

Universidad Politécnica de Cataluña
Facultad de Matemáticas y Estadística
MAMMEE
Discrete and Algorithmic Geometry

Geometric thickness of complete graphs

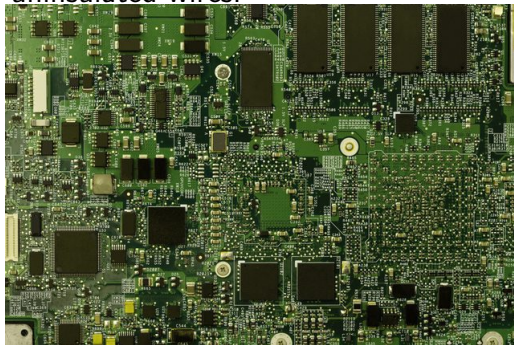
Manuel Gijón Agudo

M. Dillencourt, D. Eppstein, D. S. Hirschberg

- 1 Motivation and first definitions
- 2 Results
 - Good news
 - Upper and lower bounds
 - K_{15}
 - Bipartite graphs
 - To sum up
- 3 Conclusions and open questions

An economic motivation

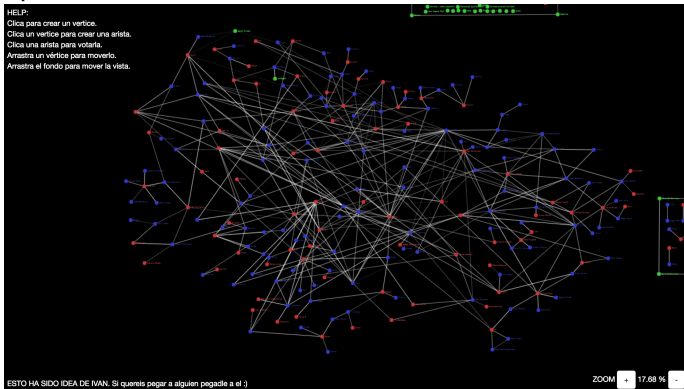
Suposse we have to print a circuit into a circuit board. Is more efficient to use uninsulated wires.



Problem: avoid dead shorts. Solution: separe the wires in different layers. This problem is equivalent to minimize the number of layers.

Improve the display of a graph

Now we imagine that we want to display information in a graph-shape.



The concept of the thickness of a graph is related with the number of colors we need to separate all the edges.

Thickness of a graph \mathcal{G}

Theoretical Thickness $\theta(\mathcal{G})$

Minimum number of planar graphs into a which a graph can be decomposed.

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Geometric Thickness $\bar{\theta}(\mathcal{G})$

Smallest value of k such that we can assign a planar point locations to the vertices of \mathcal{G} , represent each edge of \mathcal{G} as a line segment, and assign each edge to one of k layers so that no two edges on the same layer cross.

Thickness of a graph

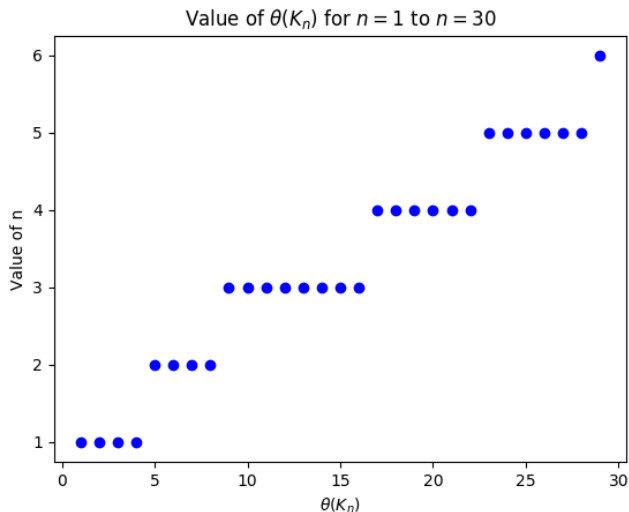
Key difference

Geometric thickness requires that the vertex placements be consistent at all layers and that straight-line edges be used, whereas graph-theoretical thickness imposes no consistency requirement between layers.

Graph-theoretical thickness for all complete graphs

$$\theta(K_n) = \begin{cases} 1, & 1 \leq n \leq 4 \\ 2, & 5 \leq n \leq 8 \\ 3, & 9 \leq n \leq 10 \\ \lceil \frac{n+2}{6} \rceil, & n > 10 \end{cases}$$

Graph-theoretical thickness for all complete graphs



Theorem 1

$$\bar{\theta}(k_n) \leq \lceil n/4 \rceil$$

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Roadmap to the proof:

- Assume that n is multiple of four, $n = 2k$ with (k even), show that n vertices can be arranged in two “rings” of k vertices, so K_n can be embedded using $k/2$ layers and with no edges on the same layer crossing.

Upper Bounds

Roadmap to the proof:

- Use the vertices of the inner ring to form a regular k -gon and consider the opposite vertices can create a zigzag path. This path has exactly one diagonal connecting diametrically opposite points.
- By continuity, we can replace by a suitably chosen common end points the infinite end points of a collection of parallel rays. Thus forming an outer ring of k vertices.
- The figure can be perturbed by moving slightly the inner ring. None of the diagonals of the polygon comprising the outer ring intersect the polygon comprising the inner ring.
- It's straightforward to verify that this is indeed a decomposition of the edges of k_n into $k/2 = n/4$ layers.

Theorem 2

For all $n \geq 1$

$$\bar{\theta}(k_n) \geq \max_{1 \leq x \leq n/2} \frac{\binom{n}{2} - 2\binom{x}{2} - 3}{3n - 2x - 7}$$

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In particular, for $n \geq 12$

$$\bar{\theta}(k_n) \geq \left\lceil \frac{3 - \sqrt{7}}{2}n + 0.342 \right\rceil \geq \left\lceil \frac{n}{5.646} + 0.342 \right\rceil$$

Theorem 3

$$\bar{\theta}(K_{15}) = 4$$

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Proof: We divide the proof in 3 cases.

- **Case I:** 3 points in the convex hull. Let \mathcal{A}, \mathcal{B} and \mathcal{C} convex hull points and let $\mathcal{A}_1, \mathcal{B}_1$ and \mathcal{C}_1 be the point furthestst from edge BC (respectively AC, AB within triangle ABC).

- **Lemma 1:** The edge $\mathcal{A}\mathcal{A}_1$ will appear in every triangulation of \mathcal{S} .

Let $\mathcal{A}_2, \mathcal{B}_2$ and \mathcal{C}_2 be the point next furthestst from edge BC (respectively AC, AB within triangle ABC).

- **Lemma 2:** At least one of the edges $\mathcal{A}_1\mathcal{A}_2$ or $\mathcal{A}\mathcal{A}_2$ will appear in every triangulation of \mathcal{S} .

Theorem 3

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Proof: We divide the problem in 3 cases.

- **Case II:** 4 points in the convex hull.
 - Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ the convex hull vertices.
 - Assume triangle \mathcal{DAB} has at least one point of \mathcal{S} in his interior (if not switch \mathcal{A} and \mathcal{C} and let \mathcal{A}_1 be the point inside furthest from line \mathcal{DB}).
 - By Lemma 2, the edge \mathcal{AA}_1 must appear in every triangulation of \mathcal{S} .
 - Since every triangulation has 38 edges three triangulations can account at most 104 edges.

Theorem 3

$$\bar{\theta}(K_{15}) = 4$$

Proof: We divide the problem in 3 cases.

- **Case III:** 5 or more points in the convex hull.
 - Let h be the number of points in the convex hull.
 - A triangulation of \mathcal{S} will have $42 - h$ edges, and all hull edges must be in each triangulation.
 - The total number of edges in three triangulations is at most $3(42 - h) + h = 126 - 2h$, that is at most 101 for $h \geq 5$.



Geometric Thickness of Complete Bipartite Graphs

Theorem 4

For the complete bipartite graph $K_{a,b}$

$$\left\lceil \frac{ab}{2a + 2b - 4} \right\rceil \leq \theta(K_{a,b}) \leq \bar{\theta}(K_{a,b}) \leq \left\lceil \frac{\min(a, b)}{2} \right\rceil$$

Geometric Thickness of Complete Bipartite Graphs

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Proof:

- First inequality: from Euler's formula since a bipartite graph which is planar with $a + b$ vertices can have at most $2a + 2b - 4$ edges
- Second inequality: assume that $a \leq b$ and a is even. Draw b vertices in a horizontal line, with $a/2$ red vertices above the line and $a/2$ vertices below. Each layer consists of all edges connecting the blue vertices with one red vertex from above the line and one red vertex from below.

Geometric Thickness of Complete Bipartite Graphs

Corollary 1:

For any integer b , $\bar{\theta}(k_{a,b}) = \theta(k_{a,b})$ provided:

$$a > \begin{cases} \frac{(b-2)^2}{2} & \text{if } b \text{ is even} \\ (b-1)(b-2) & \text{if } b \text{ is odd} \end{cases}$$

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Proof: If $a > b$, the leftmost and rightmost quantities in the expresion of the last theorem will be equal provided $ab/(2a+2b-4) > (b-2)/2$ if b is even, or provided $ab/(2a+2b-4) > (b-1)/2$ if b is odd. By simplifying this inequality holds.



Theorem 5

$$\bar{\theta}(K_{6,8}) = 3$$

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Roadmap to the proof:

- From the last theorem, second inequality: $\bar{\theta}(k_{6,8}) \leq 3$
- We need to show that $\bar{\theta}(k_{6,8}) > 2$. We proceed by contradiction: we have an embedding of $k_{6,8}$ with geometric thickness 2 and underlying points set S . A graph with 14 edges has at most 24 edges (Euler), so each layer each layer has exactly 24 edges.

Theorem 5

$$\bar{\theta}(K_{6,8}) = 3$$

Roadmap to the proof:

- We claim that must be at least one red and one blue vertex on the convex hull of \mathcal{S} . Again by contradiction, we suppose that all convex hull vertices are the same color.
 - Use bipartition and the number of vertices in each layer.
 - Contradiction! \Rightarrow there are at least one red and one blue vertex.

Geometric Thickness of Complete Bipartite Graphs

Theorem 5

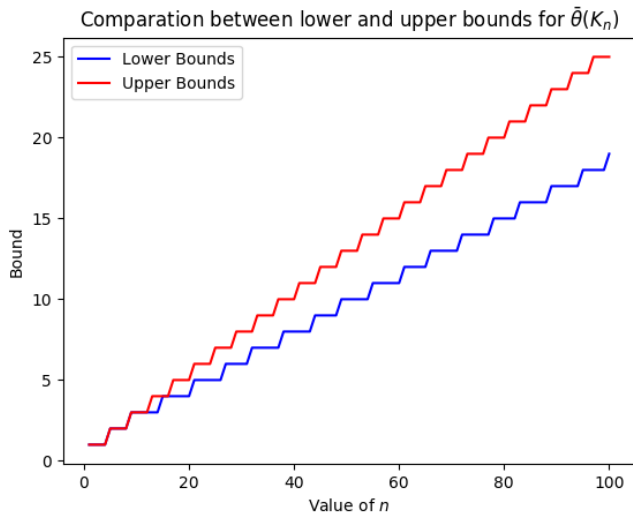
$$\bar{\theta}(k_{6,8}) = 3$$

Roadmap to the proof:

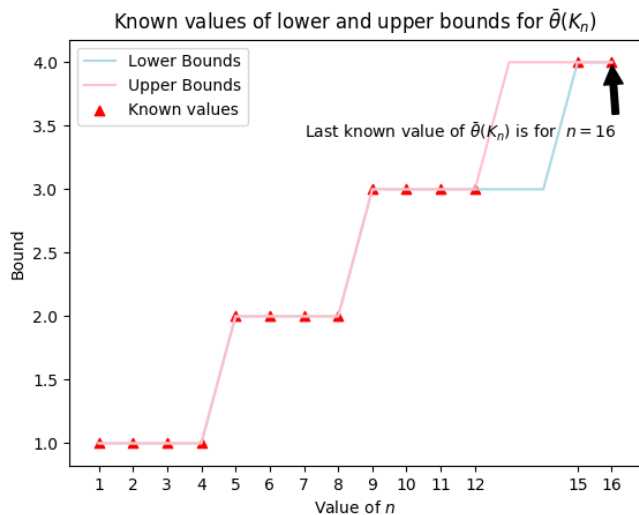
- The claim implies that one of the layers must contain a convex hull edge.
- This edge could be added to the second layer without destroying planarity..
- Since the second layer has already 14 vertices and 24 edges it's impossible, which leads us to a contradiction. $\Rightarrow \bar{\theta}(k_{6,8}) > 2$



The first 100 values of $\bar{\theta}$





The first 100 values of $\bar{\theta}$



Some open questions

- To find exact values for $\bar{\theta}(K_n)$, in particular for $n = 13, 14$.
- What is the smallest graph (G) such that $\bar{\theta}(\mathcal{G}) > \theta(\mathcal{G})$?
- What is the complexity of computing $\bar{\theta}(\mathcal{G})$ for a given graph \mathcal{G} ?

-  D.Dillencourt, D. Eppstein and D. S. Hirschberg, *Geometric thickness of complete graphs*, J. Graph Algorithmss and Applications 4 (3), 5-17, 2000.
-  P. Mutzel, T. Odenthal, M. Scharbrodt, *The thickness og graphs: a survey*, Graphs and Combinatorics, March 1998, Volume 14, Issue 1, pp 59–73.