

Hamiltonian Systems

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1 Hamiltonian Equations

1.1 Notation

We denote \mathbb{F}^n as the space of all n -dimensional vectors (all vectors are column vectors). $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ denotes the set of all linear transformations $\mathbb{F}^n \rightarrow \mathbb{F}^m$ (are sometimes identified with the set of all $m \times n$ matrices).

Functions are real and smooth unless otherwise stated; smooth means \mathcal{C}^∞ or real analytic. If $f(x)$ is a smooth function from an open set in \mathbb{R}^n to \mathbb{R}^m then $\frac{\partial f}{\partial x}$ denotes the $m \times n$ Jacobian matrix:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \cdots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

If A is a matrix, then A^T denotes its transpose, A^{-1} its inverse, and A^{-T} the inverse transpose.

If $f : \mathbb{R}^n \rightarrow \mathbb{R}$, then $\partial f / \partial x$ is a row vector. $\nabla f = \nabla_x f = f_x$ denote the column vector $(\partial f / \partial x)^T$. Df denotes the derivative of f thought of as a map from an open set in \mathbb{R} into $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. The variable t denotes a real scalar variable called time, and the symbol $\dot{f} = \partial f / \partial t$.

1.2 Hamilton's Equations

If the forces are derived from a potential function, the equations of motion of the mechanical system have many special properties, most of which follow from the fact that the equations of motion can be written as a Hamiltonian system. The Hamiltonian formalism is the natural mathematical structure in which to develop the theory of conservative mechanical systems.

A **Hamiltonian system** is a system of $2n$ ordinary differential equations of the form:

$$\begin{aligned} \dot{q} &= H_p \\ \dot{p} &= H_q \end{aligned}$$

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i}(q, p, t) \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i}(q, p, t) \quad i = 1, \dots, n \end{aligned} \tag{1}$$

where $H = H(q, p, t)$ is called **the Hamiltonian**, is a smooth real-valued function defined for $(q, p, t) \in \mathcal{O}$, an open set in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$.

The vectors $q = (q_1, q_2, \dots, q_n)$ and $p = (p_1, p_2, \dots, p_n)$ are traditionally called the **position** and **momentum** vectors, respectively, and t is called **time**, because that is what these variables represent in the classical examples. The variables q and p are said to be **conjugate variables**: p is conjugate to q . The concept of conjugate variable grows in importance as the theory develops.

The integer n is the **number of degrees of freedom** of the system.

We define the vector z as:

$$z = \begin{bmatrix} q \\ p \end{bmatrix}$$

a $2n$ vector. We define also the matrix J as the next $2n \times 2n$ skew symmetric matrix and the gradient in the next way:

$$J = J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

$$\nabla H = \begin{bmatrix} \frac{\partial H}{\partial z_1} \\ \vdots \\ \frac{\partial H}{\partial z_{2n}} \end{bmatrix}$$

where 0 is the $n \times n$ zero matrix and I_n is the $n \times n$ identity matrix. The case 2×2 matrix J_2 is a special case, it's denoted by K . In this notation the system is written as:

$$\dot{z} = J \nabla H(z, t) \quad (2)$$

[reason why:](#)

Explanation muhahahahahaha

1.3 Poisson Bracket

Let $F, G : U \subset \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ be C^r ($r \geq 1$) functions such that $(q, p, t) \mapsto F(q, p, t), G(q, p, t)$.

We define the **Poisson Bracket (PB)** as a C^{r-1} function $\{F, G\} : U \rightarrow \mathbb{R}$

$$\begin{aligned} \{F, G\} &= (\nabla_z F)^T J (\nabla_z G) \\ &= (\nabla_q F)^T (\nabla_p G) - (\nabla_p F)^T (\nabla_q G) \\ &= \sum_{i=1}^n \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right) \end{aligned}$$

Properties:

1. **Skew-symmetric:** $\{F, G\} = -\{G, F\}$

In particular: $\{F, F\}$

[proof:](#)

$$\begin{aligned}
-\{F, G\} &= -\left((\nabla_q F)^T (\nabla_p G) - (\nabla_p F)^T (\nabla_q G)\right) \\
&= (\nabla_p F)^T (\nabla_q G) - (\nabla_q F)^T (\nabla_p G) \\
&= (\nabla_q G)^T (\nabla_p F) - (\nabla_p G)^T (\nabla_q F) \\
&= \{G, F\}
\end{aligned}$$



2. **Bilinear:** $\{\alpha F_1 + \beta F_2, G\} = \alpha\{F_1, G\} + \beta\{F_2, G\}, \quad \alpha, \beta \in \mathbb{R}$

proof:

$$\begin{aligned}
\{\alpha F_1 + \beta F_2, G\} &= \left(\nabla_z(\alpha F_1 + \beta F_2)\right)^T J(\nabla_z G) \\
&= \left(\nabla_z(\alpha F_1)\right)^T J(\nabla_z G) + \left(\nabla_z(\beta F_2)\right)^T J(\nabla_z G) \\
&= \alpha \left(\nabla_z(F_1)\right)^T J(\nabla_z G) + \beta \left(\nabla_z(F_2)\right)^T J(\nabla_z G) \\
&= \alpha\{F_1, G\} + \beta\{F_2, G\}
\end{aligned}$$



3. **Leibnitz rule:** $\{F_1, F_2, G\} = F_1\{F_2, G\} + F_2\{F_1, G\}$

proof:

4. **Jacobi identity:** $\{F_1, \{F_2, F_3\}\} + \{F_3, \{F_1, F_2\}\} + \{F_2, \{F_3, F_1\}\} = 0$

proof:

2 N-Body Problem

Let's us consider N point masses in the space (\mathbb{R}^3 , the planar case \mathbb{R}^2 , the coolinear case \mathbb{R}), whit the i -th particle having a mass $m_i > 0$ and a position vector $q_i = (q_{i1}, q_{i2}, q_{i3})^t$.

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The equations of the system comes from the **Newton's law of universal gravitation**:

$$\ddot{q}_i m_i = \sum_{\substack{j=1 \\ j \neq i}}^N G m_i m_j \frac{(q_j - q_i)}{\|q_j - q_i\|^3} = \frac{\partial U}{\partial q_i} \quad I = 1, 2, \dots, N \quad (3)$$

reason why:

$$\left\| \frac{u}{\|u\|^3} \right\| = \frac{\|u\|}{\|u\|^3} = \frac{1}{\|u\|^2}$$

Where $G = 6.67408 \cdot 10^{-11} \frac{m^3}{s^2 Kg}$ is the **Gravitacional constant**.

We define the **Self potencial**, the negative of potencial energy, as:

$$U = - \sum_{1 \leq i < j \leq N} \frac{G m_i m_j}{\|q_j - q_i\|} \quad (4)$$

3 Tópico sobre el que haré el trabajo

4 Exercises

4.1 Chapter 1: Introduction to Hamiltonian systems

Make the phase portrait of the Hamiltonian system

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= x - \frac{x^3}{3}\end{aligned}$$

and compute its Hamiltonian

[Solución](#)

Make the phase portrait of the Hamiltonian system

$$\begin{aligned}\dot{x} &= x \\ \dot{y} &= -y + x^2\end{aligned}$$

and compute its Hamiltonian

[Solución](#)

(Meyer-Hall-Offin) Let x, y, z be the usual coordinates in \mathbb{R}^3 , $r = xi + yj + zk$, $X = \dot{x}$, $Y = \dot{y}$, $Z = \dot{z}$, $R = \dot{r} = Xi + Yj + Zk$.

1. Compute the three components of angular momentum $mr \times R$.
2. Compute the Poisson bracket of any two of the components of angular momentum and show that it is $\pm m$ times the third component of angular momentum.
3. Show that if a system admits two components of angular momentum as integrals, then the system admits all three components of angular momentum as integrals.

1. [adea](#)

2. [dsa](#)

3. [dadsa](#)

(Meyer-Hall-Offin) **A Lie algebra** A is a vector space with a product: $A \times A \rightarrow A$ that satisfies:

- **Anticommutative:** $ab \neq ba$
- **Distributive:** $a(b + c) = ab + ac$
- **Scalar associative:** $(\alpha a)b = \alpha(ab)$
- **Jacobis identity:** $a(bc) + b(ca) + c(ab) = 0$, $a, b, c \in A$, $\alpha \in \{\mathbb{R}, \mathbb{C}\}$

1. Show that vectors in \mathbb{R}^3 form a Lie algebra where the product $*$ is the cross product.
2. Show that smooth functions on an open set in \mathbb{R}^{2n} form a Lie algebra, where $fg = \{f, g\}$, the Poisson bracket.
3. Show that the set of all $n \times n$ matrices, $gl(n, \mathbb{R})$, is a Lie algebra, where $AB = AB - BA$, the Lie product.

1. bla

2. bla

3. bla

(Meyer-Hall-Offin) The pendulum equation is $\ddot{\theta} + \sin \theta = 0$.

1. Show that $2I = \frac{1}{2}\dot{\theta}^2 + (1 - \cos \theta) = \frac{1}{2}\dot{\theta}^2 + 2\sin^2(\theta/2)$ is an integral.
2. Sketch the phase portrait.
3. Make the substitution $y = \sin(\theta/2)$ to get $\dot{y}^2 = (1 - y^2)(I - y^2)$. Show that when $0 < I < 1$, $y = \text{sn}(t, k)$ solves this equation when $k^2 = I$ (Look at the definition of elliptic sine function of Section 1.6 of Meyer-Hall-Offin).

1. bla

2. bla

3. bla

(Meyer-Hall-Offin) Let $H : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ be a globally defined conservative Hamiltonian, and assume that $H(z) \rightarrow +\infty$ as $|z| \rightarrow +\infty$. Show that all solutions of $\dot{z} = J\nabla H(z)$ are bounded (Hint: Think like Dirichlet).

Solución

Consider a \mathcal{C}^2 Hamiltonian $H = H(q, p, t) : U \subset \mathbb{R}^{2n+1} \rightarrow \mathbb{R}$ such that $\det(\partial_p^2 H) \neq 0$ on U . Define $v = \partial_p H(q, p, t)$. Prove:

1.

$$\begin{aligned}\partial_{q_i} L(q, v, t) &= -\partial_{q_i} H(q, p, t) \\ \partial_{v_i} L(q, v, t) &= p_i \\ \partial_t L(q, v, t) &= -\partial_t H(q, p, t)\end{aligned}$$

2. The Lagrangian L is \mathcal{C}^2 and $\det(\partial_v^2 L) \neq 0$.

3. The Euler-Lagrange equations associated to L and the Hamiltonian equations $\dot{q}_i = \partial_{p_i} H$, $\dot{p}_i = -\partial_{q_i} H$ are equivalent.

1. bla

2. bla

3. bla

4.2 Chapter 2: The N-body problem

Prove that the linear momentum is a first integral and that the center of mass moves with constant velocity for the 3 body problem.

[Solución](#)

Prove that if (a_1, a_2, \dots, a_N) is a central configuration with value λ :

1. For any $\tau \in \mathbb{R}$ then $(\tau a_1, \tau a_2, \dots, \tau a_N)$ is also a central configuration with value $\frac{\lambda}{\tau^3}$.
2. If A is an orthogonal matrix, then $Aa = (Aa_1, Aa_2, \dots, Aa_N)$ is also a central configuration with the same value λ .

1. bal bla

2. bla bla

(Meyer-Hall-Offin) Draw the complete phase portrait of the collinear Kepler problem. Integrate the collinear Kepler problem.

[Solución](#)

(Meyer-Hall-Offin) Show that $\varpi^2(\epsilon^2 - 1) = 2hc^2$ for the Kepler problem. (Attention: Meyer-Hall-Offin has a typo)

[Solución](#)

(Meyer-Hall-Offin) The area of an ellipse is $\pi a^2(1 - \epsilon^2)^{1/2}$, where a is the semi-major axis. We have seen in Keplers problem that area is swept out at a constant rate of $c/2$. Prove Keplers third law: The period p of a particle in a circular or elliptic orbit ($\epsilon < 1$) of the Kepler problem is $p = (\frac{2\pi}{\sqrt{\mu}})a^{3/2}$.

[Solución](#)

4.3 Chapter 3: Linear Hamiltonian systems

4.4 Chapter 6: Symplectic Transformations

4.5 Chapter 8: Geometric Theory

4.6 Chapter 9: Continuation of solutions

(Meyer-Hall-Offin) Show that the scaling used in Section 9.4 of Meyer-Hall-Offin to obtain Hills orbits for the restricted problem works for Hills lunar problem (see previous problem) also. Why does not the scaling for comets work?

[Solución](#)

Prove Lemma 9.7.1 in Meyer-Hall-Offin. Verify that formula (9.11) is the condition for an orthogonal crossing of the line of syzygy in Delaunay elements.

[Solución](#)

4.7 Chapter 10: Normal forms

4.8 Chapter 13: Stability and KAM Theory

(Meyer-Hall-Offin) Using Poincaré elements show that the continuation of the circular orbits established in Section 6.2 (Poincaré orbits) are of twist type and hence stable.

[Solución](#)

5 Apendix

5.1 Complete examples

5.1.1 Harmonic oscillator

5.1.2 The Pendulum

This is a case of a one dEGREE OF FREEDOM of second order, bla bla bla bla

5.2 Needed resoults and definitions

5.2.1 Linear Algebra

matriz ortogonal

no singular

skew-simmetric

DEVOLVER A SU SITIO LAS FOTOS TAMAÑO CARNET

5.2.2 Calculus

teorema punto fijo de bla bla bla bla

Chain Rule: Let $F = f \circ g$, or, equivalently, $F(x) = f(g(x))$ for all x . Then:

$$(f \circ g)' = (f' \circ g) \cdot g'$$

$$F'(x) = f'(g(x)) \cdot g'(x)$$

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

The two versions of the chain rule are related; if $z = f(y)$ and $y = g(x)$:

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = f'(y) \cdot g'(x) = f'(g(x)) \cdot g'(x)$$

proof: ♠

Gradient: The gradient of a scalar function ($\mathbb{R}^n \rightarrow \mathbb{R}$) $f(x_1, \dots, x_n)$ denoted by ∇f or $\vec{\nabla} f$ denotes the vector differential operator. The gradient of f is defined as the unique vector field whose dot product with any unitvector v at each point x is the directional derivative off along v . That is,

$$(\nabla f(x)) \cdot v = D_v f(x)$$

- *Cartesian coordinates*: Lets focus in \mathbb{R}^3 , where i, j, k are the standard unit vectors in the directions of axis x, y, z .

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

- *Cylindrical coordinates*: Lets focus in \mathbb{R}^3 , where ρ is the axial distance, φ is the azimuthal or azimuth angle and z is the the axial coordinate and e_ρ, e_φ, e_z are the unit vectors pointing along the coordinate directions.

$$\nabla f(\rho, \varphi, z) = \frac{\partial f}{\partial \rho}e_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \varphi}e_\varphi + \frac{\partial f}{\partial z} = \left(\frac{\partial f}{\partial \rho}, \frac{1}{\rho} \frac{\partial f}{\partial \varphi}, \frac{\partial f}{\partial z} \right)$$

proof: ♠

- *Spherical coordinates*: Lets focus in \mathbb{R}^3 , where r is the radial distance, φ is the azimuthal angle and θ is the polar angle, and e_r, e_φ, e_θ are local unit vectors pointing in the coordinate directions.

$$\nabla f(r, \theta, \varphi) = \frac{\partial f}{\partial r}e_r + \frac{1}{r} \frac{\partial f}{\partial \theta}e_\theta + \frac{1}{r \sin(\theta)} \frac{\partial f}{\partial \varphi}e_\varphi = \left(\frac{\partial f}{\partial r}, \frac{1}{r} \frac{\partial f}{\partial \theta}, \frac{1}{r \sin(\theta)} \frac{\partial f}{\partial \varphi} \right)$$

proof: ♠

Laplace Operator: is a differential operator given by the divergence of the gradient of a function on Euclidean space.

$$\Delta f = \nabla^2 f = \nabla \cdot \nabla f = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}$$

proof:

reason why:

5.2.3 Geometry

5.3 Funcional Analysis

5.3.1 Differential forms

5.3.2 Measure Theory

5.3.3 Ordinary Differential Equations

Peano existence theorem: