Traffic Flow Problem

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Abstract

A truth that cannot be denied is that traffic networks are the backbone for modern cities and localities. All over the world, they seek to facilitate transportation from point to another; in other words, developing countries are today establishing new airports, ports, highways, and subways as well as their continuous work on improving the facilities they already have before planning new ones. Therefore, the scientific research done by engineers and modeled by mathematicians is looking forward to establish a complete description of traffic flow characteristics that would help figuring out serious issues. We started our mission from a mathematical model that basically expresses the flow of traffic using the Lighthill – Whitham–Richards Model (LWR) and some equations branching from this model to reach the final forms that relate speed and flow with other parameters. Furthermore, we integrated some basic principles from game theory with the established model to show its effectiveness in studying drivers' behaviors; then we have formed a complete model for a common real situation using game theory. The results produced by equations were not only proved using simulations but also by comparing with real data from other modern sources. Finally, we have discussed some of future work that could be introduced to the researched topic

1 Introduction

Have you ever thought about how they are arranged? Those red, yellow, and green traffic lights with their unique timing for each square and street; those bumps and zebra crossing here and there; and even the number of lanes for each road across the city. How things work and how to make them more and more efficient, that's exactly what engineers are interested in; in fact, one of the most known problems that engineers have to deal with is traffic flow optimization. Over decades, as a result of the development that mankind has reached in industry, paired with the increasing population, the number of cars has been increasing, and traffic problems have appeared.

Congestion, which directly affects people's time and their life style, is one of the present problems, but it's not considered to be a recent one. Everyday drivers, cars owners, and public transportation passengers, may have better sense of the problem as resentment usually appears on their faces; they waste most of their time stuck in congestion [1]. Is this only what we are trying to state as a problem? Traffic flow has an impact not only on individuals but also on societies, and it may be extended to say that it affects the economy of whole countries. Economically speaking, the main impact of congestion is the loss of productivity where some of employees[2], for instance, probably spend most of their time travelling to work rather than working[1]; in other words, the same thing is to be said for tourists, foreign investors, and ordinary citizens who are unable to do what they are supposed to do in certain time due to delay[2]. Missing appointments, inability to do personal activities, or late sleeping time all could be side effects of congestion on employees[1].

Area	Loss in billions note
USA	\$300 [3]
LA	\$19.2 [4]
Bangladesh	\$11.4 [5]

Furthermore, poor traffic management — wrong arrangement of lights and barriers, inappropriate speed limits, and unplanned intersections — can raise the annual number of accidents, and, consequently, the deaths from accidents. Risks also include blockage of traffic while emergency cars are passing (ambulances and fire lorries for example), and that probably leads to more deaths[6]. As a result, traffic flow optimization is one important component of most of modern development plans worldwide.

Due to the above-mentioned losses and risks resulted from traffic congestion, engineers and governmental organizations have unified their efforts to find new approaches to avoid such situations. Overall, instead of overspending large part of the national income on constructing brand new bridges and roads that may exacerbate the current problems, the idea was to digitalize the traffic system where analyzers would be able to collect a wide range of data to set on problems precisely and find optimal solutions for each case. Examples of devices used in this process include the following:

- **Cameras:** The hope was to cover the whole city with cameras to be able to detect motion of vehicles and predict the occurrence of congestion. However, cameras' recording quality was too poor to capture details at darkness, and they obtain only 2-D images while 3-D ones are also needed to handle some other valuable parameters like distance. [7]
- **LIDARS:** Lidar is light detection and ranging tool where laser is used to operate. Lidars was the optimal solution with its ability to detect 3-D images at hard situations like darkness, fog, etc.[7]

Those devices mentioned in the previous step are considered to be the hardware of the system throughout the city to convert physical characteristics into digital data. The next step is to analyze the collected data according to a couple of algorithms to control the flow of traffic and suggest alternative routes for drivers.

- **Smart Traffic Light Control (STLC):** STLC is a smart traffic light connected to internet for controlling the traffic flow according to the analyzed data to avoid any possible congestions.[8]

One feature of modern software solutions and algorithms is that they are generally based on mathematical models as engineers used to use the mathematics as an approach to deal with complex problems. In this paper, we are making use of partial differential equations to construct a mathematical model treating the traffic flow as a continuous fluid. The basic principles of fluid mechanics will be used to determine the relationships between the basic quantities: such as density (ρ), the number of vehicles per unit length in a given time; flow rate (Q), the number of vehicles passing through a fixed position per unit of time; maximum speed ($V = Q/\rho$); number of lanes on the road; the maximum number of cars in a cross-section; and the average speed. Then, the next step is to analyze the data and apply the game theory on.[9] Game theory can be used as a competitive game if the road is tightly controlled to reach Nash equilibrium, taking into account the percentage of vehicles entering and leaving the road branches. Finally, we are supposed to compare the results with reality and suggest solutions to get the flow of vehicles rapid as possible under different conditions in different locations.[9]

Traffic characteristics study is vital for designing geometric features in roadways; more specifically, they include the volume of traffic, speed of traffic, and percentage of trucks or large size vehicles like buses, etc [10]. Overall, traffic characteristics is broadly classified under 2 types: road user characteristics and vehicular characteristics. Road user characteristics includes the physical health of the driver like the aspects of sight, hearing, and reaction; moreover, it may extend to include the psychological state of the driver like emotional factors — anger, fear, and anxiety[10] . Vehicular characteristics is divided into three types as follows:

- Static characteristics: dimensions of vehicles like the length, width, wheel base, and center clearance.[10]
- Dynamic characteristics: state parameters of vehicles like speed, power, and acceleration. [10]
- Braking characteristics: noise, peak force, durability, and continuous power dissipation.[11]

2 Literature review

For nearly a century, various efforts have been devoted to studies which explore interesting traffic phenomena and aim to reveal the theory behind traffic flow; those studies can be traced back to the 1930s with Greenshields' model in 1935 [12]. In this period, traffic engineering began as a rather practical field that relied on its practitioners' common sense to find solutions to specific traffic issues; However, this perspective has been changed at the beginning of the 1950s, when the scientific field was mature enough as engineers from a variety of specializations began to flock to it[12]. Applying mathematical and statistical concepts to describe traffic flow, John Glen Wardrop is credited with creating what is known today as the traffic flow theory[12]. The previous, more "rule-of-thumb" oriented, line of reasoning was completely new to the mathematical approach, which established itself as a solid foundation of theoretical analyses during this highly active period. Moreover, the fluid-dynamics based model — developed by Michael James Lighthill, Gerald Beresford Whitham, and Paul Richards (LWR model for shortness) — beside the car-following experiments and theories developed by the research laboratory of General Motors are examples of progress made in this decade (1950s)[12]. In addition, the publication of the BMW trio model — developed by Martin J. Beckmann, Charles Bartlett McGuire, and Christopher B. Winsten — proved that the transportation economic theory was also making progress at that time. Overall, mathematical models of traffic flow problem are divided into two main sub-categories: the macroscopic models and the microscopic ones, and we are discussing both of them in detail.

Macroscopic models

Macroscopic traffic modeling requires a sufficient number of vehicles on a road. In order to treat each stream of vehicles as flowing in a tube or stream, Three factors are crucial for macroscopic traffic modeling.[13]

- $\mathbf{q}(\mathbf{x}, \mathbf{t})$: rate of flow
- $\mathbf{v}(\mathbf{x}, \mathbf{t})$: the speed of traffic flow
- $\rho(\mathbf{x}, \mathbf{t})$: the traffic density

Several macroscopic models will be discussed below.

A. LWR Model

The tradeoff between the advantages of transportation and heavy loss results from traffic congestion forced the governments to find a solution, and the start was in 1950s with the proposal of the LWR model, which is a deterministic model as its results are controlled by its laws, not any other factor.[14] The LWR model didn't fulfil the required needs as a it was paired with several issues: the deterministic nature of the model didn't give any chance for any random event like an accident that may occur and be taken into consideration, and the model neither differentiates between the hours of the day nor the days of the week. The issues result from the equilibrium state, which ties the speed only to the density without even any consideration to on-ramps and off-ramps along the road and assumes that the nearby vehicles have the same speed[14].

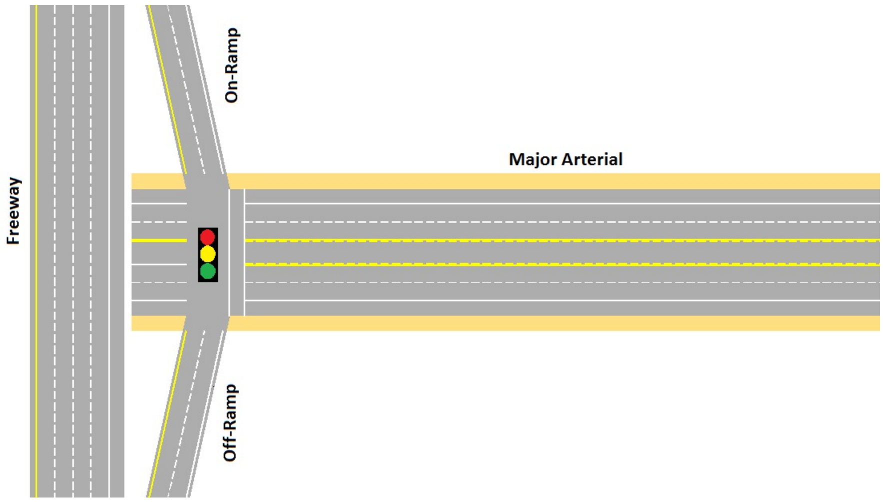


Figure 1: On-Ramp and Off-Ramp

The Stochastic model was introduced as an enhancement to the LWR model, and the forcing function $g(\rho, x, t)$ was added to the right-hand side of the conservation law as it's composed of the mean-reverting drift term $a(x,t) + b(x,t) \cdot \rho$, where $b(x,t) < 0$, to take into consideration any change in the flow due to on-ramps and off-ramps; a Brownian Sheet $W(x,t)$, which is concerned with assigning a random noise to each small rectangular area in the road to take any random events into its account; and the volatility term [14] $\sigma(x,t) \cdot W(x,t)$.

$$\rho_t + q_x = g(\rho, x, t)$$

$$q = \rho \cdot v$$

$$g(\rho, x, t) \cdot dx \cdot dt = (\alpha(x,t) + b(x,t) \cdot \rho) \cdot dx \cdot dt + \sigma(x,t) \cdot dW(x,t)$$

$$dW(x,t) = W(x + dt, t + dt) - W(x, t + dt) - W(x + dx, t) + W(x,t)$$

B.Payne's Model

The LWR model didn't take the relation between drivers and their reactions into consideration, so another approach was made by Payne in 1971 and Whitham in 1974.[13] The PW model is a higher order differential equation model which assumes that each vehicle tends to reach the mean speed while the driver's reaction is considered [13]:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \frac{1}{T}(V(\rho) - v) - \frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x}$$

$$c^2(\rho) = \frac{-1}{T}\frac{dv}{d\rho}$$

The anticipation term describes the driver's behavior, the relaxation term describes the tendency of each vehicle to reach the mean speed; the variable T represents the reaction time between vehicles; and the term $v\frac{\partial v}{\partial x}$is the ingoing and outgoing vehicles.[13]

Figure 2: Drivers' Reaction

C.ARZ Model

The PW model assumed that the traffic flow is isotropic which is an unrealistic assumption so another approach was developed by AW & Rascle (2000) and Zhang (2002) separately which is known as the ARZ model[15]. ARZ model is a second-order nonlinear hyperbolic PDE problem, it assumes that the flow is anisotropic and can deal with stop-and-go conditions like ramp metering.[15]

- **The Aw-Rascle model**

$$\partial_t\rho + \partial_x\rho v = 0$$

$$\partial_t v + (v - \rho p'(\rho))\partial_x v = \frac{V(\rho) - v}{\tau}$$

where $p(\rho) = \rho^\gamma$ and $\gamma \in R_+$

- **The Zhang Model**

$$\partial_i\rho + \partial_x(\rho v) = 0$$

$$\partial_t v + (v + \rho V'(\rho))\partial_x v = 0$$

where $p(\rho)$ is the traffic pressure and τ is the relaxation term.[15]

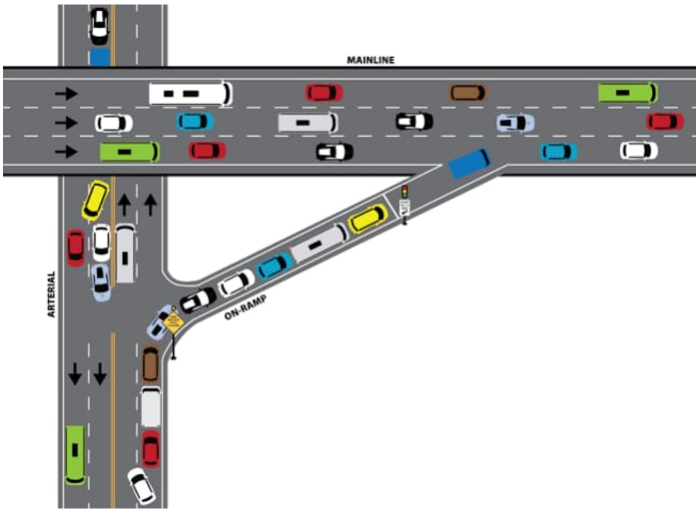


Figure 3: Ramp Metering

Microscopic models

Microscopic models of traffic flow usually describe the details of traffic flow interactions by simulating single vehicles and drivers. [13] The dynamic variables of those models represent small properties like the position and velocity of single vehicles. Those models can be divided into two categories: cell automata models, which are discrete in time and space[13], and continuous models, which are continuous in time. This information is required for detailed studies of car-following behavior and traffic instabilities[13]. A few of microscopic models will be discussed below.

A.The follow-the-leader Model

The follow-the-leader model was presented by Gazis, Herman, and Rothery.[13] In this model it is assumed that the dynamics of vehicle (i) are given by the equation of motion[13]:

$$\dot{p_i}(t) = v_i(t)$$

And the acceleration equation:

$$\dot{v_i}(t+T) = k_i[v_{i+1}(t) - v_i(t)]$$

In the above equation, the acceleration of vehicle i is reduced by the adaptation time T. Thus the following vehicle is assumed to accelerate at time $t+T$. The k_i parameter indicates the sensitivity of the driver of vehicle i. [13] The upcoming functions can be assumed for the sensitivity:

- **constant:** $k_i = a_i$
- **step function:**

$$k_i = \begin{cases} a_i & \Delta p \leq \Delta p_{crit} \\ b_i & \Delta p > \Delta p_{crit} \end{cases}$$

- **reciprocal spacing:** $k_i = c_i/\Delta p$

Here a_i, b_i and c_i are constants. In this hypothesis, the parameter k_i is assumed to be a constant.[13]

B. The Optimal Velocity Model

The optimal velocity (OV) model is one of the traffic flow models that successfully describes some realistic features. In this model, the motion of the preceding car influences the motion of the following car, so it's considered as a type of car-following models. [13]According to the OV model, the acceleration of each car is controlled in such a way as to align its velocity with the optimal velocity determined by the distance Δx between the preceding and following cars.[13] The position x of a car obeys the equation:

$$\frac{d^2x}{dt^2} = \alpha \left[V_{\text{optimal}}(\Delta x) - \frac{dx}{dt} \right]$$

where the constant α is a susceptibility.
In the original optimal velocity model, the acceleration and deceleration processes are treated symmetrically; a real vehicle, however, has a stronger deceleration ability than an acceleration.[13] The advantage of this model is that it's simple to use and calibrate while its weakness appears in the unrealistic large accelerations in some circumstances.

C.Game Theory Model

It has been a challenge to develop a general model to describe traffic flow, due to the large number of influencing factors that may also change according to the dynamics of movement through different areas: highway, urban, single-lane, or multi-lane. Hence, there're several models that focus on interactions independently. One of the most well-known models that is frequently used in academic studies is the fluid-dynamic traffic model, relating the physical flow of objects with the flow of vehicles that interact with each other and with the traffic control systems. The model is based on the assumption that drivers' behaviors are near-perfect where most of models are not able to predict unusual actions of human nature, and that can aid in self-regulation of the system. Nonetheless, this assumption can lead to limited or counterfactual outcomes if drivers are oversimplified, whose decisions probably lead to traffic jams. As a result, some more recent models have put into consideration the social aspects of drivers that could affect their interactions; One example is Wastavino's model of traffic, which studies drivers' behaviors at crossroads. The results of applying this model showed that some drivers who are careful more than usual form a source of traffic jams;[16] furthermore, that is why drivers' behaviors should be included in the study of traffic flow.

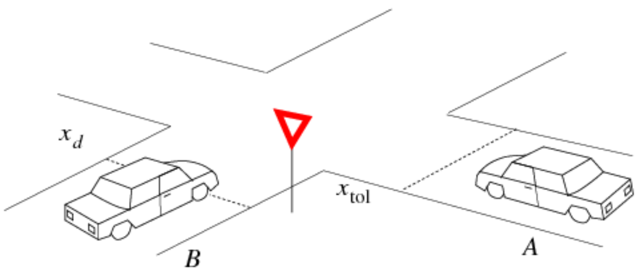


Figure 4: Modeling traffic on crossroad

Driving can be described as a set of decisions that driver makes by moving, increasing speed, stopping, or changing lanes. Those decisions mainly depend on his or her goals and the information taken from the road. Based on those decisions, the driver will be rewarded either by increasing the speed to reduce the time (required outcome), or by slowing down to increase the time (traffic congestion). Some recent studies have focused on describing many of those decisions using a game theory model and applying the results to self-driving cars. Some other studies have analyzed drivers' behaviors and their impact on traffic, and those studies was conducted by a model consisting of agents with bounded rationality who can choose between two possible decisions (cooperative or defective). Based on the results of those studies, it's possible to identify the drivers who are making right choices and who are not. Another model has envisioned how to integrate the development of cooperative behavior of a fixed group of agents into a cooperative game and then compare the performance of the original model with the that of the model associated with different behaviors.[17] On the other hand, most of studies mainly focused on presenting an equilibrium model of traffic flow with mixed autonomy, based on the theory of two-players games.

