Code Theory

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1. Memoryless resources

1.1. Sources and average word length

Definition 1: a **source** is a finite set $\mathcal{S}$ together with a set of random variables $(X_1, X_2, ...)$ whose range is $\mathcal{S}$.

If $P(X_n = \mathcal{S}_i)$ only depends on i and not on n then we say the source is **stationary** and if the X_n are independent then it's **memoryless**.

Insert example here

Definition 2: Let $\mathcal{T}$ be a finite set called **alphabet**. A map $\mathfrak{C} : \mathbb{S} \longrightarrow \mathbb{U}_{n\geq 1} T^n$ is called a **code**.

If $|T| = r$ then $\mathfrak{C}$ is a r**-ary code**.

A code extends from $\mathbb{S}$ to $T \cup T^2 \cup ...$ to $\mathbb{S} \cup \mathbb{S}^2 \cup ...$ to $T \cup T^2 \cup ...$ in obvious way.

insert example here

Definition 3: The **average word-lenght** of a code $\mathfrak{C}$ is $L(\mathfrak{C}) := \sum_{i=1}^{n} p_i l_i$ where l_i is the length of the image of the symbol of $\mathbb{S}$, which is emitted with probability p_i.

For now, we write $\mathfrak{C}$ to be the image of $\mathfrak{C}$.

1.2. Uniquely decodeable codes

Definition 4: If for any sequencies $u_1...u_n = v_1...v_m$ in $\mathfrak{C}$ implies $m = n$ and $u_i = v_i$ for $i = 1, ..., n$ then we say that $\mathfrak{C}$ is **uniquely decodeable**.

insert example here

insert example here

insert example here

Let $\mathfrak{C}_0 = \mathfrak{C}$:

- $\mathfrak{C}_n := \{\omega \in T \cup T^2 \cup ... | u\omega = v \text{ for some } u \in \mathfrak{C}_{n-1}, v \in \mathfrak{C} \text{ or } u\omega = v \text{ for some } u \in \mathfrak{C}, v \in \mathfrak{C}_{n-1}\}$
- $\mathfrak{C}_\infty := \bigcup_{k\geq 1} \mathfrak{C}_k$

Since everythig is finite either $\mathfrak{C}_m = \emptyset$ for some m and then $\mathfrak{C}_n = \emptyset$ for $n \geq m$ or it will be periodic and start repeating.

Theorem 1: $\mathfrak{C}$ is uniquely decodeable $\Longleftrightarrow \mathfrak{C} \cap \mathfrak{C}_\infty = \emptyset$.

proof: Insert proof here

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insert example here

insert example here

Definition 5: A code is a **prefix-code** if no codeword is prefix of another (ie. $\mathfrak{C}_1 = \emptyset$).

A prefix code is uniquely decodeable.

Theorem 2: (**Kraft's inequality**) $\exists$ r-ary prefix code with word lengths $l_1, l_2, .., l_q \iff$

$$\sum_{i=1}^{q} r^{-l_i} \leq 1$$

proof: Insert proof here

insert example here

Theorem 3: (**McMillan's inequality**) $\exists$ r-ary uniquely decodeable code with word lengths $l_1, l_2, .., l_q \iff$

$$\sum_{i=1}^{q} r^{-l_i} \leq 1$$

proof: Insert proof here

1.3. Optimal codes

Let be $\mathcal{S}$ a source with symbols $s_1, ..., s_q$ emitted with probabilities $p_1, ..., p_q$ and $\mathfrak{C}$ is a code which encodes s_i with a codeword length l_i. Recall $L(\mathfrak{C}) = \sum_{i=1}^{q} p_i l_i$.

Definition 6: An **optimal code** for $\mathcal{S}$ is an uniquely decodeable code $\mathfrak{D}$ such that $L(\mathfrak{C}) \geq L(\mathfrak{D})$ for all uniquel decodeable code $\mathfrak{C}$.

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Definition 7: A code constructed in this way is called a **Hoffman code**.

insert example here

Construct the r-arg Huffman code we sum together (at each step) the r smallest probabilities.

For this to work we need $q \equiv 1(r-1)$. Recall q is the number of symbols in the source. If not, then we add symbols with probabilities zero so that it is.

insert example here

Lemma 1: Every source $\mathcal{S}$ has an optimal binary code $\mathfrak{D}$ in which two of the longest codewords are **siblings**, ie. $\exists x$ (a string) such that $x_0, x_1 \in \mathfrak{D}$.

proof: Insert proof here

Theorem 4: The Huffman code is an optimal code.

proof: Insert proof here

1.4. Extension of sources

Given a source $\mathcal{S}$ we define $\mathcal{S}^n$ the source with $|\mathcal{S}|^n$ symbols, typically $s_1, ..., s_n$, emitted with $p_1, ..., p_n$ probabilities.

insert example here

2. Information and entropy

2.1. Definitions

Definition 1: the **information** coveyed by a source is a function $I : S \to [0, \infty)$ where S is a **source** [1] with the properties:

- $I(s_i)$ is a decreasing function of the propability p_i, with $I(s_i) = 0$ if $p_i = 1$.
- $I(s_i s_j) = I(s_i) + I(s_j)$, ie.the information geined by two symbols is the sum of the information obtained from each where the source has symbols $s_1, ..., s_q$ emitted with probabilities $p_1, ..., p_q$.

Lemma 1: $I(s_i) = -\log_r p_i$ for some r.

proof: Insert proof here

Definition 2: The **r-ary entropy** $H_r(S)$ of a source S is the average information coveyed by S.

$$H_r(S) := -\sum_{i=1}^{q} p_i \log_r p_i$$

, by convenction $x \log_r x$ evaluated at 0 is 0.

Insert five examples

2.2. Properties of the entropy funcion

Theorem 1: $H_r(S) \leq \log_r q$ with equality if and only iff S is the source where each symbol is emitted with probability $1/q$.

proof: Insert proof here

Theorem 2: $H_r(S) \leq L(C)$ for unique decodeable code C.

proof: Insert proof here

2.3. Shannon-Fano Code

Let S be the source with symbols s_i and probabilities p_i. Let $l_i := \lceil \log_r 1/p_i \rceil$.

Then: $\sum_{i=1}^{q} r^{-l_i} \leq \sum r^{-\log_r 1/p_i} = \sum p_i = 1$

[1] A **source** is a finite set S together with a sequence of random variables X_i whose range is S

Definition 3: by Kraft exists a prefix code with woed length $l_1, l_2, ..., l_1$. This code is called **Shannon-Fano code**.

Inert example here

Lemma 2: For the Shannon-Fano code C: $H_r(S) \leq L(C) < H_r(S) + 1$.

proof: Insert proof here

2.4. Product of sources

Let S and T be two memoryless sources, S with symbols s_i and probabilities p_i and T with symbols t_j and probabilities q_j.

Definition 4: The **product source** $S \times T$ is a source with symbols $s_i t_j$ and probabilities $p_i q_j$.

Theorem 3: $H_r(S \times T) = H_r(S) + H_r(T)$.

proof: Insert proof here

Corollary 1: $H_r(S^n) = nH_r(S)$.

Theorem 4: Noiseless Coding The average word length L_n of an optiml code of S^n satisfies:

$$\frac{L_n}{n} \longrightarrow H_r(S)\,, n \to \infty$$

proof: Insert proof here

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some examples

2.5. Markov Chains

Definition 4: A **Markov Chain** is a sequency of random variables where X_{n+1} depends only for X_n.

$$P(X_{n+1} = s_j | X_n = s_j) = p_{i,j}$$

This can be represented in a direct graph and also by a matrix $P := (p)_{i,j}$.

Suppose u_0 is the vector which describes the initial distribution, ie. the i-th coordinate of u_0 is probability we start at s_i. Probability of beeing in the i-th state after r steps is the i-th coordinate of $u_0 P^r$.

Theorem 5: if $\exists r \in \mathbb{N}$ such that P^r has no zero entries, then $u_0 P^r \longrightarrow u$, as $n \to \infty$.

Definition 5: This vector u is called the **stationary distribution**. It is normalised eigenvector of P^t with eigenvalue 1, ie. $u_j = \sum_i p_{i,j} u_i$ and $\sum_j u_j = 1$.

Definition 6: If P is the matrix of a Markov Chain and $\exists r$ such that P^r has non zero entries then we say that the Markov Chain is **regular**.

2.6. Sources with memory

Suppose S is a Markov Chain source with random variables $X_1, X_2, ...$ such that

$$P(X_{n+1} = s_j | X_n = s_j) = p_{i,j}$$

Definition 7: S is **not memoryless**, but it is stationary.

Theorem 6: suppose S is a regular Markov Chain source with stationary distribution $u = (u_1, ..., u_)$. Let S' be the stationary memoryless source with the same source elements as S (where s_i is emmitted with probability w_i). Then:

$$H_r(S) \leq H_r(S')$$

proof: Insert proof here

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3. Information channels

3.1. Channel matrix

Let $\mathcal{A}$ be a stationary memoryless source with random variables $X_1, X_2, ...$ where $P(X_n = a_i) = p_i$ for $a_i \in \mathcal{A}$.

Suppose we transmit $\mathcal{A}$ through a channel Γ.

Let $\mathcal{B}$ be a source with random variables $Y_1, Y_2, ...$ where $P(Y_n = b_j) = q_j$

For b_j emerging from the channel:

$$\mathcal{A} \overset{\Gamma}{\longrightarrow} \mathcal{B}$$

Definition 1: The **channel** is defined by a matrix (p_{ij}) where $p_{ij} = P(X_n = b_j | X_n = a_i)$ the probability we recieve b_j given that a_i was sent, p_{ij}**-forward probabilities**. The **backwards probabilities** are $q_{ij} = P(X_n = a_i | Y_n = b_j)$ and **joint prabablities** $r_{ij} = P(X_n = a_i, Y_n = b_j)$

insert example here

inser example here (binary eraure channel)

3.2. System Entropies and mutual information

Definition 2: We define the **input entropy** as:

$$H(\mathcal{A}) := -\sum_i p_i \log(p_i)$$

Definition 3: We define the **output entropy** as:

$$H(\mathcal{B}) := -\sum_j q_j \log(q_j)$$

We suppress the r (base) in the $\log_r$ but it's always the same for every one.

Given that we have recived $b_j \in \mathcal{B}$, $H(\mathcal{A}|Y_n = b_j) = -\sum_i q_{ij} \log(q_{ij})$.

This is relling us the average information of $\mathcal{A}$ knowing that $Y_n = b_j$.

If $H(\mathcal{A}|Y_n = b_j) = 0$ then $\exists m$ such that $q_{ij} = 0$ for all $i \neq m$ and $q_{ij} = 1$ if $i = m$, ie. $P(X_n = a_m | Y_n = b_j) = 1$, ie. if we recieve b_j then we know that a_m was sent.

If $H(\mathcal{A}|Y_n = b_j) = H(\mathcal{A})$ then we learn nothing about $\mathcal{A}$ when we recieve b_j and this occurs when $q_{ij} = P(X_n = a_i | Y_n = b_j) = P(X_n = a_i) = p_i$.

Definition 4: Averaging over $b_j \in \mathcal{B}$ we get the **condicional entropy**:

$$H(\mathcal{A}|\mathcal{B}) := -\sum_j P(Y_n = b_j) H(\mathcal{A}|Y_n = b_j) = -\sum_{i,j} q_j q_{ij} \log q_{ij}$$

Similary:

$$H(\mathcal{B}|\mathcal{A}) := -\sum_{i,j} p_i p_{ij} \log p_{ij}$$

Definition 5: The **joint entropy**:

$$H(\mathcal{A},\mathcal{B}) := -\sum_{i,j} r_{ij} \log r_{ij}$$

insert example here

Theorem 1: For sources $\mathcal{A}$ and $\mathcal{B}$:

$$H(\mathcal{A},\mathcal{B}) = H(\mathcal{A}|\mathcal{B}) + H(\mathcal{B}) = H(\mathcal{B}|\mathcal{A}) + H(\mathcal{A})$$

proof: Insert proof here

Definition 6: We define the **mutual information** as the amount of information about $\mathcal{A}$ we have learnt from $\mathcal{B}$ and vice-versa:

$$I(\mathcal{A},\mathcal{B}) := H(\mathcal{B}) - H(\mathcal{B}|\mathcal{A}) = H(\mathcal{A}) - H(\mathcal{A}|\mathcal{B})$$

If $H(\mathcal{A}) = H(\mathcal{A}|\mathcal{B})$ then $\mathcal{B}$ tells us nothing about $\mathcal{A}$, so $I(\mathcal{A},\mathcal{B}) = 0$. This is an unrialiable channel and useless as a mean of communication.

If $H(\mathcal{A}|\mathcal{B}) = 0$ then knowing $\mathcal{B}$ we know everythin about $\mathcal{A}$, so $I(\mathcal{A},\mathcal{B}) = H(\mathcal{A})$. This is the perfect situation because when we recive something, we know exactly what was sent.

insert example here

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11. P-adic codes

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11.1. P-adic numbers

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