# Hamiltonian Systems

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# Contents

# 1 Hamiltonian Equations

## 1.1 Notation and first definitions

## 1.2 Poisson Bracket

Let $F, G : U \subset \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ be $\mathcal{C}^r \quad (r \geq 1)$ functions such that $(q,p,t) \longmapsto F(q,p,t), G(q,p,t)$.

We define the **Poisson Bracket (PB)** as a $\mathcal{C}^{r-1}$ fuction $\{F,G\} : U \longrightarrow \mathbb{R}$

$$\begin{aligned}\{F,G\} &= (\nabla_z F)^T J (\nabla_z G) \\ &= (\nabla_q F)^T (\nabla_p G) - (\nabla_p F)^T (\nabla_q G) \\ &= \sum_{i=1}^{n} \Big( \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} \Big)\end{aligned}$$

**Properties:**

1. **Skew-simmetric**: $\{F,G\} = -\{G,F\}$
   In particular: $\{F,F\}$
   ***proof:***

$$\begin{aligned}-\{F,G\} &= -\Big( (\nabla_q F)^T (\nabla_p G) - (\nabla_p F)^T (\nabla_q G) \Big) \\ &= (\nabla_p F)^T (\nabla_q G) - (\nabla_q F)^T (\nabla_p G) \\ &= (\nabla_q G)^T (\nabla_p F) - (\nabla_p G)^T (\nabla_q F) \\ &= \{G,F\}\end{aligned}$$

   ♠

2. **Bilinear**: $\{\alpha F_1 + \beta F_2, G\} = \alpha\{F_1, G\} + \beta\{F_2, G\}, \quad \alpha, \beta \in \mathbb{R}$
   ***proof:***

$$\begin{aligned}\{\alpha F_1 + \beta F_2, G\} &= \Big( \nabla_z (\alpha F_1 + \beta F_2) \Big)^T J (\nabla_z G) \\ &= \Big( \nabla_z (\alpha F_1) \Big)^T J (\nabla_z G) + \Big( \nabla_z (\beta F_2) \Big)^T J (\nabla_z G) \\ &= \alpha \Big( \nabla_z (F_1) \Big)^T J (\nabla_z G) + \beta \Big( \nabla_z (F_2) \Big)^T J (\nabla_z G) \\ &= \alpha\{F_1, G\} + \beta\{F_2, G\}\end{aligned}$$

   ♠

3. **Leibnitz rule**: $\{F_1, F_2, G\} = F_1\{F_2, G\} + F_2\{F_1, G\}$
   ***proof:***

4. **Jacobi identity**: $\{F_1, \{F_2, F_3\}\} + \{F_3, \{F_1, F_2\}\} + \{F_2, \{F_3, F_3 1\}\} = 0$
   ***proof:***

## 2 N-Body Problem

Let's us consider $N$ point masses in the space ($\mathbb{R}^3$, the planar case $\mathbb{R}^2$, the coolinear case $\mathbb{R}$), whit the $i$-th particle having a mass $m_i > 0$ and a position vector $q_i = (q_{i1}, q_{i2}, q_{i3})^t$.

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The equations of the system comes from the **Newton's law of universal gravitation**:

$$\ddot{q}_i m_i = \sum_{\substack{j=1\\ j\neq 1}}^{N} G m_i m_j \frac{(q_j - q_i)}{||q_j - q_i||^3} = \frac{\partial U}{\partial q_i} \qquad I = 1, 2, \cdots, N \tag{1}$$

***reason why:***

$$\left\| \frac{u}{||u||^3} \right\| = \frac{||u||}{||u||^3} = \frac{1}{||u||^2}$$

Where $G = 6.67408 \cdot 10^{-11} \frac{m^3}{s^2 Kg}$ is the **Gravitacional constant**.

We define the **Self potencial**, the negative of potencial energy, as:

$$U = \sum_{1 \leq i < j \leq N} \frac{G m_i m_j}{||q_j - q_i||} \tag{2}$$

# 3 Tópico sobre el que haré el trabajo

# 4 Exercises

## 4.1 Chapter 1: Introduction to Hamiltonian systems

Make the phase portrait of the Hamiltonian system

$$\dot{x} = y$$
$$\dot{y} = x - \frac{x^3}{3}$$

and compute its Hamiltonian

Solución

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Make the phase portrait of the Hamiltonian system

$$\dot{x} = x$$
$$\dot{y} = -y + x^2$$

and compute its Hamiltonian

Solución

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(Meyer-Hall-Offin) Let $x, y, z$ be the usual coordinates in $\mathbb{R}^3$, $r = xi + yj + zk$, $X = \dot{x}$, $Y = \dot{y}$, $Z = \dot{z}$, $R = \dot{r} = Xi + Yj + Zk$.

1. Compute the three components of angular momentum $mr \times R$.
2. Compute the Poisson bracket of any two of the components of angular momentum and show that it is $\pm m$ times the third component of angular momentum.
3. Show that if a system admits two components of angular momentum as integrals, then the system admits all three components of angular momentum as integrals.

1. adea
2. dsa
3. dadsa

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(Meyer-Hall-Offin) **A Lie algebra** $A$ is a vector space with a product: $A \times A \to A$ that satisfies:

- **Anticommutative**: $ab \neq ba$
- **Distributive**: $a(b+c) = ab + ac$
- **Scalar associative**: $(\alpha a)b = \alpha(ab)$
- **Jacobis identity**: $a(bc) + b(ca) + c(ab) = 0, \quad a, b, c \in A, \quad \alpha \in \{\mathbb{R}, \mathbb{C}\}$

1. Show that vectors in $\mathbb{R}^3$ form a Lie algebra where the product $*$ is the cross product.
2. Show that smooth functions on an open set in $\mathbb{R}^{2n}$ form a Lie algebra, where $fg = \{f, g\}$, the Poisson bracket.
3. Show that the set of all $n \times n$ matrices, $gl(n, \mathbb{R})$, is a Lie algebra, where $AB = ABBA$, the Lie product.

1. bla
2. bla
3. bla

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(Meyer-Hall-Offin) The pendulum equation is $\ddot{\theta} + \sin\theta = 0$.

1. Show that $2I = \frac{1}{2}\dot{\theta}^2 + (1\cos\theta) = \frac{1}{2}\dot{\theta}^2 + 2\sin^2(\theta/2)$ is an integral.
2. Sketch the phase portrait.
3. Make the substitution $y = \sin(\theta/2)$ to get $\dot{y}^2 = (1-y^2)(I-y^2)$. Show that when $0 < I < 1$, $y = ksn(t,k)$ solves this equation when $k^2 = I$ (Look at the definition of elliptic sine function of Section 1.6 of Meyer-Hall-Offin).

1. bla
2. bla
3. bla

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(Meyer-Hall-Offin) Let $H : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ be a globally defined conservative Hamiltonian, and assume that $H(z) \to +\infty$ as $z \to +\infty$. Show that all solutions of $\dot{z} = J\nabla H(z)$ are bounded (Hint: Think like Dirichlet).

Solución

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Consider a $\mathcal{C}^2$ Hamiltonian $H = H(q,p,t) : U \subset \mathbb{R}^{2n+1} \longrightarrow \mathbb{R}$ such that $det(\partial_p^2 H) \neq 0$ on $U$. Define $v = \partial_p H(q,p,t)$. Prove:

1. 
$$\begin{aligned} \partial_{q_i} L(q,v,t) &= -\partial_{q_i} H(q,p,t) \\ \partial_{v_i} L(q,v,t) &= p_i \\ \partial_t L(q,v,t) &= -\partial_t H(q,p,t) \end{aligned}$$

2. The Lagrangian $L$ is $\mathcal{C}^2$ and $det(\partial_v^2 L) \neq 0$.
3. The Euler-Lagrange equations associated to $L$ and the Hamiltonian equations $\dot{q}_i = \partial_{p_i} H$, $\dot{p}_i = -\partial_{q_i} H$ are equivalent.

1. bla
2. bla
3. bla

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## 4.2 Chapter 2: The N-body problem

Prove that the linear momentum is a first integral and that the center of mass moves with constant velocity for the 3 body problem.

Solución

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Prove that if $(a_1, a_2, \cdots, a_N)$ is a central configuration with value $\lambda$:

1. For any $\tau \in \mathbb{R}$ then $(\tau a_1, \tau a_2, \cdots, \tau a_N)$ is also a central configuration with value $\frac{\lambda}{\tau^3}$.
2. If $A$ is an orthogonal matrix, then $Aa = (Aa_1, Aa_2, \cdots, Aa_N)$ is also a central configuration with the same value $\lambda$.

1. bal bla
2. bla bla

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(Meyer-Hall-Offin) Draw the complete phase portrait of the collinear Kepler problem. Integrate the collinear Kepler problem.

Solución

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(Meyer-Hall-Offin) Show that $\varpi^2(\epsilon^2 - 1) = 2hc2$ for the Kepler problem. (Attention: Meyer-Hall-Offin has a typo)

Solución

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(Meyer-Hall-Offin) The area of an ellipse is $\pi a^2(1-\epsilon^2)^{1/2}$, where a is the semi-major axis.We have seen in Keplers problem that area is swept out at a constant rate of $c/2$. Prove Keplers third law: The period $p$ of a particle in a circular or elliptic orbit ($\epsilon < 1$) of the Kepler problem is $p = (\frac{2\pi}{\sqrt{\mu}})a^{3/2}$.

Solución

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### 4.3 Chapter 3: Linear Hamiltonian systems

### 4.4 Chapter 6: Symplectic Transformations

### 4.5 Chapter 8: Geometric Theory

### 4.6 Chapter 9: Continuation of solutions

(Meyer-Hall-Offin) Show that the scaling used in Section 9.4 of Meyer-Hall-Offin to obtain Hills orbits for the restricted problem works for Hills lunar problem (see previous problem) also. Why does not the scaling for comets work?

Solución

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Prove Lemma 9.7.1 in Meyer-Hall-Offin. Verify that formula (9.11) is the condition for an orthogonal crossing of the line of syzygy in Delaunay elements.

Solución

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### 4.7 Chapter 10: Normal forms

### 4.8 Chapter 13: Stability and KAM Theory

(Meyer-Hall-Offin) Using Poincaré elements show that the continuation of the circular orbits established in Section 6.2 (Poincar orbits) are of twist type and hence stable.

Solución

# 5 Apendix

## 5.1 Complete examples

### 5.1.1 Harmonic oscillator

### 5.1.2 The Pendulum

This is a case of a one dEGREE OF FREEDOM of second order, bla bla bla bla

## 5.2 Needed resoults and definitions

### 5.2.1 Linear Algebra

matriz ortogonal

no singular

skew-simmetric

DEVOLVER A SU SITIO LAS FOTOS TAMAÑO CARNET

### 5.2.2 Calculus

teorema punto fijo de bla bla bla bla

**Chain Rule**: bla bla bla bla

**Gradient**: The gradient of a scalar function ($\mathbb{R}^n \to \mathbb{R}$) $f(x_1, ..., x_n)$ denoted by $\nabla f$ or $\vec{\nabla} f$ denotes the vector differential operator. The gradient of $f$ is defined as the unique vector field whose dot product with any unitvector $v$ at each point $x$ is the directional derivative off along $v$. That is,

$$(\nabla f(x)) \cdot v = D_v f(x)$$

- *Cartesian coordinates*: Lets focus in $\mathbb{R}^3$, where $i, j, k$ are the standard unit vectors in the directions of axis $x, y, z$.

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

- *Cylindrical coordinates*: Lets focus in $\mathbb{R}^3$, where $\rho$ is the axial distance, $\varphi$ is the azimuthal or azimuth angle and $z$ is the the axial coordinate and $e_\rho, e_\varphi, e_z$ are the unit vectors pointing along the coordinate directions.

$$\nabla f(\rho,\varphi,z) = \frac{\partial f}{\partial \rho} e_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \varphi} e_\varphi + \frac{\partial f}{\partial z} = \left( \frac{\partial f}{\partial \rho}, \frac{1}{\rho} \frac{\partial f}{\partial \varphi}, \frac{\partial f}{\partial z} \right)$$

- *Spherical coordinates*: Lets focus in $\mathbb{R}^3$, where $r$ is the radial distance, $\varphi$ is the azimuthal angle and $\theta$ is the polar angle, and $e_r, e_\varphi, e_\theta$ are local unit vectors pointing in the coordinate directions.

$$\nabla f(r,\theta,\varphi) = \frac{\partial f}{\partial r} e_r + \frac{1}{r}\frac{\partial f}{\partial \theta} e_\theta + \frac{1}{r\sin(\theta)}\frac{\partial f}{\partial \varphi} e_\varphi = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{1}{r\sin(\theta)}\frac{\partial f}{\partial \varphi}\right)$$

**Laplace Operator**:

$$\Delta f = \nabla^2 f = \nabla \cdot \nabla f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

***proof:***

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***reason why:***

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#### 5.2.3 Geometry

### 5.3 Funcional Analysis

#### 5.3.1 Differential forms

#### 5.3.2 Measure Theory

#### 5.3.3 Ordinary Differential Equations

**Peano existence theorem**: