

Twist Maps and Aubry-Mather Sets

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1 Introduction

Our main goal in this work is give an introduction to the Aubry-Mather theory, explain its importance in the context it is on and to expose the fundamental results of the topic. In order to arrive to this point, we first present results and definitions in the next section, being the fundamental results in the third chapter.

In order to understand this topic, in the second chapter we present results and definitions that are indispensable to understand the further results. Unlike what we have done in class, we will working mostly with maps instead of flows and this is the reason why this chapter is longer, despite it is less important, than the next one.

KAM-Theory for this kind of maps shows that for this kind of maps which are sufficiently \mathcal{C}^k -close to an integrable one, φ_0 , many of this invariant curves persist. The invariant curves are destroyed when we go too far away from the integrable situation and the Mather sets \mathcal{M}_α are the most important remnants of the invariant curves of irrational rotation number α .

The Aubry-Mather theory appears to explain what happens with perturbed systems and its main contribution is to explain what happens with the orbits in the cases that are not covered by KAM theory. This results are the key of the chapter three.

We will be working with area-preserving monotone twist maps $\varphi : S^1 \times [0, 1] \rightarrow S^1 \times [0, 1]$ preserving the boundary components. The Mather sets will be a particular φ -invariant subsets of the cylinder $S^1 \times [0, 1]$. We will like to find closed φ -invariant curves which separates $S^1 \times \{0\}$ from $S^1 \times \{1\}$ because this are related with the stability of the system $\{\varphi^n\}_{n \in \mathbb{Z}}$ in this sense:

- If such an invariant curve exists, then every orbit of φ on one of the two sides in which that curve divides the cylinder.
- If such an invariant curve does not exist, then $\exists p \in S^1 \times (0, 1)$ such that $\varphi^n(p)$ converges to $S^1 \times \{0\}$ if $n \rightarrow \infty$ and $S^1 \times \{1\}$ if $n \rightarrow -\infty$.

One note about the bibliography, this work has been done using a very similar structure that the one in [1]. Also we have consider the documents [2] and [3].

2 First definitions and results

Definition 1: $\mathbb{R}^{\mathbb{Z}} := \{x | x : \mathbb{Z} \rightarrow \mathbb{R}\}$, the space of bi-infinite sequences of real numbers (we consider it along with the product topology). An element $x = (x_i)_{i \in \mathbb{Z}}$ is sometimes called **trajectory**.

Definition 2: Convergence of a sequence $x^n \in \mathbb{R}^{\mathbb{Z}}$ to $x \in \mathbb{R}^{\mathbb{Z}}$ means that $\lim_{n \rightarrow \infty} x_i^n = x_i$ ($\forall i \in \mathbb{Z}$).

Given a function $H : \mathbb{R}^2 \rightarrow \mathbb{R}$ we can extend H to trajectories (or finite segments) by:

$$H(x_j, \dots, x_k) = \sum_{i=j}^{k-1} H(x_i, x_{i+1})$$

Definition 3:

- We say that the segment (x_j, \dots, x_k) is **minimal** with respect to H if:

$$H(x_j, \dots, x_k) = \sum_{i=j}^{k-1} H(x_i, x_{i+1}) \leq \sum_{i=j}^{k-1} H(x_i^*, x_{i+1}^*) = H(x_j^*, \dots, x_k^*) \quad (\forall (x_j^*, \dots, x_k^*) : x_j = x_j^*, x_k = x_k^*)$$

- $x \in \mathbb{R}^{\mathbb{Z}}$ is **minimal** with respect to H if every finite segment of x is minimal (with respect to H).
- We call $\mathcal{M} = \mathcal{M}(H)$ to the **set of minimal trajectories** (with respect to H), for $x \in \mathbb{R}^{\mathbb{Z}}$.

Hypotesis for H (we assume that it is continuous):

- (H1) *Periodicity condition*: $\forall (\xi, \eta) \in \mathbb{R}^2 : H(\xi + 1, \eta + 1) = H(\xi, \eta)$.
- (H2) *Condition at infinity*: $\lim_{|\eta| \rightarrow \infty} H(\xi, \xi + \eta) = \infty$ uniformly in ξ .
- (H3) *Ordering condiction*: If $\xi_* < \xi^*, \eta_* < \eta^* \Rightarrow H(\xi_*, \eta_*) + H(\xi^*, \eta^*) < H(\xi_*, \eta^*) + H(\xi^*, \eta_*)$.
- (H4) *Transversality condiction*: If $(x_{-1}, x_0, x_1) \neq (x_{-1}^*, x_0^*, x_1^*)$ are minimal and $x_0 = x_0^*$ then $(x_{-1} - x_{-1}^*)(x_1 - x_1^*) < 0$.

Definition 4: If $H \in \mathcal{C}^2$ we say that $x \in \mathbb{R}^{\mathbb{Z}}$ is **stationary** if:

$$D_2 H(x_{i-1}, x_i) + D_1 H(x_i, x_{i+1}) = 0 \quad (\forall i \in \mathbb{Z})$$

Observation 1: each $x \in \mathcal{M}(H)$ is stationary with respect to H .

Definition 5: We say that $\mathbb{R}^{\mathbb{Z}}$ is **partially ordered** by $x < x^*$ if and only if $x_i < x_i^*$ ($\forall i \in \mathbb{Z}$).

Definition 6: $x, x^* \in \mathbb{R}^{\mathbb{Z}}$ **cross**:

- (a) at $i \in \mathbb{Z}$ if $x_i = x_i^*$ and $(x_{i-1} - x_{i-1}^*)(x_{i+1} - x_{i+1}^*) < 0$.
- (b) between i and $i + 1$ if $(x_i - x_i^*)(x_{i+1} - x_{i+1}^*) < 0$.

Observation 2: According to the *transversality condiction* (H4), trajectories $x, x^* \in \mathcal{M}$ either are cross or are **comparable** ($x < x^*$, $x = x^*$ or $x > x^*$).

Definition 7: $x, x^* \in \mathbb{R}^{\mathbb{Z}}$ are:

- **α -asymptotic** if $\lim_{i \rightarrow -\infty} |x_i - x_i^*| = 0$.
- **ω -asymptotic** if $\lim_{i \rightarrow \infty} |x_i - x_i^*| = 0$.
- **asymptotic** if they are both, α -asymptotic and ω -asymptotic.

There is an action T of the group \mathbb{Z}^2 on $\mathbb{R}^{\mathbb{Z}}$ by order-preserving homeomorphisms: if $(a, b) \in \mathbb{Z}^2$ and $x \in \mathbb{R}^{\mathbb{Z}}$ then:

$$T_{(a,b)}x = x^* \quad \text{where } x_i^* = x_{i-a} + b$$

The action of $T_{(a,b)}$ on x corresponds to translation of $\text{graph}(x) \subseteq \mathbb{R}^2$ by (a, b) .

Definition 8: $x \in \mathbb{R}^{\mathbb{Z}}$ is periodic with period $(q, p) \in (\mathbb{Z} - \{0\}) \times \mathbb{Z}$ if $T_{(q,p)}x = x$.

Consequences of the hypotesis:

- From H1: We have $H(x) = H(T_{(a,b)}x)$ for every segment $(x_j, \dots, x_k), k > j$ and every $(a, b) \in \mathbb{Z}^2$. In particular $T_{(a,b)}$ maps minimal segments to minimal ones and \mathcal{M} onto itself. The continuity of H implies that \mathcal{M} is closed in $\mathbb{R}^{\mathbb{Z}}$.
- From H2: It is possible to prove that $\forall (\xi, \eta) \in \mathbb{R}^2$ and $\forall i < k$ there exists a minimal segment (x_j, \dots, x_k) with $x_j = \xi, x_k = \eta$. If (x_j, \dots, x_k) is minimal then so is every subsegment (x_l, \dots, x_m) ($l \geq j, m \leq k$).
- From H3 and H4: **Lemma 1:** Minimal trajectories cross at most once. If $x \in \mathcal{M}$ and $x^* \in \mathcal{M}$ coincide at $i \in \mathbb{Z}$ then x and x^* cross at $i \in \mathbb{Z}$.

proof:

- Second part comes from the transversality condition (H4).
- We focus on the first part. We use a contradiction argument, we assume that x and x^* cross between j and $j+1$ and between k and $k+1, j < k$. The case where one or both of the crossings take place at an integer can be treated similarly.

We consider the segments $(x_j, x_{j+1}^*, \dots, x_k^*, x_{k+1})$ and $(x_j^*, x_{j+1}, \dots, x_k, x_{k+1}^*)$. Using the ordering conditions (H3):

$$\begin{aligned} H(x_j, x_{j+1}^*, \dots, x_k^*, x_{k+1}) + H(x_j^*, x_{j+1}, \dots, x_k, x_{k+1}^*) &= H(x_j, x_{j+1}^*) + H(x_{j+1}^*, \dots, x_k^*) \\ &\quad + H(x_k^*, x_{k+1}) + H(x_j^*, x_{j+1}) \\ &\quad + H(x_{j+1}, \dots, x_k) + H(x_k, x_{k+1}^*) \\ &< H(x_j^*, x_{j+1}^*, \dots, x_{k+1}^*) \\ &\quad + H(x_j, x_{j+1}, \dots, x_{k+1}) \end{aligned}$$

This contradicts the minimality of at least one of the segments (x_j, \dots, x_{k+1}) and $(x_j^*, \dots, x_{k+1}^*)$.



Corollary: If $x \in \mathcal{M}$ and $x^* \in \mathcal{M}$ are periodic with the same period then x and x^* do not cross. If $x \in \mathcal{M}$ is periodic with minimal period (q, p) then q and p are relatively prime.

proof:

- If x and x^* have the same period and cross once, then x and x^* cross infinitely often, this contradicts the previous theorem.
- If $x \in \mathcal{M}$ is periodic with minimal period (q, p) and $(q, p) = n(a, b)$ with $(a, b) \in \mathbb{Z}^n$ and $n > 1$ then $T_{(a,b)}x \neq x$.

Since x and $T_{(a,b)}x$ do not cross, we have either $T_{(a,b)}x < x$ or $T_{(a,b)}x > x$.

By induction we get $T_{(q,p)}x < x$ (resp. $T_{(q,p)}x > x$) in contradiction to our hypothesis.



Theorem 1: $\forall (q, p) \in (\mathbb{Z} - \{0\}) \times \mathbb{Z} \exists x \in \mathcal{M}$ periodic with (q, p) .

proof:

We assume $q > 0$. The idea of the proof is to consider the set $P_{(q,p)} = \{x \in \mathbb{R}^{\mathbb{Z}} | T_{(q,p)}x = x\}$ of trajectories periodic with (q, p) and to minimize the function

$$H_{(q,p)} : P_{(q,p)} \rightarrow \mathbb{R}, \quad H_{(q,p)}(x) = H(x_0, \cdot, x_q)$$

We will prove that arbitrary segments of a minimum x of $H_{(q,p)}$ are minima, i.e. $x \in \mathcal{M}$. This will follow once we have proved that no two minima of $H_{(q,p)}$ cross.

- From properties H1 and H2: $H_{(q,p)}$ attains its infimum $H_{(q,p)}^{\min}$ on $P_{(q,p)}$.
- Assume two minima x and x^* of $H_{(q,p)}$ cross. This can only happen if $q \geq 2$.
- We define $x^+ := \max \{x_i, x_i^*\} \in \mathbb{R}^{\mathbb{Z}}$ and $x^- := \min \{x_i, x_i^*\} \in \mathbb{R}^{\mathbb{Z}}$. Then x^+ and x^- are also periodic with (q, p) .
- Using H3, we see that:

$$H_{(q,p)}(x^-) + H_{(q,p)}(x^+) \leq H_{(q,p)}(x) + H_{(q,p)}(x^*) = 2H_{(q,p)}^{\min} \quad (1)$$

which strict inequality id x and x^* cross between i and $i + 1$ for some $0 \leq i < q$.

Since $x^-, x^+ \in P_{(q,p)}$ we have equality in 1 so that x and x^* can not cross between i and $i + 1$ for any $0 \leq i < q$.

By periocity it implies that x and x^* cannot cross between i and $i + 1$ for any $i \in \mathbb{Z}$.

Hence, id x and x^* cross the cross at some $i \in \mathbb{Z}$. We may assume that $i = 1$ since $T_{(i-1,0)}x$ and $T_{(i-1,0)}x^*$ are also minimo of $H_{(q,p)}$.

In this case H4 implies that not both (x_0^-, x_1^-, x_2^-) and (x_0^+, x_1^+, x_2^+) are minimal.

- By changing x_1^- resp. x_1^+ in $P_{(q,p)}$ which coincides with x^- resp. x^+ except for $i = nq + 1$ ($n \in \mathbb{Z}$), and such that

$$H_{(q,p)}(x^-) < H_{(q,p)}(x^-) \text{ resp. } H_{(q,p)}(x^+) < H_{(q,p)}(x^+) \quad (2)$$

But 1 implies that x^- and x^+ are minima of $H_{(q,p)}$ and it contradicts 2.

- Hence x and x^* cannot cross at all. In particular, we obtain:
If x is a minimun if $H_{(q,p)}$ then x does not cross any of its translates $T_{(j,k)}x$ ($j, k \in \mathbb{Z}^2$).
- Finally, we have to prove that the last argument implies our claim: suppose $n \geq 1$, $(q^*, p^*) = n(q, p)$ and $x \in P_{(q^*, p^*)}$ is a minimun of $H_{(q^*, p^*)}$.
So the last argument appliest to x with (q, p) replaced by (q^*, p^*) .
Using the previous corollary,

$$H_{(q^*, p^*)}^{\min} = nH_{(q,p)}^{\min} \quad (3)$$

Since $H_{(q^*, p^*)}(\bar{x}) = nH_{(q,p)}(\bar{x})$ for all $\bar{x} \in P_{(q,p)}$. Equation 3 implies that every minimum x of $H_{(q,p)}$ is also a minumum of $H_{(q^*, p^*)}$ for every $(q^*, p^*) = n(q, p)$.

- In particular, if $x \in P_{(q,p)}$ is a minimun of $H_{(q,p)}$ we have:

$$(x_0, \dots, x_{nq}) \text{ is a minimal segment } \forall n \geq 1 \quad (4)$$

- Using the periodicity of x we see that 4 implies that arbitrary segments of x are minima, i.e. $x \in \mathcal{M}$.



Lemma 2: Suppose $x \in \mathcal{M}$ and $x^* \in \mathcal{M}$ are α -asymptotic (resp. ω -asymptotic) and $|x_{i+1} - x_i|$ is bounded for $i \rightarrow -\infty$ (resp. $i \rightarrow \infty$). Then x and x^* do not cross.

Theorem 2: Suppose $x \in \mathcal{M}$. Then x and $T_{(a,b)}x$ do not cross for any $(a, b) \in \mathbb{Z}^2$. Equivalently, if $x \in \mathcal{M}$ then $B_x = \{T_{(a,b)}x | (a, b) \in \mathbb{Z}^2\}$ is totally ordered.

proof:

- The case $a = 0$ is trivial, so suppose $a \neq 0$.
- We assume that $T_{(a,b)}x$ and x cross and, without loss of generality, that the crossing takes place at 0 or between 0 and 1. By previous results we know that they just cross once.
- Interchanging x and x^* if its necessary we assume that

$$x_j^* < x_j \text{ for } j < 0 \text{ and } x_j^* > x_j \text{ for } j > 0$$

- We are going to prove the case $a > 0$, the case $a < 0$ is treated in the same way.
 - The preceding equations imply: for every $j \leq 0$ the sequence $v \in \mathbb{N} \rightarrow x_{j-va} + vb$ is decreasing and for every $j > 0$ the sequence $v \in \mathbb{N} \rightarrow x_{j-va} - vb$ is decreasing.
 - We compare x to an $\bar{x} \in \mathcal{M}$ which is periodic with period (a, b) and satisfies $\bar{x}_0 < x_0$. To obtain such \bar{x} we use Theorem 1 and a translation $T_{(0,j)}$, if necessary.
 - By Lemma 1 we have $\bar{x}_j < x_j$ for $j \leq 0$ or $\bar{x}_j < x_j$ for $j \geq 0$.
 - Lets focus on the case $\bar{x}_j < x_j$ for $j \leq 0$ (the other one is analogous).
 - + For $j \leq 0$ the sequence $v \rightarrow x_{j-va} + vb$ is decreasing and bounded below by $\bar{x}_{j-va} + vb = \bar{x}_j$.
 - + Hence

$$\bar{x}_j := \lim_{v \rightarrow \infty} (x_{j-va} + vb) = \lim_{v \rightarrow \infty} (T_{(va, vb)}x)_j$$
 exists for $j \leq 0$ and $\bar{x}_{j-a} + b = \bar{x}_j$.
 - + From the periocity of $(\bar{x}_j)_{j \leq 0}$ one can conclude that both x and x^* are α -asymptotic to $(\bar{x}_j)_{j \leq 0}$ and that $|x_{i+1} - x_i|$ is bounded for $j \rightarrow \infty$.
 - + Now Lemma 2 contradicts our assumption that x and x^* cross.



Notation 1: We call \bar{B}_x the closure of $B_x := \{T_{(a,b)}x | (a, b) \in \mathbb{Z}^2\} \subset \mathbb{R}^{\mathbb{Z}}$ and $p_i : \mathbb{R}^{\mathbb{Z}} \rightarrow \mathbb{R}$ the projection $x \rightarrow x_i$ (obviousy is continuous, open and order preserving).

Lemma 3: Suppose $x \in \mathcal{M}$. Then \bar{B}_x is totally ordered. The projection p_0 maps \bar{B}_x homeomorphically onto a closed subset of \mathbb{R} .

Notation 2:

- Let G_+ denote the group of orientation-preserving homeomorphisms of the circle $S^1 = \mathbb{R}/\mathbb{Z}$.
- $\bar{G}_+ = \{f | f : \mathbb{R} \rightarrow \mathbb{R} \text{ continuous, strictly increasing, } f(x+1) = f(x) + 1\}$

Theorem 3: For every $x \in \mathcal{M}$ there exists a a circle map $f \in \bar{G}_+$ such that $x_{i+1} = f(x_i)$ ($\forall i \in \mathbb{Z}$).

proof:

- We define f on a closet set $A := p_0(\bar{B}_x)$ by $f := p_1 \circ (p_0|_{\bar{B}_x})^{-1}$ or, equivalently, by $f(x_0^*) := x_1^*$ if $x \in \bar{B}_x$.
- By Lemma 3 f is strictly increasing homeomorphism of A onto itself. Obviously f satisfies $f(t+1) = f(t) + 1$ ($\forall t \in A$).
- We extend f from A to \mathbb{R} in an affine way, i.e. if our space is $\cup(a_n, b_n)$ then $f((1-t)a_n + tb_n) := (1-t)f(a_n) + tf(b_n)$ for $t \in [0, 1]$. One easily sees that $f \in \bar{G}_+$ and $f(x_i) = f((T_{(-i,0)}x)_0) = x_{i+1}$.



