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Facultad de Matemáticas y Estadística
MAMMEE
Hamiltonian Systems

Twist Maps and Aubry-Mather Sets

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Thickness of a graph $\mathcal{G}$

Theorical Thickness $\theta(\mathcal{G})$

Minimum number of planar graphs into a which a graph can be descomposed.

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Theorical Thickness $\theta(\mathcal{G})$

Minimum number of planar graphs into a which a graph can be descomposed.

Geometric Thickness $\bar{\theta}(\mathcal{G})$

Smallest value of k such that we can assign a planar point locations to the vertices of $\mathcal{G}$, represent each edge of $\mathcal{G}$ as a line segment, and assign each edge to one of k layers so that no two edges on the same layer cross.

Thickness of a graph

Key difference

Geometric thickness requires that the vertex placements be consistent at all layers and that straight-line adges be used, whereas graph-theorical thickness imposes no consistency requirement between layers.

Graph-theorical thickness for all complete graphs

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Upper Bounds

Theorem 1

$$\bar{\theta}(k_n) \leq \lceil n/4 \rceil$$

Upper Bounds

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Roadmap to the proof:

- Assume that n is multiple of four, $n = 2k$ with (k even), show that n vertices can be arranged in two "rings" of k vertices, so K_n can be embedded using $k/2$ layers and with no edges on the same layer crossing.

Upper Bounds

Roadmap to the proof:

- Use the vertices of the inner ring to form a regular k-gon and considerer the opposite vertices con create a zigzag path. This path has exactly one diagonal connecting diametrically opposite points.
- By continuity, we can replace by a suitably chosen common end points the infinite end points of a collection of parellel rays. Thus forming an outer ring of k vertices.
- The figure can be perturbed by moving slightly the inner ring. None of the diagonals of the polygon comprinsing the outher ring intersect the polygon comprising the inner ring.
- It's straighforward to verify that this is indeed a descomposition of the edges of k_n into $k/2 = n/4$ layers.

$\blacksquare$

Lower Bounds

Theorem 2

For all $n \geq 1$

$$\bar{\theta}(k_n) \geq \max_{1 \leq x \leq n/2} \frac{\binom{n}{2} - 2\binom{x}{2} - 3}{3n - 2x - 7}$$

Lower Bounds

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For all $n \geq 1$

$$\bar{\theta}(k_n) \geq \max_{1 \leq x \leq n/2} \frac{\binom{n}{2} - 2\binom{x}{2} - 3}{3n - 2x - 7}$$

In particular, for $n \geq 12$

$$\bar{\theta}(k_n) \geq \left\lceil \frac{3-\sqrt{7}}{2} n + 0.342 \right\rceil \geq \left\lceil \frac{n}{5.646} + 0.342 \right\rceil$$

Geometric Thickness of K_{15}

Theorem 3

$$\bar{\theta}(k_{15}) = 4$$

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Proof: We divide the proof in 3 cases.

- **Case I:** 3 points in the convex hull. Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ convex hull points and let $\mathcal{A}_1, \mathcal{B}_1$ and $\mathcal{C}_1$ be the point furtherst from edge $\mathcal{BC}$ (respectively $\mathcal{AC}, \mathcal{AB}$ within triangle $\mathcal{ABC}$).
  - **Lemma 1:** The edge $\mathcal{AA}_1$ will appear in every triangulation of $\mathcal{S}$.

  Let $\mathcal{A}_2, \mathcal{B}_2$ and $\mathcal{C}_2$ be the point next furtherst from edge $\mathcal{BC}$ (respectively $\mathcal{AC}, \mathcal{AB}$ within triangle $\mathcal{ABC}$.
  - **Lemma 2:** At least one of the edges $\mathcal{A}_1\mathcal{A}_2$ or $\mathcal{AA}_2$ will appear in every triangulation of $\mathcal{S}$.

Theorem 3

$$\bar{\theta}(k_{15}) = 4$$

Proof: We divide the problem in 3 cases.

- **Case II:** 4 points in the convex hull.
  - Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ the convex hull vertices.
  - Assume triangle $\mathcal{DAB}$ has at least one point of $\mathcal{S}$ in his interior (if not switch $\mathcal{A}$ and $\mathcal{C}$ and let $\mathcal{A}_1$ be the point inside furthest from line $\mathcal{DB}$).
  - By Lemma 2, the edge $\mathcal{A}\mathcal{A}_1$ must appear in every triangulation of $\mathcal{S}$.
  - Since every triangulation has 38 edges three triangulations can account at most 104 edges.

Geometric Thickness of K_{15}

Theorem 3

$$\bar{\theta}(k_{15}) = 4$$

Proof: We divide the problem in 3 cases.

- **Case III:** 5 or more points in the convex hull.
  - Let h be the number of points in the convex hull.
  - A triangulation of $\mathcal{S}$ will have $42 - h$ edges, and all hull edges must be in each triangulation.
  - The total number of edges in three triangulatinos is at most $3(42 - 2h) + h = 126 - -5h$, that is at most 101 for $h \geq 5$.

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Geometric Thickness of Complete Bipartite Graphs

Theorem 4

For the complete bipartite graph $K_{a,b}$

$$\left\lceil \frac{ab}{2a+2b-4} \right\rceil \leq \theta(K_{a,b}) \leq \bar{\theta}(k_{a,b}) \leq \left\lceil \frac{\min(a,b)}{2} \right\rceil$$

Geometric Thickness of Complete Bipartite Graphs

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Proof:

- First inequality: from Euler's formula since a bipartite graph which is planar with $a+b$ vertices can have at most $2a+2b-4$ edges
- Second inequality: assume that $a \leq b$ and a is even. Draw b vertices in a horizontal line, with $a/2$ red vertives above the line and $a/2$ vertices below. Each layer consists of all edges connecting the blue vertices with one red vertex from above the line and one red vertex from below.

Geometric Thickness of Complete Bipartite Graphs

Corollary 1:

For any integer b, $\bar{\theta}(k_{a,b}) = \theta(k_{a,b})$ provided:

$$a > \begin{cases} \frac{(b-2)^2}{2} & \text{if b is even} \\ (b-1)(b-2) & \text{if b is odd} \end{cases}$$

Corollary 1:

For any integer b, $\bar{\theta}(k_{a,b}) = \theta(k_{a,b})$ provided:

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Proof: If $a > b$, the leftmost and rightmost quantities in the expresion of the last theorem will be equal provided $ab/(2a+2b-4) > (b-2)/2$ if b is even, or provided $ab/(2a+2b-4) > (b-1)/2$ f b is odd. By simplifing this inequality holds.

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The first 100 values of $\bar{\theta}$

Bibliography

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