

Code Theory

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1. Memoryless resources

1.1. Sources and average word length

Definition 1: a **source** is a finite set \mathcal{S} together with a set of random variables (X_1, X_2, \dots) whose range is \mathcal{S} .

If $P(X_n = \mathcal{S}_i)$ only depends on i and not on n then we say the source is **stationary** and if the X_n are independent then it's **memoryless**.

Insert example here

Definition 2: Let \mathcal{T} be a finite set called **alphabet**. A map $\mathfrak{C} : \mathbb{S} \longrightarrow \mathbb{U}_{n \geq 1} T^n$ is called a **code**.

If $|\mathcal{T}| = r$ then \mathfrak{C} is a **r -ary code**.

A code extends from \mathbb{S} to $T \cup T^2 \cup \dots$ to $\mathbb{S} \cup \mathbb{S}^2 \cup \dots$ to $T \cup T^2 \cup \dots$ in obvious way.

insert example here

Definition 3: The **average word-length** of a code \mathfrak{C} is $L(\mathfrak{C}) := \sum_{i=1}^n p_i l_i$ where l_i is the length of the image of the symbol of \mathbb{S} , which is emitted with probability p_i .

For now, we write \mathfrak{C} to be the image of \mathfrak{C} .

1.2. Uniquely decodeable codes

Definition 4: If for any sequences $u_1 \dots u_n = v_1 \dots v_m$ in \mathfrak{C} implies $m = n$ and $u_i = v_i$ for $i = 1, \dots, n$ then we say that \mathfrak{C} is **uniquely decodeable**.

insert example here

insert example here

insert example here

Let $\mathfrak{C}_0 = \mathfrak{C}$:

- $\mathfrak{C}_n := \{\omega \in T \cup T^2 \cup \dots \mid u\omega = v \text{ for some } u \in \mathfrak{C}_{n-1}, v \in \mathfrak{C} \text{ or } u\omega = v \text{ for some } u \in \mathfrak{C}, v \in \mathfrak{C}_{n-1}\}$
- $\mathfrak{C}_\infty := \bigcup_{k \geq 1} \mathfrak{C}_k$

Since everythig is finite either $\mathfrak{C}_m = \emptyset$ for some m and then $\mathfrak{C}_n = \emptyset$ for $n \geq m$ or it will be periodic and start repeating.

Theorem 1: \mathfrak{C} is uniquely decodeable $\iff \mathfrak{C} \cap \mathfrak{C}_\infty = \emptyset$.

proof: Insert proof here

insert example here

insert example here

insert example here

Definition 5: A code is a **prefix-code** if no codeword is prefix of another (ie. $\mathfrak{C}_1 = \emptyset$).

A prefix code is uniquely decodeable.

Theorem 2: (Kraft's inequality) \exists r -ary prefix code with word lengths $l_1, l_2, \dots, l_q \iff$

$$\sum_{i=1}^q r^{-l_i} \leq 1$$

proof: Insert proof here

insert example here

Theorem 3: (McMillan's inequality) \exists r -ary uniquely decodeable code with word lengths $l_1, l_2, \dots, l_q \iff$

$$\sum_{i=1}^q r^{-l_i} \leq 1$$

proof: Insert proof here

1.3. Optimal codes

Let be \mathbb{S} a source with symbols s_1, \dots, s_q emitted with probabilities p_1, \dots, p_q and \mathfrak{C} is a code which encodes s_i with a codeword length l_i . Recall $L(\mathfrak{C}) = \sum_{i=1}^q p_i l_i$.

Definition 6: An **optimal code** for \mathbb{S} is an uniquely decodeable code \mathfrak{D} such that $L(\mathfrak{C}) \geq L(\mathfrak{D})$ for all unique decodeable code \mathfrak{C} .

inset example here

insert example here

Definition 7: A code constructed in this way is called a **Huffman code**.

insert example here

1.4. Extension of sources

2. Information and entropy

2.1. Definitions

Definition 1: the **information** conveyed by a source is a function $I : S \rightarrow [0, \infty)$ where S is a **source**¹ with the properties:

- $I(s_i)$ is a decreasing function of the probability p_i , with $I(s_i) = 0$ if $p_i = 1$.
- $I(s_i s_j) = I(s_i) + I(s_j)$, ie. the information gained by two symbols is the sum of the information obtained from each where the source has symbols s_1, \dots, s_q emitted with probabilities p_1, \dots, p_q .

