Newton Rahmpson Method

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1 Introduction

Newton's method, also known as the Newton-Raphson method, is a powerful numerical method for finding the roots of a function. It is named after Sir Isaac Newton and Joseph Raphson, who independently developed the method in the 17th century.

The method involves an iterative process that uses the first derivative of a function to approximate the location of its roots. Starting with an initial guess, the method generates a sequence of increasingly accurate estimates for the root until a desired level of accuracy is achieved.

The Newton-Raphson method has many applications in various fields, including engineering, physics, and finance. It is particularly useful when analytical solutions to equations are not available, or when solving the equation analytically is too complex or time-consuming.

In this project, we will explore the theory behind Newton's method and some examples of how it can be used to solve equations in practice.

2 The Theory Behind Newton Rahmpson Method

Let x_0 be a good estimate of r, and let $r = x_0 + h$. Since the true root is r, and $h = r - x_0$, the number h measures how far the estimate x_0 is from the true roo t.

Since h is 'small,' we can use the linear (tangent line) approximation to conclude that,

$$0 = f(r) = f(x_0 + h) \approx f(x_0) + hf'(x_0)$$

and therefore, unless $f'(x_0)$ is close to 0

$$h \approx -\frac{f(x_0)}{f'(x_0)} \ .$$

It follows that

$$r = x_0 + h \approx x_0 - \frac{f(x_0)}{f'(x_0)}$$

Our new improved estimate x_1 of r is there for given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \ .$$

The next estimate x_2 is obtained from x_1 in exactly the same way as x_1 was obtained from x_0 :

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Continue in this way. If x_n is the current estimate, then the next estimate x_{n+1} is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \ .$$

3 Solving Nonlinear Equations

Nonlinear equations cannot be solved analytically, so numerical methods like Newton's method can be used to find its roots.

Example: Using the Newton method, solve the nonlinear equation $f(x) = 0$, where

$$f(x) = x^3 + 3x^2 - x - 8,$$

with an initial guess of $x_0 = 1$. Iterate the method 2 times and present the resulting estimates and include the corresponding values of $f(x)$ and $f'(x)$.

Solution: To solve this nonlinear equation we need to use Newton Method and therefore, the formula we need is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \ .$$

from the question we know that our initial guess is $x_0 = 1$. First let's plug this value to the fuction but before that we have to define $f(x)$ and $f'(x)$.

$$f(x) = x^3 + 3x^2 - x - 8$$

$$f(1) = 1 + 3 - 1 - 8 = -5,$$

$$f'(x) = 3x^2 + 6x - 1$$

$$f'(1) = 3 + 6 - 1 = 8$$

First iteration ,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - (\frac{-5}{8}) = 1,625 \ .$$

$$f(x_1) = (1,625)^3 + 3(1,625)^2 - (1,625) - 8 = 2,587891$$

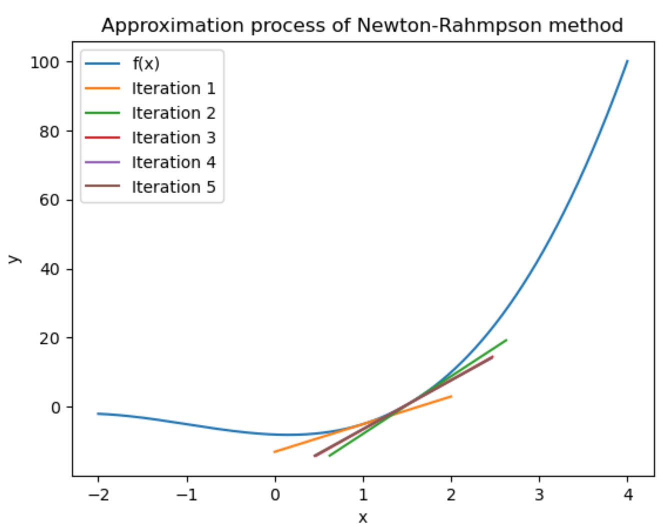
$$f'(x_1) = 3(1,625)^2 + 6(1,625) - 1 = 16,671875$$

Second iteration ,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1,625 - (\frac{2,587891}{16,671875}) = 1.4697750477975633$$

$$\cdot \ \cdot \ \cdot$$

After five iterations, Approximate's to the root $x = 1.4566783471563658$. Here's the visualization of this process (implemented on Jupyter notebook).



$$root x = 1.4566783471563658, iterations needed : 5$$