# Hamiltonian Systems

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# Contents

# 1 Hamiltonian Equations

## 1.1 Notation

We denote $\mathbb{F}^n$ as the space of all $n$-dimensional vectors (all vectors are column vectors). $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ de- notes the set of all linear transformations $\mathbb{F}^n \to \mathbb{F}^m$ (are sometimes identified with the set of all $m \times n$ matrices).

Functions are real and smooth unless otherwise stated; smooth means $\mathcal{C}^\infty$ or real analytic. If $f(x)$ s a smooth function from an open set in $\mathbb{R}^n$ to $\mathbb{R}^m$ then $\frac{\partial f}{\partial x}$ denotes the $m \times n$ Jacobian matrix:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \cdots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

If $A$ is a matrix, then $A^T$ denotes its transpose, $A^{-1}$ its inverse, and $A^{-T}$ the inverse transpose.

If $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, then $\partial f/\partial x$ is a row vector. $\nabla f = \nabla_x f = f_x$ denote the column vector $(\partial f/\partial x)^T$. $Df$ denotes the derivative of $f$ thought of as a map from an open set in $\mathbb{R}$ into $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. The variable $t$ denotes a real scalar variable called time, and the symbol $\dot{f} = \partial f/\partial t$.

## 1.2 Hamilton's Equations

If the forces are derived from a potential function, the equations of motion of the mechanical system have many special properties, most of which follow from the fact that the equations of motion can be written as a Hamiltonian system. The Hamiltonian formalism is the natural mathe- matical structure in which to develop the theory of conservative mechanical systems.

A **Hamiltonian system** is a system of 2n ordinary differential equations of the form:

$$\begin{cases} \dot{q} = H_p \\ \dot{p} = H_q \end{cases}$$

$$\begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i}(q, p, t) \\ \dot{p}_i = \dfrac{\partial H}{\partial q_i}(q, p, t) \quad i = 1, \cdots, n \end{cases} \tag{1}$$

where $H = H(q, p, t)$ is called **the Hamiltonian**, is a smooth real-valued func- tion defined for $(q, p, t) \in \mathcal{O}$, an open set in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$.

The vectors $q = (q_1, q_2, \cdots, q_n)$ and $p = (p_1, p_2, \cdots, p_n)$ are traditionally called the **position** and **momentum** vectors, respectively, and $t$ is called **time**, because that is what these variables represent in the classical examples. The variables $q$ and $p$ are said to be **conjugate variables**: $p$ is conjugate to $q$. The concept of conjugate variable grows in importance as the theory develops.

The integer $n$ is the **number of degrees of freedom** of the system.

We define the vector $z$ as:

$$z = \begin{bmatrix} q \\ p \end{bmatrix}$$

a $2n$ vector. We define also the matrix $J$ as the next $2n \times 2n$ skew symmetric matrix and the gradient in the next way:

$$J = J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

$$\nabla H = \begin{bmatrix} \frac{\partial H}{\partial z_1} \\ \vdots \\ \frac{\partial H}{\partial z_{2n}} \end{bmatrix} \tag{2}$$

where 0 is the is the $n \times n$ zero matrix and $I_n$ is the $n \times n$ identity matrix. The case $2 \times 2$ matrix $J_2$ is a special case, it's denoted by $K$. In this notation the system is written as:

$$\dot{z} = J\nabla H(z, t) \tag{3}$$

***reason why:***

Explanation muhahahahahaha

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One of the basic results from the general theory of ordinary differential equations is *the existence and uniqueness theorem*. This theorem states that for each $(z_0, t_0) \in \mathcal{O}$ , there is a unique solution $z = \Phi(z_0, t_0, t)$ of 3 defined for $t$ near $t_0$ that satisfies the initial condition $z_0 \cdot \Phi = \Phi(z_0, t_0, t_0)$ is defined on an open neighborhood $\mathcal{Q}$ of $(z_0, t_0, t_0) \in \mathbb{R}^{2n+2}$ into $\mathbb{R}^{2n}$.

The function $\Phi(z_0, t_0, t)$ is smooth in all its displayed arguments, and so $\Phi$ is $\mathcal{C}^\infty$ if the equations are $\mathcal{C}^\infty$ , and it is analytic if the equations are analytic. $\Phi(z_0, t_0, t)$ is called **general solution**

In the special case when $H$ is independent of $t$, so that $H : \mathcal{O} \to \mathbb{R}$ where $\mathcal{O}$ is an open set in $\mathbb{R}^{2n}$ , the differential equations 3 are autonomous, and the Hamiltonian system is called **conservative**.

It follows that $\Phi(z_0, 0, t - t_0) = \Phi(z_0, t_0, t)$ holds, because both sides satisfy equation 3 and the same initial conditions. Usually the $t_0$ dependence is dropped and only $\Phi(z_0, t)$ is considered, where $\Phi(z_0, t)$ is the solution of 3 satisfying $\Phi(z_0, 0) = z_0$.

The solutions are pictured as parameterized curves in $\mathcal{O} \subset \mathbb{R}^{2n}$, , and the set $\mathcal{O}$ is called the **phase space**. By the existence and uniqueness the- orem, there is a unique curve through each point in $\mathcal{O}$; and by the uniqueness theorem, two such solution curves cannot cross in $\mathcal{O}$.

An **integral** for 3 is a smooth function $F : \mathcal{O} \to \mathbb{R}$ which is constant along the solutions of 3; i.e., $F(\Phi(z_0, t)) = F(z_0)$ is constant. The classi- cal conserved quantities of energy, momentum, etc. are integrals. The level surfaces $F^{-1}(c) \subset \mathbb{R}^{2n}$, where $c$ is a constant, are **invariant sets**; i.e., they are sets such that if a solution starts in the set, it remains in the set.

In general, the **level sets** are are mainfolds of dimension $2n - 1$ and so with an integral $F$, the solutions lie on the set $F^{-1}(c)$, which is of dimension $2n - 1$. If you were so lucky as to find $2n - 1$ independent integrals, $F_1, F_2, \cdots, F_{2n-1}$, then holding all these integrals fixed would define a curve in $\mathbb{R}^{2n}$, the solution curve. In the classical sense, the problem has been integrated.

## 1.3 Poisson Bracket

Let $F, G : U \subset \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ be $\mathcal{C}^r$ $(r \geq 1)$ functions such that $(q,p,t) \longmapsto F(q,p,t), G(q,p,t)$.

We define the **Poisson Bracket (PB)** as a $\mathcal{C}^{r-1}$ fuction $\{F,G\} : U \longrightarrow \mathbb{R}$

$$\begin{aligned} \{F,G\} &= (\nabla_z F)^T J (\nabla_z G) \\ &= (\nabla_q F)^T (\nabla_p G) - (\nabla_p F)^T (\nabla_q G) \\ &= \sum_{i=1}^{n} \Big( \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \Big) \end{aligned} \tag{4}$$

**Properties:**

1. **Skew-simmetric**:

$$\{F,G\} = -\{G,F\}$$

In particular: $\{F,F\}$

***proof:***

$$\begin{aligned} -\{F,G\} &= -\Big( (\nabla_q F)^T (\nabla_p G) - (\nabla_p F)^T (\nabla_q G) \Big) \\ &= (\nabla_p F)^T (\nabla_q G) - (\nabla_q F)^T (\nabla_p G) \\ &= (\nabla_q G)^T (\nabla_p F) - (\nabla_p G)^T (\nabla_q F) \\ &= \{G,F\} \end{aligned}$$

♠

2. **Bilinear**:

$$\{\alpha F_1 + \beta F_2, G\} = \alpha\{F_1, G\} + \beta\{F_2, G\}, \quad \alpha, \beta \in \mathbb{R}$$

***proof:***

$$\begin{aligned} \{\alpha F_1 + \beta F_2, G\} &= \Big( \nabla_z (\alpha F_1 + \beta F_2) \Big)^T J (\nabla_z G) \\ &= \Big( \nabla_z (\alpha F_1) \Big)^T J (\nabla_z G) + \Big( \nabla_z (\beta F_2) \Big)^T J (\nabla_z G) \\ &= \alpha \Big( \nabla_z (F_1) \Big)^T J (\nabla_z G) + \beta \Big( \nabla_z (F_2) \Big)^T J (\nabla_z G) \\ &= \alpha\{F_1, G\} + \beta\{F_2, G\} \end{aligned}$$

♠

3. **Leibnitz rule**:

$$\{F_1, F_2, G\} = F_1\{F_2, G\} + F_2\{F_1, G\}$$

***proof:***

4. **Jacobi identity**:

$$\{F_1, \{F_2, F_3\}\} + \{F_3, \{F_1, F_2\}\} + \{F_2, \{F_3, F_31\}\} = 0 \tag{5}$$

***proof:***

Clearly $\{F, G\}$ is a smooth map $\mathcal{O} \to \mathbb{R}$.

By a common abuse of notation, let $F(t) = F(\Phi(z_0, t_0, t), t)$, where $\Phi$ is the solution of 3. By the chain rule we have that:

$$\frac{d}{dt}F(t) = \frac{\partial F}{\partial t}(\Phi(z_0, t_0, t), t) + \{F, H\}(\Phi(z_0, t_0, t), t) \tag{6}$$

Hence $dH/dt = \partial H/\partial t$.

***reason why:***

explanation

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**Theorem**: Let $F, G$, and $H$ be as above and independent of time $t$. Then:

1. $F$ is an integral for 3 if and only if $\{F, H\} = 0$.
2. $H$ is an integram for 3.
3. If $F$ and $G$ are integrals for 3, then so is $\{F, G\} = 0$.
4. $\{F, G\}$ is the time rate of change of $F$ along the solutions of 3.

***proof:***

1. Directly follows directly from the definition of an integral and from 6. Let's see why:
   COMPLETAR ESTO!!
2. Follows form (1) and from the fact that the Poisson bracket is skew-symmetric, so $\{H, H\} = 0$.
   COMPLETAR ESTO!!
3. Follows from the Jacobi identity
   Completar, poner en ecuaciones
4. Follows from 6.
   Poner en ecuaciones

♠

In many of the examples given below, the Hamiltonian $H$ is the total energy of a physical system; when it is, the theorem says that *energy is a conserved quantity.*

In the conservative case when $H$ is independent of $t$, a critical point of $H$ as a function (i.e., a point where the gradient of $H$ is zero) is an equilibrium (or critical, rest, stationary) point of the system of differential equations 2 or 3

For the rest of this section, let $H$ be independent of $t$. An **equilibrium point** $\zeta$ of system 3 is **stable** if:

$$\forall \epsilon > 0 \;\; \exists \delta > 0 : ||\zeta - \Phi(z_0, t)|| < \epsilon \;\; \forall t \text{ such that } ||\zeta - z_0|| < \delta$$

note that $\forall t$ means both positive and negative t, and that stability is for both the future and the past.

**Theorem (Dirichlet)**: If $\zeta$ is a strict local minimum or maximum of $H$, then $\zeta$ is stable.

***proof:***

Without loss of generality, assume that $\zeta = 0$ and $H(0) = 0$. ACLARAR EL POR QUÉ NO PERDEMOS GENERALIDAD.

Because $H(0) = 0$ and 0 is a strict minimun of $H$ (WHY), there is an $\eta > 0$ such that $H(z)$ is positive for $0 < ||z|| < \eta$ (we can say that $H$ is positive definite).

Let $k = \min(\epsilon, \eta)$ and $M = \min\{H(z) : ||z|| = k\}$, so $M > 0$.

Because $H(0) = 0$ and $H$ is continuos, there is a $\delta > 0$ such that $H(z) < M$ for $||z|| < \delta$.

$$||z|| < \delta \;\; \Rightarrow H(z_0) = H(\Phi(z_o, t)) < M \;\; \forall t$$

We have that $||\Phi(z_0, t)|| < k \leq \epsilon \;\; \forall t$, because if not, there is a time $t'$ when $||\Phi(z_0, t')|| = k$, and $H(\Phi(z_0, t')) \geq M$, a contradiction. RXPLICAR CON QUÉ CHOCA

♠

## 1.4 The Harmonic Oscillator

The harmonic oscillator is the second-order, linear, autonomous, ordinary differential equation:

## 1.5 The Forced Nonlinear Oscillator

## 1.6 The Elliptic Sine Function

## 1.7 Linear Flow on the Torus

## 1.8 Euler–Lagrange Equations

# 2 N-Body Problem

Let's us consider $N$ point masses in the space ($\mathbb{R}^3$, the planar case $\mathbb{R}^2$, the coolinear case $\mathbb{R}$), whit the $i$-th particle having a mass $m_i > 0$ and a position vector $q_i = (q_{i1}, q_{i2}, q_{i3})^t$.

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The equations of the system comes from the **Newton's law of universal gravitation**:

$$\ddot{q}_i m_i = \sum_{\substack{j=1 \\ j\neq 1}}^{N} G m_i m_j \frac{(q_j - q_i)}{||q_j - q_i||^3} = \frac{\partial U}{\partial q_i} \qquad I = 1, 2, \cdots, N \tag{7}$$

***reason why:***

$$\left\| \frac{u}{||u||^3} \right\| = \frac{||u||}{||u||^3} = \frac{1}{||u||^2}$$

Where $G = 6.67408 \cdot 10^{-11} \frac{m^3}{s^2 Kg}$ is the **Gravitacional constant**.

We define the **Self potencial**, the negative of potencial energy, as:

$$U = \sum_{1\leq i<j\leq N} \frac{G m_i m_j}{||q_j - q_i||} \tag{8}$$

***reason why:***

Let us check, for instance, $N = 3$ and $i = 2$.

$$U = G\frac{m_1 m_2}{||q_1 - q_2||} + G\frac{m_1 m_3}{||q_1 - q_3||} + G\frac{m_3 m_2}{||q_3 - q_2||}$$

$$U = \left(\frac{\partial U}{\partial q_{21}}, \frac{\partial U}{\partial q_{22}}, \frac{\partial U}{\partial q_{23}}\right)^T$$

Let's compute just one of the terms,

$$\begin{aligned}
\frac{\partial U}{\partial q_{21}} = \cdots &= G m_1 m_2 \Big(\frac{-1}{2}\Big) \frac{-2(q_{11} - q_{21})}{||q_1 - q_2||^3} + G m_3 m_2 \Big(\frac{-1}{2}\Big) \frac{-2(q_{31} - q_{21})}{||q_3 - q_2||^3} \\
&= \frac{G m_2 m_1}{||q_2 - q_1||^3}(q_{11} - q_{21}) + \frac{G m_2 m_3}{||q_2 - q_3||^3}(q_{31} - q_{21})
\end{aligned}$$

**Notation**: $q = (q_1, \cdots, q_N) \in \mathbb{R}^{3N}$, $M = diag(m_1, m_1, m_1, m_2, m_2, ,_2, \cdots, m_N, m_N, m_N)$

$$\Rightarrow M\ddot{q} = \frac{\partial U}{\partial q}$$

This is a system of $3N$ 2nd o.d.e.

Let's define:

- **Collision set**: $\triangle = \{q \in \mathbb{R}^{3N} : q_i = q_j, \quad i \neq j\}$
- **Position space**: $\mathbb{R}^{3N} \setminus \triangle$

Let us pass to the hamiltonian formalism: Introduce $p = (p_1, p_2, \cdots, p_N) \in \mathbb{R}^{3N}$ by $p = M\dot{q}$, or simply $p_i = m_i\dot{q}_i$, The **linear momentum** of the $i$-th particle.

We have to convert the system in a new one of first order (with $6N$ equations) to obtain:

$$\begin{cases} \dot{q}_i = \dfrac{p_i}{m_i} = \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i = \displaystyle\sum_{\substack{j=1 \\ j\neq i}}^{N} G\frac{m_i m_j}{||q_j - q_i||^3}(q_j - q_i) = -\dfrac{\partial H}{\partial q_i} \end{cases} \tag{9}$$

where:

- $H(q,p) = \underbrace{T(p)}_{\text{Kinetic Energy}} - \underbrace{U(q)}_{\text{Potencial Energy}}$
- $T(p) = \sum_{i=1}^{N} \frac{||p_i||^2}{2m_i} = p^T M p = \frac{1}{2}\sum m_i ||\dot{q}_i||^2$

This is called the **Hamiltonian with $3N$ degrees of freedom (d.o.f.)** (the number og positions or momenta).

### 2.0.1 Kepler Problem

Introducing $p = \dot{q}$ we arrive to:

$$\begin{cases} \dot{q}_i = p_i = \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i = -\mu\dfrac{q_i}{||q||^3} = -\dfrac{\partial H}{\partial q_i} \end{cases} \tag{10}$$

where $H = \frac{||p||^2}{2} - \frac{\mu}{||q||}$, 3 d.o.f.

We degine the **Angular Momentum** as:

$$A = q \times p = q \times \dot{q}$$

***reason why:***

$$A = q \times p = \begin{vmatrix} i & j & k \\ q_1 & q_2 & q_3 \\ p_1 & p_2 & p_3 \end{vmatrix} = \dots.$$
$$= q \times \dot{q}$$

**Observation**: Angular momentum is a **constant of motion**.

***reason why:***

$$\dot{A} = \dot{q} \times p + q \times \dot{p} = p \times p - \frac{u}{||q||^3} q \times q = 0$$

**Lemma**:

$$\frac{d}{dt}\left(\frac{q}{||q||}\right) = \frac{(q \times \dot{q}) \times q}{||q||^3} = \frac{A \times q}{||q||^3}$$

***proof:***

Let's remind a property of the vector product: $(A \times B) \times C = (C \cdot A)B - (C \cdot B)A$

$||q|| = \sqrt{q_1^2 + q_2^2 + q_3^2}$

$\dot{||q||} = \frac{q \cdot \dot{q}}{||q||^2}$

We call $u = \frac{q}{||q||}$, so we have:

$$q = ||q||u \to \dot{q} = \dot{||q||}u + ||q||\dot{u} = \frac{q \cdot \dot{q}}{||q||} \frac{q}{||q||} + ||q||\dot{u}$$

$$\dot{u} = \frac{\dot{q}}{||q||} - \frac{(q \cdot \dot{q})q}{||q||} = \frac{(q \cdot q)\dot{q} - (q \cdot \dot{q})q}{||q||^3} = \frac{(q \times \dot{q}) \times q}{||q||^3}$$

where: $A = q, \quad B = \dot{q}, \quad C = q.$

♠

# 3 Tópico sobre el que haré el trabajo

# 4 Exercises

## 4.1 Chapter 1: Introduction to Hamiltonian systems

Make the phase portrait of the Hamiltonian system

$$\begin{cases} \dot{x} = y \\ \dot{y} = x - \dfrac{x^3}{3} \end{cases}$$

and compute its Hamiltonian

- Plano de fases
- $$\begin{cases} \dot{x} = y = \dfrac{\partial H}{\partial y}(x,y) \\ \dot{y} = x - \dfrac{x^3}{3} = \dfrac{\partial H}{\partial x}(x,y) \end{cases}$$

  One one hand,

  $$\int \frac{\partial H}{\partial x} dx = \int x - \frac{x^3}{3} dx = \frac{x^2}{2} - \frac{x^4}{12} + C(y)$$

  On the other hand,

  $$\int \frac{\partial H}{\partial y} dy = \int y dy = \frac{y^2}{2} + C(x)$$

  So the Hamiltonian is:

  $$H(x,y) = \frac{x^2}{2} - \frac{x^4}{12} + \frac{y^2}{2}$$

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Make the phase portrait of the Hamiltonian system

$$\begin{cases} \dot{x} = x \\ \dot{y} = -y + x^2 \end{cases}$$

and compute its Hamiltonian

- Plano de fases
- $$\begin{cases} \dot{x} = x = \dfrac{\partial H}{\partial y}(x,y) \\ \dot{y} = -y + x^2 = \dfrac{\partial H}{\partial x}(x,y) \end{cases}$$

(Meyer-Hall-Offin) Let $x, y, z$ be the usual coordinates in $\mathbb{R}^3$, $r = xi + yj + zk$, $X = \dot{x}$, $Y = \dot{y}$, $Z = \dot{z}$, $R = \dot{r} = Xi + Yj + Zk$.

1. Compute the three components of angular momentum $mr \times R$.
2. Compute the Poisson bracket of any two of the components of angular momentum and show that it is $\pm m$ times the third component of angular momentum.
3. Show that if a system admits two components of angular momentum as integrals, then the system admits all three components of angular momentum as integrals.

1. adea
2. dsa
3. dadsa

(Meyer-Hall-Offin) **A Lie algebra** $A$ is a vector space with a product: $A \times A \to A$ that satisfies:

- **Anticommutative**: $ab \neq ba$
- **Distributive**: $a(b + c) = ab + ac$
- **Scalar associative**: $(\alpha a)b = \alpha(ab)$
- **Jacobis identity**: $a(bc) + b(ca) + c(ab) = 0, \quad a, b, c \in A, \quad \alpha \in \{\mathbb{R}, \mathbb{C}\}$

1. Show that vectors in $\mathbb{R}^3$ form a Lie algebra where the product $*$ is the cross product.
2. Show that smooth functions on an open set in $\mathbb{R}^{2n}$ form a Lie algebra, where $fg = \{f, g\}$, the Poisson bracket.
3. Show that the set of all $n \times n$ matrices, $gl(n, \mathbb{R})$, is a Lie algebra, where $AB = ABBA$, the Lie product.

1. bla
2. bla
3. bla

(Meyer-Hall-Offin) The pendulum equation is $\ddot{\theta} + \sin\theta = 0$.

1. Show that $2I = \frac{1}{2}\dot{\theta}^2 + (1 \cos\theta) = \frac{1}{2}\dot{\theta}^2 + 2\sin^2(\theta/2)$ is an integral.

2. Sketch the phase portrait.
3. Make the substitution $y = \sin(\theta/2)$ to get $\dot{y}^2 = (1-y^2)(I-y^2)$. Show that when $0 < I < 1$, $y = ksn(t,k)$ solves this equation when $k^2 = I$ (Look at the definition of elliptic sine function of Section 1.6 of Meyer-Hall-Offin).

1. bla
2. bla
3. bla

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(Meyer-Hall-Offin) Let $H : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ be a globally defined conservative Hamiltonian, and assume that $H(z) \to +\infty$ as $z \to +\infty$. Show that all solutions of $\dot{z} = J\nabla H(z)$ are bounded (Hint: Think like Dirichlet).

Solución

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Consider a $\mathcal{C}^2$ Hamiltonian $H = H(q,p,t) : U \subset \mathbb{R}^{2n+1} \longrightarrow \mathbb{R}$ such that $det(\partial_p^2 H) \neq 0$ on $U$. Define $v = \partial_p H(q,p,t)$. Prove:

1. 
$$\begin{aligned}
\partial_{q_i} L(q,v,t) &= -\partial_{q_i} H(q,p,t) \\
\partial_{v_i} L(q,v,t) &= p_i \\
\partial_t L(q,v,t) &= -\partial_t H(q,p,t)
\end{aligned}$$
2. The Lagrangian $L$ is $\mathcal{C}^2$ and $det(\partial_v^2 L) \neq 0$.
3. The Euler-Lagrange equations associated to $L$ and the Hamiltonian equations $\dot{q}_i = \partial_{p_i} H$, $\dot{p}_i = -\partial_{q_i} H$ are equivalent.

1. bla
2. bla
3. bla

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## 4.2 Chapter 2: The N-body problem

Prove that the linear momentum is a first integral and that the center of mass moves with constant velocity for the 3 body problem.

Solución

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Prove that if $(a_1, a_2, \cdots, a_N)$ is a central configuration with value $\lambda$:

1. For any $\tau \in \mathbb{R}$ then $(\tau a_1, \tau a_2, \cdots, \tau a_N)$ is also a central configuration with value $\frac{\lambda}{\tau^3}$.
2. If $A$ is an orthogonal matrix, then $Aa = (Aa_1, Aa_2, \cdots, Aa_N)$ is also a central configuration with the same value $\lambda$.

1. bal bla
2. bla bla

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(Meyer-Hall-Offin) Draw the complete phase portrait of the collinear Kepler problem. Integrate the collinear Kepler problem.

Solución

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(Meyer-Hall-Offin) Show that $\varpi^2(\epsilon^2 - 1) = 2hc2$ for the Kepler problem. (Attention: Meyer-Hall-Offin has a typo)

Solución

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(Meyer-Hall-Offin) The area of an ellipse is $\pi a^2(1 - \epsilon^2)^{1/2}$, where a is the semi-major axis.We have seen in Keplers problem that area is swept out at a constant rate of $c/2$. Prove Keplers third law: The period $p$ of a particle in a circular or elliptic orbit ($\epsilon < 1$) of the Kepler problem is $p = (\frac{2\pi}{\sqrt{\mu}})a^{3/2}$.

Solución

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### 4.3 Chapter 3: Linear Hamiltonian systems

### 4.4 Chapter 6: Symplectic Transformations

### 4.5 Chapter 8: Geometric Theory

### 4.6 Chapter 9: Continuation of solutions

(Meyer-Hall-Offin) Show that the scaling used in Section 9.4 of Meyer-Hall-Offin to obtain Hills orbits for the restricted problem works for Hills lunar problem (see previous problem) also. Why does not the scaling for comets work?

Solución

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Prove Lemma 9.7.1 in Meyer-Hall-Offin. Verify that formula (9.11) is the condition for an orthogonal crossing of the line of syzygy in Delaunay elements.

Solución

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### 4.7 Chapter 10: Normal forms

### 4.8 Chapter 13: Stability and KAM Theory

(Meyer-Hall-Offin) Using Poincaré elements show that the continuation of the circular orbits established in Section 6.2 (Poincar orbits) are of twist type and hence stable.

Solución

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# 5 Apendix

## 5.1 Needed resoults and definitions

### 5.1.1 Linear Algebra

matriz ortogonal

no singular

skew-simmetric

DEVOLVER A SU SITIO LAS FOTOS TAMAÑO CARNET

### 5.1.2 Calculus

**Lipschitz continuity**: We have two metric spaces $(X, d_X), (Y, d_Y)$ and a function $f : X \to Y$. We say that f is **Lipschitz continuous** if $\exists K \geq 0$ such that, for all $x_1, x_2 \in X$,

$$d_Y(f(x_1), f(x_2)) \leq K \cdot d_X(x_1, x_2)$$

where $K$ is known as **Lipschitz constant**.

teorema punto fijo de bla bla bla bla

**Chain Rule**: Let $F = f \circ g$, or, equivalently, $F(x) = f(g(x))$ for all $x$. Then:

$$(f \circ g)' = (f' \circ g) \cdot g'$$

$$F'(x) = f'(g(x)) \cdot g'(x)$$

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

The two versions of the chain rule are related; if $z = f(y)$ and $y = g(x)$:

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = f'(y) \cdot g'(x) = f'(g(x)) \cdot g'(x)$$

***proof:*** ♠

**Gradient**: The gradient of a scalar function $(\mathbb{R}^n \to \mathbb{R})$ $f(x_1, ..., x_n)$ denoted by $\nabla f$ or $\vec{\nabla} f$ denotes the vector differential operator. The gradient of $f$ is defined as the unique vector field whose dot product with any unitvector $v$ at each point $x$ is the directional derivative off along $v$. That is,

$$(\nabla f(x)) \cdot v = D_v f(x)$$

- *Cartesian coordinates*: Lets focus in $\mathbb{R}^3$, where $i, j, k$ are the standard unit vectors in the directions of axis $x, y, z$.

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

- *Cylindrical coordinates*: Lets focus in $\mathbb{R}^3$, where $\rho$ is the axial distance, $\varphi$ is the azimuthal or azimuth angle and $z$ is the the axial coordinate and $e_\rho, e_\varphi, e_z$ are the unit vectors pointing along the coordinate directions.

$$\nabla f(\rho,\varphi,z) = \frac{\partial f}{\partial \rho} e_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \varphi} e_\varphi + \frac{\partial f}{\partial z} = \left( \frac{\partial f}{\partial \rho}, \frac{1}{\rho}\frac{\partial f}{\partial \varphi}, \frac{\partial f}{\partial z} \right)$$

  ***proof:*** ♠

- *Spherical coordinates*: Lets focus in $\mathbb{R}^3$, where $r$ is the radial distance, $\varphi$ is the azimuthal angle and $\theta$ is the polar angle, and $e_r, e_\varphi, e_\theta$ are local unit vectors pointing in the coordinate directions.

$$\nabla f(r,\theta,\varphi) = \frac{\partial f}{\partial r} e_r + \frac{1}{r}\frac{\partial f}{\partial \theta} e_\theta + \frac{1}{r\sin(\theta)}\frac{\partial f}{\partial \varphi} e_\varphi = \left( \frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{1}{r\sin(\theta)}\frac{\partial f}{\partial \varphi} \right)$$

  ***proof:*** ♠

**Laplace Operator**: is a differential operator given by the divergence of the gradient of a function on Euclidean space.

$$\Delta f = \nabla^2 f = \nabla \cdot \nabla f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

#### 5.1.3 Geometry

#### 5.1.4 Funcional Analysis

#### 5.1.5 Differential forms

#### 5.1.6 Measure Theory

### 5.2 Ordinary Differential Equations

#### 5.2.1 Peano existence theorem

**Arzelà–Ascoli Theorem**:

***proof:***

♠

**Peano existence Theorem**: Let $I = [a,b] \subset \mathbb{R}$, $D \subset \mathbb{R}^n$, $f : I \times D \to \mathbb{R}^n$ be a continuos function. We need to assume that $(t_0, x_0) \in I \times D$ and $C > 0$, $T > 0$ are given such that $[t_0 - T, t_0 + T] \times B_C(x_0) \subseteq I \times D$, $(B_C(x_0) = \{x \in \mathbb{R}^n : d(x, x_0) < C\})$ then the system:

$$\begin{cases} x(t_0) = x_0 \\ x'(t) = f(t, x(t)), \quad t \in [t_0 - \gamma, t_0 + \gamma] \end{cases}$$

possesses at least one solution $x$, where $\gamma \leq \min\{T, C/M\}$, $M = \max_{(t,x)\in[t_0-T,t_0+T]\times B_C(x_0)} |f(t,x)|$

***proof:***

♠

### 5.2.2 Picard–Lindelöf theorem

Consider the initial values problem:

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0$$

**Picard–Lindelöf Theorem**: let's suppose that $f$ is uniformly Lipschitz continuous in y (meaning the Lipschitz constant can be taken independent of t) and continuous in t, then for some value $\epsilon > 0$, there exists a unique solution $y(t)$ to the initial value problem on the interval $[t_0 - \epsilon, t_0 + \epsilon]$.

***proof:***

♠

### 5.2.3 Linear systems

We always can reduce a equation of order higher than 1 to a system of equations of order 1 in the next way:

$$x^{(n)} = f(t, x^{'}, x^{''}, \cdots, x^{n-1})$$

We introduce new variables in the next way:

$$x_1 = x,\ x_2 = x',\ \cdots,\ x_n = x^{n-1}$$

$$\begin{cases} x_1^{'} = x_2 \\ x_2^{'} = x_3 \\ \ \vdots \\ x_n^{'} = f(t, x^{'}, x^{''}, \cdots, x^{n-1}) \end{cases}$$

Now we have a system of $n$ equations of first orden. We can use vectoral notation, we call $x = (x_1, x_2 \cdots, x_n)^T$, $F = (F_1, F_2, \cdots, F_n)^T$ where $F_{i-1} = F_{i-1}(t, x) = x_i$, so then:

$$x' = F$$

About the inital condions problem,

The **general form** of a linear system on $n$ equations is the next one:

$$\begin{cases} x_1^{'} = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + b_1(t) \\ x_2^{'} = a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + b_2(t) \\ \quad \vdots \\ x_n^{'} = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + b_n(t) \end{cases} \tag{11}$$

If $b_i(t) = 0 \;\; i = 1, \cdots, n$ then we say that the system is **homogenous**, otherwise it's called **no homogenous**.

We can write this system in **matrix form**:

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \qquad x'(t) = \begin{pmatrix} x_1^{'}(t) \\ x_2^{'}(t) \\ \vdots \\ x_n^{'}(t) \end{pmatrix}$$

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix} \qquad b(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{pmatrix}$$

So obviosly we can write 11 as:

$$x'(t) = A(t)x(t) + b(t) \tag{12}$$

note that if the system is homogenous then $b(t) = 0$ and $x'(t) = A(t)x(t)$.

**Theorem**: If every component of $A(t)$ and $b(t)$ are continuous in the open interval $(a, b) \subset \mathbb{R}$, then for each $t_0 \in (a, b)$ and for all vector $x_0 \in \mathbb{R}^n$ such that $x(t_0) = x_0$, the system

$$x'(t) = A(t)x(t) + b(t)$$

has an unique solution in the interval $(a, b)$.

***proof:***

♠