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MAMMEE
Discrete and Algorithmic Geometry

Geometric thickness of complete graphs

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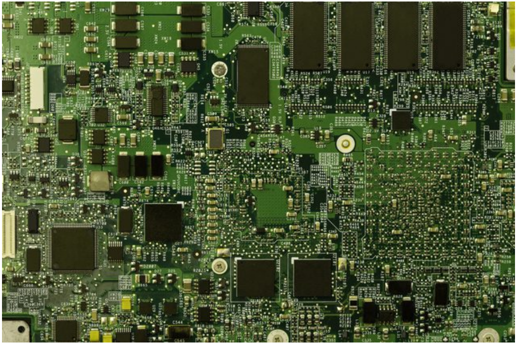
Index

An economic motivation

Suposse we have to print a circuit into a circuit board. Is less expensive to use uninsulated wires, so let's try to do it.



An economic motivation

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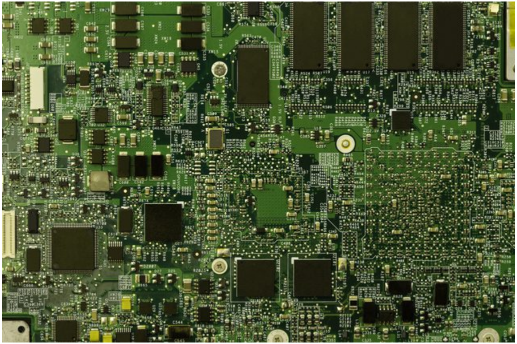

The problem is that if they are uninsulated the only way to avoid dead shorts is to separate the wires in diferent layers. This problem is equivalent to mimimize the number of layers.

Improve the visualitation of a graph

Now we imagine that we want to visualize information in a graph-shape.

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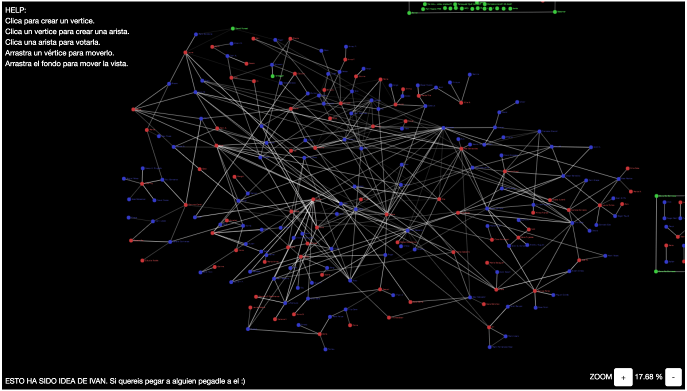
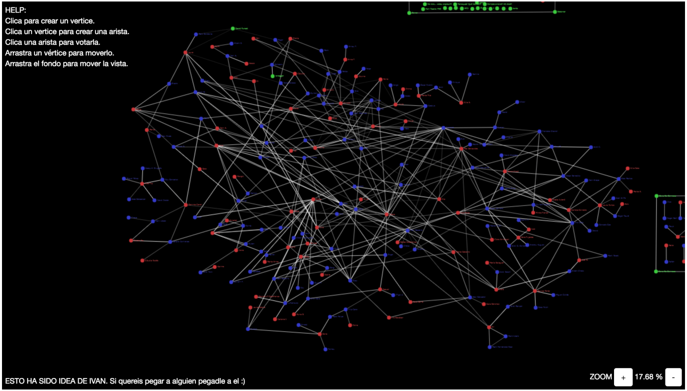


Improve the visualitation of a graph

Now we imagine that we want to visualize information in a graph-shape.



The concept of the thickness of a graph is related with the number of colors we need to separate oll the edges (think that each color means one layer).

Thickness of a graph $\mathcal{G}$

Theorical Thickness $\theta(\mathcal{G})$

Minimum number of planar graphs into a which a graph can be descomposed.

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Theorical Thickness $\theta(\mathcal{G})$

Minimum number of planar graphs into a which a graph can be descomposed.

Geometric Thickness $\bar{\theta}(\mathcal{G})$

Smallest value of k such that we can assign a planar point locations to the vertices of $\mathcal{G}$, represent each edge of $\mathcal{G}$ as a line segment, and assign each edge to one of k layers so that no two edges on the same layer cross.

Thickness of a graph

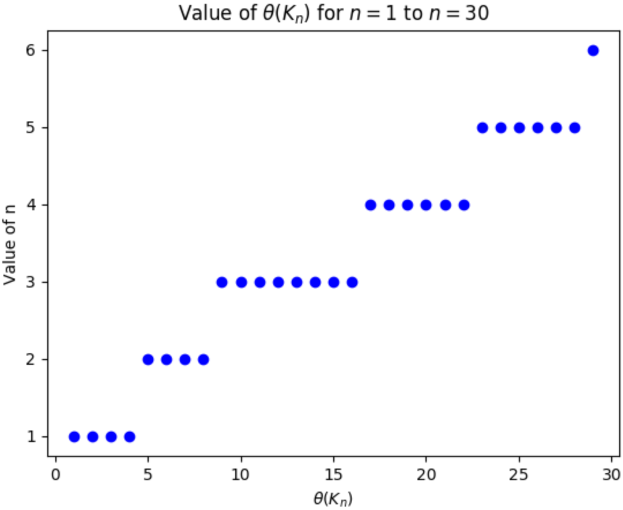
Key difference

Geometric thickness requires that the vertex placements be consistent at all layers and that straight-line adges be used, whereas graph-theorical thickness imposes no consistency requirement between layers.

Graph-theorical thickenss for all complete graphs

$$\boxed{\theta(K_n) = \begin{cases} 1, & 1 \leq n \leq 4 \\ 2, & 5 \leq n \leq 8 \\ 3, & 9 \leq n \leq 10 \\ \lceil \frac{n+2}{6} \rceil, & n > 10 \end{cases}}$$

Graph-theorical thickenss for all complete graphs



Upper Bounds

Theorem 1

$$\bar{\theta}(k_n) \leq \lceil n/4 \rceil$$

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Roadmap to the proof:

- Assume that n is multiple of four, $n = 2k$ with (k even) to show that n vertices can be arranged in two "rings" of k vertices each, so K_n can be embedded using only $k/2$ layers and with no edges on the same layer crossing. An inner and an outher rings are formed.

Upper Bounds

Proof:

- Use the vertices of the inner ring to form a regular k-gon and considerer the opposite vertices con create a zigzag path. This path has exactly one diagonal connecting diametrically opposite points.
- By continuity, we can replace by a suitably chosen common endpoints the infinite endpoints of a collection of parellel rays. Thus foring an outer ring of k vertices.
- The figure can be perturbed by moving slightly the inner ring so that none of the diagonals of the polygon comprinsing the outher ring intersect the polygon comprising the inner ring.
- It's straighforward to verify that this is indeed a descomposition of the edges og k_n into $k/2 = n/4$ layers.

∎

Lower Bounds

Theorem 2

For all $n \geq 1$

$$\bar{\theta}(k_n) \geq \max_{1 \leq x \leq n/2} \frac{\binom{n}{2} - 2\binom{x}{2} - 3}{3n - 2x - 7}$$

Lower Bounds

Theorem 2

For all $n \geq 1$

$$\bar{\theta}(k_n) \geq \max_{1 \leq x \leq n/2} \frac{\binom{n}{2} - 2\binom{x}{2} - 3}{3n - 2x - 7}$$

In particular, for $n \geq 12$

$$\bar{\theta}(k_n) \geq \left\lceil \frac{3 - \sqrt{7}}{2} n + 0.342 \right\rceil \geq \left\lceil \frac{n}{5.646} + 0.342 \right\rceil$$

Geometric Thickness of K_{15}

Theorem 3

$$\bar{\theta}(k_{15}) = 4$$

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Proof: We divide the problem in 3 cases.

- **Case I:** 3 points in the convex hull. Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ convex hull points and let $\mathcal{A}_1, \mathcal{B}_1$ and $\mathcal{C}_1$ be the point furtherst from edge $\mathcal{BC}$ (respectively $\mathcal{AC}, \mathcal{AB}$ within triangle $\mathcal{ABC}$.
  - **Lemma 1:** The edge $\mathcal{AA}_1$ will appear in every triangulation of $\mathcal{S}$.

  Let $\mathcal{A}_2, \mathcal{B}_2$ and $\mathcal{C}_2$ be the point next furtherst from edge $\mathcal{BC}$ (respectively $\mathcal{AC}, \mathcal{AB}$ within triangle $\mathcal{ABC}$.
  - **Lemma 2:** At least one of the edges $\mathcal{A}_1\mathcal{A}_2$ or $\mathcal{AA}_2$ will appear in every triangulation of $\mathcal{S}$.

Geometric Thickness of K_{15}

Theorem 3

$$\bar{\theta}(k_{15}) = 4$$

Proof: We divide the problem in 3 cases.

- **Case II:** 4 points in the convex hull.
    - Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ the four convex hull vertices.
    - Assume triangle $\mathcal{DAB}$ has ta least one point of $\mathcal{S}$ in his interior (if not switch $\mathcal{A}$ and $\mathcal{C}$ and let $\mathcal{A}_1$ be the point inside furthest from line $\mathcal{DB}$.
    - By Lemma 2, the edge $\mathcal{AA}_1$ must appear in every triangulation of $\mathcal{S}$.
    - Since every triangulation has 38 edges three triangulations can account at most 104 edges.

Geometric Thickness of K_{15}

Theorem 3

$$\bar{\theta}(k_{15}) = 4$$

Proof: We divide the problem in 3 cases.

- **Case III:** 5 or more points in the convex hull.
  - Let h be the number og points in the convex hull.
  - A triangulation of $\mathcal{S}$ will have $42 - h$ edges, and all hull edges must be in each triangulation.
  - The total number of edges in three triangulatinos is at most $3(42 - 2h) + h = 126 - -5h$, whis is at most 101 for $h \geq 5$.

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Theorem 4

For the complete bipartite graph $K_{a,b}$

$$\left\lceil \frac{ab}{2a+2b-4} \right\rceil \leq \theta(K_{a,b}) \leq \bar{\theta}(k_{a,b}) \leq \left\lceil \frac{\min(a,b)}{2} \right\rceil$$

Geometric Thickness of Complete Bipartite Graphs

Theorem 4

For the complete bipartite graph $K_{a,b}$

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Proof:

- The first inequeality follows from eulre formula since a bipartite graph which is planaer with $a+b$ vertices can have at mosr $2a+2b-4$ edges
- The second inequality, assume that $a \leq b$ and a is even. Draw b vertices in a horizontal, whit $a/2$ red vertives above the line and $a/2$ vertices below. Each layer consists od all edges connecting the blue vertices with one red vertex from above the line and one red vertex from below.

Corollary 1:

For any integer b, $\bar{\theta}(k_{a,b}) = \theta(k_{a,b})$ provided:

$$a > \begin{cases} \frac{(b-2)^2}{2} & \text{if b is even} \\ (b-1)(b-2) & \text{if b is odd} \end{cases}$$

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Proof: If $a > b$, the leftmost and rightmost quantities in the expresion of the last theorem will be equal provided $ab/(2a+2b-4) > (b-2)/2$ if b is even, or provided $ab/(2a+2b-4) > (b-1)/2$ f b is odd. By simplifing this inequality holds.

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Geometric Thickness of Complete Bipartite Graphs

Theorem 5

$$\bar{\theta}(k_{6,8}) = 3$$

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Proof:

- From the last theorem, second inequality: $\bar{\theta}(k_{6,8}) \leq 3$
- We need to show that $\bar{\theta}(k_{6,8}) > 2$. We proceed by contradiction: we soposse that we have an embedding of $k_{6,8}$ with geometric thickness 2 and underlying points set S. $k_{6,8}$ has 14 vetices and 48 edges. By Euler's formula a graph with 14 edges has at most 24 edges, so each layer each layer has exactly 24 edges and each face of the layer is a quadrilateral.

Geometric Thickness of Complete Bipartite Graphs

Theorem 5

$$\bar{\theta}(k_{6,8}) = 3$$

Proof:

- Two-color points od $\mathcal{S}$ acordit to re bipartition of the graph. We claim that must be at least one red and one blue vertex on the convex hull of $\mathcal{S}$. Again by contradriction, we suppose that all convex hull vertices are the same color.
  - Because each layer is bipartite and the convez hull contains at least three vertivces the outher face in either layer would consist if at least 6 vertices.
  - Contradiction: it's impossible beacause each face is bounded by a qyeadrulateral $\Rightarrow$ there are at least one red and one blue vertex on the convex hull of $\mathcal{S}$

Geometric Thickness of Complete Bipartite Graphs

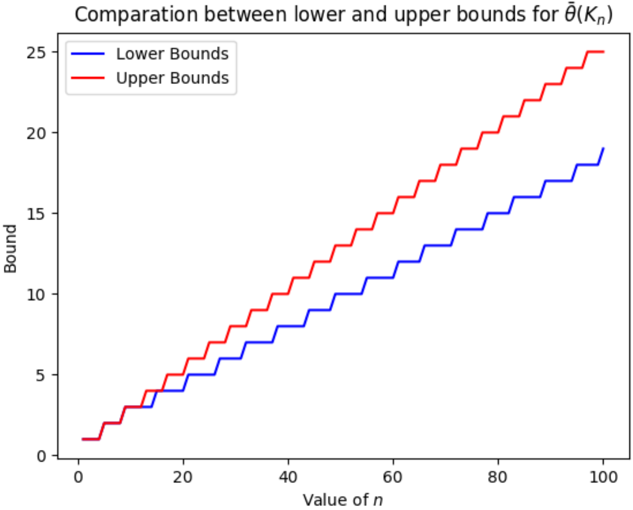
Theorem 5

$$\bar{\theta}(k_{6,8}) = 3$$

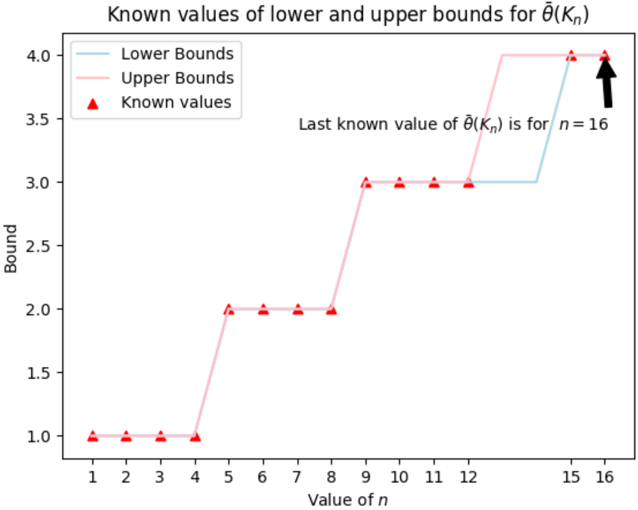
- The claim implies that one of the layers must contain (say the first) musd contaun a convex hull edge.
- Then this edge could be added to the second layer without destroying either planarity or bipartiteness.
- Since the second layer has already 14 verties and 24 edges it's imposible, which leads us to a contradiction. $\Rightarrow \bar{\theta}(k_{6,8}) > 2$

■

The first 100 values of $\bar{\theta}$



The first 100 values of $\bar{\theta}$



Some open questions

- To find exact values for $\bar{\theta}(K_n)$, in particular for $n = 13, 14$.
- What is the smallest graph (G) such that $\bar{\theta}(\mathcal{G}) > \theta(\mathcal{G})$?
- We know that it's true for complete graphs, but it's in general true that $\bar{\theta}(\mathcal{G}) = O(\theta(\mathcal{G}))$?
- What is the complexity of computing $\bar{\theta}(\mathcal{G})$ for a given graph $\mathcal{G}$?.

Bibliography

D.Dillencourt, D. Eppstein and D. S. Hirschberg, *Geometric thickness of complete graphs*, J. Graph Algorithmss and Applications 4 (3), 5-17, 2000.