Twist Maps and Aubry-Mather Sets

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1 Introduction

Caso 2 dimensional

Se trataría de dar una introducción y comparar con la teoría KAM explicada por Marcel, o sea que él te podrá dar más detalles. También en el último capítulo de Meyer Offin hay una aproximación variacional.

Our objetive in this work is give an introduccion to the Aubry-Mathers theory, explain its importance in the context it is on and expose the fundamental resoutls of the topic. In order to arrive to this point, we first present resoults and definitions in the next section.

2 First definitions and resoults

Definition 1: $\mathbb{R}^{\mathbb{Z}} := \{x | x : \mathbb{Z} \to \mathbb{R}\}$, the space of bi-infinite sequences of real numbers (we consider it along with the product topology). An element $x = (x_i)_{i\in\mathbb{Z}}$ is sometimes called **trajectory**.

Definition 2: **Convergence** of a sequence $x^n \in \mathbb{R}^{\mathbb{Z}}$ to $x \in \mathbb{R}^{\mathbb{Z}}$ means that $\lim_{n\to\infty} x_i^n = x_i$ $(\forall i \in \mathbb{Z})$.

Given a dunction $H : \mathbb{R}^2 \to \mathbb{R}$ we can extend H to trajectories (or finite segments) by:

$$H(x_j, \cdots, x_k) = \sum_{i=j}^{k-1} H(x_i, x_{i+1})$$

Definition 3:

- We say that the segment $(x_j, \cdots, x_k)$ is **minimal** with respect to H if:

$$H(x_j, \cdots, x_k) = \sum_{i=j}^{k-1} H(x_i, x_{i+1}) \leq \sum_{i=j}^{k-1} H(x_i^*, x_{i+1}^*) = H(x_j^*, \cdots, x_k^*)\ (\forall (x_j^*, \cdots, x_k^*)\ :\ x_j = x_j^*, x_k = x_k^*)$$

- $x \in \mathbb{R}^{\mathbb{Z}}$ is **minimal** with respect to H) if every finite segment of x is minimal (with respect to H).
- We call $\mathcal{M} = \mathcal{M}(H)$ to the **set of minimal trajectories** (with respect to H), for $x \in \mathbb{R}^{\mathbb{Z}}$.

Hypotesis for H (we assume that it is continuous):

(H1) *Periodicity condition*: $\forall (\xi, \eta) \in \mathbb{R}^2 :\ H(\xi + 1, \eta + 1) = H(\xi, \eta)$.

(H2) *Condition at infinity*: $\lim_{|\eta|\to\infty} H(\xi, \xi + \eta) = \infty$ uniformly in ξ.

(H3) *Ordering condiction*: If $\xi_* < \xi^*, \eta_* < \eta^* \Rightarrow H(\xi_*, \eta_*) + H(\xi^*, \eta^*) < H(\xi_*, \eta^*) + H(\xi^*, \eta_*)$.

(H4) *Transversality condiction*: If $(x_{-1}, x_0, x_1) \neq (x_{-1}^*, x_0^*, x_1^*)$ are minimal and $x_0 = x_0^*$ then $(x_{-1} - x_{-1}^*)(x_1 - x_{-1}^*) < 0$.

Definition 4: If $H \in \mathcal{C}^2$ we say that $x \in \mathbb{R}^{\mathbb{Z}}$ is **stationary** if:

$$D_2 H(x_{i-1}, x_i) + D_1 H(x_i, x_{i+1}) = 0\ \ (\forall i \in \mathbb{Z})$$

Observation 1: each $x \in \mathcal{M}(H)$ is stationary with respect to H.

Definition 5: We say that $\mathbb{R}^{\mathbb{Z}}$ is **partially ordered** by $x < x^*$ if and only if $x_i < x_i^*$ $(\forall i \in \mathbb{Z})$.

Definition 6: $x, x^* \in \mathbb{R}^{\mathbb{Z}}$ **cross**:

(a) at $i \in \mathbb{Z}$ if $x_i = x_i^*$ and $(x_{i-1} - x_{i-1}^*)(x_{i+1} - x_{i+1}^*) < 0$.

(b) between i and $i+1$ if $(x_i - x_i^*)(x_{i+1} - x_{i+1}^*) < 0$.

Observation 2: Acording to the *transversality condition* (H4), trajectories $x, x^* \in \mathcal{M}$ either are cross or are **comparable** ($x < x^*$, $x = x^*$ or $x > x^*$).

Definition 7: $x, x^* \in \mathbb{R}^{\mathbb{Z}}$ are:

- α**-asymptotic** if $\lim_{i\to-\infty} |x_i - x_i^*| = 0$.
- ω**-asymptotic** if $\lim_{i\to\infty} |x_i - x_i^*| = 0$.
- **asymptotic** if they are both, α-asymptotic and ω-asymptotic.

There is an action T on the group $\mathbb{Z}^2$ on $\mathbb{R}^{\mathbb{Z}}$ by order-preserving homeomorphisms: if $(a, b) \in \mathbb{Z}^2$ and $x \in \mathbb{R}^{\mathbb{Z}}$ then:

$$T_{(a,b)}x = x^* \quad \text{where } x_i^* = x_{i-a} + b$$

The action of $T_{(a,b)}$ on x corresponds to translation of graph(x) $\subseteq \mathbb{R}^2$ by (a, b).

Definition 8: $x \in \mathbb{R}^{\mathbb{Z}}$ is periodic with period $(q, p) \in \big(\mathbb{Z} - \{0\}\big) \times \mathbb{Z}$ if $T_{(a,b)}x = x$.

Consecuences of the hypotesis:

- From H1: We have $H(x) = H(T_{(a,b)}x)$ for every segment $(x_j, \cdots, x_k), k > j$ and every $(a, b) \in \mathbb{Z}^2$. In particular $T_{(a,b)}$ maps minimal segments to minimal ones and $\mathcal{M}$ onto itself. The continuity of H implies that $\mathcal{M}$ is closed in $\mathbb{R}^{\mathbb{Z}}$.
- From H2: It is possible prove that $\forall(\xi, \eta) \in \mathbb{R}^2$ and $\forall i < k$ there exits a minimal segment $(x_j, \cdots, x_k)$ with $x_j = \xi, x_k = \eta$. If $(x_j, \cdots, x_k)$ is minimal then si us evert subsegment $(x_l, \cdots, x_m)$ $(l \geq j, m \leq k)$.
- From H3 and H4: **Lemma 1**: Minimal trajectories cross at most once. If $x \in \mathcal{M}$ and $x^* \in \mathcal{M}$ conincide at $i \in \mathbb{Z}$ then x and x^* cross at $i \in \mathbb{Z}$.

  proof:

  - Second part comes from the transversality condition (H4).
  - We focus on the first part. We use a contradiction argument, we assume that x and x^* cross between j and $j+1$ and between k and $k+1$, $j < k$. The case where one or both of the crossings take place at an integer can be treated similarly.

    We consider the segments $(x_j, x_{j+1}^*, \cdots, x_k^*, x_{k+1})$ and $(x_j^*, x_{j+1}, \cdots, x_k, x_{k+1}^*)$. Using the ordering conditions (H3):

$$\begin{aligned} H(x_j, x^*_{j+1}, \cdots, x^*_k, x_{k+1}) + H(x^*_j, x_{j+1}, \cdots, x_k, x^*_{k+1}) &= H(x_j, x^*_{j+1}) + H(x^*_{j+1}, \cdots, x^*_k) \\ &+ H(x^*_k, x_{k+1}) + H(x^*_j, x_{j+1}) \\ &+ H(x_{j+1}. \cdots, x_k) + H(x_k, x^*_{k+1}) \\ &< H(x^*_j, x^*_{j+1}, \cdots, x^*_{k+1}) \\ &+ H(x_j, x_{j+1}, \cdots, x_{k+1}) \end{aligned}$$

This contradicts the minimality of at least one of the segments $(x_j, \cdots, x_{k+1})$ and $(x^*_j, \cdots, x^*_{k+1})$.

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Corollary: If $x \in \mathcal{M}$ and $x^* \in \mathcal{M}$ are pericodic with the same period then x and x^* do not cross. If $x \in \mathcal{M}$ is periodic with minimal period (q,p) then q and p are relatively prime.

proof:

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Theorem 1: $\forall (q,p) \in \big(\mathbb{Z} - \{0\}\big) \times \mathbb{Z} \;\; \exists x \in \mathcal{M}$ periodic with (q,p).

proof:

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Lemma 2: Suppose $x \in \mathcal{M}$ and $x^* \in \mathcal{M}$ are α-asymptotic (resp. ω-asymptotic) and $|x_{i+1} - x_i|$ is bounded for $i \to -\infty$ (resp. $i \to \infty$). Then x and x^* do not cross.

Theorem 2: Suppose $x \in \mathcal{M}$. Then x and $T_{(a,b)}x$ do not cross for any $(a,b) \in \mathbb{Z}^2$.

proof:

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Notation 1: We call $\bar{B}_x$ the closure of $B_x := \{T_{(a,b)}x | (a,b) \in \mathbb{Z}^2\} \subset \mathbb{R}^{\mathbb{Z}}$ and $p_i : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}$ the projection $x \to x_i$ (obviousy is continuous, open and order preserving).

Lemma 3: Suppose $x \in \mathcal{M}$. Then $\bar{B}_x$ is totally ordered. The projection p_0 maps $\bar{B}_x$ homeomorphically onto a closed subset of $\mathbb{R}$.

Notation 2:

- Let G_+ denote the group of orientation-preserving homeomorphisms of the circle $S^1 = \mathbb{R}/\mathbb{Z}$.
- $\bar{G_+} = \{f | f : \mathbb{R} \to \mathbb{R} \;$ continuous, strictly increasing, $\; f(x+1) = f(x) + 1\}$

Theorem 3: For every $x \in \mathcal{M}$ there exists a a circle map $f \in \bar{G_+}$ such that $x_{i+1} = f(x_i) \;\; (\forall i \in \mathbb{Z})$.

proof:

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DEFINIMOS AQWUI LO QUE VIENE A SER EL ROTACIONAL MUHAHAHAHHHA

Theorem 4: $\forall \alpha \in \mathbb{R}$ the set $\mathcal{M}_\alpha := \{x \in \mathcal{M} | \alpha(x) = \alpha$ is not empty.

proof:

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3 Main resoults

3.1 Structure of the set of minimal trajectories with irrational rotation number

Definition 9: $\mathcal{M}_\alpha$ can be describe by a single circle map, and in particular, every $\bar{B}_x \subseteq \mathcal{M}_\alpha$ contains the set $\mathcal{M}_\alpha^{\text{rec}}$ of **recurrent trajectories** in $\mathcal{M}_\alpha$,

$$\mathcal{M}_\alpha^{\text{rec}} := \{x \in \mathcal{M}_\alpha | \exists k_i \in \left(\mathbb{Z}^2 - \{0\}\right) \text{ such that } x = \lim_{i\to\infty} T_{k_i} x\}$$

Theorem 5: Suppose α is irrational. Then $\mathcal{M}_\alpha$ is totally ordered.

proof:

ESTO ES LARGO DE COJONES! MUHAHAHAHAHA

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3.2 Structure of the set of minimal trajectories with rational rotation number

For rational α, say $\alpha = \frac{p}{q}$ with p and q relatively prime.

Notation 3:

- $P_{q,p} := \{x \in \mathbb{R}^{\mathbb{Z}} | T_{(q,p)} x = x\}$.
- $H_{q,p} := H(x_0, \cdots, x_q)$.

Theorem 6: $\mathcal{M}_\alpha^{\text{rec}}$ is non-empty, closed and totally ordered. Every $x \in \mathcal{M}_\alpha^{\text{rec}}$ has a minimal period (q,p). If $x \in \mathcal{M}_\alpha^{\text{rec}}$ then x is a minimun of $H_{q,p} : P_{q,p} \to \mathbb{R}$, in particular $H_{q,p}^{\min}$ $(\forall x \in \mathcal{M}_\alpha^{\text{rec}})$.

proof:

ESTO ES LARGO DE COJONES! MUHAHAHAHAHA

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4 Conclusions

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References

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