Hamiltonian Systems

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Contents

1 Hamiltonian Equations

1.1 Notation

We denote $\mathbb{F}^n$ as the space of all n-dimensional vectors (all vectors are column vectors). $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ de- notes the set of all linear transformations $\mathbb{F}^n \to \mathbb{F}^m$ (are sometimes identified with the set of all $m \times n$ matrices).

Functions are real and smooth unless otherwise stated; smooth means $\mathcal{C}^\infty$ or real analytic. If $f(x)$ s a smooth function from an open set in $\mathbb{R}^n$ to $\mathbb{R}^m$ then $\frac{\partial f}{\partial x}$ denotes the $m \times n$ Jacobian matrix:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \cdots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

If A is a matrix, then A^T denotes its transpose, A^{-1} its inverse, and A^{-T} the inverse transpose.

If $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, then $\partial f/\partial x$ is a row vector. $\nabla f = \nabla_x f = f_x$ denote the column vector $(\partial f/\partial x)^T$. Df denotes the derivative of f thought of as a map from an open set in $\mathbb{R}$ into $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. The variable t denotes a real scalar variable called time, and the symbol $\dot{f} = \partial f/\partial t$.

1.2 Hamilton's Equations

If the forces are derived from a potential function, the equations of motion of the mechanical system have many special properties, most of which follow from the fact that the equations of motion can be written as a Hamiltonian system. The Hamiltonian formalism is the natural mathe- matical structure in which to develop the theory of conservative mechanical systems.

A **Hamiltonian system** is a system of 2n ordinary differential equations of the form:

$$\begin{cases} \dot{q} = H_p \\ \dot{p} = H_q \end{cases}$$

$$\begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i}(q, p, t) \\ \dot{p}_i = \dfrac{\partial H}{\partial q_i}(q, p, t) \quad i = 1, \cdots, n \end{cases} \tag{1}$$

where $H = H(q, p, t)$ is called **the Hamiltonian**, is a smooth real-valued func- tion defined for $(q, p, t) \in \mathcal{O}$, an open set in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$.

The vectors $q = (q_1, q_2, \cdots, q_n)$ and $p = (p_1, p_2, \cdots, p_n)$ are traditionally called the **position** and **momentum** vectors, respectively, and t is called **time**, because that is what these variables represent in the classical examples. The variables q and p are said to be **conjugate variables**: p is conjugate to q. The concept of conjugate variable grows in importance as the theory develops.

The integer n is the **number of degrees of freedom** of the system.

We define the vector z as:

$$z = \begin{bmatrix} q \\ p \end{bmatrix}$$

a $2n$ vector. We define also the matrix J as the next $2n \times 2n$ skew symmetric matrix and the gradient in the next way:

$$J = J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

$$\nabla H = \begin{bmatrix} \frac{\partial H}{\partial z_1} \\ \vdots \\ \frac{\partial H}{\partial z_{2n}} \end{bmatrix} \tag{2}$$

where 0 is the is the $n \times n$ zero matrix and I_n is the $n \times n$ identity matrix. The case 2×2 matrix J_2 is a special case, it's denoted by K. In this notation the system is written as:

$$\dot{z} = J\nabla H(z,t) \tag{3}$$

details:

First:

$$\begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i}(q,p,t) \\ \dot{p}_i = \dfrac{\partial H}{\partial q_i}(q,p,t) \quad i = 1, \cdots, n \end{cases}$$

$$\begin{aligned} \nabla H(p,q,t) &= H(q_1, q_2, \cdots, q_n, p_1, p_2, \cdots, p_n, t) \\ &= \left(\frac{\partial H}{\partial q_1}, \frac{\partial H}{\partial q_2}, \cdots, \frac{\partial H}{\partial q_n} \frac{\partial H}{\partial p_1}, \frac{\partial H}{\partial p_2}, \cdots, \frac{\partial H}{\partial p_n}, \frac{\partial H}{\partial t} \right) \\ &= \left(\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p}, \frac{\partial H}{\partial t} \right) \end{aligned}$$

It's and abuse of notation, but when we write ∇H usually we are refering to $\nabla_z H$.

$$\nabla_z H = \left(\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p} \right)$$

Second:

$$\dot{z} = \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = J\nabla H(z,t) = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \cdot \left(\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p} \right) = \begin{bmatrix} \frac{\partial H}{\partial p} \\ -\frac{\partial H}{\partial q} \end{bmatrix}$$

One of the basic results from the general theory of ordinary differential equations is *the existence and uniqueness theorem*. This theorem states that for each $(z_0, t_0) \in \mathcal{O}$, there is a unique solution $z = \Phi(z_0, t_0, t)$ of 3 defined for t near t_0 that satisfies the initial condition $z_0 \cdot \Phi = \Phi(z_0, t_0, t_0)$ is defined on an open neighborhood $\mathcal{Q}$ of $(z_0, t_0, t_0) \in \mathbb{R}^{2n+2}$ into $\mathbb{R}^{2n}$.

The function $\Phi(z_0, t_0, t)$ is smooth in all its displayed arguments, and so Φ is $\mathcal{C}^\infty$ if the equations are $\mathcal{C}^\infty$, and it is analytic if the equations are analytic. $\Phi(z_0, t_0, t)$ is called **general solution**

In the special case when H is independent of t, so that $H : \mathcal{O} \to \mathbb{R}$ where $\mathcal{O}$ is an open set in $\mathbb{R}^{2n}$, the differential equations 3 are autonomous, and the Hamiltonian system is called **conservative**.

It follows that $\Phi(z_0, 0, t - t_0) = \Phi(z_0, t_0, t)$ holds, because both sides satisfy equation 3 and the same initial conditions. Usually the t_0 dependence is dropped and only $\Phi(z_0, t)$ is considered, where $\Phi(z_0, t)$ is the solution of 3 satisfying $\Phi(z_0, 0) = z_0$.

The solutions are pictured as parameterized curves in $\mathcal{O} \subset \mathbb{R}^{2n}$, , and the set $\mathcal{O}$ is called the **phase space**. By the existence and uniqueness the- orem, there is a unique curve through each point in $\mathcal{O}$; and by the uniqueness theorem, two such solution curves cannot cross in $\mathcal{O}$.

An **integral** for 3 is a smooth function $F : \mathcal{O} \to \mathbb{R}$ which is constant along the solutions of 3; i.e., $F(\Phi(z_0, t)) = F(z_0)$ is constant. The classi- cal conserved quantities of energy, momentum, etc. are integrals. The level surfaces $F^{-1}(c) \subset \mathbb{R}^{2n}$, where c is a constant, are **invariant sets**; i.e., they are sets such that if a solution starts in the set, it remains in the set.

In general, the **level sets** are are mainfolds of dimension $2n - 1$ and so with an integral F, the solutions lie on the set $F^{-1}(c)$, which is of dimension $2n - 1$. If you were so lucky as to find $2n - 1$ independent integrals, $F_1, F_2, \cdots, F_{2n-1}$, then holding all these integrals fixed would define a curve in $\mathbb{R}^{2n}$, the solution curve. In the classical sense, the problem has been integrated.

1.3 Poisson Bracket

Let $F, G : U \subset \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ be $\mathcal{C}^r$ $(r \geq 1)$ functions such that $(q, p, t) \longmapsto F(q, p, t), G(q, p, t)$.

We define the **Poisson Bracket (PB)** as a $\mathcal{C}^{r-1}$ fuction $\{F, G\} : U \longrightarrow \mathbb{R}$

$$\begin{aligned} \{F, G\} &= (\nabla_z F)^T J (\nabla_z G) \\ &= (\nabla_q F)^T (\nabla_p G) - (\nabla_p F)^T (\nabla_q G) \\ &= \sum_{i=1}^{n} \Big(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \Big) \end{aligned} \tag{4}$$

Properties:

1. **Skew-simmetric**:

$$\{F, G\} = -\{G, F\}$$

   In particular: $\{F, F\}$

   proof:

$$\begin{aligned}
-\{F,G\} &= -\Big((\nabla_q F)^T(\nabla_p G) - (\nabla_p F)^T(\nabla_q G)\Big) \\
&= (\nabla_p F)^T(\nabla_q G) - (\nabla_q F)^T(\nabla_p G) \\
&= (\nabla_q G)^T(\nabla_p F) - (\nabla_p G)^T(\nabla_q F) \\
&= \{G,F\}
\end{aligned}$$

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2. **Bilinear**:

$$\{\alpha F_1 + \beta F_2, G\} = \alpha\{F_1,G\} + \beta\{F_2,G\}, \quad \alpha,\beta \in \mathbb{R}$$

proof:

$$\begin{aligned}
\{\alpha F_1 + \beta F_2, G\} &= \Big(\nabla_z(\alpha F_1 + \beta F_2)\Big)^T J(\nabla_z G) \\
&= \Big(\nabla_z(\alpha F_1)\Big)^T J(\nabla_z G) + \Big(\nabla_z(\beta F_2)\Big)^T J(\nabla_z G) \\
&= \alpha\Big(\nabla_z(F_1)\Big)^T J(\nabla_z G) + \beta\Big(\nabla_z(F_2)\Big)^T J(\nabla_z G) \\
&= \alpha\{F_1,G\} + \beta\{F_2,G\}
\end{aligned}$$

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3. **Leibnitz rule**:

$$\{F_1, F_2, G\} = F_1\{F_2,G\} + F_2\{F_1,G\}$$

proof:

4. **Jacobi identity**:

$$\{F_1,\{F_2,F_3\}\} + \{F_3,\{F_1,F_2\}\} + \{F_2,\{F_3,F_31\}\} = 0 \tag{5}$$

proof:

Clearly $\{F,G\}$ is a smooth map $\mathcal{O} \to \mathbb{R}$.

By a common abuse of notation, let $F(t) = F(\Phi(z_0,t_0,t),t)$, where Φ is the solution of 3. By the chain rule we have that:

$$\frac{d}{dt}F(t) = \frac{\partial F}{\partial t}(\Phi(z_0,t_0,t),t) + \{F,H\}(\Phi(z_0,t_0,t),t) \tag{6}$$

Hence $dH/dt = \partial H/\partial t$.

reason why:

explanation

Theorem: Let F, G, and H be as above and independent of time t. Then:

1. F is an integral for 3 if and only if $\{F, H\} = 0$.
2. H is an integram for 3.
3. If F and G are integrals for 3, then so is $\{F, G\} = 0$.
4. $\{F, G\}$ is the time rate of change of F along the solutions of 3.

proof:

1. Directly follows directly from the definition of an integral and from 6. Let's see why:
   COMPLETAR ESTO!!
2. Follows form (1) and from the fact that the Poisson bracket is skew-symmetric, so $\{H, H\} = 0$.
   COMPLETAR ESTO!!
3. Follows from the Jacobi identity
   Completar, poner en ecuaciones
4. Follows from 6.
   Poner en ecuaciones

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In many of the examples given below, the Hamiltonian H is the total energy of a physical system; when it is, the theorem says that *energy is a conserved quantity.*

In the conservative case when H is independent of t, a critical point of H as a function (i.e., a point where the gradient of H is zero) is an equilibrium (or critical, rest, stationary) point of the system of differential equations 2 or 3

For the rest of this section, let H be independent of t. An **equilibrium point** ζ of system 3 is **stable** if:

$$\forall \epsilon > 0 \;\; \exists \delta > 0 : ||\zeta - \Phi(z_0, t)|| < \epsilon \;\; \forall t \text{ such that } ||\zeta - z_0|| < \delta$$

note that $\forall t$ means both positive and negative t, and that stability is for both the future and the past.

Theorem (Dirichlet): If ζ is a strict local minimum or maximum of H, then ζ is stable.

proof:

Without loss of generality, assume that $\zeta = 0$ and $H(0) = 0$. ACLARAR EL POR QUÉ NO PERDEMOS GENERALIDAD.

Because $H(0) = 0$ and 0 is a strict minimun of H (WHY), there is an $\eta > 0$ such that $H(z)$ is positive for $0 < ||z|| < \eta$ (we can say that H is positive definite).

Let $k = \min(\epsilon, \eta)$ and $M = \min\{H(z) : ||z|| = k\}$, so $M > 0$.

Because $H(0) = 0$ and H is continuos, there is a $\delta > 0$ such that $H(z) < M$ for $||z|| < \delta$.

$$||z|| < \delta \;\Rightarrow H(z_0) = H(\Phi(z_o, t)) < M \quad \forall t$$

We have that $||\Phi(z_0,t)|| < k \leq \epsilon \ \ \forall t$, because if not, there is a time t' when $||\Phi(z_0,t')|| = k$, and $H(\Phi(z_0,t')) \geq M$, a contradiction. RXPLICAR CON QUÉ CHOCA

♠

1.4 Relationship bethween Hamiltonian Systems and Poisson Bracket

Let $\phi(t,t_0,z_0)$ be a solution of $\dot{z} = J\nabla H(t,z)$. By an abuse of notation, let $F(t) = F(t,\phi(t,t_0,z_0))$, then:

$$\frac{d}{dt}F(t) = \frac{dF}{dt}(t,t_0,z_0) + \{F,H\}(t,t_0,z_0)$$

, hence $\frac{\partial H}{\partial t} = \frac{dH}{dt}$.

details:

We have to apply the chain rule:

$$\frac{d}{dt}F(t) =$$

COMPLETAR ESTA PARTE

Example:

1.5 The Harmonic Oscillator

The harmonic oscillator is the second-order, linear, autonomous, ordinary differential equation:

$$\ddot{x} + \omega^2 x = 0 \tag{7}$$

where ω is a positive constant. It can be written as a system of two first order equations by introducing the conjugate variable $u = \dot{x}/\omega$ and as a Hamilto- nian system by letting (energy in physical problems):

$$H = \frac{\omega}{2}(x^2 + u^2)$$

The equations become:

$$\begin{cases} \dot{x} = \omega u = \dfrac{\partial H}{\partial u} \\ \dot{u} = -\omega x = -\dfrac{\partial H}{\partial x} \end{cases} \tag{8}$$

details: Detalles de esta cosa

The variable u is a scaled velocity, and thus the x, u plane is essentially the position-velocity plane, or the phase space of physics. The basic existence and uniqueness theorem of differential equations asserts that through each point (x_0, u_0) in the plane, there is a unique solution passing through this point at any particular epoch t_0. The general solutions are given by the formula:

$$\begin{bmatrix} x(t, t_0, x_0, u_0) \\ u(t, t_0, x_0, u_0) \end{bmatrix} = \begin{bmatrix} \cos\Big(\omega(t-t_0)\Big) & -\sin\Big(\omega(t-t_0)\Big) \\ \sin\Big(\omega(t-t_0)\Big) & \cos\Big(\omega(t-t_0)\Big) \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \tag{9}$$

The solution curves are parameterized circles. The reason that one intro- duces the scaled velocity instead of using the velocity itself, as is usually done, is so that the solution curves become circles instead of ellipses. In dynamical systems the geometry of this family of curves in the plane is of prime impor- tance. Because the system is independent of time, it admits H as an integral ($\dot{H} = \omega x\dot{x} + \omega u\dot{u} = 0$). Because a solution lies in the set where $H =$ constant , which is a circle in the x, u plane, the integral alone gives the geometry of the solution curves in the plane. Note that the origin is a local minimun of H and it's stable.

We introduce the polar coordinates $r^2 = x^2 + u^2,\ \theta = tan^{-1}(u\ x)$, so 8 becomes:

$$\begin{cases} \dot{r} = 0 \\ \dot{\theta} = -\omega \end{cases} \tag{10}$$

This shows again that the solutions lie on circles about the origin because $\dot{r} = 0$. The circles are swept out with constant angular velocity.

details: Detalles del cambio de variables y demás

1.6 The Forced Nonlinear Oscillator

Consider the system:

$$\ddot{x} + f(x) = g(t) \tag{11}$$

where x is a scalar and f and g are smooth real-valued functions of a scalar variable. A mechanical system that gives rise to this equation is a spring-mass system. Here, x is the displacement of a particle of mass 1. The particle is connected to a nonlinear spring with restoring force $-f(x)$ and is subject to an external force $g(t)$. One assumes that these are the only forces acting on the particle and, in particular, that there are no velocity-dependent forces acting such as a frictional force.

An electrical system that gives rise to this equation is an LC circuit with an external voltage source. In this case, x represents the charge on a nonlinear capacitor in a series circuit that contains a linear inductor and an external electromotive force $g(t)$. In this problem, assume that there is no resistance in the circuit, and so there are no terms in $\dot{x}$.

This equation is equivalent to the system:

$$\begin{cases} \dot{x} = y = \dfrac{\partial H}{\partial y} \\ \dot{y} = -f(x) + g(t) = -\dfrac{\partial H}{\partial x} \end{cases} \tag{12}$$

where:

$$H = \frac{1}{2}y^2 + F(x) - xg(t), \qquad F(x) = \int_0^x f(s)ds$$

details: Deducción de esta cosa

Many named equations are of this form, for example:

1. The harmonic oscillator:
$$\ddot{x} + \omega^2 x = 0$$
2. The pendulum equation:
$$\ddot{\theta} + \sin\theta = 0$$
3. The forced Duffing's equation:
$$\ddot{x} + x + \alpha x^3 = \cos(\omega t)$$

details:

1. The harmonic oscillator:
$$\ddot{x} + \omega^2 x = 0$$
2. The pendulum equation:
$$\ddot{\theta} + \sin\theta = 0$$
3. The forced Duffing's equation:
$$\ddot{x} + x + \alpha x^3 = \cos(\omega t)$$

In the case when the forcing term g is absent ($g \equiv 0$), H is an integral, and the solutions lie in the level curves of H. Therefore, the phase portrait is easily obtained by plotting the level curves. In fact, these equations are integrable in the classical sense that they can be solved "up to a quadrature"; i.e., they are completely solved after one integration or quadrature.

Let $h = H(x_0, y_0)$. Solve $H = h$ for y and separate the variables to obtain:

$$y = \frac{dx}{dt} = \pm\sqrt{2h - 2F(x)}, \qquad t - t_0 = \pm \int_{x_0}^{x} \frac{d\tau}{\sqrt{2h - 2F(\tau)}}, \tag{13}$$

details:

Detalles de todo esto

1.7 The Elliptic Sine Function

details:

1.8 Linear Flow on the Torus

In order to show that the solutions of $\dot{\theta} = -\omega, \dot{\phi} = -\mu$ are dense when ω/μ is irrational, we need a few lemmas from number theory.

Lemma: Let δ be any irrational number. Then for every $\epsilon > 0$, there exist integers p, q such that

$$|q\delta - p| < \epsilon$$

proof:

- Case $0 < \delta < 1$:

  Let $N \geq 2$ be an integer and $S_N = \{s\delta - r : 1 \leq s, r \leq N\}$. For eache element if this set we have $|s\delta - r| < N$.

  Because δ is irrational, there are N^2 distinct members in the set S_N; so at least one pair is less than $N/4$ apart (If not, the total length would be greater than $(N^2 - 1)4/N > 2N$).

  Call this pair $s\delta - r$ and $s'\delta - r'$. Thus

  $$0 < |(s - s')\delta - (r - r')| < \frac{4}{N} < \frac{4}{|s - s'|}$$

  Take $N > 4/\epsilon,\ q = s - s',\ p = r' - r$ to finish this case.

- Case $-1 < \delta < 0$: apply the above to $-\delta$.
- Case $|\delta| > 1$: apply the above to δ^{-1}.

♠

Lemma: Let δ be any irrational number and ξ any real number. Then, for every $\epsilon > 0$ there exist integers p, q such that

$$|q\delta - p - \xi| < \epsilon$$

proof:

Let p', q' be as given in the previous lemma, so $\eta = q'\delta - p'$ satisfies $0 < |\eta| < \epsilon$.

$$\Rightarrow \exists m \in \mathbb{Z} : |m\eta - \xi| < \epsilon$$

The prove is finished taking $q = mq', p = mp'$.

♠

Theorem: Let ω/μ be irrational. Then the solutions curves defined by equations $\dot{\theta} = -\omega, \dot{\phi} = -\mu$ are dense in the torus.

proof:

Measure the angles in revolutions instead of radians so that the angles θ and ϕ are defined in modulo 1 instead of 2π.

The solutions of the equations through $\theta = \phi = 0$ at $t = 0$ is $\theta(t) = \omega t$ and $\phi(t) = \mu t$.

Let $\epsilon > 0$ and ξ given. Then $\theta \equiv \xi$ and $\phi \equiv 0 \pmod 1$ is an arbitrary point on the circle $\phi \equiv 0 \pmod 1$ on the torus.

Let $\delta = \omega/\mu$ and p, q as in the previous lemma. Let $\tau = q/\mu$, so $\theta(\tau) = \delta q$ and $\phi(\tau) = q$.

Thus, $|\theta(\tau) - p - \xi| < \epsilon$, but because p is an integer, this means that $\theta(\tau)$ is within ϵ and ξ; so the solution through the origin is dense on the circle $\phi \equiv 0 \pmod 1$.

The remainder of the proof follows by translation.

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1.9 Euler–Lagrange Equations

2 N-Body Problem

Let's us consider N point masses in the space ($\mathbb{R}^3$, the planar case $\mathbb{R}^2$, the coolinear case $\mathbb{R}$), whit the i-th particle having a mass $m_i > 0$ and a position vector $q_i = (q_{i1}, q_{i2}, q_{i3})^t$.

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The equations of the system comes from the **Newton's law of universal gravitation**:

$$\ddot{q}_i m_i = \sum_{\substack{j=1 \\ j\neq 1}}^{N} G m_i m_j \frac{(q_j - q_i)}{||q_j - q_i||^3} = \frac{\partial U}{\partial q_i} \qquad I = 1, 2, \cdots, N \tag{14}$$

reason why:

$$\left\| \frac{u}{||u||^3} \right\| = \frac{||u||}{||u||^3} = \frac{1}{||u||^2}$$

Where $G = 6.67408 \cdot 10^{-11} \frac{m^3}{s^2 Kg}$ is the **Gravitacional constant**.

We define the **Self potencial**, the negative of potencial energy, as:

$$U = \sum_{1 \leq i < j \leq N} \frac{G m_i m_j}{||q_j - q_i||} \tag{15}$$

reason why:

Let us check, for instance, $N = 3$ and $i = 2$.

$$U = G \frac{m_1 m_2}{||q_1 - q_2||} + G \frac{m_1 m_3}{||q_1 - q_3||} + G \frac{m_3 m_2}{||q_3 - q_2||}$$

$$U = \left(\frac{\partial U}{\partial q_{21}}, \frac{\partial U}{\partial q_{22}}, \frac{\partial U}{\partial q_{23}} \right)^T$$

Let's compute just one of the terms,

$$\begin{aligned} \frac{\partial U}{\partial q_{21}} = \cdots &= G m_1 m_2 \Big(\frac{-1}{2} \Big) \frac{-2(q_{11} - q_{21})}{||q_1 - q_2||^3} + G m_3 m_2 \Big(\frac{-1}{2} \Big) \frac{-2(q_{31} - q_{21})}{||q_3 - q_2||^3} \\ &= \frac{G m_2 m_1}{||q_2 - q_1||^3} (q_{11} - q_{21}) + \frac{G m_2 m_3}{||q_2 - q_3||^3} (q_{31} - q_{21}) \end{aligned}$$

Notation: $q = (q_1, \cdots, q_N) \in \mathbb{R}^{3N}$, $M = diag(m_1, m_1, m_1, m_2, m_2, ,_2, \cdots, m_N, m_N, m_N)$

$$\Rightarrow M\ddot{q} = \frac{\partial U}{\partial q}$$

This is a system of $3N$ 2nd o.d.e.

Let's define:

- **Collision set**: $\triangle = \{q \in \mathbb{R}^{3N} : q_i = q_j, \quad i \neq j\}$
- **Position space**: $\mathbb{R}^{3N} \setminus \triangle$

Let us pass to the hamiltonian formalism: Introduce $p = (p_1, p_2, \cdots, p_N) \in \mathbb{R}^{3N}$ by $p = M\dot{q}$, or simply $p_i = m_i \dot{q}_i$, The **linear momentum** of the i-th particle.

We have to convert the system in a new one of first order (with $6N$ equations) to obtain:

$$\begin{cases} \dot{q}_i = \dfrac{p_i}{m_i} = \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i = \displaystyle\sum_{\substack{j=1 \\ j\neq i}}^{N} G \frac{m_i m_j}{||q_j - q_i||^3}(q_j - q_i) = -\dfrac{\partial H}{\partial q_i} \end{cases} \tag{16}$$

where:

- $H(q,p) = \underbrace{T(p)}_{\text{Kinetic Energy}} - \underbrace{U(q)}_{\text{Potencial Energy}}$
- $T(p) = \sum_{i=1}^{N} \frac{||p_i||^2}{2m_i} = p^T M p = \frac{1}{2}\sum m_i ||\dot{q}_i||^2$

This is called the **Hamiltonian with $3N$ degrees of freedom (d.o.f.)** (the number og positions or momenta).

2.0.1 Kepler Problem

Introducing $p = \dot{q}$ we arrive to:

$$\begin{cases} \dot{q}_i = p_i = \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i = -\mu \dfrac{q_i}{||q||^3} = -\dfrac{\partial H}{\partial q_i} \end{cases} \tag{17}$$

where $H = \frac{||p||^2}{2} - \frac{\mu}{||q||}$, 3 d.o.f.

We degine the **Angular Momentum** as:

$$A = q \times p = q \times \dot{q}$$

reason why:

$$A = q \times p = \begin{vmatrix} i & j & k \\ q_1 & q_2 & q_3 \\ p_1 & p_2 & p_3 \end{vmatrix} = \ldots. \\ = q \times \dot{q}$$

Observation: Angular momentum is a **constant of motion**.

reason why:

$$\dot{A} = \dot{q} \times p + q \times \dot{p} = p \times p - \frac{u}{||q||^3} q \times q = 0$$

Lemma:

$$\frac{d}{dt}\left(\frac{q}{||q||}\right) = \frac{(q \times \dot{q}) \times q}{||q||^3} = \frac{A \times q}{||q||^3}$$

proof:

Let's remind a property of the vector product: $(A \times B) \times C = (C \cdot A)B - (C \cdot B)A$

$||q|| = \sqrt{q_1^2 + q_2^2 + q_3^2}$

$\dot{||q||} = \frac{q \cdot \dot{q}}{||q||^2}$

We call $u = \frac{q}{||q||}$, so we have:

$$q = ||q||u \rightarrow \dot{q} = \dot{||q||}u + ||q||\dot{u} = \frac{q \cdot \dot{q}}{||q||} \frac{q}{||q||} + ||q||\dot{u}$$

$$\dot{u} = \frac{\dot{q}}{||q||} - \frac{(q \cdot \dot{q})q}{||q||} = \frac{(q \cdot q)\dot{q} - (q \cdot \dot{q})q}{||q||^3} = \frac{(q \times \dot{q}) \times q}{||q||^3}$$

where: $A = q, \quad B = \dot{q}, \quad C = q.$

♠

Now we discuss the different cases:

- If $A = 0$, from the lemma $\frac{q}{||q||}$ = constant vector $\Rightarrow$ coolinear movement.

  $$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = a \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow \text{our hamiltonian has just 1 d.o.f. and it's easily solvable (problem 3.1)}$$

- If $A \neq 0$, p and q are othogonal to A, which is constant, so the motion takes place in the plane $A \cdot q = 0$, which is called **invariant plane**.

  We change the reference system in such a way that $A = (0, 0, 0)^T$ and the motion takes place in the orthogonal plane $q = (x, y, o)^T, \dot{q} = (\dot{x}, \dot{y}, 0)^T$.

  We can introduce **polar coordinates**:

  $$q = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix}$$

$$\dot{q} = \begin{bmatrix} \dot{r}\cos(\theta) - r\sin(\theta)\cdot \dot{theta} \\ \dot{r}\sin(\theta) + r\cos(\theta)\cdot\dot{\theta} \end{bmatrix}$$

$$A = q \times \dot{q} = \begin{bmatrix} 0 \\ 0 \\ r^2\dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}$$

Interpretation of the angular moment

Resoult: the particle sweeps the area at constant rat (*"The areolar velocity is constant"*).

reason why:

Let's compute the area of the space $\mathcal{S}$ sweept out by the particle. For that we use Fubunni, change of variables and we desceribe at some point the trajectory as $(r(t), \theta(t))$, so $d\theta = \theta' dt$ and $\theta_1 = \theta(t_1), \theta_2 = \theta(t_2)$. Remind that $c = r^2\dot{\theta}$

$$\begin{aligned} Area(\mathcal{S}) &= \int\int_{\mathcal{S}} dxdy = \int_{\theta_1}^{\theta_2} d\theta \int_0^{r(\theta)} rdr \\ &= \int_{\theta_1}^{\theta_2} \frac{r^2(\theta)}{2} d\theta \\ &= \int_{t_1}^{t_2} \frac{r^2\dot{\theta}}{2} dt = \int_{t_1}^{t_2} \frac{c}{2} dt \\ &= \frac{c}{2}(t_2 - t_1) \end{aligned}$$

3 Twist maps and Aubry-Mather Theory

Definition: We assume $0 < r_1 < r_2 < \infty$, throughout, and denote (*the finite annulus*) $\mathfrak{U} = \{(\mathcal{V}, r) : \mathcal{V} \in \mathbb{T} = \mathbb{R}/\mathbb{Z}, r_1 < r < r_2\}$. A **twist map** of the finite annulus is an orientation preserving homeomorphism $F = (\mathcal{V}F, rF)$ of $\mathfrak{U}$, that satisfies the "twist" condition: *For every $\mathcal{V}_0$ the function $\mathcal{V}F(\mathcal{V}_0, r)$is strictly monotone in r.*

Definition: The **Haussdorf metric** is defined by setting:

$$d(A, B) := \sup \{d(x, B) | x \in A\} + \sup \{d(A, y) | y \in B\}$$

for any two closed sets A, B. We refer to a limit with respect to the topology induced by the Haussdorf metric as a Haussdorf limit.

We have to make two observations:

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-

página 425

4 Exercises

4.1 Chapter 1: Introduction to Hamiltonian systems

Make the phase portrait of the Hamiltonian system

$$\begin{cases} \dot{x} = y \\ \dot{y} = x - \dfrac{x^3}{3} \end{cases}$$

and compute its Hamiltonian

- Plano de fases
- $$\begin{cases} \dot{x} = y = \dfrac{\partial H}{\partial y}(x, y) \\ \dot{y} = x - \dfrac{x^3}{3} = \dfrac{\partial H}{\partial x}(x, y) \end{cases}$$

  One one hand,

  $$\int \frac{\partial H}{\partial x} dx = \int x - \frac{x^3}{3} dx = \frac{x^2}{2} - \frac{x^4}{12} + C(y)$$

  On the other hand,

  $$\int \frac{\partial H}{\partial y} dy = \int y dy = \frac{y^2}{2} + C(x)$$

  So the Hamiltonian is:

  $$H(x, y) = \frac{x^2}{2} - \frac{x^4}{12} + \frac{y^2}{2}$$

Make the phase portrait of the Hamiltonian system

$$\begin{cases} \dot{x} = x \\ \dot{y} = -y + x^2 \end{cases}$$

and compute its Hamiltonian

- Plano de fases
- $$\begin{cases} \dot{x} = x = \dfrac{\partial H}{\partial y}(x, y) \\ \dot{y} = -y + x^2 = \dfrac{\partial H}{\partial x}(x, y) \end{cases}$$

(Meyer-Hall-Offin) Let x, y, z be the usual coordinates in $\mathbb{R}^3$, $r = xi + yj + zk$, $X = \dot{x}$, $Y = \dot{y}$, $Z = \dot{z}$, $R = \dot{r} = Xi + Yj + Zk$.

1. Compute the three components of angular momentum $mr \times R$.
2. Compute the Poisson bracket of any two of the components of angular momentum and show that it is $\pm m$ times the third component of angular momentum.
3. Show that if a system admits two components of angular momentum as integrals, then the system admits all three components of angular momentum as integrals.

1. adea
2. dsa
3. dadsa

(Meyer-Hall-Offin) **A Lie algebra** A is a vector space with a product: $A \times A \to A$ that satisfies:

- **Anticommutative**: $ab \neq ba$
- **Distributive**: $a(b + c) = ab + ac$
- **Scalar associative**: $(\alpha a)b = \alpha(ab)$
- **Jacobis identity**: $a(bc) + b(ca) + c(ab) = 0, \quad a, b, c \in A, \quad \alpha \in \{\mathbb{R}, \mathbb{C}\}$

1. Show that vectors in $\mathbb{R}^3$ form a Lie algebra where the product $*$ is the cross product.
2. Show that smooth functions on an open set in $\mathbb{R}^{2n}$ form a Lie algebra, where $fg = \{f, g\}$, the Poisson bracket.
3. Show that the set of all $n \times n$ matrices, $gl(n, \mathbb{R})$, is a Lie algebra, where $AB = ABBA$, the Lie product.

1. bla
2. bla
3. bla

(Meyer-Hall-Offin) The pendulum equation is $\ddot{\theta} + \sin\theta = 0$.

1. Show that $2I = \frac{1}{2}\dot{\theta}^2 + (1 \cos\theta) = \frac{1}{2}\dot{\theta}^2 + 2\sin^2(\theta/2)$ is an integral.

2. Sketch the phase portrait.
3. Make the substitution $y = \sin(\theta/2)$ to get $\dot{y}^2 = (1-y^2)(I-y^2)$. Show that when $0 < I < 1$, $y = ksn(t,k)$ solves this equation when $k^2 = I$ (Look at the definition of elliptic sine function of Section 1.6 of Meyer-Hall-Offin).

1. bla
2. bla
3. bla

(Meyer-Hall-Offin) Let $H : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ be a globally defined conservative Hamiltonian, and assume that $H(z) \to +\infty$ as $z \to +\infty$. Show that all solutions of $\dot{z} = J\nabla H(z)$ are bounded (Hint: Think like Dirichlet).

Solución

Consider a $\mathcal{C}^2$ Hamiltonian $H = H(q,p,t) : U \subset \mathbb{R}^{2n+1} \longrightarrow \mathbb{R}$ such that $det(\partial_p^2 H) \neq 0$ on U. Define $v = \partial_p H(q,p,t)$. Prove:

1.
$$\begin{aligned}\partial_{q_i} L(q,v,t) &= -\partial_{q_i} H(q,p,t)\\ \partial_{v_i} L(q,v,t) &= p_i\\ \partial_t L(q,v,t) &= -\partial_t H(q,p,t)\end{aligned}$$
2. The Lagrangian L is $\mathcal{C}^2$ and $det(\partial_v^2 L) \neq 0$.
3. The Euler-Lagrange equations associated to L and the Hamiltonian equations $\dot{q}_i = \partial_{p_i} H$, $\dot{p}_i = -\partial_{q_i} H$ are equivalent.

1. bla
2. bla
3. bla

4.2 Chapter 2: The N-body problem

Prove that the linear momentum is a first integral and that the center of mass moves with constant velocity for the 3 body problem.

Solución

Prove that if $(a_1, a_2, \cdots, a_N)$ is a central configuration with value λ:

1. For any $\tau \in \mathbb{R}$ then $(\tau a_1, \tau a_2, \cdots, \tau a_N)$ is also a central configuration with value $\frac{\lambda}{\tau^3}$.
2. If A is an orthogonal matrix, then $Aa = (Aa_1, Aa_2, \cdots, Aa_N)$ is also a central configuration with the same value λ.

1. bal bla
2. bla bla

(Meyer-Hall-Offin) Draw the complete phase portrait of the collinear Kepler problem. Integrate the collinear Kepler problem.

Solución

(Meyer-Hall-Offin) Show that $\varpi^2(\epsilon^2 - 1) = 2hc2$ for the Kepler problem. (Attention: Meyer-Hall-Offin has a typo)

Solución

(Meyer-Hall-Offin) The area of an ellipse is $\pi a^2(1-\epsilon^2)^{1/2}$, where a is the semi-major axis.We have seen in Keplers problem that area is swept out at a constant rate of $c/2$. Prove Keplers third law: The period p of a particle in a circular or elliptic orbit ($\epsilon < 1$) of the Kepler problem is $p = (\frac{2\pi}{\sqrt{\mu}})a^{3/2}$.

Solución

(Meyer-Hall-Offin) Let

$$K = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Then:

$$e^{Kt} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

Find a circular solution of the two-dimensional Kepler problem of the form $q = e^{Kt}a$ where a is a constant vector.

Solución

(Meyer-Hall-Offin) Assume that a particular solution of the N-body problem exists for all $t > 0$ with $h > 0$. . Show that $U \to \infty$ as $t \to \infty$. Does this imply that the distance between one pair of particles goes to infinity? (No.)

Solución

(Meyer-Hall-Offin) Hills lunar problem is defined by the Hamiltonian

$$H = \frac{||y||^2}{2} - x^T K y - \frac{1}{||x||} - \frac{1}{2}(3x_1^2 - ||x||^2)$$

where $x, y \in \mathbb{R}^2$.

1. Write the equations of motion.
2. Show that there are two equilibrium points on the x_1-axis.
3. Sketch the Hills regions for Hills lunar problem.
4. Why did Hill say that the motion of the moon was bounded? (He had the Earth at the origin, and an infinite sun infinitely far away and x was the position of the moon in this ideal system. What can you say if x and y are small?
5. Show that the linearized system at these equilibrium points are saddle-centers; i.e., it has one pair of real eigenvalues and one pair of imaginary eigenvalues.

1. bla
2. bla
3. bla
4. bla
5. bla

4.3 Chapter 3: Linear Hamiltonian systems

Let $\lambda \neq 0$ be an eigenvalue of a symplectic matrix A. Prove that $\overline{\lambda}, \lambda^{-1}$ and $\overline{\lambda}^{-1}$ are also eigenvalues of A.

Solución

Prove Lemma 3.3.6 of Meyer-Hall-Offin.

Solución

Prove Lemma 3.3.7 of Meyer-Hall-Offin.

Solución

Prove Lemma 3.3.8 of Meyer-Hall-Offin.

Solución

(Meyer-Hall-Offin) Prove that the two symplectic matrices

$$A = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \qquad B = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

are not symplectically similar.

Solución

(Meyer-Hall-Offin) Consider the system

$$M\ddot{q} + Vq = 0$$

where M and V are $n \times n$ symmetric matrices and M is positive definite. From matrix theory there is a nonsingular matrix P such that $P^T M P = I$ and an orthogonal matrix R such that $R^T(P^T V P)R = \Lambda = diag(\lambda_1, \cdots, \lambda_n)$.

Show that the above equation can be reduced to $\ddot{p} + \Lambda p = 0$. Discuss the stability and asymptotic behavior of these systems. Write equation 1 as a Hamiltonian system with Hamiltonian matrix $A = J \cdot diag(V, M^{-1})$. Use the above results to obtain a symplectic matrix T such that

$$T^{-1}AT = \begin{bmatrix} 0 & I \\ -\Lambda & 0 \end{bmatrix}$$

(Hint: Try $T = diag(PR, P^T R)$)

Solución

(Meyer-Hall-Offin) Let M and V be as in the previous exercise.

1. Show that if V has one negative eigenvalue, then some solutions of $M\ddot{q} + Vq = 0$ tend to infinity as $t \to \infty$.
2. Consider the system
$$M\ddot{q} + \nabla U(q) = 0$$
where M is positive definite and $U : \mathbb{R}^n \longrightarrow \mathbb{R}$ is smooth. Let q_0 be a critical point of U such that the Hessian of U at q_0 has one negative eigenvalue (so q_0 is not a local minimum of U). Show that q_0 is an unstable critical point for the system.

1. bla bla bla bla
2. bla bla bla bla

4.4 Chapter 6: Symplectic Transformations

(Meyer-Hall-Offin) Show that if you scale time by $t \to \mu t$, then you should scale the Hamiltonian by $H \to \mu^{-1}H$.

Solución

(Meyer-Hall-Offin) Scale the Hamiltonian of the N-body problem in rotating coordinates so that $\omega = 1$

Solución

(Meyer-Hall-Offin) Consider the Restricted 3-body problem. To investigate solutions near ∞, scale by $x \to \epsilon^{-2}x, y \to \epsilon y$. Show that the Hamiltonian becomes

$$H(x, y) = -xKy + \epsilon^3 \left(\frac{||y||^2}{2} - \frac{1}{||x||} \right) + \mathcal{O}(\epsilon^2)$$

Justify this result on physical grounds.

Solución

(Meyer-Hall-Offin) Consider the Restricted 3-body problem. To investigate solutions near one of the primaries first shift the origin to one primary. Then scale $x \to \epsilon^2 x, y \to \epsilon^{-1} y, t \to \epsilon^3 t$.

Solución

(Meyer-Hall-Offin) Write the functions $r^{2k}, r^{2k+5}\cos(5\theta)$ and $r^{2k+5}\sin(5\theta)$ in rectangular coordinates. Sketch the level curves of $r^2 + r^5\cos(5\theta)$.

Solución

(Meyer-Hall-Offin) Consider the Kepler problem written in polar coordinates. Since the angular momentum G is a first integral, set $G = c$. investigate the equation for r

$$\ddot{r} = \dot{R} = -\frac{c^2}{r^3} + \frac{\mu}{r^2}$$

using geometric methods.

Solución

(Meyer-Hall-Offin) The regularized Kepler problem has three first integrals. Denote them E_1, E_2 and A (as in Section 7.6.1 of Meyer-Hall-Offin). Compute the total algebra of integrals of the regularized Kepler problem.

Solución

4.5 Chapter 8: Geometric Theory

Consider the vector fields X and Y and their flows $\phi(t,x)$ and $\psi(t,x)$. Assume there exists an homeomorphism h which gives a topological equivalence between them. Prove that:

1. p is a fixed point of X if and only if $h(p)$ is a fixed point of Y.
2. $\gamma = \{\phi(t,x) : t \in [0,T]\}$ is a periodic orbit of X if and only if $h(\gamma)$ is a periodic orbit of Y. What can you say about the period of γ and $h(\gamma)$?
3. Prove that if h is a conjugation the periods of γ and $h(\gamma)$ are the same.

1.
2.
3.

(Meyer-Hall-Offin) Let $\{\phi_t\}$ be a smooth dynamical system; i.e., $\{\phi_t\}$ atisfies (8.5). Prove that $\phi(t,\xi) = \phi_t(\xi)$ is the general solution of an autonomous differential equation.

Solución

(Meyer-Hall-Offin) Let ψ be a diffeomorphism of $\mathbb{R}^m$; so, it defines a discrete dynamical system. A non-fixed point is called an ordinary point. So $p \in \mathbb{R}^m$ is an ordinary point if $\psi(p) \neq p$. Prove that there are local coordinates x at an ordinary point p and coordinates y at $q = \psi(p)$ such that in these local coordinates $y_1 = x_1, \cdots, y_m = x_m + 1$ (this is the analog of the flow box theorem for discrete systems).

Solución

(Meyer-Hall-Offin) Let ψ be as in Problem 2. Let p be a fixed point of ψ. The eigenvalues of $\partial_x \psi(p)$ are called the (characteristic) multipliers of p. If all the multipliers are different from $+1$, then p is called an elementary fixed point of psi. Prove that elementary fixed points are isolated.

Solución

(Meyer-Hall-Offin)

1. Let $0 < a < b$ and $\xi \in \mathbb{R}^m$be given. Show that there is a smooth nonnegative function $\gamma : \mathbb{R}^m \to \mathbb{R}$ which is identically $+1$ on the ball $||x - \xi|| < a$ and identically zero for $||x - \xi|| > b$.
2. Let $\mathcal{O}$ be any closed set in $\mathbb{R}^m$. Show that there exists a smooth, nonnegative function $\delta : \mathbb{R}^m \to \mathbb{R}$ which is zero exactly on $\mathcal{O}$.

1.
2.

(Meyer-Hall-Offin) Let $H(q_1, \cdots, q_N, p_1, \cdots, p_N)$, $\ q_i, p_i \in \mathbb{R}^3 \ \ \forall i, j$ be invariant under translation; so, $H(q_1+s, \cdots, q_N+s, p_1, \cdots, p_N) = H(q_1, \cdots, q_N, p_1, \cdots, p_N)$, $\ \forall s \in \mathbb{R}^3$. Show that total linear momentum, $L = \sum p_i$, is an integral. This is another consequence of the Noether theorem.

Solución

(Meyer-Hall-Offin) An $m \times m$ nonsingular matrix T is such that $T^2 = I$ is a discrete symmetry of (or a reflection for) $\dot{x} = f(x)$ if and only if $f(Tx) = -Tf(x), \quad \forall x \in \mathbb{R}^m$. This equation is also called reversible in this case.

1. (Meyer-Hall-Offin) Prove: If T is a discrete symmetry of $M\ddot{q} + Vq = 0$, then $\phi(t, T\xi) = T\phi(-t, \xi)$, where $\phi(t, \xi)$ is the general solution of $\dot{x} = f(x)$.
2. (Meyer-Hall-Offin) Consider the 2×2 case and let $T = diag(1, -1)$. What does $f(Tx) = -Tf(x)$ mean about the parity of f_1, f_2? Show that the first item means that a reflection of a solution in the x_1 axis is a solution.

1. bla bla
2. bla bla

4.6 Chapter 9: Continuation of solutions

(Meyer-Hall-Offin) Consider a periodic system of equations of the form $\dot{x} = f(t, x, \nu)$ where ν is a parameter and f is T-periodic in t. Let $\phi(t, \xi, \nu)$ be the general solution, $\phi(t, \xi, \nu) = \xi$.

1. Show that $\phi(t, \xi', \nu')$ is T-periodic if and only if $\phi(T, \xi', \nu') = \xi'$.
2. A T-periodic solution $\phi(t, \xi', \nu')$ can be continued if there is a smooth function $\bar{\xi}(\nu)$ such that $\bar{\xi}(\nu') = \xi'$ and $\phi(t, \bar{\xi}(\nu), \nu)$ is T-periodic. The multipliers of the T-periodic solution $\phi(t, \xi', \nu')$ are the eigenvalues of $\partial_\xi \phi(T, \xi', \nu')$. Show that a T-periodic solu- tion can be continued if all its multipliers are different from $+1$.

1. bla bla bla bla
2. bla bla bla bla

(Meyer-Hall-Offin) Consider the classical Duffing's equation $\ddot{x} + x + \gamma x^3 = A\cos(\omega t)$, which is Hamiltonian with respect to

$$H(x, y, t) = \frac{1}{2}(x^2 + y^2) + \gamma \frac{x^4}{4} - A\cos(\omega t)$$

where $y = \dot{x}$. Show that if $\omega^{-1} \neq 0, \pm 1, \pm 2, \pm 3, \cdots$, then for small forcing A and small nonlinearity γ there is a small periodic solution of the forced Duffing equation with the same period as the external forcing, $T = 2\pi/\omega$.

Solución

(Meyer-Hall-Offin) Hill's lunar problem is defined by the Hamiltonian

$$H = \frac{||y||^2}{2} - x^T K y - \frac{1}{||x||} - \frac{1}{2}(3x_1^2 - ||x||^2)$$

where $x, y \in \mathbb{R}^2$. Show that it has two equilibrium points on the x_1 axis. Linearize the equations of motion about these equilibrium points, and discuss how the Lyapunov's center and the stable manifold theorem apply.

Solución

(Meyer-Hall-Offin) Show that the scaling used in Section 9.4 of Meyer-Hall-Offin to obtain Hills orbits for the restricted problem works for Hills lunar problem (see previous problem) also. Why does not the scaling for comets work?

Solución

Prove Lemma 9.7.1 in Meyer-Hall-Offin. Verify that formula (9.11) is the condition for an orthogonal crossing of the line of syzygy in Delaunay elements.

Solución

4.7 Chapter 10: Normal forms

(Meyer-Hall-Offin) Consider a Hamiltonian of two degrees of freedom of the form (10.32) in Meyer-Hall-Offin, $x \in \mathbb{R}^4$. Let $H_0(x)$ be the Hamiltonian of two harmonic oscillators. Change to action–angle variables $(I_1, I_2, \phi_1, \phi_2)$ and let $H_0 = \omega_1 I_1 + \omega_2 I_2$. Use Theorem 10.4.1 to show that the terms in the normal form are of the form $aI_1^{p/2} I_2^{q/2} \cos(r\phi_1 + s\phi_2)$ or $bI_1^{p/2} I_2^{q/2} \sin(r\phi_1 + s\phi_2)$, where a, b are constants, , if and only if $r\omega_1 + s\omega_2 = 0$, and the terms have the d'Alembert character.

Solución

4.8 Chapter 13: Stability and KAM Theory

(Meyer-Hall-Offin) Using Poincaré elements show that the continuation of the circular orbits established in Section 6.2 (Poincar orbits) are of twist type and hence stable.

Solución

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5 Apendix

5.1 Needed resoults and definitions

5.1.1 Linear Algebra

matriz ortogonal

no singular

skew-simmetric

DEVOLVER A SU SITIO LAS FOTOS TAMAÑO CARNET

5.1.2 Calculus

Lipschitz continuity: We have two metric spaces $(X, d_X), (Y, d_Y)$ and a function $f : X \to Y$. We say that f is **Lipschitz continuous** if $\exists K \geq 0$ such that, for all $x_1, x_2 \in X$,

$$d_Y(f(x_1), f(x_2)) \leq K \cdot d_X(x_1, x_2)$$

where K is known as **Lipschitz constant**.

teorema punto fijo de bla bla bla bla

Chain Rule: Let $F = f \circ g$, or, equivalently, $F(x) = f(g(x))$ for all x. Then:

$$(f \circ g)' = (f' \circ g) \cdot g'$$

$$F'(x) = f'(g(x)) \cdot g'(x)$$

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

The two versions of the chain rule are related; if $z = f(y)$ and $y = g(x)$:

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = f'(y) \cdot g'(x) = f'(g(x)) \cdot g'(x)$$

proof: ♠

Gradient: The gradient of a scalar function ($\mathbb{R}^n \to \mathbb{R}$) $f(x_1, ..., x_n)$ denoted by ∇f or $\vec{\nabla} f$ denotes the vector differential operator. The gradient of f is defined as the unique vector field whose dot product with any unitvector v at each point x is the directional derivative off along v. That is,

$$(\nabla f(x)) \cdot v = D_v f(x)$$

- *Cartesian coordinates*: Lets focus in $\mathbb{R}^3$, where i, j, k are the standard unit vectors in the directions of axis x, y, z.

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

- *Cylindrical coordinates*: Lets focus in $\mathbb{R}^3$, where ρ is the axial distance, φ is the azimuthal or azimuth angle and z is the the axial coordinate and e_ρ, e_φ, e_z are the unit vectors pointing along the coordinate directions.

$$\nabla f(\rho,\varphi,z) = \frac{\partial f}{\partial \rho} e_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \varphi} e_\varphi + \frac{\partial f}{\partial z} = \left(\frac{\partial f}{\partial \rho}, \frac{1}{\rho} \frac{\partial f}{\partial \varphi}, \frac{\partial f}{\partial z} \right)$$

proof: ♠

- *Spherical coordinates*: Lets focus in $\mathbb{R}^3$, where r is the radial distance, φ is the azimuthal angle and θ is the polar angle, and e_r, e_φ, e_θ are local unit vectors pointing in the coordinate directions.

$$\nabla f(r,\theta,\varphi) = \frac{\partial f}{\partial r} e_r + \frac{1}{r} \frac{\partial f}{\partial \theta} e_\theta + \frac{1}{r \sin(\theta)} \frac{\partial f}{\partial \varphi} e_\varphi = \left(\frac{\partial f}{\partial r}, \frac{1}{r} \frac{\partial f}{\partial \theta}, \frac{1}{r \sin(\theta)} \frac{\partial f}{\partial \varphi} \right)$$

proof: ♠

Laplace Operator: is a differential operator given by the divergence of the gradient of a function on Euclidean space.

$$\Delta f = \nabla^2 f = \nabla \cdot \nabla f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

5.1.3 Geometry

5.1.4 Funcional Analysis

5.1.5 Differential forms

5.1.6 Measure Theory

5.2 Ordinary Differential Equations

5.2.1 Peano existence theorem

Arzelà–Ascoli Theorem:

proof:

♠

Peano existence Theorem: Let $I = [a,b] \subset \mathbb{R}$, $D \subset \mathbb{R}^n$, $f : I \times D \to \mathbb{R}^n$ be a continuos function. We need to assume that $(t_0, x_0) \in I \times D$ and $C > 0$, $T > 0$ are given such that $[t_0 - T, t_0 + T] \times B_C(x_0) \subseteq I \times D$, $(B_C(x_0) = \{x \in \mathbb{R}^n : d(x, x_0) < C\})$ then the system:

$$\begin{cases} x(t_0) = x_0 \\ x'(t) = f(t, x(t)), \quad t \in [t_0 - \gamma, t_0 + \gamma] \end{cases}$$

possesses at least one solution x, where $\gamma \leq \min\{T, C/M\}$, $M = \max_{(t,x)\in[t_0-T,t_0+T]\times B_C(x_0)} |f(t,x)|$

proof:

We are going to reduce the proof to an application of the Arzelà-Ascoli Theorem. We are going to define a set of functions that is uniformly bounded and equicontinuos, then pick a suitable sequence within that set and take the Arzelà–Ascoli Theorem to prove the existence of a convergent subsequence. The limit will be our desired function.

For each $r \in (0, \gamma)$, we define $x_r : [t_0 - r, t_o + T]$ as follows:

$$x_r(t) = \begin{cases} x_0 & t \in [t_0 - r, t_0] \\ x_0 + \int_{t_0}^{t} f(s, x_r(s-r))ds & t \geq t_0 \end{cases}$$

This formula inductively defines x_r in all of the $[t_0 - r, t_0 + T]$: if we know x_r on the interval $[t_0 - r, t_0 + kr]$, we can use this to comute the value of x_r in the interval $[t_0 - r, t_0 + (k+1)r]$.

Furthermore, also by induction on k we can prove that in fact, that $x_r(t)$ is contained within $B_r(x_0)$ for $t \in [t_0 - r, t_0 + kr]$; we need this in order for the integral formula to make sense in the first place.

This we do like this: Assume the claim is true for a k, and let $t \in [t_0 - r, t_0 + (k+1)r]$. Then:

$$\begin{aligned} ||x_0 - x_0 + \int_{t_0}^{t} f(s, x_r(s-r))ds|| &\leq \int_{t_0}^{t} ||f(s, x_r(s-r))||dr \\ &\leq \int_{t_0}^{t} M dr \\ &\leq M \cdot C/M = C \end{aligned}$$

Next we extend x_r to $[t_0 - \gamma, t_0]$ as follows:

$$x_r(t) := \begin{cases} x_0 & t \in [t_0 - r, t_0] \\ x_0 + \int_{t}^{t_0} f(s, x_r(s+r))ds & t \geq t_0 \end{cases}$$

This is a continuous extension of the "old" x_r, since both the old and the new parts of the function are continuous and coincide on all of $[t_0 - r, t_0]$ In this case, the very same arguments for well-definedness apply, and we get the same estimate of $||x_0 - x_r(t)||$ as above, which hence holds on all of $[t_0 - \gamma, t_0 + \gamma]$. Thus, by the triangle inequality, we obtain uniform boundedness as follows:

$$||x_r||_\infty \leq ||x_r - x_0||_\infty + ||x_0||_\infty$$

where we identified x_0 with the vector-valued constant function that is constantly x_0 on $[t_0 - \gamma, t_0 + \gamma]$.

We now prove equicontinuity. Let thus $|t_1 - t_2| \leq \delta$, where δ is to be specified later. There are three cases to be considered:

1. $t_1, t_2 \in [t_0 - \gamma, t_0]$
2. $t_1 \in [t_0 - \gamma, t_0], \quad t_2 \in [t_0, t_0 + \gamma]$
3. $t_1, t_2 \in [t_0, t_0 + \gamma]$

We will only do the first two cases, the third is analogous.

1.
$$\begin{aligned} ||x(t_1) - x(t_2)|| &\leq \int_{t_1}^{t_2} ||f(s, x_r(s+r))||ds \\ &\leq |t_2 - t_1| \cdot M \end{aligned}$$

Hence, choosing $\delta 1/M_\epsilon$ suffices to get $||x(t_1) - x(t_2)|| \leq \epsilon$.

2.
$$\begin{aligned} ||x(t_1) - x(t_2)|||| &\leq \int_{t_0}^{t_1} f(s, x_r(s+r))ds|| + || \leq \int_{t_0}^{t_2} f(s, x_r(s-r))ds|| \\ &\leq \int_{t_1}^{t_2} \Bigg(||f(s, x_r(s+r))|| + ||f(s, x_r(s-r))|| \Bigg) ds \\ &\leq |t_2 - t_1| 2M \end{aligned}$$

where we replaced $f(s, x_r(s+r))$ or $f(s, x_r(s-r))$ respectively with zero where it is not defined. Hence, choosing $\delta \leq 1/(2M)_\epsilon$ suffices to get $||x(t_1) - x(t_2)|| \leq \epsilon$.

Hence, we are given equicontinuity. Now we seek to apply the Arzelá–Ascoli theorem. To this end, we define

$$r_n := \frac{1}{n+N}$$

where N is sufficiently large such that $r_n \in (0, \gamma)$ for all $n \in \mathbb{N}$. Then the Arzelá–Ascoli theorem states that there exists a subsequence of the sequence x_{r_n} which converges uniformly to a certain limit function $x(t)$ (which must hence be continuous, as uniform convergence preserves continuity). Call this sequence $y_k \to x$. For all $t \in [t_0, t_0 + \gamma]$ we get the equation:

$$y_k(t) = x_0 + \int_{t_0}^{t} f\Bigg(s, y_k\Big(s - \frac{1}{n_k - N}\Big)\Bigg) ds$$

and if we pass to the limit $k \to \infty$, we obtain that x solves the problem on $[t_0, t_0 + \gamma]$. Indeed, this follows from $y_k\Big(s - \frac{1}{n_k - N}\Big) \to x(s)$ uniformly, as uniform convergence allows us to interchange limits and integration. In the same manner, we get that x solves the problem on $[t_0 - \gamma, t_0]$, and hence we have indeed constructed a solution on $[t_0 - \gamma, t_0 + \gamma]$

♠

5.2.2 Picard–Lindelöf theorem

Consider the initial values problem:

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0$$

Picard–Lindelöf Theorem: let's suppose that f is uniformly Lipschitz continuous in y (meaning the Lipschitz constant can be taken independent of t) and continuous in t, then for some value $\epsilon > 0$, there exists a unique solution $y(t)$ to the initial value problem on the interval $[t_0 - \epsilon, t_0 + \epsilon]$.

proof:

♠

5.2.3 Linear systems

We always can reduce a equation of order higher than 1 to a system of equations of order 1 in the next way:

$$x^{(n)} = f(t, x^{'}, x^{''}, \cdots, x^{n-1})$$

We introduce new variables in the next way:

$$x_1 = x,\ x_2 = x',\ \cdots,\ x_n = x^{n-1}$$

$$\begin{cases} x_1^{'} = x_2 \\ x_2^{'} = x_3 \\ \quad\vdots \\ x_n^{'} = f(t, x^{'}, x^{''}, \cdots, x^{n-1}) \end{cases}$$

Now we have a system of n equations of first orden. We can use vectoral notation, we call $x = (x_1, x_2 \cdots, x_n)^T$, $F = (F_1, F_2, \cdots, F_n)^T$ where $F_{i-1} = F_{i-1}(t, x) = x_i$, so then:

$$x' = F$$

About the inital condions problem,

The **general form** of a linear system on n equations is the next one:

$$\begin{cases} x_1^{'} = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + b_1(t) \\ x_2^{'} = a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + b_2(t) \\ \quad\vdots \\ x_n^{'} = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + b_n(t) \end{cases} \tag{18}$$

If $b_i(t) = 0 \quad i = 1, \cdots, n$ then we say that the system is **homogenous**, otherwise it's called **no homogenous**.

We can write this system in **matrix form**:

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \qquad x'(t) = \begin{pmatrix} x_1^{'}(t) \\ x_2^{'}(t) \\ \vdots \\ x_n^{'}(t) \end{pmatrix}$$

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix} \qquad b(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{pmatrix}$$

So obviosly we can write 18 as:

$$x'(t) = A(t)x(t) + b(t) \tag{19}$$

note that if the system is homogenous then $b(t) = 0$ and $x'(t) = A(t)x(t)$.

Theorem: If every component of $A(t)$ and $b(t)$ are continuous in the open interval $(a, b) \subset \mathbb{R}$, then for each $t_0 \in (a, b)$ and for all vector $x_0 \in \mathbb{R}^n$ such that $x(t_0) = x_0$, the system

$$x'(t) = A(t)x(t) + b(t)$$

has an unique solution in the interval (a, b).

proof:

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