

Twist Maps and Aubry-Mather Sets

Manuel Gijón Agudo

May, 2019

Contents

1	Introduction	2
2	First definitions and basic resoutls	2
3	Main resoutls	2
4	Conclusions	4

1 Introduction

Caso 2 dimensional

Se trataría de dar una introducción y comparar con la teoría KAM explicada por Marcel, o sea que él te podrá dar más detalles. También en el último capítulo de Meyer Offin hay una aproximación variacional.

Our objective in this work is give an introduction to the Aubry-Mathers theory, explain its importance in the context it is on and expose the fundamental results of the topic. In order to arrive to this point, we first present results and definitions in the next section.

2 First definitions and basic results

bla bla bla bla

3 Main results

Definition: a monotone **twist map** is an orientation preserving \mathcal{C}^1 -diffeomorphism $\varphi : S^1 \times [0, 1] \longrightarrow S^1 \times [0, 1]$ of an annulus which admits a lift $\bar{\varphi} = (f, g) : \mathbb{R} \times [0, 1] \longrightarrow \mathbb{R} \times [0, 1]$ with the following properties:

- (a) $\bar{\varphi}$ preserves (Lebesgue) area.
- (b) **Twist condition:** $D_2 f > 0$.
- (c) $g(\xi, 0) = 0, g(\xi, 1) = 1$.

Notes:

- Instead of (a) we could require $\det(\bar{\varphi}') = 1$.
- Condition (c) means that φ does not commute the boundary components.
- We are assuming that the translation $T_{(1,0)}, T_{(1,0)}(\xi, y) = (\xi + 1, y)$, generates the group of covering transformations. Then $f(\xi + 1, y) = f(\xi, y) + 1, g(\xi + 1, y) = g(\xi, y)$.

A fundamental property of the monotone twist maps is that it can be globally described by a generating function:

$$H : D \rightarrow \mathbb{R}, \text{ where } D := \{(\xi, \eta) \in \mathbb{R}^2 | f(\xi, 0) \leq \eta \leq f(\xi, 1)\}$$

up to an additive constant, H is uniquely determined by:

$$\bar{\varphi}(x_0, y_0) = (x_1, y_1) \Leftrightarrow \begin{cases} -D_1 H(x_0, y_0) & = y_0 \\ D_2 H(x_0, y_0) & = y_1 \end{cases} \quad (1)$$

To construct H let $a, b : D \rightarrow \mathbb{R}$ be defined by:

$$a(\xi, f(\xi, y)) := y, \quad b(\xi, f(\xi, \eta)) := g(\xi, a(\xi, \eta))$$

Then, 1 is equivalent to:

$$dH = -ad\xi + bd\eta$$

From $\det(\bar{\varphi}') = 1$ can be concluded that the 1-form $\omega = -ad\xi + bd\eta$ is closed.

Since D is simply connexted, there exists $H : D \rightarrow \mathbb{R}$ such that 1 is true. H is \mathcal{C}^2 in the interior of D and D_1H, D_2H and D_2D_1H extend continuously to D .

Moreover, from (b) in the previous definition implies $D_2D_1H = -D_2a \leq \delta$ for some $\delta > 0$. Finally $H(\xi + 1, \eta + 1) = H(\xi, \eta)$, i.e. ω is even exact on the cylinder D modulo translations $T_{(j,j)} \in \mathbb{Z}'$: ω vanishes along the curve $\xi \rightarrow (\xi, f(\xi, 0))$ from $(0, f(0, 0))$ to $(1, f(0, 0) + 1)$.

PENDIENTE!!!!

Taking 1 as definition we can extend $\bar{\varphi}$ resp. φ to \mathbb{R}^2 resp. $S^1 \times \mathbb{R}$. According to this equation, we have a variational principle for the orbits of the extend $\bar{\varphi}$. We recall that a sequence $(x_i)_{i \in \mathbb{Z}}$ is stationary with respect to H if $D_2H(x_{i-1}, x_i) + D_1(x_i, x_{i+1}) = 0 \quad \forall i \in \mathbb{Z}$.

Resout 1: If $(x_i)_{i \in \mathbb{Z}}$ is stationary with respect to H then $(x_i, y_i)_{i \in \mathbb{Z}}$, $y_i = -D_1H(x_i, x_{i+1})$, is an orbit of $\bar{\varphi}$, i.e. $\bar{\varphi}(x_i, y_i) = (x_{i+1}, y_{i+1})$. Conversely, if $(x_i, y_i)_{i \in \mathbb{Z}}$ is an orbit of $\bar{\varphi}$ then $(x_i)_{i \in \mathbb{Z}}$ is stationary with respect to H .

Observation: Since the $(x_i)_{i \in \mathbb{Z}} \in \mathcal{M}(H)$ are stationary with respect to H every statement on minimal trajectories can be interpreted as a statement on certain types of orbits of the extended $\bar{\varphi}$. We can return to the original $\bar{\varphi} : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R} \times [0, 1]$ in the following way: Let $f_0, f_1 \in \bar{G}_+$ be defined by $f_0(\xi) = f(\xi, 0)$, $f_1(\xi) = f(\xi, 1)$ and let α_0, α_1 be the rotation number of f_0, f_1 . The interval $[\alpha_0, \alpha_1]$ is called the **twist interval** of $\bar{\varphi}$. Later we will see that $\alpha_0 < \alpha_1$.

Lemma: Suppose $x \in \mathcal{M}_\alpha$ and $\alpha \in (\alpha_0, \alpha_1)$, i.e. α is the interior of the twist interval of $\bar{\varphi}$. Then the orbit $(x_i, y_i)_{i \in \mathbb{Z}}$, $y_i = -D_1H(x_i, x_{i+1})$, is contained in $\mathbb{R} \times (0, 1)$.

Theorem 1: For every irrational $\alpha \in (\alpha_0, \alpha_1)$ there exists a φ -invariant set $\mathcal{M} \subseteq S^1 \times (0, 1)$ with the following properties:

- (a) M_α is the graph of a Lipschitz function $\psi_\alpha : A_\alpha \rightarrow (0, 1)$ defined on a closet set $A_\alpha \subseteq S^1$.
- (b) φ has rotation number $\alpha \bmod (\mathbb{Z})$ on M_α , i.e. there exists $h \in \mathcal{G}_+$ with $\alpha(h) \equiv \alpha \bmod (\mathbb{Z})$ such that $h(A_\alpha) = A_\alpha$ and

$$\varphi(\xi, \psi_\alpha(\xi)) = (h(\xi), \psi_\alpha(h(\xi))) \quad \forall \xi \in A_\alpha$$

- (c) The set $\mathcal{M}_\alpha^{\text{rec}}$ of recurrent points in \mathcal{M}_α projects either to a cantor set in S^1 or to all S^1 . In the latter case \mathcal{M}_α is a φ -invariant Lipschitz curve winding once around the annulus $S^1 \times [0, 1]$ and $\varphi|_{\mathcal{M}_\alpha}$ is topologically conjugate to a rotation.

proof: (we prove (a) - (c) for $\bar{\varphi}$ instead of φ ; everything is invariant counting on $(\xi, \eta) \rightarrow (\xi + 1, \eta)$). Considering the previous lemma, we define:

$$\bar{\mathcal{M}}_\alpha := \{(\xi, \eta) \in \mathbb{R} \times (0, 1) | \exists x \in \mathcal{M}_\alpha \text{ such that } \xi = x_0, \eta = -D_1H(x_0, x_1)\}$$

According to 1 and the resoult 1 the set $\bar{\mathcal{M}}_\alpha$ is φ -invariant. According to the resoultls INSERTAR AQUI LAS REFERENCIAS DE MIERDA MUHAHAHAHA the set $\bar{\mathcal{M}}_\alpha$ has one-to-one projection onto a closed set $\bar{A}_\alpha \subseteq \mathbb{R}$.

Considering REFERENCIA 3.19 MUHAHAHAHHA there exists a Lipschitz continuous $\bar{h} \in \bar{\mathcal{G}}_+$ of a rotation number α such that $\bar{h}(x_0) = x_1 \quad \forall x \in \mathcal{M}_\alpha$. Hence $\bar{\psi}_\alpha(\xi) := -D_1 H(\xi, \bar{h}(\xi))$ for $\epsilon \in \bar{A}_\alpha$ is Lipschitz as well.

This proves (a) and (b). Now (c) is a consequence of the properties of circle maps.



Theorem 2: If C is a φ -invariant curve winding once around $S^1 \times [0, 1]$ and if $\varphi|C$ has rotation number α then $C \subseteq \mathcal{M}_\alpha$ and $C = \mathcal{M}_\alpha$ if α is irrational.

Lemma: For every monotone twist map the twist interval has non-empty interior.

Corollary: For a standard diffeomorphism $\varphi : S^1 \times \mathbb{R} \rightarrow S^1 \times \mathbb{R}$ from (b) in the Theorem 1 choose the function s such that $s'(\xi_0) < -2$ for some $\xi_0 \in \mathbb{R}$. Then there is no φ -invariant curve winding around $S^1 \times \mathbb{R}$.

proof:

We know from the Theorem 2 that the existence of a φ -invariant curve winding around $S^1 \times \mathbb{R}$ implies the existence of an $x \in \mathcal{M}(H)$ with $x_0 = \xi_0$ where $H(\xi, \eta) = \frac{1}{2}(\xi - \eta)^2 + S(\xi)$, $S' = s$. Now we have:

$$D_{22}H(x_{-1}, x_0, x_1) = D_{22}H(x_{-1}, x_0) + D_{11}H(x_0, x_1) = 2 + s'(\xi_0) < 0$$

But this contradicts the minimality of the segment (x_{-1}, x_0, x_1) .



Observation: all this resoultls are true (except the Lypschrift continuity in the Theorem 1) are also true if we relax the conditions of first definition (on a monotone twist map φ) in the following way:

- (a) It is sufficient that φ is a homeomorphism instead of being a \mathcal{C}^1 -diffeomorphism.
- (b) Instead of $D_2 f > 0$ it is suffices that $y \rightarrow f(\xi, y)$ is strictly increasing $\forall \xi \in \mathbb{R}$.

4 Conclusions

[1]

References

- [1] bal bla bla bla