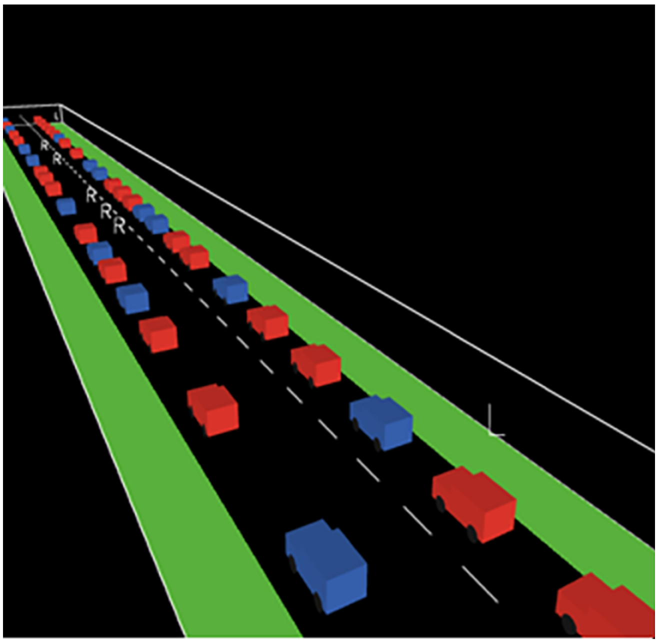


Figure 5: cooperative and defective vehicles

3 Mathematical Modeling

As it is repeatedly mentioned before, traffic congestion has its drawbacks on the economy and societies' quality of life. The first approach to solve the problem was to construct new roads, but it was costly, and its results were disappointing in the long term. The second approach, the engineering one, was to meter the rate of flow in each road and control the flow at intersections depending on the collected data; in addition, one of the most commonly used models is the one based on fluid-dynamics and tries to state analogy between the flow of fluids and the traffic flow. In this paper, we are using the fluid-dynamic model as we are interested in studying continuous roads, and we will make use of game theory to take into consideration drivers' behaviors.

3.1 The Fluid-Dynamic model

LWR model

Traffic flow modeling at the macro level requires the presence of a sufficient number of vehicles on a roadway. To treat each stream of vehicles as flowing in a tube, there are three essential factors [18] :

- $q(x,t)$: is the flux at location x and time t, which is defined as the number of vehicles passing through the location in a unit of time.
- $v(x,t)$: is the average speed of the vehicles at location x and time t.
- $\rho(x,t)$: is the density at location x and time t, which is defined as the number of vehicles in a unit distance.

The fundamental Relationship

The preceding three variables are not independent, and equation (1) is the fundamental equation relating them together.

$$q = \rho \cdot v \tag{1}$$

To justify equation (1), let $\phi(x,t,u)$ represents the density of vehicles with speed u at location x and time t, so $\int_{u1}^{u2} \phi(x,t,u)\, du$represents the number of vehicles with speed between u1 and u2 per unit distance.

$$\mathrm{p(x,t)} = \int_{O}^{\infty} \phi(\mathrm{x,t,u})\mathrm{du} \tag{2}$$

and

$$\mathrm{v(x,t)} = \frac{\int_0^\infty \phi(\mathrm{x,t,u})\mathrm{udu}}{\int_0^x \phi(\mathrm{x,t,u})\mathrm{du}} \tag{3}$$

Then equation (4) represents the total flux passing through x.

$$\mathrm{q(x,t)} = \frac{\int_0^\infty \phi(\mathrm{x,t,u})\mathrm{udtdu}}{\mathrm{dt}} = \int_0^\infty \phi(\mathrm{x,t,u})\mathrm{udu} \mid \tag{4}$$

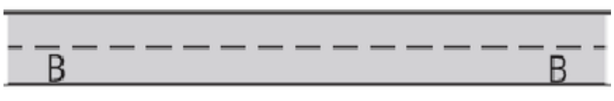
The average speed and flux are easier to be obtained than density, so the new measurements concern with those two component and density could be calculated from equation (1).

The Conservation Law in traffic flow model

The second main relation in the macroscopic traffic flow is the conservation law equation (5)

$$\rho_t + q_x = 0 \tag{5}$$

where the road is assumed to be isotropic with no on-ramps or off-ramps as for segment BB the number of ongoing vehicles equals the number of outgoing ones.



The justification of the conservation law starts with segment BB where the increment of vehicles between t1 and t2 results from the difference between ingoing and outgoing vehicles

$$\int_{x_1}^{x_2} \rho\left(x,t_2\right) dx - \int_{x_1}^{x_2} \rho\left(x,t_1\right) dx = \int_{t_1}^{t_2} q\left(x_1,t\right) dt - \int_{t_1}^{t_2} q\left(x_2,t\right) dt \tag{6}$$

From the fundamental theorem of calculus

$$\int_{x_1}^{x_2} \int_{t_1}^{t_2} \rho_t dt dx = - \int_{t_1}^{t_2} \int_{x_1}^{x_2} q_x dx dt \tag{7}$$

Which can be reduced to

$$\int_{x_1}^{x_2} \int_{t_1}^{t_2} (\rho_t + q_x) \, dt dx = 0 \tag{8}$$

Where equation (8) represents the conservation law.

The Speed-Density Relationship

The speed-density relationship completes the macroscopic traffic flow model where conservation law could be solved and generates road's characteristics.

Equilibrium model is a model at which the average speed is assumed to reach the equilibrium one whatever how fast the density changes, and LWR model is an equilibrium one.

$$v = V(\rho) \tag{9}$$

Greenshields Model

Greenshields expressed the speed-density function as a linear relation [19] in equation (10)

$$v = V_f \left(1 - \frac{\rho}{\rho_j}\right) \tag{10}$$

Where Vf represents the free flow velocity — maximum road speed — and ρ_j represents the jam density — maximum road density — depending on the road characteristics. From equation (1), we have

$$v = \frac{q}{\rho} \tag{11}$$

Then by substituting equation (11) to equation (10), we have

$$\frac{q}{\rho} = V_f * \left(1 - \frac{\rho}{\rho_j}\right) \tag{12}$$

$$q = Vf * \left(\rho - \frac{\rho^2}{\rho_j}\right) \tag{13}$$

To reach the maximum flow (q_m, ρ_c), critical density is to be found

$$\frac{\partial q}{\partial \rho} = v_{ff} - \frac{2.v_{ff}}{\rho_j}\rho_m = 0 \tag{14}$$

$$\rho_m = \frac{\rho_j}{2} \tag{15}$$

By substituting equation (15) to equation (13)

$$q_m = \frac{\rho_j \cdot v_{ff}}{4} \tag{16}$$

From equation (1)

$$\rho = \frac{q}{v} \tag{17}$$

By substituting (17) to equation (10)

$$v = v_{ff} - \frac{v_{ff}}{\rho_j} \cdot \frac{q}{v} \tag{18}$$

$$\frac{v_{ff}}{\rho_j} \cdot \frac{q}{v} = v_{ff} - v \tag{19}$$

$$q = \rho_j \cdot v - \frac{\rho_j}{v_{ff}} v^2 \tag{20}$$

Equation (20) shows the relationship between flow and speed, and the maximum flow q_m is reached when v_c – critical speed – is reached.

$$\frac{\partial q}{\partial v} = \rho_j - \frac{2 \cdot \rho_j}{v_{ff}} \cdot v_m = 0 \tag{21}$$

$$v_m = \frac{v_{ff}}{2} \tag{22}$$

By substituting equation (22) to (20), q_m can be formulated as shown in equation (16). Thus, q_m can be reached when $v = v(c)$ and $\rho = \rho(c)$.

