Hamiltonian Systems

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1 Hamiltonian Equations

1.1 Notation

We denote $\mathbb{F}^n$ as the space of all n-dimensional vectors (all vectors are column vectors). $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ de- notes the set of all linear transformations $\mathbb{F}^n \to \mathbb{F}^m$ (are sometimes identified with the set of all $m \times n$ matrices).

Functions are real and smooth unless otherwise stated; smooth means $\mathcal{C}^\infty$ or real analytic. If $f(x)$ s a smooth function from an open set in $\mathbb{R}^n$ to $\mathbb{R}^m$ then $\frac{\partial f}{\partial x}$ denotes the $m \times n$ Jacobian matrix:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \cdots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

If A is a matrix, then A^T denotes its transpose, A^{-1} its inverse, and A^{-T} the inverse transpose.

If $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, then $\partial f / \partial x$ is a row vector. $\nabla f = \nabla_x f = f_x$ denote the column vector $(\partial f / \partial x)^T$. Df denotes the derivative of f thought of as a map from an open set in $\mathbb{R}$ into $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. The variable t denotes a real scalar variable called time, and the symbol $\dot{f} = \partial f / \partial t$.

1.2 Hamilton's Equations

If the forces are derived from a potential function, the equations of motion of the mechanical system have many special properties, most of which follow from the fact that the equations of motion can be written as a Hamiltonian system. The Hamiltonian formalism is the natural mathe- matical structure in which to develop the theory of conservative mechanical systems.

A **Hamiltonian system** is a system of 2n ordinary differential equations of the form:

$$\begin{aligned} \dot{q} &= H_p \\ \dot{p} &= H_q \end{aligned}$$

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i}(q, p, t) \\ \dot{p}_i &= \frac{\partial H}{\partial q_i}(q, p, t) \quad i = 1, \cdots, n \end{aligned} \tag{1}$$

where $H = H(q, p, t)$ is called **the Hamiltonian**, is a smooth real-valued func- tion defined for $(q, p, t) \in \mathcal{O}$, an open set in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$.

The vectors $q = (q_1, q_2, \cdots, q_n)$ and $p = (p_1, p_2, \cdots, p_n)$ are traditionally called the **position** and **momentum** vectors, respectively, and t is called **time**, because that is what these variables represent in the classical examples. The variables q and p are said to be **conjugate variables**: p is conjugate to q. The concept of conjugate variable grows in importance as the theory develops.

The integer n is the **number of degrees of freedom** of the system.

We define the vector z as:

$$z = \begin{bmatrix} q \\ p \end{bmatrix}$$

a $2n$ vector. We define also the matrix J as the next $2n \times 2n$ skew symmetric matrix and the gradient in the next way:

$$J = J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

$$\nabla H = \begin{bmatrix} \frac{\partial H}{\partial z_1} \\ \vdots \\ \frac{\partial H}{\partial z_{2n}} \end{bmatrix}$$

where 0 is the is the $n \times n$ zero matrix and I_n is the $n \times n$ identity matrix. The case 2×2 matrix J_2 is a special case, it's denoted by K. In this notation the system is written as:

$$\dot{z} = J \nabla H(z, t) \tag{2}$$

reason why:

Explanation muhahahahahaha

One of the basic results from the general theory of ordinary differential equations is *the existence and uniqueness theorem*. This theorem states that for each $(z_0, t_0) \in \mathcal{O}$, there is a unique solution $z = \Phi(z_0, t_0, t)$ of 2 defined for t near t_0 that satisfies the initial condition $z_0 \cdot \Phi = \Phi(z_0, t_0, t_0)$ is defined on an open neighborhood $\mathcal{Q}$ of $(z_0, t_0, t_0) \in \mathbb{R}^{2n+2}$ into $\mathbb{R}^{2n}$.

The function $\Phi(z_0, t_0, t)$ is smooth in all its displayed arguments, and so Φ is $\mathcal{C}^\infty$ if the equations are $\mathcal{C}^\infty$, and it is analytic if the equations are analytic. $\Phi(z_0, t_0, t)$ is called **general solution**

In the special case when H is independent of t, so that $H : \mathcal{O} \to \mathbb{R}$ where $\mathcal{O}$ is an open set in $\mathbb{R}^{2n}$, the differential equations 2 are autonomous, and the Hamiltonian system is called **conservative**.

It follows that $\Phi(z_0, 0, t - t_0) = \Phi(z_0, t_0, t)$ holds, because both sides satisfy equation 2 and the same initial conditions. Usually the t_0 dependence is dropped and only $\Phi(z_0, t)$ is considered, where $\Phi(z_0, t)$ is the solution of 2 satisfying $\Phi(z_0, 0) = z_0$.

The solutions are pictured as parameterized curves in $\mathcal{O} \subset \mathbb{R}^{2n}$, , and the set $\mathcal{O}$ is called the **phase space**. By the existence and uniqueness the- orem, there is a unique curve through each point in $\mathcal{O}$; and by the uniqueness theorem, two such solution curves cannot cross in $\mathcal{O}$.

An **integral** for 2 is a smooth function $F : \mathcal{O} \to \mathbb{R}$ which is constant along the solutions of 2; i.e., $F(\Phi(z_0, t)) = F(z_0)$ is constant. The classi- cal conserved quantities of energy, momentum, etc. are integrals. The level surfaces $F^{-1}(c) \subset \mathbb{R}^{2n}$, where c is a constant, are **invariant sets**; i.e., they are sets such that if a solution starts in the set, it remains in the set.

In general, the **level sets** are are mainfolds of dimension $2n - 1$ and so with an integral F, the solutions lie on the set $F^{-1}(c)$, which is of dimension $2n - 1$. If you were so lucky as to find $2n - 1$

independent integrals, $F_1, F_2, \cdots, F_{2n-1}$, then holding all these integrals fixed would define a curve in $\mathbb{R}^{2n}$, the solution curve. In the classical sense, the problem has been integrated.

1.3 Poisson Bracket

Let $F, G : U \subset \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ be $\mathcal{C}^r \quad (r \geq 1)$ functions such that $(q, p, t) \longmapsto F(q, p, t), G(q, p, t)$.

We define the **Poisson Bracket (PB)** as a $\mathcal{C}^{r-1}$ fuction $\{F, G\} : U \longrightarrow \mathbb{R}$

$$\begin{aligned} \{F, G\} &= (\nabla_z F)^T J(\nabla_z G) \\ &= (\nabla_q F)^T (\nabla_p G) - (\nabla_p F)^T (\nabla_q G) \\ &= \sum_{i=1}^{n} \Big(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \Big) \end{aligned} \tag{3}$$

Properties:

1. **Skew-simmetric**: $\{F, G\} = -\{G, F\}$

   In particular: $\{F, F\}$

   proof:

$$\begin{aligned} -\{F, G\} &= -\Big((\nabla_q F)^T (\nabla_p G) - (\nabla_p F)^T (\nabla_q G) \Big) \\ &= (\nabla_p F)^T (\nabla_q G) - (\nabla_q F)^T (\nabla_p G) \\ &= (\nabla_q G)^T (\nabla_p F) - (\nabla_p G)^T (\nabla_q F) \\ &= \{G, F\} \end{aligned}$$

   ♠

2. **Bilinear**: $\{\alpha F_1 + \beta F_2, G\} = \alpha\{F_1, G\} + \beta\{F_2, G\}, \quad \alpha, \beta \in \mathbb{R}$

   proof:

$$\begin{aligned} \{\alpha F_1 + \beta F_2, G\} &= \Big(\nabla_z(\alpha F_1 + \beta F_2) \Big)^T J(\nabla_z G) \\ &= \Big(\nabla_z(\alpha F_1) \Big)^T J(\nabla_z G) + \Big(\nabla_z(\beta F_2) \Big)^T J(\nabla_z G) \\ &= \alpha \Big(\nabla_z(F_1) \Big)^T J(\nabla_z G) + \beta \Big(\nabla_z(F_2) \Big)^T J(\nabla_z G) \\ &= \alpha\{F_1, G\} + \beta\{F_2, G\} \end{aligned}$$

   ♠

3. **Leibnitz rule**: $\{F_1, F_2, G\} = F_1\{F_2, G\} + F_2\{F_1, G\}$

   proof:

4. **Jacobi identity**: $\{F_1, \{F_2, F_3\}\} + \{F_3, \{F_1, F_2\}\} + \{F_2, \{F_3, F_3 1\}\} = 0$

   proof:

2 N-Body Problem

Let's us consider N point masses in the space ($\mathbb{R}^3$, the planar case $\mathbb{R}^2$, the coolinear case $\mathbb{R}$), whit the i-th particle having a mass $m_i > 0$ and a position vector $q_i = (q_{i1}, q_{i2}, q_{i3})^t$.

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The equations of the system comes from the **Newton's law of universal gravitation**:

$$\ddot{q}_i m_i = \sum_{\substack{j=1 \\ j\neq 1}}^{N} G m_i m_j \frac{(q_j - q_i)}{||q_j - q_i||^3} = \frac{\partial U}{\partial q_i} \qquad I = 1, 2, \cdots, N \tag{4}$$

reason why:

$$\left\| \frac{u}{||u||^3} \right\| = \frac{||u||}{||u||^3} = \frac{1}{||u||^2}$$

Where $G = 6.67408 \cdot 10^{-11} \frac{m^3}{s^2 Kg}$ is the **Gravitacional constant**.

We define the **Self potencial**, the negative of potencial energy, as:

$$U = \sum_{1 \leq i < j \leq N} \frac{G m_i m_j}{||q_j - q_i||} \tag{5}$$

3 Tópico sobre el que haré el trabajo

4 Exercises

4.1 Chapter 1: Introduction to Hamiltonian systems

Make the phase portrait of the Hamiltonian system

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= x - \frac{x^3}{3}\end{aligned}$$

and compute its Hamiltonian

Solución

Make the phase portrait of the Hamiltonian system

$$\begin{aligned}\dot{x} &= x \\ \dot{y} &= -y + x^2\end{aligned}$$

and compute its Hamiltonian

Solución

(Meyer-Hall-Offin) Let x, y, z be the usual coordinates in $\mathbb{R}^3$, $r = xi + yj + zk$, $X = \dot{x}$, $Y = \dot{y}$, $Z = \dot{z}$, $R = \dot{r} = Xi + Yj + Zk$.

1. Compute the three components of angular momentum $mr \times R$.
2. Compute the Poisson bracket of any two of the components of angular momentum and show that it is $\pm m$ times the third component of angular momentum.
3. Show that if a system admits two components of angular momentum as integrals, then the system admits all three components of angular momentum as integrals.

1. adea
2. dsa
3. dadsa

(Meyer-Hall-Offin) **A Lie algebra** A is a vector space with a product: $A \times A \to A$ that satisfies:

- **Anticommutative**: $ab \neq ba$
- **Distributive**: $a(b+c) = ab + ac$
- **Scalar associative**: $(\alpha a)b = \alpha(ab)$
- **Jacobis identity**: $a(bc) + b(ca) + c(ab) = 0, \quad a, b, c \in A, \quad \alpha \in \{\mathbb{R}, \mathbb{C}\}$

1. Show that vectors in $\mathbb{R}^3$ form a Lie algebra where the product $*$ is the cross product.
2. Show that smooth functions on an open set in $\mathbb{R}^{2n}$ form a Lie algebra, where $fg = \{f, g\}$, the Poisson bracket.
3. Show that the set of all $n \times n$ matrices, $gl(n, \mathbb{R})$, is a Lie algebra, where $AB = ABBA$, the Lie product.

1. bla
2. bla
3. bla

(Meyer-Hall-Offin) The pendulum equation is $\ddot{\theta} + \sin\theta = 0$.

1. Show that $2I = \frac{1}{2}\dot{\theta}^2 + (1 \cos\theta) = \frac{1}{2}\dot{\theta}^2 + 2\sin^2(\theta/2)$ is an integral.
2. Sketch the phase portrait.
3. Make the substitution $y = \sin(\theta/2)$ to get $\dot{y}^2 = (1 - y^2)(I - y^2)$. Show that when $0 < I < 1$, $y = ksn(t, k)$ solves this equation when $k^2 = I$ (Look at the definition of elliptic sine function of Section 1.6 of Meyer-Hall-Offin).

1. bla
2. bla
3. bla

(Meyer-Hall-Offin) Let $H : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ be a globally defined conservative Hamiltonian, and assume that $H(z) \to +\infty$ as $z \to +\infty$. Show that all solutions of $\dot{z} = J\nabla H(z)$ are bounded (Hint: Think like Dirichlet).

Solución

Consider a $\mathcal{C}^2$ Hamiltonian $H = H(q, p, t) : U \subset \mathbb{R}^{2n+1} \longrightarrow \mathbb{R}$ such that $det(\partial_p^2 H) \neq 0$ on U. Define $v = \partial_p H(q, p, t)$. Prove:

1.
$$\begin{aligned}\partial_{q_i} L(q,v,t) &= -\partial_{q_i} H(q,p,t)\\ \partial_{v_i} L(q,v,t) &= p_i\\ \partial_t L(q,v,t) &= -\partial_t H(q,p,t)\end{aligned}$$

2. The Lagrangian L is $\mathcal{C}^2$ and $det(\partial_v^2 L) \neq 0$.
3. The Euler-Lagrange equations associated to L and the Hamiltonian equations $\dot{q}_i = \partial_{p_i} H$, $\dot{p}_i = -\partial_{q_i} H$ are equivalent.

1. bla
2. bla
3. bla

4.2 Chapter 2: The N-body problem

Prove that the linear momentum is a first integral and that the center of mass moves with constant velocity for the 3 body problem.

Solución

Prove that if $(a_1, a_2, \cdots, a_N)$ is a central configuration with value λ:

1. For any $\tau \in \mathbb{R}$ then $(\tau a_1, \tau a_2, \cdots, \tau a_N)$ is also a central configuration with value $\frac{\lambda}{\tau^3}$.
2. If A is an orthogonal matrix, then $Aa = (Aa_1, Aa_2, \cdots, Aa_N)$ is also a central configuration with the same value λ.

1. bal bla
2. bla bla

(Meyer-Hall-Offin) Draw the complete phase portrait of the collinear Kepler problem. Integrate the collinear Kepler problem.

Solución

(Meyer-Hall-Offin) Show that $\varpi^2(\epsilon^2 - 1) = 2hc2$ for the Kepler problem. (Attention: Meyer-Hall-Offin has a typo)

Solución

(Meyer-Hall-Offin) The area of an ellipse is $\pi a^2(1-\epsilon^2)^{1/2}$, where a is the semi-major axis.We have seen in Keplers problem that area is swept out at a constant rate of $c/2$. Prove Keplers third law: The period p of a particle in a circular or elliptic orbit ($\epsilon < 1$) of the Kepler problem is $p = (\frac{2\pi}{\sqrt{\mu}})a^{3/2}$.

Solución

4.3 Chapter 3: Linear Hamiltonian systems

4.4 Chapter 6: Symplectic Transformations

4.5 Chapter 8: Geometric Theory

4.6 Chapter 9: Continuation of solutions

(Meyer-Hall-Offin) Show that the scaling used in Section 9.4 of Meyer-Hall-Offin to obtain Hills orbits for the restricted problem works for Hills lunar problem (see previous problem) also. Why does not the scaling for comets work?

Solución

Prove Lemma 9.7.1 in Meyer-Hall-Offin. Verify that formula (9.11) is the condition for an orthogonal crossing of the line of syzygy in Delaunay elements.

Solución

4.7 Chapter 10: Normal forms

4.8 Chapter 13: Stability and KAM Theory

(Meyer-Hall-Offin) Using Poincaré elements show that the continuation of the circular orbits established in Section 6.2 (Poincar orbits) are of twist type and hence stable.

Solución

5 Apendix

5.1 Complete examples

5.1.1 Harmonic oscillator

5.1.2 The Pendulum

This is a case of a one dEGREE OF FREEDOM of second order, bla bla bla bla

5.2 Needed resoults and definitions

5.2.1 Linear Algebra

matriz ortogonal

no singular

skew-simmetric

DEVOLVER A SU SITIO LAS FOTOS TAMAÑO CARNET

5.2.2 Calculus

teorema punto fijo de bla bla bla bla

Chain Rule: Let $F = f \circ g$, or, equivalently, $F(x) = f(g(x))$ for all x. Then:

$$(f \circ g)' = (f' \circ g) \cdot g'$$

$$F'(x) = f'(g(x)) \cdot g'(x)$$

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

The two versions of the chain rule are related; if $z = f(y)$ and $y = g(x)$:

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = f'(y) \cdot g'(x) = f'(g(x)) \cdot g'(x)$$

proof: ♠

Gradient: The gradient of a scalar function ($\mathbb{R}^n \to \mathbb{R}$) $f(x_1, ..., x_n)$ denoted by ∇f or $\overrightarrow{\nabla} f$ denotes the vector differential operator. The gradient of f is defined as the unique vector field whose dot product with any unitvector v at each point x is the directional derivative off along v. That is,

$$(\nabla f(x)) \cdot v = D_v f(x)$$

- *Cartesian coordinates*: Lets focus in $\mathbb{R}^3$, where i, j, k are the standard unit vectors in the directions of axis x, y, z.

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

- *Cylindrical coordinates*: Lets focus in $\mathbb{R}^3$, where ρ is the axial distance, φ is the azimuthal or azimuth angle and z is the the axial coordinate and e_ρ, e_φ, e_z are the unit vectors pointing along the coordinate directions.

$$\nabla f(\rho,\varphi,z) = \frac{\partial f}{\partial \rho} e_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \varphi} e_\varphi + \frac{\partial f}{\partial z} = \left(\frac{\partial f}{\partial \rho}, \frac{1}{\rho}\frac{\partial f}{\partial \varphi}, \frac{\partial f}{\partial z} \right)$$

  proof: ♠

- *Spherical coordinates*: Lets focus in $\mathbb{R}^3$, where r is the radial distance, φ is the azimuthal angle and θ is the polar angle, and e_r, e_φ, e_θ are local unit vectors pointing in the coordinate directions.

$$\nabla f(r,\theta,\varphi) = \frac{\partial f}{\partial r} e_r + \frac{1}{r}\frac{\partial f}{\partial \theta} e_\theta + \frac{1}{r\sin(\theta)}\frac{\partial f}{\partial \varphi} e_\varphi = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{1}{r\sin(\theta)}\frac{\partial f}{\partial \varphi} \right)$$

  proof: ♠

Laplace Operator: is a differential operator given by the divergence of the gradient of a function on Euclidean space.

$$\Delta f = \nabla^2 f = \nabla \cdot \nabla f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

proof:

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reason why:

5.2.3 Geometry

5.3 Funcional Analysis

5.3.1 Differential forms

5.3.2 Measure Theory

5.3.3 Ordinary Differential Equations

Peano existence theorem: