# Graph Theory .- Spectral Graph Theory

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## Exercise 10, Part 2

Let $G \Box H$ denote the cartesian product of $G$ and $H$. The vertex set is $V(G) \times V(H)$ and $(x, y) \sim (z, y)$ if one of the coordinates agree and the ohter one is a pair of adyacent vertices.

- (a) Show that the Laplace eigenvalues of $G \Box H$ are precisely $\mu_i(G) + \mu_j(H)$, for all $i, j$.
- (b) The n-cube $Q_n$ is defined as $Q_1 = K_2$ and $Q_n = K_2 \Box Q_{n-1}$ for $n \geq 2$. Determine $\mu_2(Q_n)$.

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### (a)

#### Using the Kronecker product

We know that $L(G)X_i = \lambda_i X_i$ and $L(H)Y_j = \mu_j Y_j$, then

$$\begin{aligned} L(G \Box H)(X_i \otimes Y_j) &= (L(G) \otimes I_m + I_n \otimes L(H))(X_i \otimes Y_j) \\ &= L(G)X_i \otimes I_m Y_j + I_n X_i \otimes L(H) Y_j \\ &= \lambda_i X_i \otimes Y_j + \mu_j X_i \otimes Y_j \\ &= (\lambda_i + \mu_j)(X_i \otimes Y_j) \end{aligned}$$

#### Directly

**Theorem .1** *Fiedler (1973)* [1]*: Let be $\{\lambda_1, ..., \lambda_n\}$ and $\{x_1, ..., x_n\}$ and $\{\mu_1, ..., \mu_m\}$ and $\{y_1, ..., y_m\}$ the sets of Laplacian eigenvalues and Laplacian eigenvectors of $G$ and $H$ respectively. Then, for each $1 \leq i \leq n$ and $1 \leq j \leq m$, $G \Box H$ has an Laplacian eigenvector $z$ of eigenvalue $\lambda_i + \mu_j$ such that $z(u, v) = x_i(u) y_j(v)$.*

*Proof: Let $L$ denote the Laplacian of $G \Box H$, $d_u$ the degree of $u$ in $G$ and $d_v$ the degree of $v$ in $H$. Let $E$ be the set of edges of $G$ and $F$ the set of edges for $H$.*

[1] Miroslav Fiedler, *"Algebraic connectivity of graphs"*, Czechoslovak Mathematical Journal, Vol.23 (1973), No.2, 298-305

$$
\begin{aligned}
(Lz)(u,v) &= (d_u + d_v)x_i(u)y_j(v) - \sum_{(u,u_2)\in E} x_i(u_2)y_j(v) - \sum_{(v,v_2)\in F} x_i(u)y_j(v_2) \\
&= \Big(d_u(x_i(u)y_j(v)) - \sum_{(u,u_2)\in E} x_i(u_2)y_j(v)\Big) + \Big(d_v(x_i(u)y_j(v)) - \sum_{(v,v_2)\in F} x_i(u)y_j(v_2)\Big) \\
&= y_j(v)\Big(d_u x_i(u) - \sum_{(u,u_2)\in E} x_i(u_2)\Big) + x_i(u)\Big(d_v y_j(v) - \sum_{(v,v_2)\in F} y_j(v_2)\Big) \\
&= y_j(v)\lambda_i x_i(u) + x_i(u)\mu_j y_j(v) = (\lambda_i + \mu_j)(x_i(u)y_j(v))
\end{aligned}
$$

■

## (b)

*First of all observe that:*

$$
\begin{aligned}
Q_n &= K_2 \square Q_{n-1} \\
&= K_2 \square (K_2 \square Q_{n-2}) \\
&= ... \\
&= (K_2 \square K_2)^{n-1} \square Q_1 \\
&= (K_2 \square K_2)^n
\end{aligned}
$$

*We are going to use the previous resoult, so we need to calculate the eigenvalues of $K_2$. The Laplacian matrix of $K_2$ is:*

$$
\begin{aligned}
L_{K_2} = L_{Q_1} &= D_{Q_1} - A_{Q_1} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
\end{aligned} \tag{1}
$$

*We calculate the eigenvalues and we get $\mu_1(Q_1) = 0$ y $\mu_2(Q_1) = 2$.*

*It's possible but ot necessary to compute all the eigenvelues for every $Q_n$, but now we can see how the secuence of eigenvalues starts $(0, 2, 4, ...)$. So for every $n$ the answer is $\mu_2(Q_n) = 2$.*

# Appendix

## (a)

***Definition 1. Kronecker product:*** *Let $\mathcal{A}$ be a $m \times n$ matrix and $\mathcal{B}$ a $p \times q$ matrix, then the Kronecker matrix $A \otimes B$ is the $mp \times nq$ block matrix:*

$$\mathcal{A} \otimes \mathcal{B} := \begin{pmatrix} a_{1,1}\mathcal{B} & a_{1,2}\mathcal{B} & \cdots & a_{1,n}\mathcal{B} \\ a_{2,1}\mathcal{B} & a_{2,2}\mathcal{B} & \cdots & a_{2,n}\mathcal{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1}\mathcal{B} & a_{m,2}\mathcal{B} & \cdots & a_{m,n}\mathcal{B} \end{pmatrix}$$

***Observation 1:*** *Of the set of properties of the Kronecker product, we will be interesting in two in partiular.*

- ***Kronecker sum:*** *If $\mathcal{A}$ is a $n \times n$ matrix and $\mathcal{B}$ is a $m \times m$ matrix:*

$$\mathcal{A} \oplus \mathcal{B} = \mathcal{A} \otimes \mathcal{I}_m + \mathcal{I}_n \otimes \mathcal{B}$$

- ***Mixed-product property:***

$$(\mathcal{A} \otimes \mathcal{B})(\mathcal{C} \otimes \mathcal{D}) = (\mathcal{A}\mathcal{C}) \otimes (\mathcal{B}\mathcal{D})$$

## (b)

*We can generalize the form of the Laplacian matrix for the hypercube from (1) in the next way:*

$$\begin{aligned} L_{Q_2} = L_{K_2 \square K_2} = D_{Q_2} - A_{Q_2} &= D_{K_2 \square K_2} - A_{K_2 \square K_2} \\ &= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix} \end{aligned} \tag{2}$$

*Observe the block structure of the matix:*

$$\begin{aligned} (2) &= \begin{bmatrix} 2I & 0 \\ 0 & 2I \end{bmatrix} - \begin{bmatrix} A_{Q_1} & I \\ I & A_{Q_1} \end{bmatrix} \\ &= \begin{bmatrix} 2I - A_{Q_1} & -I \\ -I & 2I - A_{Q_1} \end{bmatrix} \end{aligned}$$

*In general we have:*

$$
\begin{aligned}
L_{Q_n} = D_{Q_n} - A_{Q_n} &= \begin{bmatrix} nI & 0 \\ 0 & nI \end{bmatrix} - \begin{bmatrix} A_{Q_{n-1}} & I \\ I & A_{Q_{n-1}} \end{bmatrix} \\
&= \begin{bmatrix} nI - A_{Q_{n-1}} & -I \\ -I & nI - A_{Q_{n-1}} \end{bmatrix} \qquad (3)
\end{aligned}
$$

*Since $Q_n$ is $n$-regular, then $D_n = nI$ and so:*

$$L_{Q_n} = D_{Q_n} - A_{Q_n} = nI - A_{Q_n}$$

*Then for $Q_{n-1}$ we have:*

$$
\begin{aligned}
L_{Q_{n-1}} &= (n-1)I - A_{n-1} \\
&= nI - I - A_{n-1} \\
&\Rightarrow L_{Q_{n-1}} + I = nI - A_{n-1} \qquad (4)
\end{aligned}
$$

*Applying (4) in (3) we have a recursive formula for the Laplacian matrix of the hypercube:*

$$L_{Q_n} \begin{bmatrix} L_{Q_{n-1}} + I & -I \\ -I & L_{Q_{n-1}} + I \end{bmatrix}$$