Graph Theory .- Coloring

Manuel Gijón Agudo

October 2017

Exercise 13

`Prove that a regular graph of odd order satisfies` $\acute{\chi}(\mathcal{G}) = \Delta(\mathcal{G}) + 1$ `(namely,` $\mathcal{G}$ `is of class two).`

proof:

Step 1. Every regular graph of odd order is overfull.

A graph is overfull if:

$$|E| = m > \Delta(\mathcal{G}) \left\lfloor \frac{n}{2} \right\rfloor = \Delta(\mathcal{G}) \left\lfloor \frac{|V|}{2} \right\rfloor$$

We apply the Handshaking lemma:

$$\sum_{v \in V} d(v) = 2|E| = 2m$$

We use that in a regular graph the degree of all vertices is the same.

$$\sum_{v \in V} d(v) = nd(v) = n\Delta(\mathcal{G}) = 2m \Rightarrow m = \Delta(\mathcal{G})\frac{n}{2}$$

Step 2. If $\mathcal{G}$ is overfull then is of Class 2.

proof: make a proof by contradiccion.

The contrary of the fact that every graph like this is in a class is just to find one that is not (so, it is in class 1, what means that $\acute{\chi}(\mathcal{G}) = \Delta(\mathcal{G})$.

Let $\mathcal{G}$ be a graph whit n vertices and m edges. $\mathcal{G}$ is an overall graph ($|E| = m > \left\lfloor \frac{n}{2} \right\rfloor$).

Then any $\Delta(\mathcal{G})$-coloring of edges partitions the set of edges into $\Delta(\mathcal{G})$ independent subsets. But the number of edges in each independent subset can not be larger than $\left\lfloor \frac{n}{2} \right\rfloor$, since otherwise two of these edges would be adjacent.

It follows that $m \leq \Delta(\mathcal{G}) \left\lfloor \frac{n}{2} \right\rfloor$ leading us to a contradiction.

■

Appendix

Definition.1 Let $\mathcal{G} = (V, E)$ be a simple graph, a *proper edge-coloring* of it is a map $c : E \to \{1, ..., k\}$ such that two incident edges recive distinct colors.

Definition.2 A *regular graph* is a graph that has $d(x) = N(x) \forall x \in V$.

Observation.1 In a regular graph we have $\Delta(\mathcal{G}) = \delta(\mathcal{G}) = N(x)\ \forall x \in V$

Definition.2 The *edge-chromatic number* $\acute{\chi}(\mathcal{G})$ of a graph $\mathcal{G}$ is the minimum positive integer k for which $\mathcal{G}$ admits a proper k-edge coloring.

Definition.3 A graph $\mathcal{G}$ is called *overfull* if:

$$|E| = m > \Delta(\mathcal{G}) \left\lfloor \frac{n}{2} \right\rfloor = \Delta(\mathcal{G}) \left\lfloor \frac{|V|}{2} \right\rfloor$$

Theorem .1 ***Vizing (1964)****: The edges of every simple undirected graph may be colored using a number of colors no bigger than $\Delta(\mathcal{G}) + 1$, the maximun degree of the graph plus one. The lower bound $\Delta(\mathcal{G}) \leq \acute{\chi}(\mathcal{G})$ is trivial.*

So, we can divide the graps into two categories, class 1 (their have an edge-chromatic numbar equal to $\Delta(\mathcal{G})$) and class 2 (when it is equal to $\Delta(\mathcal{G}) + 1$).