Lemma 1: $I(s_i) = -\log_r p_i$ for some r .

proof: Insert proof here

Definition 2: The r -ary **entropy** $H_r(S)$ of a source S is the average information conveyed by S .

$$H_r(S) := - \sum_{i=1}^q p_i \log_r p_i$$

, by convention $x \log_r x$ evaluated at 0 is 0.

Insert five examples

2.2. Properties of the entropy function

Theorem 1: $H_r(S) \leq \log_r q$ with equality if and only iff S is the source where each symbol is emitted with probability $1/q$.

proof: Insert proof here

Theorem 2: $H_r(S) \leq L(C)$ for unique decodeable code C .

proof: Insert proof here

2.3. Shannon-Fano Code

Let S be the source with symbols s_i and probabilities p_i . Let $l_i := \lceil \log_r 1/p_i \rceil$.

Then: $\sum_{i=1}^q r^{-l_i} \leq \sum r^{-\log_r 1/p_i} = \sum p_i = 1$

¹A **source** is a finite set S together with a sequence of random variables X_i whose range is S

Definition 3: by Kraft exists a prefix code with word length l_1, l_2, \dots, l_1 . This code is called **Shannon-Fano code**.

Inert example here

Lemma 2: For the Shannon-Fano code C : $H_r(S) \leq L(C) < H_r(S) + 1$.

proof: Insert proof here

2.4. Product of sources

Let S and T be two memoryless sources, S with symbols s_i and probabilities p_i and T with symbols t_j and probabilities q_j .

Definition 4: The **product source** $S \times T$ is a source with symbols $s_i t_j$ and probabilities $p_i q_j$.

Theorem 3: $H_r(S \times T) = H_r(S) + H_r(T)$.

proof: Insert proof here

Corollary 1: $H_r(S^n) = nH_r(S)$.

Theorem 4: Noiseless Coding The average word length L_n of an optimal code of S^n satisfies:

$$\frac{L_n}{n} \rightarrow H_r(S), n \rightarrow \infty$$

proof: Insert proof here

some examples

2.5. Markov Chains

Definition 4: A **Markov Chain** is a sequence of random variables where X_{n+1} depends only for X_n .

$$P(X_{n+1} = s_j | X_n = s_i) = p_{i,j}$$

This can be represented in a direct graph and also by a matrix $P := (p)_{i,j}$.

Suppose u_0 is the vector which describes the initial distribution, ie. the i -th coordinate of u_0 is probability we start at s_i . Probability of being in the i -th state after r steps is the i -th coordinate of $u_0 P^r$.

Theorem 5: if $\exists r \in \mathbb{N}$ such that P^r has no zero entries, then $u_0 P^r \rightarrow u$, as $n \rightarrow \infty$.

Definition 5: This vector u is called the **stationary distribution**. It is normalised eigenvector of P^t with eigenvalue 1, ie. $u_j = \sum_i p_{i,j} u_i$ and $\sum_j u_j = 1$.

Definition 6: If P is the matrix of a Markov Chain and $\exists r$ such that P^r has non zero entries then we say that the Markov Chain is **regular**.

2.6. Sources with memory

Suppose S is a Markov Chain source with random variables X_1, X_2, \dots such that

$$P(X_{n+1} = s_j | X_n = s_j) = p_{i,j}$$

Definition 7: S is **not memoryless**, but it is stationary.

Theorem 6: suppose S is a regular Markov Chain source with stationary distribution $u = (u_1, \dots, u_n)$. Let S' be the stationary memoryless source with the same source elements as S (where s_i is emitted with probability w_i). Then:

$$H_r(S) \leq H_r(S')$$

proof: Insert proof here

3. Information channels

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3.4. Decision rules

3.5. Improving reliability

3.6. Rates of transmission and Hamming distance

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5.3. Asymptotically good codes

6. Linear codes

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6.2. Syndrom decoding

6.3. Dual code and Mc Williams identities

6.4. The Griesmer bound

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7.1. Introduction

7.2. Quadratic residue codes

7.3. BCH Codes

Decision problem, yes/no problem

8. Maximun distance separable codes

8.1. Syngleton bound

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