Underwood model

Greenshields expressed the Speed-Density function as an exponential relation [20] equation (23)

$$v = v_{ff} \cdot e^{-\frac{\rho}{\rho_m}} \tag{23}$$

Where V_f represents the free flow velocity — maximum road speed, and ρ_m represents the maximum jam density — critical road density — depending on the road characteristics. From equation (1)

The following equation (24) shows the relationship between speed and density in logarithmic natural form.

$$\ln v = \ln v_{ff} - \frac{\rho}{\rho_m} \tag{24}$$

The flow density relationship can be formulated using the basic equation (1) and by substituting equation (11) into (23)

$$\frac{q}{\rho} = v_{ff} \cdot e^{-\frac{\rho}{\rho_m}} \tag{25}$$

$$q = \rho \cdot v_{ff} \cdot e^{-\frac{\rho}{\rho_m}} \tag{26}$$

Equation (26) shows the relationship between flow and density, and to obtain ρ_m we have

$$\frac{\partial q}{\partial \rho} = v_{ff} e^{\overline{\rho/m}} - \frac{v_{ff}\rho}{\rho_m} e^{-\frac{\rho}{\rho m}} = 0 \tag{27}$$

The flow speed relationship can be formulated using basic equation (1) and by substituting equation [17] into equation [23]

$$v = v_{ff} \cdot e^{-\frac{q}{v \cdot \rho_m}} \tag{28}$$

$$\ln v = \ln v_{ff} - \frac{q}{v \cdot \rho_m} \tag{29}$$

$$\frac{q}{v \cdot \rho_m} = \ln v_{ff} - \ln v \tag{30}$$

$$q = v \cdot \rho_m \cdot (\ln v_{ff} - \ln v) \tag{31}$$

Equation (31) expressed the relationship between flow and speed. The maximum flow q(m) is reached as $v = v(c)$ where $v(c)$ is formulated in equation (35).

$$\frac{\partial q}{\partial v} = \rho_m (\ln v_{ff} - \ln \rho_m) + \rho_m \cdot v_m \left(-\frac{1}{v_m} \right) = 0 \tag{32}$$

$$\rho_m (\ln v_{ff} - \ln v_m) - \rho_m = 0 \tag{33}$$

$$(\ln v_{ff} - \ln v_m) = 1 \tag{34}$$

$$v_m = e^{\ln v_{ff} - 1} \tag{35}$$

3.2 Applying Game theory:

Applying basic principles from game theory, in this model, we are interested in studying continuous roads, taking into consideration roadblocks—random police checkpoints for example—and shortcuts that might affect drivers' behaviors. Game theory is a mathematical decision-making tool that is useful to analyze the interaction between multiple rational agents (drivers in this case) to determine the optimal course of action for all agents and estimate the pay-off (time delay) for each one; agents who act rationally want to maximize their pay-off by making the best decision with respect to other players' decisions. In our case, studying the traffic flow at certain points of intersection by applying the concept of game theory, we are trying to figure out the number of cars going through a shortcut compared with those who will choose the continuous road with potential congestion. Using a mixed strategy analysis to reach the Nash equilibrium for this case, in this model, we started by assuming that all agents can choose between one of two roads with no restrictions and with adequate knowledge of road characteristics to reach a multi-agent game that can be simplified by the symmetry between agents to a game between two groups of agents; each group has the freedom to choose either road to reach the destination. The probability of choosing one of the two roads (shortcut or main road) results in equal utility for both parties. Moreover, from symmetry, we can generalize this ratio to be applied for each vehicle individually, taking into account the difference in time spent across the main road with respect to the density while passage of each car across the road; then, $f(2\rho)$ is replaced with $f(\rho + d\rho)$. Hence, the same equation could be used to study the decision of each agent separately. 3.2.

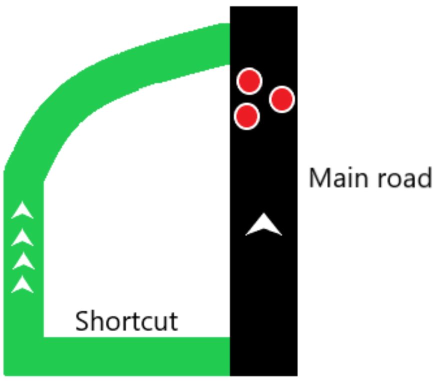


Figure 6: Main road and shortcuts

1st player \2nd player	shortcut	impedance
shortcut	$\Delta t_0, \Delta t_0$	$\Delta t_0, f(\rho)$
impedance	$f(\rho), \Delta t_0$	$f(2\rho), f(2\rho)$

Table 1: Mixed strategy

where:

- Δt_0: time it takes to cross the shortcut
- $f(\rho)$: Equation of the time it takes to cross the barrier with the change in density

- $f(\rho) < \Delta to < f(2 * \rho)$
- h: Probability of passing through the shortcut

$$u_s = \Delta t_0(h + (1 - h)) = \Delta t_0 \quad (36)$$

$$\begin{aligned} u_i &= f(\rho)^* h + f(2\rho) * (1 - h) \\ &= (f(\rho) - f(2\rho))^* h + f(2\rho) \end{aligned} \quad (37)$$

$$u_s = u_i \quad (38)$$

$$\Delta t_0 = (f(\rho) - f(2\rho))^* h + f(2\rho) \quad (39)$$

$$h = \frac{\Delta t_0 - f(2\rho)}{f(\rho) - f(2\rho)} \quad (40)$$

In fact, the idea stated above — shortcuts may have negative impacts on drivers' pay-off — introduces what is technically known as Braess's paradox. Braess's paradox hinges on the very reasonable assumption that drivers will try to find a route that minimizes their own personal travel time. Whenever a new road is opened, drivers may flood that road in order to take advantage of it. However, if everyone does the same thing, that new route might be clogged. As a result of this, drivers are likely to try a different route the next day as they want to avoid traffic. After a few days or weeks, people find the route that seems fastest for them. Traffic settles into a state of equilibrium. Braess showed that there are examples where, when all the dust settles, there's a unique equilibrium where everyone is now taking longer to get to work than they did before [21]. In terms of competitive games, each rational agent tends to choose the shortcut to get higher pay-off; as a result, shortcuts get congested and all players' pay-off is at its lowest level as the travel time through the shortcut or the main road is directly proportional to the density of vehicles: more vehicles cause much delay.

- ρ in the main road $= (1 - h) * \rho_T$
- ρ in the shortcut $= h * \rho_T$

4 Experimental work & Results analysis

In this section, we are concerned with testing the LWR model and the game theory model for different test cases to establish conclusions about how those models work and the scope each model is powerful for. Generally, the LWR model aims to describe the flow in continuous roads with no entries or exits that may affect the flow; in fact, the model is based on the assumption that the number of vehicles entering the road is equal to the number of vehicles leaving it. On the other hand, the game theory specifically is suited for finding the best decisions for drivers at intersections or exits from the main road, and that is what we are trying to demonstrate in our experiments.

4.1 LWR Experiment

The traffic flow model depends mainly on describing the characteristics of the road, so the start was by searching for Speed-Density functions to get the most accurate results; Through this model, Greenshields and Underwood functions were taken into consideration to compare them. We have conducted two experiments -using Eclipse SUMO software simulation program- on a road of length 4 km with no ramps; moreover, we have assumed that the maximum velocity is equal to 100 km/hour for both experiments. In the first experiment, the maximum density is small (40 vehicles/km), while in the second one the maximum density is large (250 vehicles/km)

The road we are conducting the experiments on:

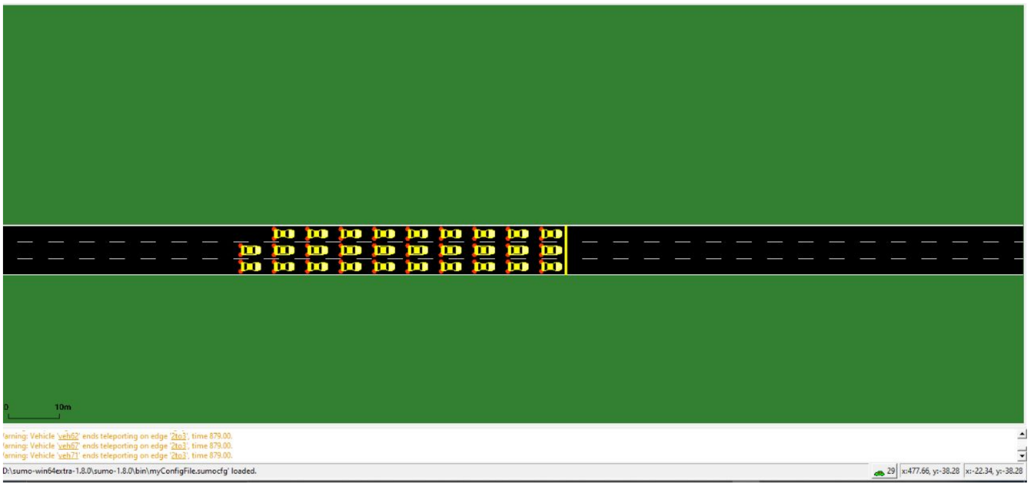
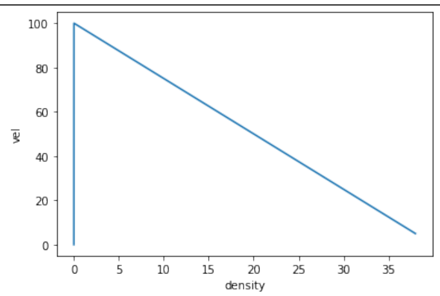
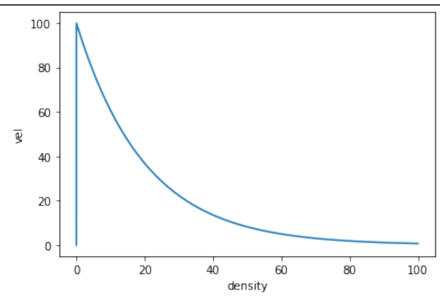
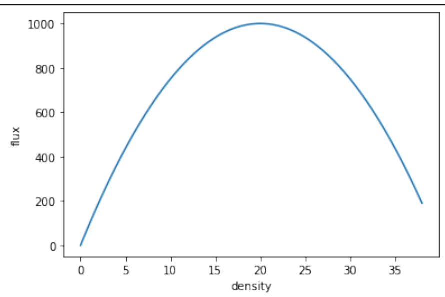
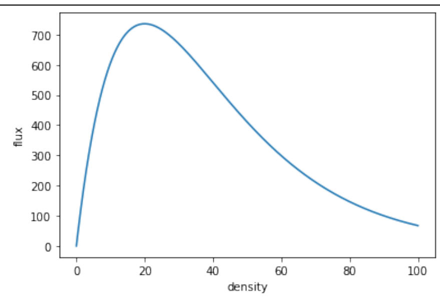


