Hamiltonian Systems

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Contents

1 Hamiltonian Equations

1.1 Notation and first definitions

1.2 Poisson Bracket

Let $F, G : U \subset \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ be $\mathcal{C}^r \quad (r \geq 1)$ functions such that $(q,p,t) \longmapsto F(q,p,t), G(q,p,t)$.

We define the **Poisson Bracket (PB)** as a $\mathcal{C}^{r-1}$ fuction $\{F,G\} : U \longrightarrow \mathbb{R}$

$$\begin{aligned}\{F,G\} &= (\nabla_z F)^T J(\nabla_z G) \\ &= (\nabla_q F)^T(\nabla_p G) - (\nabla_p F)^T(\nabla_q G) \\ &= \sum_{i=1}^{n}\Big(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\Big)\end{aligned}$$

Properties:

1. **Skew-simmetric**: $\{F,G\} = -\{G,F\}$

   In particular: $\{F,F\}$

   proof:

$$\begin{aligned}-\{F,G\} &= -\Big((\nabla_q F)^T(\nabla_p G) - (\nabla_p F)^T(\nabla_q G)\Big) \\ &= (\nabla_p F)^T(\nabla_q G) - (\nabla_q F)^T(\nabla_p G) \\ &= (\nabla_q G)^T(\nabla_p F) - (\nabla_p G)^T(\nabla_q F) \\ &= \{G,F\}\end{aligned}$$

   ♠

2. **Bilinear**: $\{\alpha F_1 + \beta F_2, G\} = \alpha\{F_1,G\} + \beta\{F_2,G\}, \quad \alpha,\beta \in \mathbb{R}$

   proof:

$$\begin{aligned}\{\alpha F_1 + \beta F_2, G\} &= \Big(\nabla_z(\alpha F_1 + \beta F_2)\Big)^T J(\nabla_z G) \\ &= \Big(\nabla_z(\alpha F_1)\Big)^T J(\nabla_z G) + \Big(\nabla_z(\beta F_2)\Big)^T J(\nabla_z G) \\ &= \alpha\Big(\nabla_z(F_1)\Big)^T J(\nabla_z G) + \beta\Big(\nabla_z(F_2)\Big)^T J(\nabla_z G) \\ &= \alpha\{F_1,G\} + \beta\{F_2,G\}\end{aligned}$$

   ♠

3. **Leibnitz rule**: $\{F_1, F_2, G\} = F_1\{F_2,G\} + F_2\{F_1,G\}$

   proof:

4. **Jacobi identity**: $\{F_1,\{F_2,F_3\}\} + \{F_3,\{F_1,F_2\}\} + \{F_2,\{F_3,F_31\}\} = 0$

   proof:

2 N-Body Problem

Let's us consider N point masses in the space ($\mathbb{R}^3$, the planar case $\mathbb{R}^2$, the coolinear case $\mathbb{R}$), whit the i-th particle having a mass $m_i > 0$ and a position vector $q_i = (q_{i1}, q_{i2}, q_{i3})^t$.

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The equations of the system comes from the **Newton's law of universal gravitation**:

$$\ddot{q}_i m_i = \sum_{\substack{j=1 \\ j\neq 1}}^{N} G m_i m_j \frac{(q_j - q_i)}{||q_j - q_i||^3} = \frac{\partial U}{\partial q_i} \qquad I = 1, 2, \cdots, N \tag{1}$$

reason why:

$$\left\| \frac{u}{||u||^3} \right\| = \frac{||u||}{||u||^3} = \frac{1}{||u||^2}$$

Where $G = 6.67408 \cdot 10^{-11} \frac{m^3}{s^2 Kg}$ is the **Gravitacional constant**.

We define the **Self potencial**, the negative of potencial energy, as:

$$U = \sum_{1 \leq i < j \leq N} \frac{G m_i m_j}{||q_j - q_i||} \tag{2}$$

3 Tópico sobre el que haré el trabajo

4 Exercises

4.1 Chapter 1: Introduction to Hamiltonian systems

Make the phase portrait of the Hamiltonian system

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= x - \frac{x^3}{3}\end{aligned}$$

and compute its Hamiltonian

Solución

Make the phase portrait of the Hamiltonian system

$$\begin{aligned}\dot{x} &= x\\ \dot{y} &= -y + x^2\end{aligned}$$

and compute its Hamiltonian

Solución

(Meyer-Hall-Offin) Let x, y, z be the usual coordinates in $\mathbb{R}^3$, $r = xi + yj + zk$, $X = \dot{x}$, $Y = \dot{y}$, $Z = \dot{z}$, $R = \dot{r} = Xi + Yj + Zk$.

1. Compute the three components of angular momentum $mr \times R$.
2. Compute the Poisson bracket of any two of the components of angular momentum and show that it is $\pm m$ times the third component of angular momentum.
3. Show that if a system admits two components of angular momentum as integrals, then the system admits all three components of angular momentum as integrals.

1. adea
2. dsa
3. dadsa

(Meyer-Hall-Offin) **A Lie algebra** A is a vector space with a product: $A \times A \to A$ that satisfies:

- **Anticommutative**: $ab \neq ba$
- **Distributive**: $a(b+c) = ab + ac$
- **Scalar associative**: $(\alpha a)b = \alpha(ab)$
- **Jacobis identity**: $a(bc) + b(ca) + c(ab) = 0, \quad a, b, c \in A, \quad \alpha \in \{\mathbb{R}, \mathbb{C}\}$

1. Show that vectors in $\mathbb{R}^3$ form a Lie algebra where the product $*$ is the cross product.
2. Show that smooth functions on an open set in $\mathbb{R}^{2n}$ form a Lie algebra, where $fg = \{f, g\}$, the Poisson bracket.
3. Show that the set of all $n \times n$ matrices, $gl(n, \mathbb{R})$, is a Lie algebra, where $AB = ABBA$, the Lie product.

1. bla
2. bla
3. bla

(Meyer-Hall-Offin) The pendulum equation is $\ddot{\theta} + \sin\theta = 0$.

1. Show that $2I = \frac{1}{2}\dot{\theta}^2 + (1\cos\theta) = \frac{1}{2}\dot{\theta}^2 + 2\sin^2(\theta/2)$ is an integral.
2. Sketch the phase portrait.
3. Make the substitution $y = \sin(\theta/2)$ to get $\dot{y}^2 = (1-y^2)(I-y^2)$. Show that when $0 < I < 1$, $y = ksn(t, k)$ solves this equation when $k^2 = I$ (Look at the definition of elliptic sine function of Section 1.6 of Meyer-Hall-Offin).

1. bla
2. bla
3. bla

(Meyer-Hall-Offin) Let $H : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ be a globally defined conservative Hamiltonian, and assume that $H(z) \to +\infty$ as $z \to +\infty$. Show that all solutions of $\dot{z} = J\nabla H(z)$ are bounded (Hint: Think like Dirichlet).

Solución

Consider a $\mathcal{C}^2$ Hamiltonian $H = H(q, p, t) : U \subset \mathbb{R}^{2n+1} \longrightarrow \mathbb{R}$ such that $det(\partial_p^2 H) \neq 0$ on U. Define $v = \partial_p H(q, p, t)$. Prove:

1.
$$\begin{aligned} \partial_{q_i} L(q,v,t) &= -\partial_{q_i} H(q,p,t) \\ \partial_{v_i} L(q,v,t) &= p_i \\ \partial_t L(q,v,t) &= -\partial_t H(q,p,t) \end{aligned}$$

2. The Lagrangian L is $\mathcal{C}^2$ and $det(\partial_v^2 L) \neq 0$.
3. The Euler-Lagrange equations associated to L and the Hamiltonian equations $\dot{q}_i = \partial_{p_i} H$, $\dot{p}_i = -\partial_{q_i} H$ are equivalent.

1. bla
2. bla
3. bla

4.2 Chapter 2: The N-body problem

Prove that the linear momentum is a first integral and that the center of mass moves with constant velocity for the 3 body problem.

Solución

Prove that if $(a_1, a_2, \cdots, a_N)$ is a central configuration with value λ:

1. For any $\tau \in \mathbb{R}$ then $(\tau a_1, \tau a_2, \cdots, \tau a_N)$ is also a central configuration with value $\frac{\lambda}{\tau^3}$.
2. If A is an orthogonal matrix, then $Aa = (Aa_1, Aa_2, \cdots, Aa_N)$ is also a central configuration with the same value λ.

1. bal bla
2. bla bla

(Meyer-Hall-Offin) Draw the complete phase portrait of the collinear Kepler problem. Integrate the collinear Kepler problem.

Solución

(Meyer-Hall-Offin) Show that $\varpi^2(\epsilon^2 - 1) = 2hc2$ for the Kepler problem. (Attention: Meyer-Hall-Offin has a typo)

Solución

(Meyer-Hall-Offin) The area of an ellipse is $\pi a^2(1-\epsilon^2)^{1/2}$, where a is the semi-major axis.We have seen in Keplers problem that area is swept out at a constant rate of $c/2$. Prove Keplers third law: The period p of a particle in a circular or elliptic orbit ($\epsilon < 1$) of the Kepler problem is $p = (\frac{2\pi}{\sqrt{\mu}})a^{3/2}$.

Solución

4.3 Chapter 3: Linear Hamiltonian systems

4.4 Chapter 6: Symplectic Transformations

4.5 Chapter 8: Geometric Theory

4.6 Chapter 9: Continuation of solutions

(Meyer-Hall-Offin) Show that the scaling used in Section 9.4 of Meyer-Hall-Offin to obtain Hills orbits for the restricted problem works for Hills lunar problem (see previous problem) also. Why does not the scaling for comets work?

Solución

Prove Lemma 9.7.1 in Meyer-Hall-Offin. Verify that formula (9.11) is the condition for an orthogonal crossing of the line of syzygy in Delaunay elements.

Solución

4.7 Chapter 10: Normal forms

4.8 Chapter 13: Stability and KAM Theory

(Meyer-Hall-Offin) Using Poincaré elements show that the continuation of the circular orbits established in Section 6.2 (Poincar orbits) are of twist type and hence stable.

Solución

5 Apendix

5.1 Complete examples

5.1.1 Harmonic oscillator

5.1.2 The Pendulum

This is a case of a one dEGREE OF FREEDOM of second order, bla bla bla bla

5.2 Needed resoults and definitions

5.2.1 Linear Algebra

matriz ortogonal

no singular

skew-simmetric

DEVOLVER A SU SITIO LAS FOTOS TAMAÑO CARNET

5.2.2 Calculus

teorema punto fijo de bla bla bla bla

Chain Rule: bla bla bla bla

proof:

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reason why:

5.3 Geometry

5.4 Funcional Analysis

5.4.1 Differential forms

5.4.2 Measure Theory

5.4.3 Ordinary Differential Equations