = current_time - waiting_time_start
                total_waiting_times.append(waiting_time)
                service_time = sample_exponential(service_rate)
                counters_free_at[finished_counter] = current_time + service_time
                counter_usage[finished_counter] += service_time
                total_system_times.append(waiting_time + service_time)
            else:
                counters_free_at[finished_counter] = 0  # Mark counter as free
        else:
            break  # End simulation if no future events are scheduled

    avg_waiting_time = np.mean(total_waiting_times) if total_waiting_times else 0
    avg_total_system_time = np.mean(total_system_times) if total_system_times else 0
    usage_percentages = [(time / simulation_time) * 100 for time in counter_usage]
    return avg_waiting_time, avg_total_system_time, usage_percentages

num_counters = 3
arrival_rate = 2
service_rate = 3
simulation_time = 5000
avg_waiting, avg_system_time, usage = simulate_bank(num_counters, arrival_rate, service_rate,
simulation_time)
print(f"Average Waiting Time: {avg_waiting:.2f}")
print(f"Counter Usage (%): {usage}")
print(f"Average Time in System: {avg_system_time:.2f}")
```

The simulation was conducted using the following input parameters:

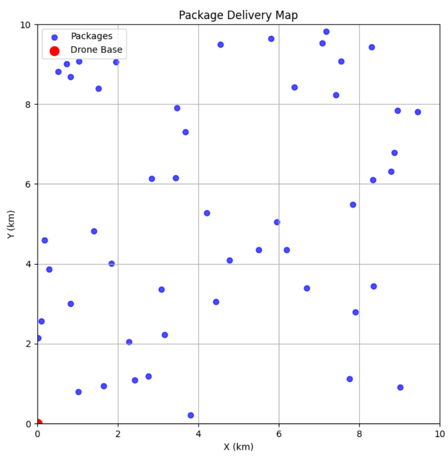
- Number of Counters: 3
- Arrival Rate (λ_a): 2 customers per unit time
- Service Rate (λ_s): 3 customers per unit time
- Simulation Time: 5000 time units

Based on the input parameters, the simulation produced the following results:

- Average Waiting Time: 0.14 time units.
- Counter Usage: 40.03% , 19.41% ,6.67%
- Average Time in System: 0.34 time units.

7 Exercise 7

```python
import numpy as np
import matplotlib.pyplot as plt

NUM_DRONES = 3
AREA_SIZE = 10
BATTERY_RANGE = 15
MAX_PAYLOAD = 5
DRONE_SPEED = 50
NUM_PACKAGES = 50

def generate_packages(num_packages, area_size):
    locations = np.random.uniform(0, area_size, size=(num_packages, 2))
    weights = np.random.uniform(1, MAX_PAYLOAD, num_packages)
    arrival_times = np.cumsum(np.random.exponential(scale=1, size=num_packages))
    return locations, weights, arrival_times

def calculate_distance(point1, point2):
    return np.sqrt(np.sum((point1 - point2) ** 2))

def assign_drone(drone_positions, drone_battery, package_location, package_weight):
    for idx, (position, battery) in enumerate(zip(drone_positions, drone_battery)):
        distance = calculate_distance(position, package_location)
        if battery >= 2 * distance and package_weight <= MAX_PAYLOAD:
            return idx, distance
    return None, None

def simulate_drones(num_drones, packages, battery_range, speed):
    locations, weights, arrival_times = packages
    drone_positions = np.zeros((num_drones, 2))
    drone_battery = np.full(num_drones, battery_range)
    total_distance = 0
    delivery_times = []
    current_time = 0

    for package_idx, package_time in enumerate(arrival_times):
        current_time = package_time
        package_location = locations[package_idx]
        package_weight = weights[package_idx]

        drone_idx, distance = assign_drone(drone_positions, drone_battery, package_location,
package_weight)
        if drone_idx is not None:
            travel_time = (2 * distance) / speed
            total_distance += 2 * distance
            delivery_times.append(current_time + travel_time)
            drone_positions[drone_idx] = package_location
            drone_battery[drone_idx] -= 2 * distance

    avg_delivery_time = np.mean(delivery_times)
    return avg_delivery_time, total_distance, locations

def plot_simulation(locations, area_size):
    plt.figure(figsize=(8, 8))
    plt.scatter(locations[:, 0], locations[:, 1], c='blue', label='Packages', alpha=0.7)
    plt.scatter(0, 0, c='red', label='Drone Base', s=100)
    plt.xlim(0, area_size)
    plt.ylim(0, area_size)
    plt.title('Package Delivery Map')
    plt.xlabel('X (km)')
    plt.ylabel('Y (km)')
    plt.legend()
    plt.grid(True)
    plt.show()

packages = generate_packages(NUM_PACKAGES, AREA_SIZE)
avg_delivery_time, total_distance, locations = simulate_drones(NUM_DRONES, packages, BATTERY_RANGE,
DRONE_SPEED)
print(f"Average Delivery Time: {avg_delivery_time:.2f} hours")
print(f"Total Distance Traveled by All Drones: {total_distance:.2f} km")
plot_simulation(locations, AREA_SIZE)
```



- Average Delivery Time: 11.06 hours
- Total Distance Traveled by All Drones: 41.04 km