Graph Theory .- Random Walks in Graphs, Ej 1, MAMME Q1 2017

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Exercise 1

- (a) Show that the mixing rate of the n-cube with a loop at every vertex is $(n-1)/n+1$ (considering adding loops to every vertex).
- (b) What is the number of steps of a random graph in Q_4^L (Q_4 with a loop at every vertex) such that the propability of being at a vertex is $1/16 \pm 10^{-3}$?

(a)

Remind the definition of the n-cube: $Q_n = K_2 \square Q_{n-1} = (K_2 \square K_2)^n$.

Let K_2^L be the graph K_2 with two extra edges, a loop over each one of the verteces. We can define the n-dimensional cube as the graph $Q_n^L = K_2^L \square Q_{n-1}^L = (K_2^L \square K_2^L)^n$. Notice that this graph is a $(n+1)$-regular graph.

We need to remember the eigenvalues of the Laplacian matrix of the hypercube (Exercise 10 from the second part of the Spectral Graph Theory chapter, the key points are written down in the appendix of this paper).

- Using the definition of M, the transition matrix : $M = D^{-1}A = \frac{1}{r}IA$ (because the graph is r-regular).
- Realise that the first eigenvalue positive in absolute value is 2, so: $n+1-2 = n-1$.
- The answer is:

$$\boxed{\frac{n-1}{n+1}}$$

(b)

The graph is 5-regular, so $d(i) = 5(\forall i)$

We know that the stationary distribution in Q_4 is $\pi = \overbrace{(1/16, ..., 1/16)}^{16 \text{ times}}$.

$$\begin{aligned} \pi = (d(1)/2m, ..., d(n)/2m) &= (5/2m, ..., 5/2m) \\ &= (1/16, ..., 1/16) \\ \Rightarrow \frac{5}{2m} = \frac{1}{16} \Rightarrow m = \frac{80}{2} = 40 & \end{aligned} \tag{1}$$

The eigenvalues are (by exercise 10, second part in spectral graph theory): $0, \pm 2, \pm 4, \pm 6, \pm 8, \pm 10, \pm 12, \pm 14, \pm 16$. In this case the second largest eigenvalue is, by (a), $2/3$.

Remind that $(v_1^T) = (\sqrt{d(1)/2m}, ..., \sqrt{d(n)/2m})$.

$$(v_l)_i (v_l)_j = \frac{d(i)}{2m} = \frac{5}{80} = \frac{1}{16} \tag{2}$$

We are going to use an inequality that appears in the demostration of the theorem 1.4.

$$|p_k(i,j) - \pi(j)|^{1/k} \leq \lambda \left| \sum_{l=2}^{n} (v_l)_i (v_l)_j \sqrt{\frac{d(i)}{d(j)}} \right|^{1/k} \tag{3}$$

We use (1), (2) and the expresion of the eigenvalue.

$$\begin{aligned} \left| \frac{1}{16} \pm 10^{-3} - \frac{1}{16} \right|^{1/k} = \left| 10^{-3} \right|^{1/k} &\leq \lambda \left| 15 \frac{1}{16} \right|^{1/k} \\ &\leq \lambda \end{aligned}$$

$$10^{-3} \leq \lambda^k = \left(\frac{2}{3} \right)^k \tag{4}$$

With $k = 18$ steps the result is equal to $0{,}0006766395 = 6{,}766395 * 10^{-4}$.

Appendix

(a)

Let $K_2^L = Q_1^L$ be the graph K_2 with a loop at every vertex. Observe that this is a 2-regular graph.

For this graph, the adjacency matrix A^L is:

$$A_1^L = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = A_1 + I_2$$

,where A_1 is the adjacency matrix of $K_2 = Q_1$ and I_2 the identity matrix of dimension 2.

In order to realize the structure of the adjacency matrix for any dimension, let's check the matrix for $K_2^L \square K_2^L = Q_2^L$:

$$A_2^L = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} A_1^L & I_2 \\ I_2 & A_1^L \end{pmatrix} = A_2 + I_{2^2}$$

,where A_2 is the adjacency matrix of Q_2 and I_{2^2} the identity matrix of dimension 2^2.

So, the structure of the adjacency matrix of A_n^L will be the next one:

$$A_n^L = \begin{pmatrix} A_{n-1}^L & I_{2^{n-1}} \\ I_{2^{n-1}} & A_{n-1}^L \end{pmatrix} = A_n + I_{2^n}$$

,where A_n is the adjacency matrix of Q_n and I_{2^n} the identity matrix of dimension 2^n.

Now we have to remind that the Laplacian Matrix of a graph is defined as $L = D - A$, and also the main result of the exercise 10 of the second part of the chapter dedicated to Spectral Graph Theory: the serie of eigenvalues for the Laplacian matrix of the hypercube starts with $0, 2, 4, 6, \ldots$. So the first non-zero eigenvalue is 2 for all dimensions.