Figure 7: Simulating the road using Eclipse SUMO software

First Experiment ($\rho_{\max} = 40$)

General Relation between v,ρ,Q

Relation Between:	Greenshields	Underwood
Velocity-Density plot		
Flux-Density plot		

The simulation Velocity-Time graph

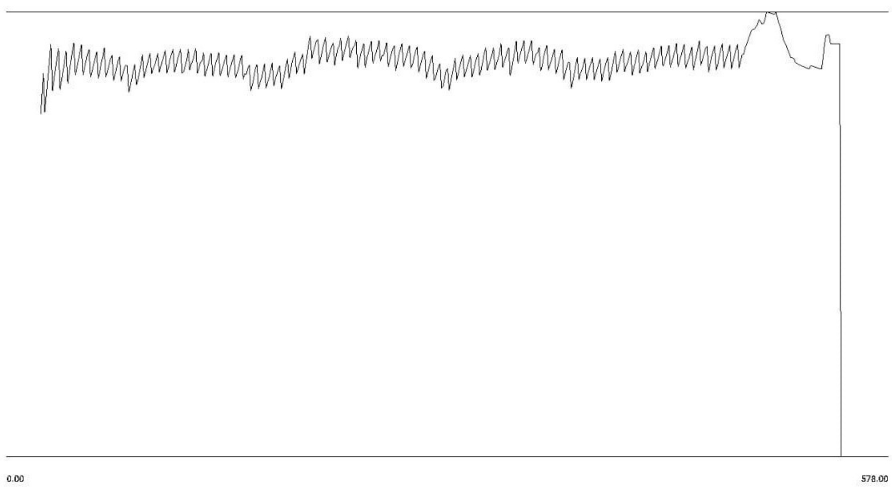


Figure 8: Velocity-Time plot for first experiment

In this simulation, it's assumed that ρ is smaller than ρ_{jam}, and this results in velocity being relatively constant around v_{max}.

The two models' Velocity-Time plots

In both plots, it's assumed that $\rho_0 = 38$ and $\rho_{jam} = 0$.

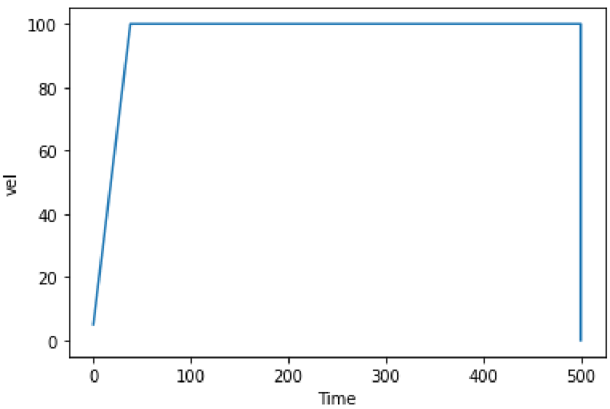


Figure 9: GreenShield Velocity-Time plot

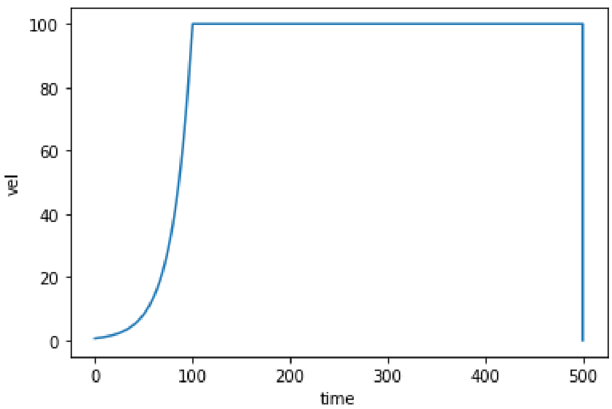


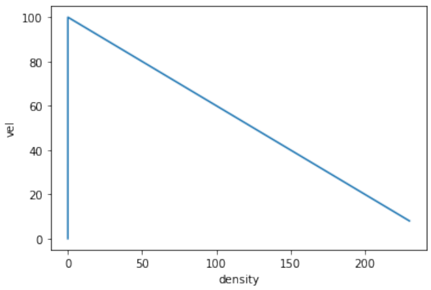
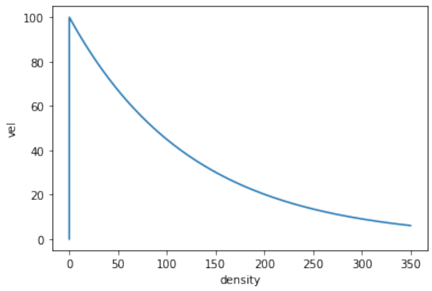
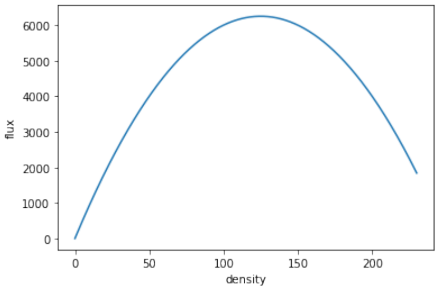
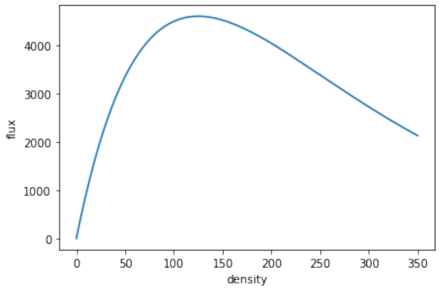
Figure 10: UnderWood Velocity-Time plot

Observations

The two models correctly predict that when $\rho <= \rho_{jam}$ the velocity will become constant at $v = v_{max}$, which shows that they are both valid models, however, it's noticed that the Greenshields model predicts that the speed increases linearly until it reaches the v_{max}, while in the underwood's model, the velocity increases exponentially.

Second Experiment ($\rho_{\max} = 250$)

General Relation between v,ρ,Q

Relation Between:	Greenshields	Underwood
Velocity-Density plot		
Flux-Density plot		

The simulation Velocity-Time graph

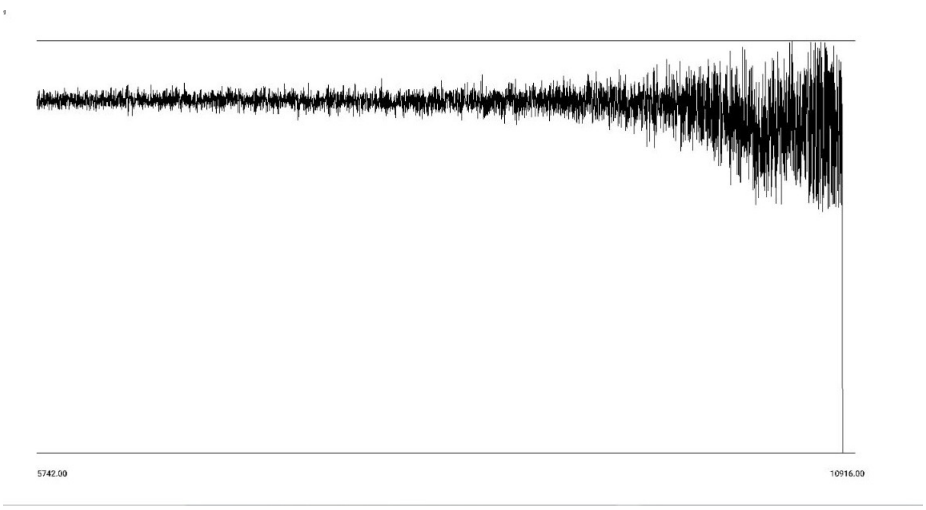


Figure 11: Velocity-Time plot for second experiment

In this simulation, it's also assumed that ρ is smaller than ρ_{jam}, and this results in velocity being relatively constant around v_max like the previous case.

The two models' Velocity-Time plots

In both plots it's assumed that $\rho_0 = 150$ and $\rho_{jam} = 0$

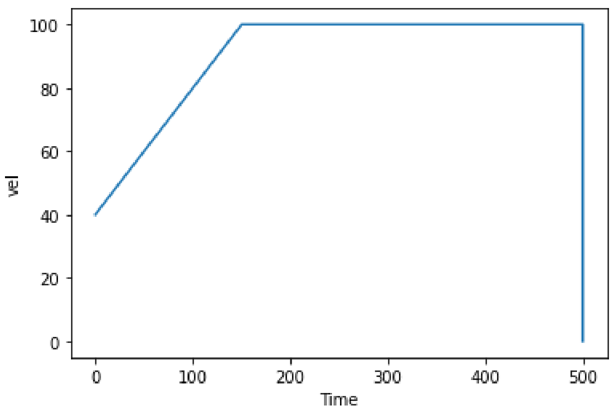


Figure 12: GreenShield Velocity-Time plot

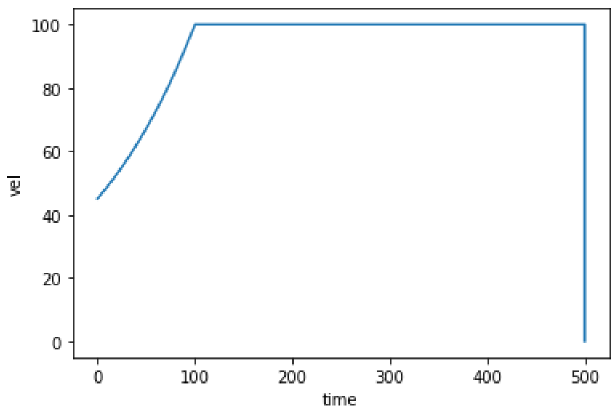


Figure 13: UnderWood Velocity-Time plot

Observations

The observations are identical to the ones of the first experiment.

Analysis

In this study, the aim was to Compare the Greenshield, and the Underwood traffic flow models. We verified that the two models correctly describe the traffic flow by testing it on a road that has linear characteristics using the Eclipse SUMO software simulation program, however, we noticed that due to the exponential nature of the underwood equation (equation 23), the error produced by it increases with the increase of ρ_{max}, and we observed that the linearity in the Greenshields equation (equation 10) works fine with a road that has linear characteristics.

Greenshields	Underwood
$v = v_f\left(1 - \frac{\rho}{\rho_{max}}\right)$	$v = v_f e^{-\rho/\rho_{crit}}$
Linear Speed-Density relation	Exponential Speed-Density relation
Function in maximum density	Function in critical density
Useful when dealing with a road that has linear characteristics .	Useful when dealing with small densities

4.2 Game theory Experiment

Trying to put the game theory model into action, in this experiment, we have monitored the traffic flow in Helwan city between two points — Al-Shuhada' square and Helwan metro station — by comparing the time spent across two different roadsto reach certain destination; the data presented in this experiment is taken from Google maps service at different day hours on 4th of December, 2022. Mansour Street is the first option and the most common one for driversto reach the metro station; however, that street is usually crammed with public transportation buses due to the bus station located there, and the number of buses across the street is directly affecting the traffic flow. The second option is Ahmed Raghib Basha Street, which usually has lower density than that of the first street; in contrast, that street is slightly longer. The results are shown in the table below:

	Time in minutes		Probability	
	K1	K2	h	h*
8:00	5	4	0.25	0.25
9:00	4	4	-1	0
10:00	3	4	-2.25	0
11:00	3	4	-2.25	0
12:00	3	4	-2.25	0
13:00	3	4	-2.25	0
14:00	5	4	0.25	0.25
15:00	5	4	0.25	0.25
16:00	5	4	0.25	0.25
17:00	5	4	0.25	0.25
18:00	5	4	0.25	0.25
19:00	6	4	1.5	1
20:00	6	4	1.5	1
21:00	4	4	-1	0
22:00	4	4	-1	0
23:00	4	4	-1	0
Distance	602.56m	944.65m		

Table 2: time spent across the two streets

- K1: Mansour Street
- K2: Ahmed Raghib Basha Street
- h: probability of going through the shortcut (K2)
- h*: actual probability where negative values (extreme tendency to take K1) are considered to be zero, and values more than 1 (extreme tendency to take K2) are considered to be 1

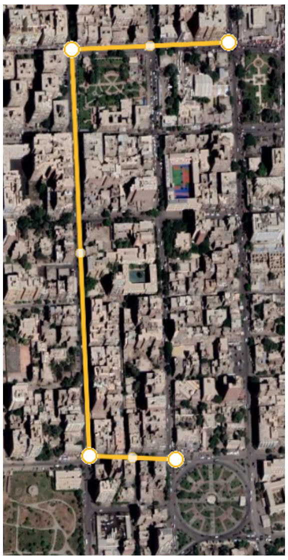


Figure 14: Ahmed Raghib Basha Street(944.65)

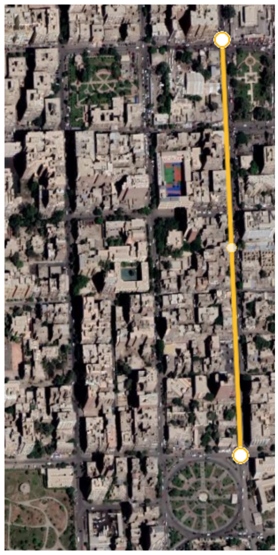


Figure 15: Mansour Street (602.56m)

Applying the mixed strategy analysis from game theory model, we were able to relate the probability (h) of preferring K2 over K1 to reach equilibrium at different densities. Moreover, we have used the LWR model and Underwood equation to calculate the velocity at certain densities, and then we can estimate the time spent from the equation:

$$T = d/V(\rho)$$

The results are shown below:

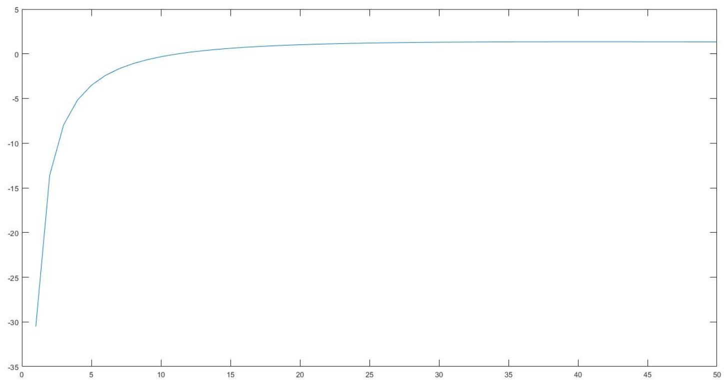


Figure 16: probabilty (h) vs. density (ρ)

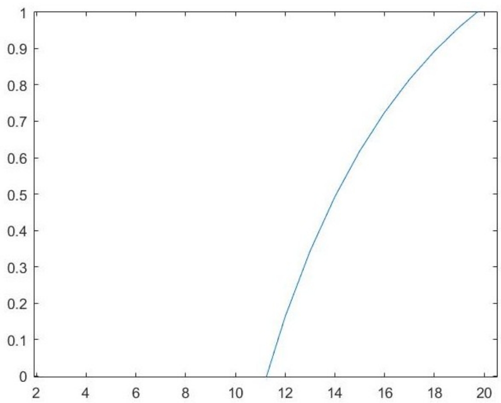


Figure 17: h vs. ρ from 11 to 20

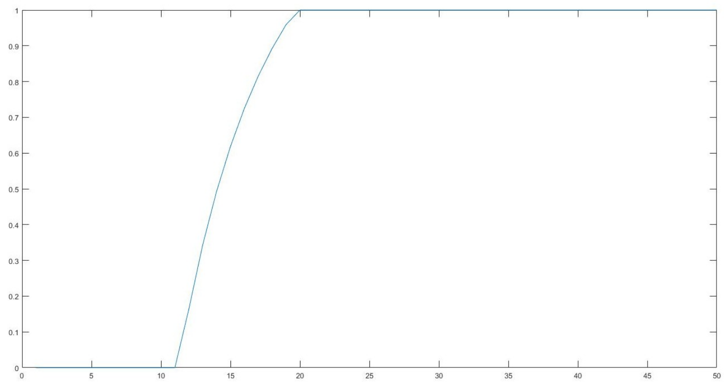


Figure 18: Normalized graph

As shown in figure16, the used equation may lead to values that are out of the probability range (0:1), and that could be roughly understood as the numbers greater than one express extreme tendency to take the shortcut, and the shortcut is running at the maximum capacity in this case. Similarly, the negative results is considered to be demonstrating extreme tendency to continue through the main road, and the shortcut seems to be useless for most drivers in this case. Figure17 emphasize that the output of the function is valid without any modification for the densities in the range from 11 to 20. Moreover, in figure 18, we have normalized the values greater than 1 to 1 as well as the negative values to 0.

Simulation

Using AnyLogic traffic simulation software and applying the boundary conditions on both roads, we have obtained the probability of taking the shortcut, and it has been noticed that the results is very similar to the probability estimated before. The results at certain densities are shown below:

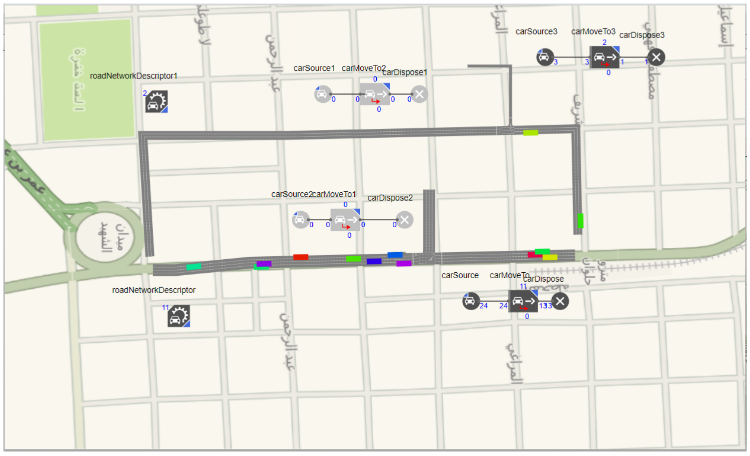


Figure 19: AnyLogic simulation for no impedance in Mansour Street (h=0)

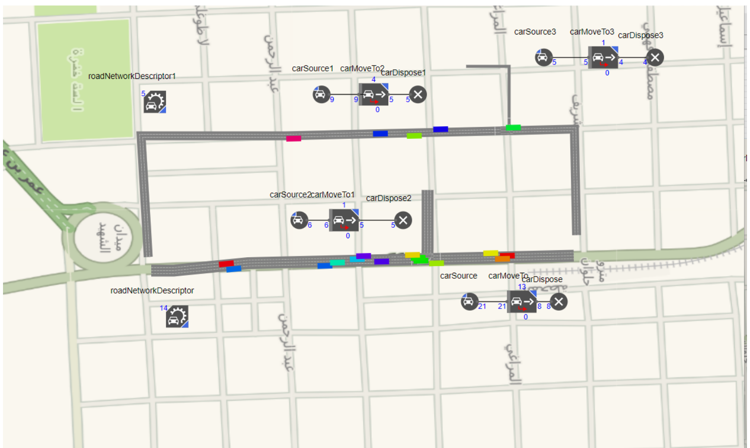


Figure 20: AnyLogic simulation for no impedance in Mansour Street (h=0.25)

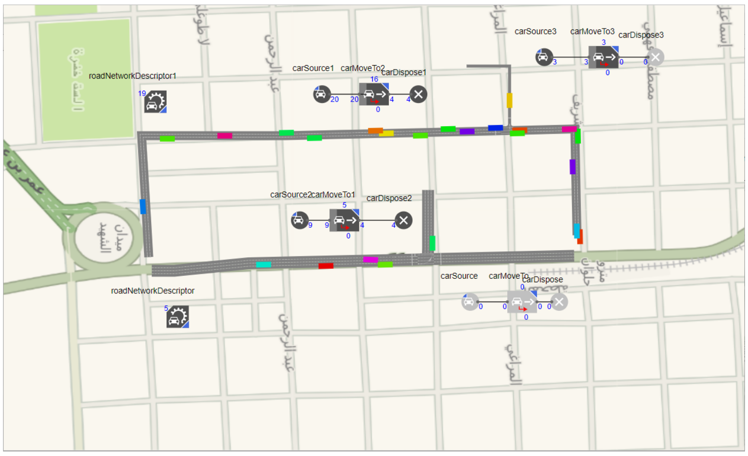


Figure 21: AnyLogic simulation for no impedance in Mansour Street (h=1)

In addition, we have also simulated the traffic flow in the same street (Mansour Street) compared by another alternative shortcut. The following test case is technically possible as long as we assume that the roads are completely homogeneous with, for example, no change in the number of lanes. Again, the simulation matches the probability calculated before. The results are shown below:

	Time in minutes		Probability	
	K1	K2	h	h*
8:00	9	8	1.5556	1
9:00	9	9	1	1
10:00	6	9	-1.5	0
11:00	7	8	0.2857	0.2857
12:00	7	8	0.2857	0.2857
13:00	8	9	0.357	0.375
14:00	9	9	1	1
15:00	9	10	0.4444	0.4444
16:00	8	9	0.375	0.375
17:00	8	9	0.375	0.375
18:00	9	10	0.4444	0.4444
19:00	9	9	1	1
20:00	11	9	1.909	1
21:00	11	9	1.909	1
22:00	8	8	1	1
23:00	8	9	0.375	0.38
Distance	1657m	1940m		

Table 3: time spent across the two streets

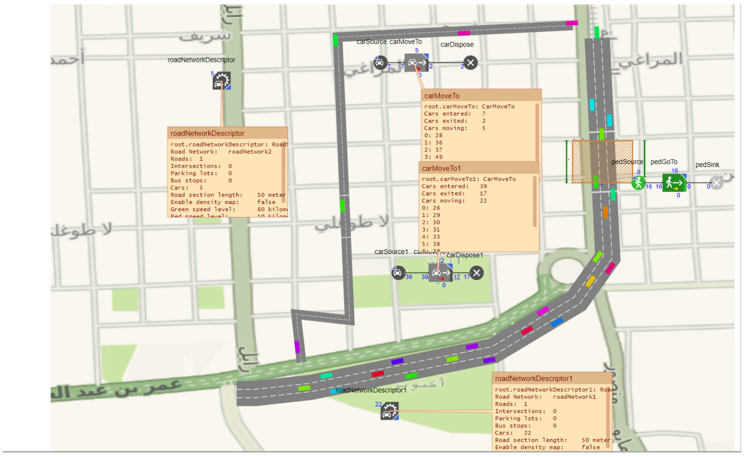


Figure 22: AnyLogic simulation for no impedance in Mansour Street (h=0)

Figure 23: AnyLogic simulation for no impedance in Mansour Street (h=0.44)

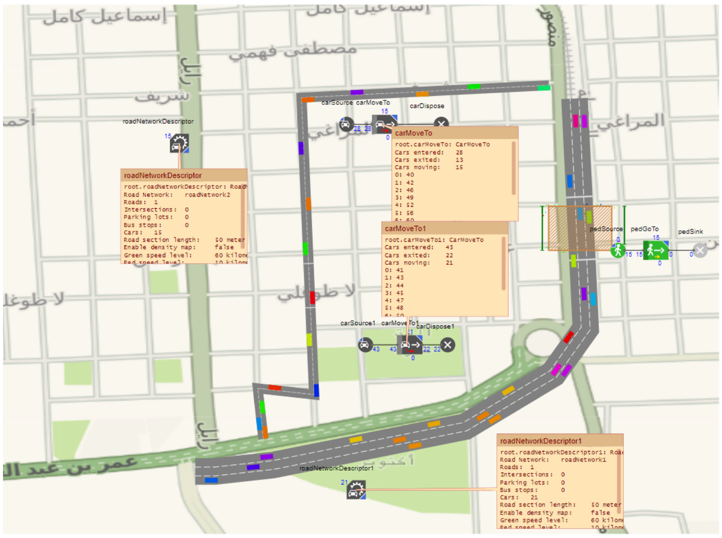


Figure 24: AnyLogic simulation for no impedance in Mansour Street (h=1)

Survey

Beside the simulations, we have conducted an online survey, targeting the drivers and aiming to test their tendency to go through shortcut rather than the main road at different daytimes. To avoid inaccurate results, the survey was delivered only to 73 drivers in Giza and Cairo, and they were given a detailed description of the road specifications and its status in each situation before answering the questions; moreover, we have phrased the survey questions carefully to avoid potential bias — confirmation bias for example. Also, it's to be mentioned that the whole survey was in Arabic due to location restrictions. The results were unsurprising and didn't really differ from the theoretical analysis we have done before; even more, the results are kind of confirming our thoughts. For the first situation, with no impedance in the main road at all, most of respondents answered that they extremely don't prefer the shortcut; the mean portability is 0.19, and the mode is 0. For the second situation and the most complicated one with low impedance and close arriving time for both roads, responses varied from 0 to 1; in fact, that is what we define as an equilibrium. In this case, there's no significant advantage for a road over another, so that some of drivers would prefer to go through the shortcut, and some other will not. Consequently, neither of the roads is overloaded or reaching the critical density. The mean is 0.53. Finally, for the last situation, with high impedance in the main road, most of drivers preferred the shortcut over the main road, and the mean was 0.88 while the mode is 1. Survey results (scaled from 0 to 5) are shown below:

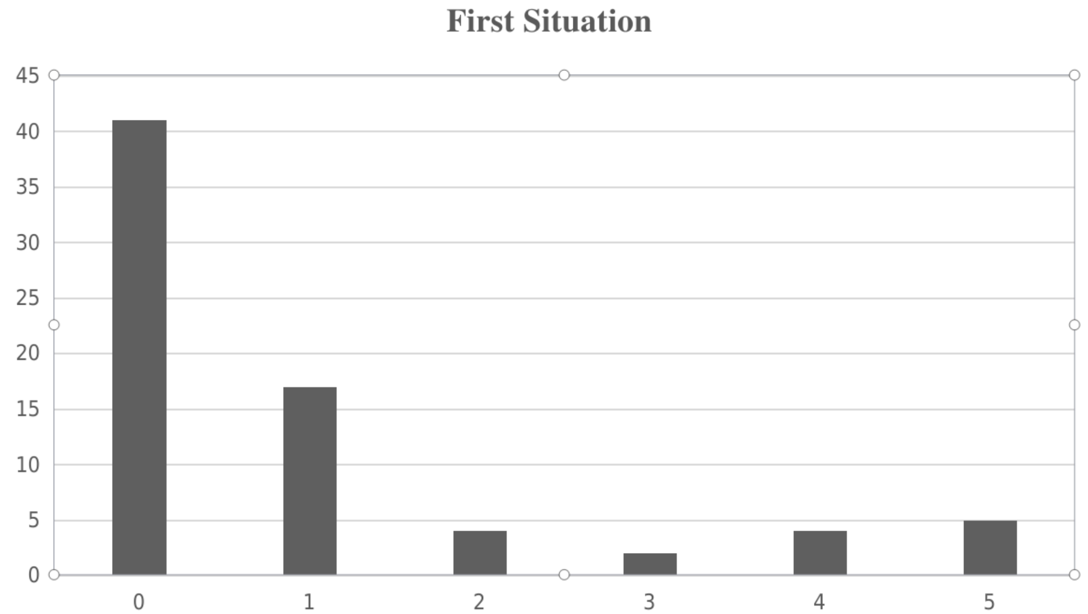


Figure 25: Responses to the first situation (no impedance)

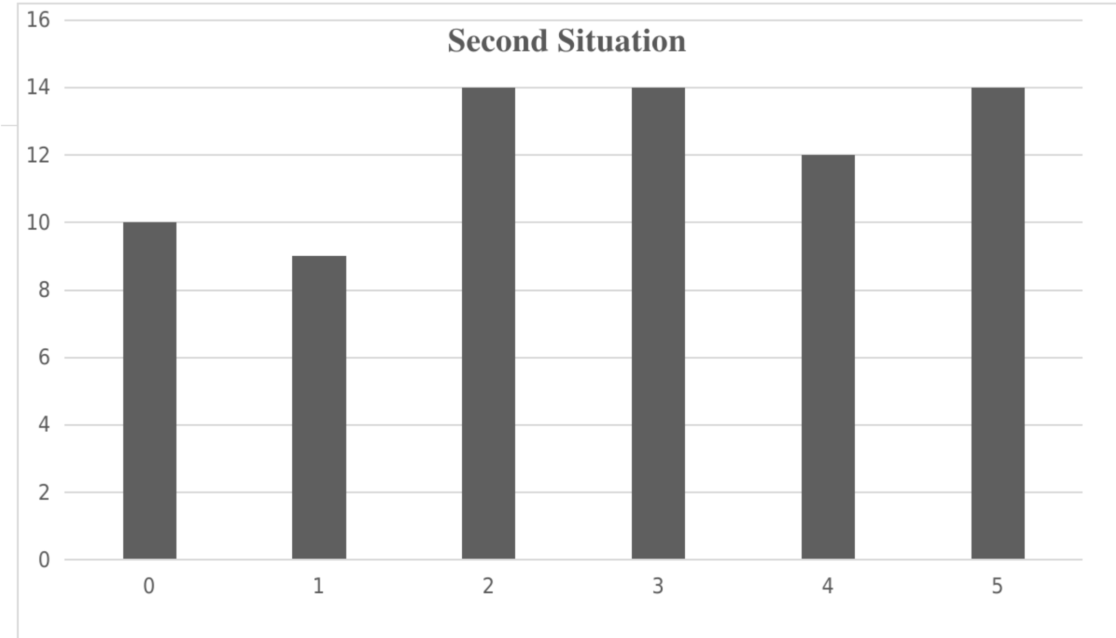


Figure 26: Responses to the second situation (low impedance)

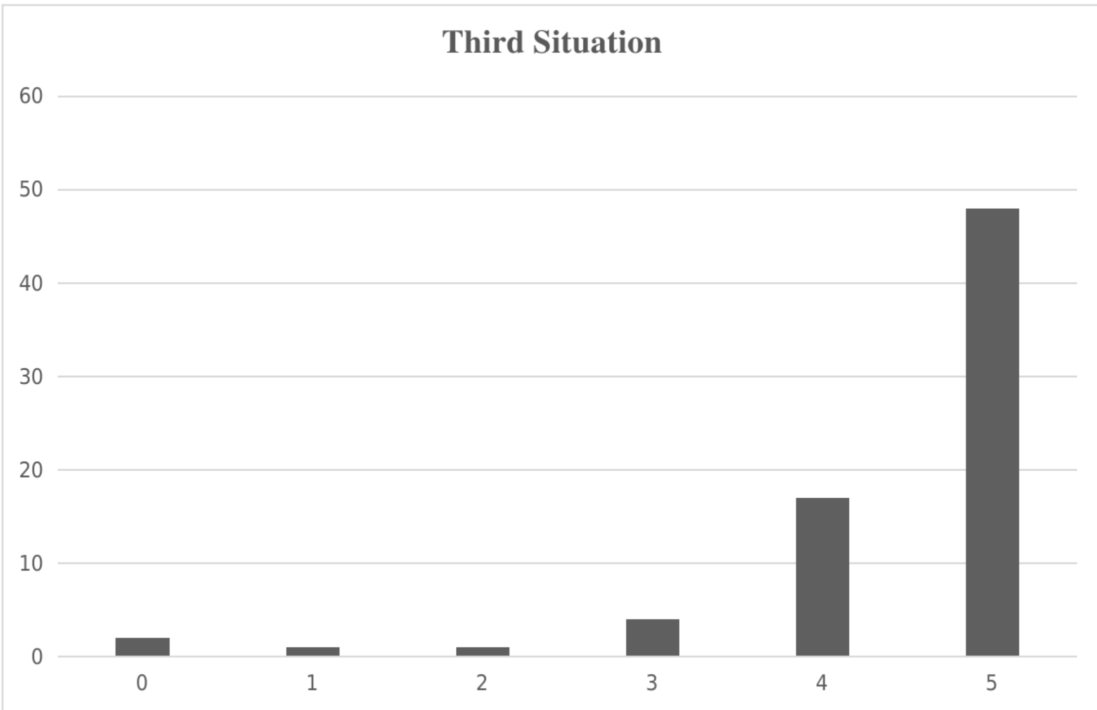


Figure 27: Responses to the second situation (high impedance)

Analysis

The game theory model is concerned with studying the decisions of drivers (treated as packets) who are completely aware of all available roads and alternatives to certain destination beside the time spent in each. In other words, experienced drivers who are familiar with shortcuts and used to drive through the congested main road usually satisfy the theoretical results of the game theory model. According to figure16, the negative values of h is corresponding to smooth flow across the main road with no obstacles at all, and the shortcut is almost unused in this case. Then, the value of h will continue to increase with the density of the main road. At certain point, equilibrium takes place when the main road is partially congested, and some drivers decide to take the shortcut. The final state is when h approaches 1, and all drivers would prefer to take the shortcut in this case; in fact, it happens when exceeding the critical density of the main road. Thus, the controlling parameter in our model is the density of the main continuous road.

5 Conclusion & future work

In conclusion, Greenshields's equations can be generally applied to roads with linear characteristics for any density less than the maximum density. On the other hand, regarding the underwood equation, the resulting values were almost equal to the values out of the simulations at low levels of density; accordingly, this equation expresses inverse relationship between velocity and density according to the exponential form equation. Based on those aspects, road characteristics are the parameters to determine which equation is the best describing a particular road. For example, Greenshields's equation perfectly describes roads with linear characteristics, and the Underwood equation is associated with low density. When arriving to a road junction that leads to the same ending point, the game theory model comes into play. To establish the Nash equilibrium of the mixed strategy game, the probability (h) that describes the balanced distribution of vehicles among branches and maximizes the utility for all drivers is to be calculated. At equilibrium, all drivers seek to achieve maximum utility, and this leads to a steady state where each driver is aware about the alternative roads and takes his or her decision carefully.

Numerous additional topics can be researched to address real traffic problems using the traffic flow model and its applications. For example, we may be able to examine the impact of drivers' strategies on traffic flow by integrating the game theory model with another macroscopic and microscopic traffic models. Inheriting the realistic dynamics accomplished by a two-lane traffic model, the resulting model aims to deal with phenomena carried on by driver-driver interactions. Future work may include adding a stochastic model to measure the randomness occurring in traffic to enhance the flow. Furthermore, complete traffic smart systems could be constructed based on smart traffic lights and self-driving cars that communicate to each other to take the best decisions in different situations. Also, there are several other potential projects that can be developed under governmental supervision; for example, designing new legislations supported with modern technologies of radars to prevent defective behaviors that, according to game theory, would have a negative impact on traffic; on the other hand, cooperative behaviors could be taken into consideration to reach Shapley's value. Selfish behaviors include increasing the speed, changing the lane, reverse driving, and violating traffic lights. In conclusion, traffic flow models can be studied further and applied much further to set on the solution of traffic problems, and that is what traffic engineers do today, and several development plans focus on.

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