Definition 9: Let $\varphi \in G_+$. The **Poincaré rotation number** $\alpha(\varphi) \in S^1$ can be interpreted geometrically as the “average angle” by which φ rotates S^1 .

Observation 3: homeomorphisms $\varphi \in G_+$ with $\alpha(\varphi)$ irrational differ radically from those with rational $\alpha(\varphi)$.

- We have $\alpha(\varphi) \in \mathbb{Q}/\mathbb{Z}$ if and only if φ has a periodic point.
- If $\alpha(\varphi)$ is irrational the limit set of every orbit $\{\varphi^i(z) | i \in \mathbb{Z}\}$ is the unique smallest closed non-empty φ -invariant subset of S^1 , i.e. the unique minimal set of φ . We denote it $Rec(\varphi)$ and the points $z \in Rec(\varphi)$ are called **recurrent**.
 - (i) Every orbit is dense in S^1 , i.e. $Rec(\varphi) = S^1$. This is the case if and only if there exists an $h \in G_+$ such that $h \circ \varphi \circ h^{-1}$ is a rotation by $\alpha(\varphi)$.
 - (ii) $Rec(\varphi)$ is a Cantor set.

Corollary: there exists a continuous map $\bar{\alpha} : \mathcal{M} \rightarrow \mathbb{R}$ with the following properties:

- (a) $\forall x \in \mathcal{M}, i \in \mathbb{Z}$ we have $|x_i - x_0 - i\bar{\alpha}(x)| < 1$, in particular

$$\bar{\alpha}(x) = \lim_{|i| \rightarrow \infty} \frac{x_i}{i}$$

- (b) If $x \in \mathcal{M}$ is periodic with period (q, p) , then $\bar{\alpha}(x) = \frac{p}{q}$.

- (c) $\bar{\alpha}$ is invariant under T ; i.e. $\bar{\alpha}(T_{(a,b)}x) = \bar{\alpha}(x) \quad (\forall (a,b) \in \mathbb{Z}^2)$.

Theorem 4: $\forall \alpha \in \mathbb{R}$ the set $\mathcal{M}_\alpha := \{x \in \mathcal{M} | \alpha(x) = \alpha\}$ is not empty.

proof:

- By Theorem 1 and the previous Corollary, we know that $\mathcal{M}_\alpha \neq \emptyset$ if $\alpha \in \mathbb{Q}$.
- Choose a sequence $\alpha_n \in \mathbb{Q}$ such that $\lim_n \alpha_n = \alpha$ and $x^n \in \mathcal{M}_{\alpha_n}$ with $x_0^n \in [0, 1]$. Then $|\alpha_n| \leq C$ for some $C > 0$. From (a) in the previous Corollary:

$$|x_i^n| \leq 2 + |i|C \quad (\forall n \in \mathbb{N}, \forall i \in \mathbb{Z})$$

- For the first results, we know that we have $x^* \in \mathcal{M}$ as a point of accumulation of the sequence x^n . By the continuity of $\bar{\alpha}$ we have $\bar{\alpha}(x^*) = \alpha$.



3 Main results

3.1 Structure of the set of minimal trajectories with irrational rotation number

Definition 10: \mathcal{M}_α can be described by a single circle map, and in particular, every $\bar{B}_x \subseteq \mathcal{M}_\alpha$ contains the set $\mathcal{M}_\alpha^{\text{rec}}$ of **recurrent trajectories** in \mathcal{M}_α ,

$$\mathcal{M}_\alpha^{\text{rec}} := \{x \in \mathcal{M}_\alpha \mid \exists k_i \in (\mathbb{Z}^2 - \{0\}) \text{ such that } x = \lim_{i \rightarrow \infty} T_{k_i} x\}$$

Theorem 5: Suppose α is irrational. Then \mathcal{M}_α is totally ordered.

proof:

- Choose $x \in \mathcal{M}_\alpha$ and $f \in \bar{G}_+$ according to the Theorem 3, i.e. $x_i = f^i(x_0)$, $\forall i \in \mathbb{Z}$.
- First we are going to show that every recurrent orbit of f is in \mathcal{M}_α , i.e. if $x_0^* \in \text{Rec}(f)$ then $x_i^* = f^i(x_0^*)$ defines an element of \mathcal{M}_α .
 - According to results about circle homeomorphisms (that one can found here [1]), there exists a sequence $(i_n, k_n) \in \mathbb{Z}^2$ such that

$$x_0^* = \lim_{n \rightarrow \infty} (x_{i_n} + k_n)$$

- Then

$$x_i^* = f^i(x_0^*) = \lim_{n \rightarrow \infty} f^i(x_{i_n} + k_n) = \lim_{n \rightarrow \infty} (x_{i_n+i} + k_n)$$

- So that $x^* = \lim_{n \rightarrow \infty} T_{(-i_n, k_n)} x \in \mathcal{M}$.
- Now suppose $x^0, x^1 \in \mathcal{M}_\alpha$ and f_0, f_1 are corresponding maps in \bar{G}_+ .
 - By a previous Lemma we know that minimal trajectories cross only one. So the preceding argument shows that no two recurrent orbits of f_0 and f_1 can intersect more than once.
 - Another result (we use this precise one along all the demonstration) of circle homeomorphisms ([1]) implies that f_0 and f_1 coincide on $\text{Rec}(f_0) = \text{Rec}(f_1)$.
 - So, if $x_0^0 \in \text{Rec}(f_0)$ and $x_0^1 \in \text{Rec}(f_1)$ the trajectories will not cross.
 - In the general case let $x_0^\pm, x_1^\pm : \mathbb{R} \rightarrow \mathbb{R}$ denote the strictly increasing maps constructed in this result on circle homeomorphisms we are working with from orbits $x_j^0 = f_0^j(x_0^0)$ and $x_j^1 = f_1^j(x_0^1)$. By the proof of the same result we know that $\exists c \in \mathbb{R}$ such that $x_0^\pm(t+c) = x_1^\pm(t)$ for all $t \in \mathbb{R}$.
 - By the definitions of x_0^\pm and x_1^\pm

$$x_0^-(j\alpha) \leq x_j^- \leq x_0^+(j\alpha)$$

and

$$x_1^-(j\alpha) \leq x_j^1 \leq x_1^+(j\alpha)$$

- Hence x^0 and x^1 could only cross if $c = 0$, i.e. if $x_0^\pm = x_1^\pm$. In this case x^0 and x^1 would be asymptotic since

$$\sum_{j \in \mathbb{Z}} |x_j^1 - x_j^0| \leq \sum_{j \in \mathbb{Z}} (x_0^+(j\alpha) - x_0^-(j\alpha)) \leq x_0^+(1) - x_0^-(0) = 1$$

- According to Lemma 2, x^0 and x^1 do not cross.



Important consequences of Theorem 5: these provide a picture of \mathcal{M}_α for α irrational.

T5.1) There exists a circle map $f \in \bar{G}_+$ with $\bar{\alpha}(f) = \alpha$ and a closed f -invariant set $A \subseteq \mathbb{R}$ such that \mathcal{M}_α consists of the orbits of f contained in A_α ; i.e. $x \in \mathcal{M}_\alpha$ if and only if $x_0 \in A_\alpha$ and $x_i = f^i(x_0)$ ($\forall i \in \mathbb{Z}$). The projection p_0 maps \mathcal{M}_α homeomorphically onto A_α .

T5.2 $x \in \mathcal{M}_\alpha$ is recurrent if and only if x_0 is a recurrent point for f , i.e. $p_0(\mathcal{M}_\alpha^{\text{rec}}) = \text{Rec}(f)$. There are a few alternatives:

- (a) $\text{Rec}(f) = \mathbb{R}$.
- (b) $\text{Rec}(f)$ is a Cantor set. In this case we also have different kinds of trayectories:
 - (b.1) Trajectories that can be approximated by elements of $\mathcal{M}_\alpha^{\text{rec}}$ from above and from below.
 - (b.2) Trajectories that can be approximated by elements of $\mathcal{M}_\alpha^{\text{rec}}$ only from above.
 - (b.3) Trajectories that can be approximated by elements of $\mathcal{M}_\alpha^{\text{rec}}$ only from below.

T5.3) Suppose that $\mathcal{M}_\alpha^{\text{rec}}$ is homeomorphic to a Cantor set. We define:

$$a_\alpha := \max \{|x_i - x_i^*| : x \in \mathcal{M}_\alpha^{\text{rec}} \text{ and } x^* \in \mathcal{M}_\alpha^{\text{rec}} \text{ asymptotic}, i \in \mathbb{Z}\}$$

If $y, z \in \mathcal{M}_\alpha^{\text{rec}}$ are not asymptotic and $y < z$ then

$$\inf (z_i - y_i) > 0, \limsup_{i \rightarrow \infty} (z_i - y_i) > a_\alpha, \text{ and } \limsup_{i \rightarrow -\infty} (z_i - y_i) > a_\alpha$$

T5.4) Every $x \in \mathcal{M}_\alpha^{\text{rec}}$ can be approximated by periodic minimal trajectories.

3.2 Structure of the set of minimal trajectories with rational rotation number

For rational α , say $\alpha = \frac{p}{q}$ with p and q relatively prime.

Notation 3:

- $P_{q,p} := \{x \in \mathbb{R}^{\mathbb{Z}} | T_{(q,p)}x = x\}$.
- $H_{q,p} := H(x_0, \dots, x_q)$.

Theorem 6: $\mathcal{M}_\alpha^{\text{rec}}$ is non-empty, closed and totally ordered. Every $x \in \mathcal{M}_\alpha^{\text{rec}}$ has a minimal period (q, p) . If $x \in \mathcal{M}_\alpha^{\text{rec}}$ then x is a minimum of $H_{q,p} : P_{q,p} \rightarrow \mathbb{R}$, in particular $H_{q,p}^{\min} (\forall x \in \mathcal{M}_\alpha^{\text{rec}})$.

proof:

- According with Theorem 1 we have $\mathcal{M}_\alpha^{\text{rec}} \neq \emptyset$.
- We also know that every $x \in \mathcal{M}_\alpha^{\text{rec}}$ has a minimal period (q, p) .
- We know that $\mathcal{M}_\alpha^{\text{rec}}$ is totally ordered.
- By contradiction, assume that exists $x \in P_{(q,p)}$ and $x^* \in \mathcal{M}_\alpha^{\text{rec}}$ such that

$$\epsilon := H(x_0^*, \dots, x_q^*) - H(x_0, \dots, x_q) > 0$$

- Choose $n \in \mathbb{N}$ so large that

$$n\epsilon > H(x_0^* - x_1) - H(x_0, x_1) + H(x_{q-1}, x_q^*) - H(x_{q-1}, x_q)$$

- Using $H(x_{q-1}, x_q^*) = H(x_{nq-1}, x_{nq}^*)$ and $H(x_{q-1}, x_q) = H(x_{nq-1}, x_{nq})$ we obtain

$$H(x_0^*, x_1, \dots, x_{nq-1}, x_{nq}^*) < H(x_0, x_1, \dots, x_{nq-1}, x_{nq}) + n\epsilon = H(x_0^*, \dots, x_{nq}^*)$$

and this contradicts the minimality of x^* .



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