

Hamiltonian Systems

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1 Hamiltonian Equations

1.1 Notation

We denote \mathbb{F}^n as the space of all n -dimensional vectors (all vectors are column vectors). $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ denotes the set of all linear transformations $\mathbb{F}^n \rightarrow \mathbb{F}^m$ (are sometimes identified with the set of all $m \times n$ matrices).

Functions are real and smooth unless otherwise stated; smooth means \mathcal{C}^∞ or real analytic. If $f(x)$ is a smooth function from an open set in \mathbb{R}^n to \mathbb{R}^m then $\frac{\partial f}{\partial x}$ denotes the $m \times n$ Jacobian matrix:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \cdots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

If A is a matrix, then A^T denotes its transpose, A^{-1} its inverse, and A^{-T} the inverse transpose.

If $f : \mathbb{R}^n \rightarrow \mathbb{R}$, then $\partial f / \partial x$ is a row vector. $\nabla f = \nabla_x f = f_x$ denote the column vector $(\partial f / \partial x)^T$. Df denotes the derivative of f thought of as a map from an open set in \mathbb{R} into $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. The variable t denotes a real scalar variable called time, and the symbol $\dot{f} = \partial f / \partial t$.

1.2 Hamilton's Equations

If the forces are derived from a potential function, the equations of motion of the mechanical system have many special properties, most of which follow from the fact that the equations of motion can be written as a Hamiltonian system. The Hamiltonian formalism is the natural mathematical structure in which to develop the theory of conservative mechanical systems.

A **Hamiltonian system** is a system of $2n$ ordinary differential equations of the form:

$$\begin{cases} \dot{q} = H_p \\ \dot{p} = H_q \end{cases} \quad \begin{cases} \dot{q}_i = \frac{\partial H}{\partial p_i}(q, p, t) \\ \dot{p}_i = -\frac{\partial H}{\partial q_i}(q, p, t) \quad i = 1, \dots, n \end{cases} \quad (1)$$

where $H = H(q, p, t)$ is called the **Hamiltonian**, is a smooth real-valued function defined for $(q, p, t) \in \mathcal{O}$, an open set in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$.

The vectors $q = (q_1, q_2, \dots, q_n)$ and $p = (p_1, p_2, \dots, p_n)$ are traditionally called the **position** and **momentum** vectors, respectively, and t is called **time**, because that is what these variables represent in the classical examples. The variables q and p are said to be **conjugate variables**: p is conjugate to q . The concept of conjugate variable grows in importance as the theory develops.

The integer n is the **number of degrees of freedom** of the system.

We define the vector z as:

$$z = \begin{bmatrix} q \\ p \end{bmatrix}$$

a $2n$ vector. We define also the matrix J as the next $2n \times 2n$ skew symmetric matrix and the gradient in the next way:

$$J = J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

$$\nabla H = \begin{bmatrix} \frac{\partial H}{\partial z_1} \\ \vdots \\ \frac{\partial H}{\partial z_{2n}} \end{bmatrix} \quad (2)$$

where 0 is the $n \times n$ zero matrix and I_n is the $n \times n$ identity matrix. The case 2×2 matrix J_2 is a special case, it's denoted by K . In this notation the system is written as:

$$\dot{z} = J \nabla H(z, t) \quad (3)$$

reason why:

Explanation muhahahahahaha

One of the basic results from the general theory of ordinary differential equations is *the existence and uniqueness theorem*. This theorem states that for each $(z_0, t_0) \in \mathcal{O}$, there is a unique solution $z = \Phi(z_0, t_0, t)$ of 3 defined for t near t_0 that satisfies the initial condition $z_0 \cdot \Phi = \Phi(z_0, t_0, t_0)$ is defined on an open neighborhood \mathcal{Q} of $(z_0, t_0) \in \mathbb{R}^{2n+2}$ into \mathbb{R}^{2n} .

The function $\Phi(z_0, t_0, t)$ is smooth in all its displayed arguments, and so Φ is \mathcal{C}^∞ if the equations are \mathcal{C}^∞ , and it is analytic if the equations are analytic. $\Phi(z_0, t_0, t)$ is called **general solution**

In the special case when H is independent of t , so that $H : \mathcal{O} \rightarrow \mathbb{R}$ where \mathcal{O} is an open set in \mathbb{R}^{2n} , the differential equations 3 are autonomous, and the Hamiltonian system is called **conservative**.

It follows that $\Phi(z_0, 0, t - t_0) = \Phi(z_0, t_0, t)$ holds, because both sides satisfy equation 3 and the same initial conditions. Usually the t_0 dependence is dropped and only $\Phi(z_0, t)$ is considered, where $\Phi(z_0, t)$ is the solution of 3 satisfying $\Phi(z_0, 0) = z_0$.

The solutions are pictured as parameterized curves in $\mathcal{O} \subset \mathbb{R}^{2n}$, and the set \mathcal{O} is called the **phase space**. By the existence and uniqueness theorem, there is a unique curve through each point in \mathcal{O} ; and by the uniqueness theorem, two such solution curves cannot cross in \mathcal{O} .

An **integral** for 3 is a smooth function $F : \mathcal{O} \rightarrow \mathbb{R}$ which is constant along the solutions of 3; i.e., $F(\Phi(z_0, t)) = F(z_0)$ is constant. The classical conserved quantities of energy, momentum, etc. are integrals. The level surfaces $F^{-1}(c) \subset \mathbb{R}^{2n}$, where c is a constant, are **invariant sets**; i.e., they are sets such that if a solution starts in the set, it remains in the set.

In general, the **level sets** are manifolds of dimension $2n - 1$ and so with an integral F , the solutions lie on the set $F^{-1}(c)$, which is of dimension $2n - 1$. If you were so lucky as to find $2n - 1$ independent integrals, $F_1, F_2, \dots, F_{2n-1}$, then holding all these integrals fixed would define a curve in \mathbb{R}^{2n} , the solution curve. In the classical sense, the problem has been integrated.

1.3 Poisson Bracket

Let $F, G : U \subset \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ be \mathcal{C}^r ($r \geq 1$) functions such that $(q, p, t) \longmapsto F(q, p, t), G(q, p, t)$.

We define the **Poisson Bracket (PB)** as a \mathcal{C}^{r-1} function $\{F, G\} : U \longrightarrow \mathbb{R}$

$$\begin{aligned} \{F, G\} &= (\nabla_z F)^T J (\nabla_z G) \\ &= (\nabla_q F)^T (\nabla_p G) - (\nabla_p F)^T (\nabla_q G) \\ &= \sum_{i=1}^n \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right) \end{aligned} \tag{4}$$

Properties:

1. **Skew-symmetric:**

$$\{F, G\} = -\{G, F\}$$

In particular: $\{F, F\}$

proof:

$$\begin{aligned} -\{F, G\} &= -\left((\nabla_q F)^T (\nabla_p G) - (\nabla_p F)^T (\nabla_q G) \right) \\ &= (\nabla_p F)^T (\nabla_q G) - (\nabla_q F)^T (\nabla_p G) \\ &= (\nabla_q G)^T (\nabla_p F) - (\nabla_p G)^T (\nabla_q F) \\ &= \{G, F\} \end{aligned}$$



2. **Bilinear:**

$$\{\alpha F_1 + \beta F_2, G\} = \alpha \{F_1, G\} + \beta \{F_2, G\}, \quad \alpha, \beta \in \mathbb{R}$$

proof:

$$\begin{aligned} \{\alpha F_1 + \beta F_2, G\} &= \left(\nabla_z (\alpha F_1 + \beta F_2) \right)^T J (\nabla_z G) \\ &= \left(\nabla_z (\alpha F_1) \right)^T J (\nabla_z G) + \left(\nabla_z (\beta F_2) \right)^T J (\nabla_z G) \\ &= \alpha \left(\nabla_z (F_1) \right)^T J (\nabla_z G) + \beta \left(\nabla_z (F_2) \right)^T J (\nabla_z G) \\ &= \alpha \{F_1, G\} + \beta \{F_2, G\} \end{aligned}$$



3. **Leibnitz rule:**

$$\{F_1, F_2, G\} = F_1 \{F_2, G\} + F_2 \{F_1, G\}$$

proof:

4. **Jacobi identity:**

$$\{F_1, \{F_2, F_3\}\} + \{F_3, \{F_1, F_2\}\} + \{F_2, \{F_3, F_1\}\} = 0 \quad (5)$$

proof:

Clearly $\{F, G\}$ is a smooth map $\mathcal{O} \rightarrow \mathbb{R}$.

By a common abuse of notation, let $F(t) = F(\Phi(z_0, t_0, t), t)$, where Φ is the solution of 3. By the chain rule we have that:

$$\frac{d}{dt}F(t) = \frac{\partial F}{\partial t}(\Phi(z_0, t_0, t), t) + \{F, H\}(\Phi(z_0, t_0, t), t) \quad (6)$$

Hence $dH/dt = \partial H/\partial t$.

reason why:

explanation

Theorem: Let F, G , and H be as above and independent of time t . Then:

1. F is an integral for 3 if and only if $\{F, H\} = 0$.
2. H is an integrand for 3.
3. If F and G are integrals for 3, then so is $\{F, G\} = 0$.
4. $\{F, G\}$ is the time rate of change of F along the solutions of 3.

proof:

1. Directly follows directly from the definition of an integral and from 6. Let's see why:
COMPLETAR ESTO!!
2. Follows from (1) and from the fact that the Poisson bracket is skew-symmetric, so $\{H, H\} = 0$.
COMPLETAR ESTO!!
3. Follows from the Jacobi identity
Completar, poner en ecuaciones
4. Follows from 6.
Poner en ecuaciones



In many of the examples given below, the Hamiltonian H is the total energy of a physical system; when it is, the theorem says that *energy is a conserved quantity*.

In the conservative case when H is independent of t , a critical point of H as a function (i.e., a point where the gradient of H is zero) is an equilibrium (or critical, rest, stationary) point of the system of differential equations 2 or 3

For the rest of this section, let H be independent of t . An **equilibrium point** ζ of system 3 is **stable** if:

$$\forall \epsilon > 0 \quad \exists \delta > 0 : \|\zeta - \Phi(z_0, t)\| < \epsilon \quad \forall t \text{ such that } \|\zeta - z_0\| < \delta$$

note that $\forall t$ means both positive and negative t , and that stability is for both the future and the past.

Theorem (Dirichlet): If ζ is a strict local minimum or maximum of H , then ζ is stable.

proof:

Without loss of generality, assume that $\zeta = 0$ and $H(0) = 0$. ACLARAR EL POR QUÉ NO PERDEMOS GENERALIDAD.

Because $H(0) = 0$ and 0 is a strict minimum of H (WHY), there is an $\eta > 0$ such that $H(z)$ is positive for $0 < \|z\| < \eta$ (we can say that H is positive definite).

Let $k = \min(\epsilon, \eta)$ and $M = \min\{H(z) : \|z\| = k\}$, so $M > 0$.

Because $H(0) = 0$ and H is continuous, there is a $\delta > 0$ such that $H(z) < M$ for $\|z\| < \delta$.

$$\|z\| < \delta \Rightarrow H(z_0) = H(\Phi(z_0, t)) < M \quad \forall t$$

We have that $\|\Phi(z_0, t)\| < k \leq \epsilon \quad \forall t$, because if not, there is a time t' when $\|\Phi(z_0, t')\| = k$, and $H(\Phi(z_0, t')) \geq M$, a contradiction. RXPLICAR CON QUÉ CHOCA



1.4 The Harmonic Oscillator

The harmonic oscillator is the second-order, linear, autonomous, ordinary differential equation:

1.5 The Forced Nonlinear Oscillator

1.6 The Elliptic Sine Function

1.7 Linear Flow on the Torus

1.8 Euler–Lagrange Equations

2 N-Body Problem

Let's us consider N point masses in the space (\mathbb{R}^3 , the planar case \mathbb{R}^2 , the coolinear case \mathbb{R}), whit the i -th particle having a mass $m_i > 0$ and a position vector $q_i = (q_{i1}, q_{i2}, q_{i3})^t$.

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The equations of the system comes from the **Newton's law of universal gravitation**:

$$\ddot{q}_i m_i = \sum_{\substack{j=1 \\ j \neq i}}^N G m_i m_j \frac{(q_j - q_i)}{\|q_j - q_i\|^3} = \frac{\partial U}{\partial q_i} \quad I = 1, 2, \dots, N \quad (7)$$

reason why:

$$\left\| \frac{u}{\|u\|^3} \right\| = \frac{\|u\|}{\|u\|^3} = \frac{1}{\|u\|^2}$$

Where $G = 6.67408 \cdot 10^{-11} \frac{m^3}{s^2 Kg}$ is the **Gravitacional constant**.

We define the **Self potencial**, the negative of potencial energy, as:

$$U = \sum_{1 \leq i < j \leq N} \frac{G m_i m_j}{\|q_j - q_i\|} \quad (8)$$

3 Tópico sobre el que haré el trabajo

4 Exercises

4.1 Chapter 1: Introduction to Hamiltonian systems

Make the phase portrait of the Hamiltonian system

$$\begin{cases} \dot{x} = y \\ \dot{y} = x - \frac{x^3}{3} \end{cases}$$

and compute its Hamiltonian

- Plano de fases

-

$$\begin{cases} \dot{x} = y = \frac{\partial H}{\partial y}(x, y) \\ \dot{y} = x - \frac{x^3}{3} = \frac{\partial H}{\partial x}(x, y) \end{cases}$$

One one hand,

$$\int \frac{\partial H}{\partial x} dx = \int x - \frac{x^3}{3} dx = \frac{x^2}{2} - \frac{x^4}{12} + C(y)$$

On the other hand,

$$\int \frac{\partial H}{\partial y} dy = \int y dy = \frac{y^2}{2} + C(x)$$

So the Hamiltonian is:

$$H(x, y) = \frac{x^2}{2} - \frac{x^4}{12} + \frac{y^2}{2}$$

Make the phase portrait of the Hamiltonian system

$$\begin{cases} \dot{x} = x \\ \dot{y} = -y + x^2 \end{cases}$$

and compute its Hamiltonian

- Plano de fases

-

$$\begin{cases} \dot{x} = x = \frac{\partial H}{\partial y}(x, y) \\ \dot{y} = -y + x^2 = \frac{\partial H}{\partial x}(x, y) \end{cases}$$

(Meyer-Hall-Offin) Let x, y, z be the usual coordinates in \mathbb{R}^3 , $r = xi + yj + zk$, $X = \dot{x}$, $Y = \dot{y}$, $Z = \dot{z}$, $R = \dot{r} = Xi + Yj + Zk$.

1. Compute the three components of angular momentum $mr \times R$.
2. Compute the Poisson bracket of any two of the components of angular momentum and show that it is $\pm m$ times the third component of angular momentum.
3. Show that if a system admits two components of angular momentum as integrals, then the system admits all three components of angular momentum as integrals.

1. [adea](#)

2. [dsa](#)

3. [dadsa](#)

(Meyer-Hall-Offin) **A Lie algebra** A is a vector space with a product: $A \times A \rightarrow A$ that satisfies:

- **Anticommutative:** $ab \neq ba$
- **Distributive:** $a(b + c) = ab + ac$
- **Scalar associative:** $(\alpha a)b = \alpha(ab)$
- **Jacobis identity:** $a(bc) + b(ca) + c(ab) = 0$, $a, b, c \in A$, $\alpha \in \{\mathbb{R}, \mathbb{C}\}$

1. Show that vectors in \mathbb{R}^3 form a Lie algebra where the product $*$ is the cross product.
2. Show that smooth functions on an open set in \mathbb{R}^{2n} form a Lie algebra, where $fg = \{f, g\}$, the Poisson bracket.
3. Show that the set of all $n \times n$ matrices, $gl(n, \mathbb{R})$, is a Lie algebra, where $AB = AB - BA$, the Lie product.

1. [bla](#)

2. [bla](#)

3. [bla](#)

(Meyer-Hall-Offin) The pendulum equation is $\ddot{\theta} + \sin \theta = 0$.

1. Show that $2I = \frac{1}{2}\dot{\theta}^2 + (1 - \cos \theta) = \frac{1}{2}\dot{\theta}^2 + 2\sin^2(\theta/2)$ is an integral.

2. Sketch the phase portrait.
3. Make the substitution $y = \sin(\theta/2)$ to get $\dot{y}^2 = (1 - y^2)(I - y^2)$. Show that when $0 < I < 1$, $y = k \operatorname{sn}(t, k)$ solves this equation when $k^2 = I$ (Look at the definition of elliptic sine function of Section 1.6 of Meyer-Hall-Offin).

1. bla

2. bla

3. bla

(Meyer-Hall-Offin) Let $H : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ be a globally defined conservative Hamiltonian, and assume that $H(z) \rightarrow +\infty$ as $|z| \rightarrow +\infty$. Show that all solutions of $\dot{z} = J\nabla H(z)$ are bounded (Hint: Think like Dirichlet).

Solución

Consider a \mathcal{C}^2 Hamiltonian $H = H(q, p, t) : U \subset \mathbb{R}^{2n+1} \rightarrow \mathbb{R}$ such that $\det(\partial_p^2 H) \neq 0$ on U . Define $v = \partial_p H(q, p, t)$. Prove:

1.

$$\partial_{q_i} L(q, v, t) = -\partial_{q_i} H(q, p, t)$$

$$\partial_{v_i} L(q, v, t) = p_i$$

$$\partial_t L(q, v, t) = -\partial_t H(q, p, t)$$

2. The Lagrangian L is \mathcal{C}^2 and $\det(\partial_v^2 L) \neq 0$.
3. The Euler-Lagrange equations associated to L and the Hamiltonian equations $\dot{q}_i = \partial_{p_i} H$, $\dot{p}_i = -\partial_{q_i} H$ are equivalent.

1. bla

2. bla

3. bla

4.2 Chapter 2: The N-body problem

Prove that the linear momentum is a first integral and that the center of mass moves with constant velocity for the 3 body problem.

Solución

Prove that if (a_1, a_2, \dots, a_N) is a central configuration with value λ :

1. For any $\tau \in \mathbb{R}$ then $(\tau a_1, \tau a_2, \dots, \tau a_N)$ is also a central configuration with value $\frac{\lambda}{\tau^3}$.
2. If A is an orthogonal matrix, then $Aa = (Aa_1, Aa_2, \dots, Aa_N)$ is also a central configuration with the same value λ .

1. bal bla

2. bla bla

(Meyer-Hall-Offin) Draw the complete phase portrait of the collinear Kepler problem. Integrate the collinear Kepler problem.

Solución

(Meyer-Hall-Offin) Show that $\varpi^2(\epsilon^2 - 1) = 2hc^2$ for the Kepler problem. (Attention: Meyer-Hall-Offin has a typo)

Solución

(Meyer-Hall-Offin) The area of an ellipse is $\pi a^2(1 - \epsilon^2)^{1/2}$, where a is the semi-major axis. We have seen in Keplers problem that area is swept out at a constant rate of $c/2$. Prove Keplers third law: The period p of a particle in a circular or elliptic orbit ($\epsilon < 1$) of the Kepler problem is $p = (\frac{2\pi}{\sqrt{\mu}})a^{3/2}$.

Solución

4.3 Chapter 3: Linear Hamiltonian systems**4.4 Chapter 6: Symplectic Transformations****4.5 Chapter 8: Geometric Theory****4.6 Chapter 9: Continuation of solutions**

(Meyer-Hall-Offin) Show that the scaling used in Section 9.4 of Meyer-Hall-Offin to obtain Hills orbits for the restricted problem works for Hills lunar problem (see previous problem) also. Why does not the scaling for comets work?

[Solución](#)

Prove Lemma 9.7.1 in Meyer-Hall-Offin. Verify that formula (9.11) is the condition for an orthogonal crossing of the line of syzygy in Delaunay elements.

[Solución](#)

4.7 Chapter 10: Normal forms**4.8 Chapter 13: Stability and KAM Theory**

(Meyer-Hall-Offin) Using Poincaré elements show that the continuation of the circular orbits established in Section 6.2 (Poincaré orbits) are of twist type and hence stable.

[Solución](#)

5 Apendix

5.1 Needed resoults and definitions

5.1.1 Linear Algebra

matriz ortogonal

no singular

skew-simmetric

DEVOLVER A SU SITIO LAS FOTOS TAMAÑO CARNET

5.1.2 Calculus

Lipschitz continuity: We have two metric spaces $(X, d_X), (Y, d_Y)$ and a function $f : X \rightarrow Y$. We say that f is **Lipschitz continuous** if $\exists K \geq 0$ such that, for all $x_1, x_2 \in X$,

$$d_Y(f(x_1), f(x_2)) \leq K \cdot d_X(x_1, x_2)$$

where K is known as **Lipschitz constant**.

teorema punto fijo de bla bla bla bla

Chain Rule: Let $F = f \circ g$, or, equivalently, $F(x) = f(g(x))$ for all x . Then:

$$(f \circ g)' = (f' \circ g) \cdot g'$$

$$F'(x) = f'(g(x)) \cdot g'(x)$$

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

The two versions of the chain rule are related; if $z = f(y)$ and $y = g(x)$:

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = f'(y) \cdot g'(x) = f'(g(x)) \cdot g'(x)$$

[proof:](#) ♠

Gradient: The gradient of a scalar function $(\mathbb{R}^n \rightarrow \mathbb{R}) f(x_1, \dots, x_n)$ denoted by ∇f or $\vec{\nabla} f$ denotes the vector differential operator. The gradient of f is defined as the unique vector field whose dot product with any unitvector v at each point x is the directional derivative off along v . That is,

$$(\nabla f(x)) \cdot v = D_v f(x)$$

- *Cartesian coordinates*: Lets focus in \mathbb{R}^3 , where i, j, k are the standard unit vectors in the directions of axis x, y, z .

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

- *Cylindrical coordinates*: Lets focus in \mathbb{R}^3 , where ρ is the axial distance, φ is the azimuthal or azimuth angle and z is the the axial coordinate and e_ρ, e_φ, e_z are the unit vectors pointing along the coordinate directions.

$$\nabla f(\rho, \varphi, z) = \frac{\partial f}{\partial \rho}e_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \varphi}e_\varphi + \frac{\partial f}{\partial z} = \left(\frac{\partial f}{\partial \rho}, \frac{1}{\rho} \frac{\partial f}{\partial \varphi}, \frac{\partial f}{\partial z} \right)$$

proof: ♠

- *Spherical coordinates*: Lets focus in \mathbb{R}^3 , where r is the radial distance, φ is the azimuthal angle and θ is the polar angle, and e_r, e_φ, e_θ are local unit vectors pointing in the coordinate directions.

$$\nabla f(r, \theta, \varphi) = \frac{\partial f}{\partial r}e_r + \frac{1}{r} \frac{\partial f}{\partial \theta}e_\theta + \frac{1}{r \sin(\theta)} \frac{\partial f}{\partial \varphi}e_\varphi = \left(\frac{\partial f}{\partial r}, \frac{1}{r} \frac{\partial f}{\partial \theta}, \frac{1}{r \sin(\theta)} \frac{\partial f}{\partial \varphi} \right)$$

proof: ♠

Laplace Operator: is a differential operator given by the divergence of the gradient of a function on Euclidean space.

$$\Delta f = \nabla^2 f = \nabla \cdot \nabla f = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}$$

5.1.3 Geometry

5.1.4 Funcional Analysis

5.1.5 Differential forms

5.1.6 Measure Theory

5.2 Ordinary Differential Equations

5.2.1 Peano existence theorem

Arzelà–Ascoli Theorem:

proof:



Peano existence Theorem: Let $I = [a, b] \subset \mathbb{R}$, $D \subset \mathbb{R}^n$, $f : I \times D \rightarrow \mathbb{R}^n$ be a continuous function. We need to assume that $(t_0, x_0) \in I \times D$ and $C > 0$, $T > 0$ are given such that $[t_0 - T, t_0 + T] \times B_C(x_0) \subseteq I \times D$, $(B_C(x_0) = \{x \in \mathbb{R}^n : d(x, x_0) < C\})$ then the system:

$$\begin{cases} x(t_0) = x_0 \\ x'(t) = f(t, x(t)), \quad t \in [t_0 - \gamma, t_0 + \gamma] \end{cases}$$

possesses at least one solution x , where $\gamma \leq \min \{T, C/M\}$, $M = \max_{(t,x) \in [t_0-T, t_0+T] \times B_C(x_0)} |f(t, x)|$

proof:



5.2.2 Picard–Lindelöf theorem

Consider the initial values problem:

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0$$

Picard–Lindelöf Theorem: let's suppose that f is uniformly Lipschitz continuous in y (meaning the Lipschitz constant can be taken independent of t) and continuous in t , then for some value $\epsilon > 0$, there exists a unique solution $y(t)$ to the initial value problem on the interval $[t_0 - \epsilon, t_0 + \epsilon]$.

proof:



5.2.3 Linear systems

We always can reduce a equation of order higher than 1 to a system of equations of order 1 in the next way:

$$x^{(n)} = f(t, x', x'', \dots, x^{n-1})$$

We introduce new variables in the next way:

$$x_1 = x, \quad x_2 = x', \quad \dots, \quad x_n = x^{n-1}$$

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ \vdots \\ x_n' = f(t, x_1, x_2, \dots, x_n) \end{cases}$$

Now we have a system of n equations of first orden. We can use vectoral notation, we call $x = (x_1, x_2, \dots, x_n)^T$, $F = (F_1, F_2, \dots, F_n)^T$ where $F_{i-1} = F_{i-1}(t, x) = x_i$, so then:

$$x' = F$$

About the initial condions problem,

The **general form** of a linear system on n equations is the next one:

$$\begin{cases} x_1' = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + b_1(t) \\ x_2' = a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + b_2(t) \\ \vdots \\ x_n' = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + b_n(t) \end{cases} \quad (9)$$

If $b_i(t) = 0 \quad i = 1, \dots, n$ then we say that the system is **homogenous**, otherwise it's called **no homogenous**.

We can write this system in **matrix form**:

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \quad x'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{pmatrix}$$

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix} \quad b(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{pmatrix}$$

So obviously we can write 9 as:

$$x'(t) = A(t)x(t) + b(t) \quad (10)$$

note that if the system is homogenous then $b(t) = 0$ and $x'(t) = A(t)x(t)$.

Theorem: If every component of $A(t)$ and $b(t)$ are continuous in the open interval $(a, b) \subset \mathbb{R}$, then for each $t_0 \in (a, b)$ and for all vector $x_0 \in \mathbb{R}^n$ such that $x(t_0) = x_0$, the system

$$x'(t) = A(t)x(t) + b(t)$$

has an unique solution in the interval (a, b) .

proof:

